The asymptotic Borel map in ultraholomorphic classes

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Introduction

Theorem (E. Borel, 1895; Peano, 1884)

For every $(a_p)_{p\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ there is $f \in C^\infty(\mathbb{R})$ such that $f^{(p)}(0) = a_p$ for all $p \in \mathbb{N}$.

- Let Σ be the Riemann surface of the logarithm. For $\gamma > 0$ we set

$$S_\gamma = \{z \in \Sigma \,|\, |\operatorname{Arg} z| < \frac{\pi\gamma}{2}\}.$$

Theorem (Ritt, 1916)

For every $(a_p)_{p\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ there is $f \in \mathcal{O}(S_\gamma)$ such that $f(z) \sim \sum_{p=0}^{\infty} a_p z^p$ as $z \to 0$ in S_γ, i.e.

$$\limsup_{z\to 0, z\in S_\gamma} |z^{-p}(f(z) - \sum_{q=0}^{p-1} a_q z^q)| < \infty, \qquad \forall p \in \mathbb{Z}_+.$$

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Weight sequences

- Let $M = (M_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. We set $m_p = M_p/M_{p-1}$, $p \in \mathbb{Z}_+$.
- We introduce the following conditions on M:

  (lc) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$.

  (dc) $M_{p+1} \leq CH^p M_p$, $p \in \mathbb{N}$.

  (mg) $M_{p+q} \leq CH^{p+q} M_p M_q$, $p, q \in \mathbb{N}$.

  (nq) $\displaystyle\sum_{p=1}^{\infty} \frac{1}{pm_p} < \infty$.

  (snq) $\displaystyle\sum_{q=p}^{\infty} \frac{1}{qm_q} \leq \frac{C}{m_p}$, $p \in \mathbb{Z}_+$.

- The Gevrey sequences $p!^\alpha$, $\alpha > 0$, satisfy all the above conditions.
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The condition (β_2)

- We introduce the following condition on M:

$$(\beta_2)\ \ \forall \varepsilon > 0\, \exists n \geq 2 \,:\, \limsup_{p\to\infty} \left(\frac{M_{np}}{M_p}\right)^{\frac{1}{p(n-1)}} \frac{1}{m_{np}} \leq \varepsilon.$$

- (β_2) expresses that the sequence $(m_p)_{p\in\mathbb{N}}$ grows fast.
- The condition

$$\lim_{p\to\infty} \frac{m_{np}}{m_p} = \infty, \text{ for some } n \in \mathbb{Z}_+,$$

  implies (β_2). The converse holds true if M satisfies some mild regularity condition (there is $n \in \mathbb{Z}_+$ such that the set of finite limit points of $\{m_{n^l}/m_{n^{l-1}} \,|\, l \in \mathbb{N}\}$ is bounded).
- The Gevrey sequences $p!^\alpha$ do not satisfy (β_2). More generally, $(mg) \Rightarrow \neg(\beta_2)$.
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- We define $\mathcal{E}^{(M)}(\mathbb{R})$ ($\mathcal{E}^{\{M\}}(\mathbb{R})$) as the space consisting of all $f \in C^\infty(\mathbb{R})$ such that for all $h > 0$ (for some $h > 0$)

$$\sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|f^{(p)}(x)|}{h^p p! M_p} < \infty.$$

- We define $\Lambda^{(M)}$ ($\Lambda^{\{M\}}$) as the space consisting of all $(c_p)_{p \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that for all $h > 0$ (for some $h > 0$)

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- The Borel map

$$\mathcal{B}^{[M]} : \mathcal{E}^{[M]}(\mathbb{R}) \to \Lambda^{[M]}, \ f \mapsto (f^{(p)}(0))_{p \in \mathbb{N}}$$

is well-defined and continuous.

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Theorem (Petzsche, 1988)

Let M satisfy (lc) and (nq). Then,

- *FSAE:*
  - (i) $\mathcal{B}^{(M)} : \mathcal{E}^{(M)}(\mathbb{R}) \to \Lambda^{(M)}$ *is surjective.*
  - (ii) $\mathcal{B}^{(M)} : \mathcal{E}^{(M)}(\mathbb{R}) \to \Lambda^{(M)}$ *admits a continuous linear right inverse.*
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- $\mathcal{B}^{\{M\}} : \mathcal{E}^{\{M\}}(\mathbb{R}) \to \Lambda^{\{M\}}$ *is surjective iff M satisfies (snq).*
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- Elementary methods: Ingenious use of Taylor's theorem and ideas from Hörmander's real analysis proof of the Denjoy-Carleman theorem.

The Borel problem in $\mathcal{E}^{[M]}(\mathbb{R})$

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Uniform ultraholomorphic classes

- Let $\gamma > 0$. We define $\mathcal{A}^{(M)}(S_\gamma)$ ($\mathcal{A}^{\{M\}}(S_\gamma)$) as the space consisting of all $f \in \mathcal{O}(S_\gamma)$ such that for all $h > 0$ (for some $h > 0$)

$$\sup_{p\in\mathbb{N}} \sup_{z\in S_\gamma} \frac{|f^{(p)}(z)|}{h^p p! M_p} < \infty.$$

- For $f \in \mathcal{A}^{[M]}(S_\gamma)$ we may define

$$f^{(p)}(0) := \lim_{z\to 0, z\in S_\gamma} f^{(p)}(z) \in \mathbb{C}, \qquad p \in \mathbb{N}.$$

Then, for all $h > 0$ (for some $h > 0$)

$$\sup_{p\in\mathbb{N}} \sup_{z\in S_\gamma} \frac{1}{h^p M_p} |z^{-p}(f(z) - \sum_{q=0}^{p-1} \frac{f^{(q)}(0)}{q!} z^q)| < \infty.$$

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The asymptotic Borel map

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$$\mathcal{B}^{[M]}_{\gamma} : \mathcal{A}^{[M]}(S_{\gamma}) \to \Lambda^{[M]}, f \mapsto (f^{(p)}(0))_{p\in\mathbb{N}}$$

is well-defined and continuous.

Main question

Characterize the surjectivity and the existence of a continuous linear right inverse of $\mathcal{B}^{[M]}_{\gamma}$ in terms of M and γ.

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Characterize the surjectivity and the existence of a continuous linear right inverse of $\mathcal{B}_{\gamma}^{[M]}$ in terms of M and γ.

The Borel-Ritt problem in $\mathcal{A}^{\{p!^\alpha\}}(S_\gamma)$

Theorem (Ramis, 1978)

$\mathcal{B}_\gamma^{\{p!^\alpha\}} : \mathcal{A}^{\{p!^\alpha\}}(S_\gamma) \to \Lambda^{\{p!^\alpha\}}$ *is surjective iff* $\gamma < \alpha$.

- Explicit construction using a truncated Laplace transform with respect to the kernel $e^{-1/z^{1/\alpha}}$.
- Let $\gamma < \alpha$. Then,

$$C^{-1} e^{-\kappa^{-1}/|z|^{1/\alpha}} \leq |e^{-1/z^{1/\alpha}}| \leq C e^{-\kappa/|z|^{1/\alpha}}, \qquad z \in S_\gamma.$$

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The growth index $\gamma(M)$

- For $\beta > 0$ we introduce the following condition on M:

$$(\gamma_\beta) \quad \sum_{q=p}^{\infty} \frac{1}{m_q^{1/\beta}} \leq \frac{Cp}{m_p^{1/\beta}}, \; p \in \mathbb{Z}_+.$$

- We define

$$\gamma(M) := \sup\{\beta > 0 \,|\, M \text{ satisfies } (\gamma_\beta)\}.$$

- If M satisfies (lc), then M satisfies (snq) if and only if $\gamma(M) > 0$.
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The Borel-Ritt problem in $\mathcal{A}^{[M]}(S_\gamma)$: Sufficient conditions

Theorem (Thilliez, 2003)

Let M satisfy (lc), (mg) and (snq). If $0 < \gamma < \gamma(M)$, then $\mathcal{B}_\gamma^{[M]} : \mathcal{A}^{[M]}(S_\gamma) \to \Lambda^{[M]}$ is surjective.

- Roumieu case: (1) Reduction to Whitney type extension result for ultradifferentiable functions (2) Construction of $F \in \mathcal{O}(S_\gamma)$ such that

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- Reduction to the Borel problem in Denjoy-Carleman classes.
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The Borel-Ritt problem in $\mathcal{A}^{[M]}(S_1)$

- $S_1 = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

Theorem (D., 2019)

Let M satisfy (lc), (dc) and (snq). Then,

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The Borel-Ritt problem in $\mathcal{A}^{[M]}(S_\gamma)$: Sufficient conditions

Theorem (Jiménez-Garrido, Sanz, Schindl, 2020)

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Let M satisfy (lc), (dc) and (snq). Let $0 < \gamma < \gamma(M)$. Then,

- $\mathcal{B}_\gamma^{(M)} : \mathcal{A}^{(M)}(S_\gamma) \to \Lambda^{(M)}$ *admits a continuous linear right inverse.*
- $\mathcal{B}_\gamma^{\{M\}} : \mathcal{A}^{\{M\}}(S_\gamma) \to \Lambda^{\{M\}}$ *is surjective.*
- *If M satisfies (β_2), $\mathcal{B}_\gamma^{\{M\}} : \mathcal{A}^{\{M\}}(S_\gamma) \to \Lambda^{\{M\}}$ admits a continuous linear right inverse.*

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The Borel-Ritt problem in $\mathcal{A}^{[M]}(S_\gamma)$: Necessary conditions

Theorem (Jiménez-Garrido, Sanz, Schindl, 2018)

Let M satisfy (lc), (mg) and (snq). If $0 < \gamma \in \mathbb{Q}$ and $\mathcal{B}_\gamma^{[M]} : \mathcal{A}^{[M]}(S_\gamma) \to \Lambda^{[M]}$ is surjective, then $\gamma < \gamma(M)$.

- Schmets and Valdivia (2000): $\gamma \in \mathbb{N}$. Reduction to the Borel problem in Denjoy-Carleman classes.
- Refinement of the method of Schmets and Valdivia.

Open problem

Show that

$$\mathcal{B}_\gamma^{[M]} : \mathcal{A}^{[M]}(S_\gamma) \to \Lambda^{[M]} \text{ is surjective } \Rightarrow \ \gamma < \gamma(M)$$

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Non-uniform ultraholomorphic classes and the asymptotic Borel map

- Let $\gamma > 0$. We define

$$\widetilde{\mathcal{A}}^{[M]}(S_\gamma) := \bigcap_{\lambda<\gamma} \mathcal{A}^{[M]}(S_\lambda).$$

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$$\widetilde{\mathcal{B}}_\gamma^{[M]} : \widetilde{\mathcal{A}}^{[M]}(S_\gamma) \to \Lambda^{[M]}, f \mapsto (f^{(p)}(0))_{p\in\mathbb{N}}$$

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The Borel-Ritt problem in $\widetilde{\mathcal{A}}^{\{p!^\alpha\}}(S_\gamma)$

Theorem (Ramis, 1978)

$\widetilde{\mathcal{B}}_\gamma^{\{p!^\alpha\}} : \widetilde{\mathcal{A}}^{\{p!^\alpha\}}(S_\gamma) \to \Lambda^{\{p!^\alpha\}}$ *is surjective iff* $\gamma \leq \alpha$.

- Explicit construction using a truncated Laplace transform with respect to the kernel $e^{-1/z^{1/\alpha}}$.
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Theorem (D., 2020)

Let M satisfy (lc), (mg) and (snq). Let $\gamma > 0$. FSAE:

(i) $\widetilde{\mathcal{B}}_\gamma^{(M)} : \widetilde{\mathcal{A}}^{(M)}(S_\gamma) \to \Lambda^{(M)}$ *is surjective.*

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Main ideas of the proof

- Surjectivity: Mittag-Leffler procedure (recall that $\mathcal{B}_{\lambda}^{(M)} : \mathcal{A}^{(M)}(S_{\gamma}) \to \Lambda^{(M)}$ is surjective for each $\gamma < \gamma(M)$). Show that the inclusion mapping

$$\ker \mathcal{B}_{\gamma_2}^{(M)} \to \ker \mathcal{B}_{\gamma_1}^{(M)}, \qquad \gamma_1 < \gamma_2 < \gamma(M),$$

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