Topological properties of convolutor spaces via the short-time Fourier transform

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(Joint work with Jasson Vindas)

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The space of integrable distributions (1)

- The space $\mathcal{B}$ consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\partial^\alpha \varphi\|_{L^\infty} < \infty, \qquad \forall \alpha \in \mathbb{N}^d.$$

- The space $\mathcal{B}$ is a Fréchet space.
- The space $\dot{\mathcal{B}}$ is given by the closure of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{B}$, i.e. it consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\lim_{|x|\to\infty} \partial^\alpha \varphi(x) = 0, \qquad \forall \alpha \in \mathbb{N}^d.$$

- The space $\dot{\mathcal{B}}$ is a Fréchet space.
- The space $\mathcal{D}'_{L^1}$ of integrable distributions is given by the topological dual of $\dot{\mathcal{B}}$.

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Theorem (Schwartz, 1950)

Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then, $f \in \mathcal{D}'_{L^1}$ if and only if $f * \varphi \in L^1$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

- Two natural topologies on $\mathcal{D}'_{L^1}$:
  1. The strong topology $b(\mathcal{D}'_{L^1}, \dot{\mathcal{B}})$.
  2. The initial topology *op* w.r.t. the mapping

$$\mathcal{D}'_{L^1} \to L_b(\mathcal{D}(\mathbb{R}^d), L^1) : f \to (\varphi \to f * \varphi).$$

Theorem (Schwartz, 1950)

The spaces $\mathcal{D}'_{L^1,b}$ and $\mathcal{D}'_{L^1,op}$ have the same bounded sets and null sequences.

- Do the topologies b and op coincide on $\mathcal{D}'_{L^1}$?

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The space of rapidly decreasing distributions (1)

- The space $\mathcal{O}_C$ consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that there is $N \in \mathbb{N}$ for which
$$\sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{(1+|x|)^N} < \infty, \qquad \forall \alpha \in \mathbb{N}^d.$$
- $\mathcal{O}_C$ is an (LF)-space (countable inductive limit of Fréchet spaces).
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Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then, $f \in \mathcal{O}'_C$ if and only if, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$(1 + |\cdot|)^N (f * \varphi) \in L^1, \qquad \forall N \in \mathbb{N}.$$

- $\mathcal{O}'_C$ is sometimes called the space of convolutors.
- Define the topologies b and op on $\mathcal{O}'_C$ as before.

Theorem (Grothendieck, 1955)

The space $\mathcal{O}'_{C,op}$ is complete, semi-reflexive, and bornological. Consequently, $\mathcal{O}'_{C,b} = \mathcal{O}'_{C,op}$ and the (LF)-space $\mathcal{O}_C$ is complete.

- He showed that $\mathcal{O}'_{C,op}$ is isomorphic to a complemented subspace of $s \widehat{\otimes} s'$ and proved that $s \widehat{\otimes} s'$ is bornological. Moreover, he showed that $(\mathcal{O}'_{C,op})'_b = \mathcal{O}_C$.

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Goals

- Show the full topological identity $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$ and extend it to weighted $\mathcal{D}'_{L^1}$ spaces (unified approach for $\mathcal{D}'_{L^1}$ and $\mathcal{O}'_C$).
- To this end, we study the structural and topological properties of a general class of weighted L^1 convolutor spaces.
- Our arguments are based on the mapping properties of the short-time Fourier transform.

C. Bargetz, N. Ortner, *Characterization of L. Schwartz' convolutor and multiplier spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$ by the short-time Fourier transform*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **108** (2014), 833–847.

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The short-time Fourier transform (STFT)

- $T_x f := f(\cdot - x)$ and $M_\xi f := e^{2\pi i \xi t} f(t)$ for $x, \xi \in \mathbb{R}^d$.
- The STFT of $f \in L^2(\mathbb{R}^d)$ w.r.t. a window function $\psi \in L^2(\mathbb{R}^d) \setminus \{0\}$ is defined as

$$V_\psi f(x,\xi) := (f, M_\xi T_x \psi)_{L^2} = \int_{\mathbb{R}^d} f(t)\overline{\psi(t-x)} e^{-2\pi i \xi t} \mathrm{d}t, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

- The mapping $V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is continuous.
- The adjoint of V_ψ is given by the weak integral

$$V_\psi^* F = \int\int_{\mathbb{R}^{2d}} F(x,\xi) M_\xi T_x \psi \mathrm{d}x\mathrm{d}\xi, \qquad F \in L^2(\mathbb{R}^{2d}).$$

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$$\frac{1}{\|\psi\|_{L^2}^2} V_\psi^* \circ V_\psi = \mathrm{id}_{L^2(\mathbb{R}^d)}.$$

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General strategy

- Suppose that $E_1, E_2 \subset \mathcal{D}'(\mathbb{R}^d)$ (with continuous inclusion) and one wants to show that $E_1 = E_2$.
- Find $F \subset \mathcal{D}'(\mathbb{R}^d_x)\widehat{\otimes}\mathcal{S}'(\mathbb{R}^d_\xi)$ such that

$$V_\psi : E_i \to F \qquad \text{and} \qquad V_\psi^* : F \to E_i$$

  are well-defined mappings for $i = 1, 2$.
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The equality $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$

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- $C_{\mathrm{pol}}(\mathbb{R}^d)$ is an (LB)-space.

Theorem

Let $\psi \in \mathcal{D}(\mathbb{R}^d)\backslash\{0\}$ and let $\tau = b$ or op. Then,

$$V_\psi : \mathcal{D}'_{L^1,\tau} \to L^1(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon C_{\mathrm{pol}}(\mathbb{R}^d_\xi)$$

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Weighted inductive limits of smooth functions

- Let $\mathcal{W} = (w_N)_N$ be an increasing sequence of continuous functions.
- Define $\mathcal{B}_{w_N}$ as the space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{w_N(x)} < \infty, \qquad \forall \alpha \in \mathbb{N}^d.$$

- $\mathcal{B}_{w_N}$ is a Fréchet space.
- The space $\dot{\mathcal{B}}_{w_N}$ is defined as the closure of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{B}_{w_N}$.
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Completeness of $\mathcal{B}_{\mathcal{W}}$ and $\dot{\mathcal{B}}_{\mathcal{W}}$

- Assume that $\mathcal{W} = (w_N)_N$ satisfies

$$\forall N \, \exists M \geq N \, \exists C > 0 \, \forall x \in \mathbb{R}^d : \sup_{y \in [-1,1]^d} w_N(x+y) \leq C w_M(x).$$

Theorem (D., Vindas, 2018)

TFAE:

- *$\mathcal{W}$ satisfies the condition (Ω), i.e.*

$$\forall N \, \exists M \geq N \, \forall K \geq M \, \exists \theta \in (0,1) \, \exists C > 0 \, \forall x \in \mathbb{R}^d :$$
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- *$\dot{\mathcal{B}}_{\mathcal{W}}$ is complete.*
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- An (LF)-space is complete if and only if it is boundedly stable and satisfies (wQ) (Vogt and Wengenroth).

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Weighted L^1 convolutor spaces

- Define $L^1_{\mathcal{W}}$ as the space consisting of all measurable functions f on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} f(x) w_N(x) \mathrm{d}x < \infty, \qquad \forall N \in \mathbb{N}.$$

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$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) := \{ f \in \mathcal{D}'(\mathbb{R}^d) \,|\, f * \varphi \in L^1_{\mathcal{W}} \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d) \}$$

and endow it with the initial topology w.r.t. the mapping

$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) \to L_b(\mathcal{D}(\mathbb{R}^d), L^1_{\mathcal{W}}) : f \to (\varphi \to f * \varphi).$$

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The main result

Theorem (D., Vindas, 2018)

The equality $(\dot{\mathcal{B}}_{\mathcal{W}})' = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ always holds algebraically. TFAE:

- *$\mathcal{W}$ satisfies the condition (Ω).*
- *$\dot{\mathcal{B}}_{\mathcal{W}}$ and $\mathcal{B}_{\mathcal{W}}$ are complete.*
- *$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$ is bornological.*
- *$(\dot{\mathcal{B}}_{\mathcal{W}})'_b = \mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$.*

In such a case, the bidual of $\dot{\mathcal{B}}_{\mathcal{W}}$ is topologically isomorphic to $\mathcal{B}_{\mathcal{W}}$.

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