On the Borel, the Borel-Ritt and the Stieltjes moment problem

Andreas Debrouwere

Ghent University

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Outline of the talk

1. Introduction.
2. The Borel problem in spaces of ultradifferentiable functions.
3. The Borel-Ritt problem in spaces of ultraholomorphic functions.
4. The Stieltjes moment problem in Gelfand-Shilov spaces.

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The Borel problem

Theorem (E. Borel, 1895; Peano, 1894)

For every $(a_p)_{p\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ *there is* $\varphi \in C^\infty(\mathbb{R})$ *such that* $\varphi^{(p)}(0) = a_p$ *for all* $p \in \mathbb{N}$.

The Borel-Ritt problem

- Let Σ be the Riemann surface of the logarithm. Set

$$S_\lambda = \{z \in \Sigma \,|\, |\operatorname{Arg} z| < \frac{\pi\lambda}{2}\}, \qquad \lambda > 0.$$

- $S_1 = \{z \in \mathbb{C} \,|\, \operatorname{Re} z > 0\}$.
- $\mathcal{A}(S_\lambda) := \{\varphi \in \mathcal{O}(S_\lambda) \,|\, \sup_{z \in S_\lambda} |\varphi^{(p)}(z)| < \infty, \forall p \in \mathbb{N}\}$.
- For $\varphi \in \mathcal{A}(S_\lambda)$ we may define

$$\varphi^{(p)}(0) := \lim_{z \to 0; z \in S_\lambda} \varphi^{(p)}(z), \qquad \forall p \in \mathbb{N}.$$

Theorem (Ritt, 1916)

Let $\lambda > 0$. For every $(a_p)_{p \in \mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ there is $\varphi \in \mathcal{A}(S_\lambda)$ such that $\varphi^{(p)}(0) = a_p$ for all $p \in \mathbb{N}$.

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The Stieltjes moment problem for positive measures

Theorem (Stieltjes, 1894)

Let $(a_p)_{p\in\mathbb{N}} \subset \mathbb{R}_+^{\mathbb{N}}$. *There is a positive measure* μ *such that*

$$\int_0^\infty x^p \mathrm{d}\mu(x) = a_p, \qquad \forall p \in \mathbb{N},$$

if and only if

$$\det \Delta_p > 0 \qquad \textit{and} \qquad \det \Delta_p^{(1)} > 0, \qquad \forall p \in \mathbb{N}.$$

- Stieltjes integral, Stieltjes transform, ...
- M. Riesz (1922): Extension of positive linear functionals.

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The unrestricted Stieltjes moment problem

Theorem (Boas, Pólya, independently, 1939)

For every $(a_p)_{p\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ *there is a complex measure* μ *such that*

$$\int_0^{\infty} x^p \mathrm{d}\mu(x) = a_p, \qquad \forall p \in \mathbb{N}.$$

Theorem (A.J. Durán,1989)

For every $(a_p)_{p\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$ *there is* $\varphi \in \mathcal{S}(0,\infty)$ *such that*

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- A.L. Durán, Estrada (1994): Reduction to Borel-Ritt problem on the right half-plane via Laplace transform.

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Spaces of ultradifferentiable functions

- Let $M = (M_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers.
- For $h > 0$ and $R > 0$ we define

$$\mathcal{E}^{M,h}([-R,R]) := \{\varphi \in C^\infty([-R,R]) \mid \sup_{p\in\mathbb{N}} \sup_{x\in[-R,R]} \frac{|\varphi^{(p)}(x)|}{h^p M_p} < \infty\}.$$

- Set

$$\mathcal{E}^{(M)}(\mathbb{R}) := \bigcap_{R>0} \bigcap_{h>0} \mathcal{E}^{M,h}([-R,R])$$

and

$$\mathcal{E}^{\{M\}}(\mathbb{R}) := \bigcap_{R>0} \bigcup_{h>0} \mathcal{E}^{M,h}([-R,R]).$$

- $\mathcal{E}^{\{(p!)_p\}}(\mathbb{R})$ is the space of real analytic functions on $\mathbb{R}$.
- $\mathcal{E}^{((p!)_p)}(\mathbb{R})$ is the space of entire functions on $\mathbb{R}$.
- The spaces $\mathcal{E}^{\{(p!^\alpha)_p\}}(\mathbb{R})$, $\alpha > 1$, were introduced by Gevrey (around 1910) to analyze the regularity of solutions to PDE's.

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Conditions on weight sequences

- Consider the following conditions on M:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $\forall p \geq 1$.

$(M.2)'$ $M_{p+1} \leq AH^p M_p$, $\forall p \in \mathbb{N}$, for some $A, H > 0$.

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$(M.3)'$ $\displaystyle\sum_{p=1}^{\infty} \frac{1}{M_p/M_{p-1}} < \infty$.

- The sequence $(p!)_p$ satisfies the above conditions except for $(M.3)'$.
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- For simplicity, we shall only consider the Beurling spaces $\mathcal{E}^{(M)}(\mathbb{R})$. All results have a counterpart for the Roumieu spaces $\mathcal{E}^{\{M\}}(\mathbb{R})$.

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The Borel problem in $\mathcal{E}^{(M)}(\mathbb{R})$

- We define the sequence space

$$\Lambda^{(M)} := \{(c_p)_{p\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{p\in\mathbb{N}} \frac{|c_p|}{h^p M_p} < \infty \text{ for all } h > 0\}.$$

- The Borel mapping

$$\mathcal{B}^{(M)} : \mathcal{E}^{(M)}(\mathbb{R}) \to \Lambda^{(M)} : \varphi \to (\varphi^{(p)}(0))_{p\in\mathbb{N}}$$

is well-defined and continuous.

Problem

Characterize the surjectivity of $\mathcal{B}^{(M)}$ in terms of M.

- Denjoy-Carleman theorem: $\mathcal{B}^{(M)}$ is injective if and only if M is quasianalytic (= does not satisfy $(M.3)'$).
- If M is quasianalytic and non-entire ($\mathcal{E}^{(p!)}(\mathbb{R}) \subsetneq \mathcal{E}^{(M)}(\mathbb{R})$), then $\mathcal{B}^{(M)}$ is never surjective (Roumieu case: Carleman (1923)).

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$$\Lambda^{(M)} := \{(c_p)_{p\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{p\in\mathbb{N}} \frac{|c_p|}{h^p M_p} < \infty \text{ for all } h > 0\}.$$

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Characterize the surjectivity of $\mathcal{B}^{(M)}$ in terms of M.

- Denjoy-Carleman theorem: $\mathcal{B}^{(M)}$ is injective if and only if M is quasianalytic (= does not satisfy $(M.3)'$).
- If M is quasianalytic and non-entire ($\mathcal{E}^{(p!)}(\mathbb{R}) \subsetneq \mathcal{E}^{(M)}(\mathbb{R})$), then $\mathcal{B}^{(M)}$ is never surjective (Roumieu case: Carleman (1923)).

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- Consider the strong non-quasianalyticity condition

$$(\gamma_1)\ \sum_{q=p}^{\infty} \frac{1}{M_p/M_{p-1}} \leq C \frac{p}{M_p/M_{p-1}}, \forall p \geq 1, \text{ for some } C > 0.$$

- The Gevrey sequences $(p!^{\alpha})_p$, $\alpha > 1$, and the q-Gevrey sequences $(q^{p^2})_p$, $q > 1$, both satisfy (γ_1).

Theorem (Petzsche, 1988)

Let M satisfy $(M.1)$ and $(M.3)'$. FSAE:

(i) *M satisfies (γ_1).*

(ii) *$\mathcal{B}^{(M)}$ is surjective.*

(iii) *$\mathcal{B}^{(M)}$ admits a continuous linear right inverse, that is, there is a continuous linear mapping $R : \Lambda^{(M)} \to \mathcal{E}^{(M)}(\mathbb{R})$ such that $\mathcal{B}^{(M)} \circ R = \mathrm{id}$.*

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- Technical masterpiece.
- Sufficiency: Explicit construction of $\chi_j \in \mathcal{E}^{(M)}(\mathbb{R})$, $j \in \mathbb{N}$, such that $\chi_j^{(p)}(0) = \delta_{j,p}$ and $R(c) = \sum_{j=0}^{\infty} c_j \chi_j \in \mathcal{E}^{(M)}(\mathbb{R})$ for all $c = (c_j)_j \in \Lambda^{(M)}$.
- Necessity: Ingenuous use of Taylor's formula combined with ideas from Hörmander's real analysis proof of the Denjoy-Carleman theorem.
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Spaces of ultraholomorphic functions and the asymptotic Borel mapping

- Let $\lambda > 0$. We define

$$\mathcal{A}^{(M)}(S_\lambda) := \{\varphi \in \mathcal{O}(S_\lambda) \mid \sup_{p\in\mathbb{N}} \sup_{z\in S_\lambda} \frac{|\varphi^{(p)}(z)|}{h^p M_p} < \infty \text{ for all } h > 0\}.$$

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The Borel-Ritt problem in $\mathcal{A}^{(M)}(S_\lambda)$ (1)

Theorem (Thilliez, 2003)

Let $\lambda > 0$ and let M satisfy $(M.1)$, $(M.2)$ and $(M.3)$. If M satisfies $(\gamma_{\lambda+1})$, then $\mathcal{B}_\lambda^{(M)}$ is surjective.

- Based upon Whitney type extension results for ultradifferentiable functions.
- Ramis (1978): For the Gevrey sequences $(p!^\alpha)_p$, $\alpha > 1$, by using the (truncated) Laplace transform (Roumieu case).
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The Borel-Ritt problem in $\mathcal{A}^{(M)}(S_\lambda)$ (2)

Theorem (Schmets, Valdivia, 2000)

Let $n \in \mathbb{N}$ and let M satisfy $(M.1)$, $(M.3)'$ and (γ_{n+1}). For $\lambda < n$, $\mathcal{B}_\lambda^{(M)}$ admits a continuous linear right inverse.

- Reduction to Petzsche's result on the Borel problem via Laplace transform.
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Theorem (Jiménez-Garrido, Sanz, Schindl, 2018)

Let $\lambda > 0$ and let M satisfy $(M.1)$, $(M.2)$ and $(M.3)$. If $\lambda \in \mathbb{Q}$ and $\mathcal{B}_\lambda^{(M)}$ is surjective, then M satisfies $(\gamma_{\lambda+1})$.

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without assuming that $\lambda \in \mathbb{Q}$.

- Show

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- Show the existence of a continuous linear right inverse of $\mathcal{B}_\lambda^{(M)} : \mathcal{A}^{(M)}(S_\lambda) \to \Lambda^{(M)}$; even open for the Gevrey sequences. Possible approach: apply (DN)-(Ω) splitting theorem. Does

$$\ker \mathcal{B}_\lambda^{(M)} = \{\varphi \in \mathcal{A}^{(M)}(S_\lambda) \mid \varphi^{(p)}(0) = 0 \text{ for all } p \in \mathbb{N}\}$$

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Gelfand-Shilov spaces and the Stieltjes moment mapping

- We define $\mathcal{S}^{(M)}(\mathbb{R})$ as the space of all $\varphi \in C^\infty(\mathbb{R})$ such that

$$\max_{k \leq n} \sup_{p \in \mathbb{N}} \frac{|x^p \varphi^{(k)}(x)|}{h^p M_p} < \infty \qquad \forall n \in \mathbb{N}, h > 0.$$

- Set $\mathcal{S}^{(M)}(0,\infty) := \{\varphi \in \mathcal{S}^{(M)}(\mathbb{R}) \mid \operatorname{supp} \varphi \subseteq [0,\infty)\}$.
- If M satisfies $(M.2)'$, the Stieltjes moment mapping

$$\mathcal{M}^{(M)} : \mathcal{S}^{(M)}(0,\infty) \to \Lambda^{(M)} : \varphi \to \left(\int_0^\infty x^p \varphi(x) \mathrm{d}x \right)_{p \in \mathbb{N}}$$

is well-defined and continuous.

Problem

Characterize the surjectivity of $\mathcal{M}^{(M)}$ in terms of M.

Gelfand-Shilov spaces and the Stieltjes moment mapping

- We define $\mathcal{S}^{(M)}(\mathbb{R})$ as the space of all $\varphi \in C^\infty(\mathbb{R})$ such that

$$\max_{k \leq n} \sup_{p \in \mathbb{N}} \frac{|x^p \varphi^{(k)}(x)|}{h^p M_p} < \infty \qquad \forall n \in \mathbb{N}, h > 0.$$

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Characterize the surjectivity of $\mathcal{M}^{(M)}$ in terms of M.

Stieltjes moment problem in $\mathcal{S}^{(M)}(0,\infty)$ (1)

Theorem (cf. Lastra and Sanz, 2009)

Let M satisfy $(M.1)$, $(M.2)'$ and $(M.3)'$. Then, $\mathcal{M}^{(M)}$ is surjective (admits a continuous linear right inverse) if and only if $\mathcal{B}_1^{(M)}$ does so.

- Extension of the method of Durán and Estrada (Laplace transform) to Gelfand-Shilov spaces.

Corollary

Let M satisfy $(M.1)$, $(M.2)'$ and $(M.3)'$.

- *If in addition M satisfies $(M.2)$ and $(M.3)$, then $\mathcal{M}^{(M)} : \mathcal{S}^{(M)}(0,\infty) \to \Lambda^{(M)}$ is surjective if and only if M satisfies (γ_2).*
- *If M satisfies (γ_3), then $\mathcal{M}^{(M)} : \mathcal{S}^{(M)}(0,\infty) \to \Lambda^{(M)}$ admits a continuous linear right inverse.*

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Stieltjes moment problem in $\mathcal{S}^{(M)}(0,\infty)$ (2)

Theorem (D., 2018)

Let M satisfy $(M.1)$, $(M.2)'$ and $(M.3)'$. FSAE:

(i) M *satisfies* (γ_2).

(ii) $\mathcal{M}^{(M)} : \mathcal{S}^{(M)}(0,\infty) \to \Lambda^{(M)}$ *is surjective.*

(iii) $\mathcal{M}^{(M)} : \mathcal{S}^{(M)}(0,\infty) \to \Lambda^{(M)}$ *admits a continuous linear right inverse.*

- Reduction to Petzsche's result on the Borel problem via Fourier transform and abstract functional analysis.
- By the result of Lastra and Sanz this also completely settles the Borel-Ritt problem on the right half-plane (under $(M.1)$, $(M.2)'$ and $(M.3)'$).

Stieltjes moment problem in $\mathcal{S}^{(M)}(0,\infty)$ (2)

Theorem (D., 2018)

Let M satisfy $(M.1)$, $(M.2)'$ and $(M.3)'$. FSAE:

(i) M satisfies (γ_2).

(ii) $\mathcal{M}^{(M)} : \mathcal{S}^{(M)}(0,\infty) \to \Lambda^{(M)}$ is surjective.

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