Factorization of ultradifferentiable vectors

Andreas Debrouwere

LSU

Harmonic Analysis Seminar

Joint work with Bojan Prangoski and Jasson Vindas

Outline of the talk

1. Factorization results in analysis.
2. Ultradifferentiable functions.
3. Main result.
4. Examples: Factorization of weighted convolution algebras of smooth functions.

Factorization of L^1

Theorem (Rudin, 1957)

For every $f \in L^1(\mathbb{R})$ there are $g, h \in L^1(\mathbb{R})$ such that $f = g * h$. This means that the convolution algebra $(L^1(\mathbb{R}), *)$ factorizes as follows

$$L^1(\mathbb{R}) = L^1(\mathbb{R}) * L^1(\mathbb{R}).$$

Theorem (Cohen, 1959)

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Factorization properties

- Let $\mathcal{M}$ be a (left) module over a (non-unital) algebra $\mathcal{A}$. We set

$$\mathcal{A} \cdot \mathcal{M} = \{a \cdot m \,|\, a \in \mathcal{A},\ m \in \mathcal{M}\}.$$

- $\mathcal{M}$ is said to satisfy the weak factorization property if

$$\mathcal{M} = \mathrm{span}(\mathcal{A} \cdot \mathcal{M}).$$

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Cohen-Hewitt factorization theorem (1)

- Let $\mathcal{A}$ be a Banach algebra. A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be a bounded (left) approximate identity if

$$\sup_{n\in\mathbb{N}} \|a_n\|_{\mathcal{A}} < \infty \qquad \text{and} \qquad \lim_{n\to\infty} a_n \cdot a = a, \qquad \forall a \in \mathcal{A}.$$

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Let $\mathcal{A}$ be a Banach algebra having a bounded approximate identity. Then, $\mathcal{A}$ has the factorization property, that is, $\mathcal{A} = \mathcal{A} \cdot \mathcal{A}$.

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Cohen-Hewitt factorization theorem (2)

Theorem (Hewitt, 1964)

Let $\mathcal{A}$ be a Banach algebra having a bounded approximate identity $(a_n)_{n\in\mathbb{N}}$ and let $\mathcal{M}$ be a Banach module over $\mathcal{A}$. For every $u \in \mathcal{M}$ such that

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there are $a \in \mathcal{A}$ and $v \in \mathcal{M}$ such that $u = a \cdot v$. In particular, if (1) holds for all $u \in \mathcal{M}$, then $\mathcal{M}$ has the factorization property, that is, $\mathcal{M} = \mathcal{A} \cdot \mathcal{M}$.

- The Cohen-Hewitt factorization theorem can be generalized to Fréchet algebras having a uniformly bounded approximate identity. However, non-unital reflexive Fréchet algebras do not even have bounded approximate identities...

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Factorization in convolution algebras of smooth functions

Ehrenpreis' problem 1960

Does the convolution algebra $(\mathcal{D}(\mathbb{R}^d), *)$ has the weak/strong factorization property?

- (Rubel, Squires and Taylor, 1978) $\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$.
- (Dixmier and Malliavin, 1978) Let $d \geq 2$. $\mathcal{D}(\mathbb{R}^d) \subsetneq \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$.
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Representations of Lie groups

- Let G be a real Lie group and E be a lcHs (= locally convex Hausdorff space). A representation (π, E) of G in E is a homomorphism $\pi : G \to \mathcal{L}(E)$ such that

$$G \times E \to E \ : \ (g, e) \to \pi(g)e$$

is continuous.

- The orbit map of $e \in E$ is given by the continuous E-valued map

$$\gamma_e : G \to E : g \to \pi(g)e$$

- Th space E^∞ of smooth vectors is defined as

$$E^\infty := \{e \in E \mid \gamma_e \in C^\infty(G; E)\}.$$

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The induced action of π

- Assume that E is a Fréchet space. The representation (π, E) induces an action of the convolution algebra $(C_c^\infty(G), *)$ on E via

$$\Pi(\varphi)e := \int_G \varphi(g)\pi(g)e\,dg, \qquad \varphi \in C_c^\infty(G), e \in E.$$

- Π restricts to an action on E^∞,i.e., E^∞ is a module over $(C_c^\infty(G), *)$.

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Examples

- Let $G = (\mathbb{R}^d, +)$ and let $E = L^p$, $1 \leq p < \infty$.
- Consider the representation of $(\mathbb{R}^d, +)$ in L^p via translation, i.e.,

$$\pi(x)f = f(\cdot - x), \qquad x \in \mathbb{R}^d, f \in L^p.$$

- Then,

$$\Pi(\varphi)f = \varphi * f, \qquad \varphi \in C_c^\infty(\mathbb{R}^d), f \in L^p.$$

  and $(L^p)^\infty$ is equal to the Schwartz space $\mathcal{D}_{L^p}$.
- If $E = C_0$, then $E^\infty = \dot{\mathcal{B}}$.
- If $E = \varprojlim_N (1 + |\cdot|)^{-N} C_0$, then $E^\infty = \mathcal{S}$.

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Let G be a Lie group and let (π, E) be a representation of G in a Fréchet space E. Then, E^∞ has the weak factorization property, that is,

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- Examples: Let $X = \mathcal{D}_{L^p}$, $\dot{\mathcal{B}}$, or $\mathcal{S}$. Then,

$$X = \mathrm{span}(C_c^\infty(\mathbb{R}^d) * X)$$

- In 2011 Gimperlein, Krötz and Lienau proved, under some technical conditions on (π, E), that the space E^ω of analytic vectors satisfies the weak factorization property over an appropriate convolution algebra. Main difficulty: Absence of compactly supported analytic functions on G.

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Our goals

- We generalize the result of $G - K - L$ in the following ways for $G = (\mathbb{R}^d, +)$:
  - Allow E to be a general quasi-complete lcHs.
  - Find appropriate growth conditions on (π, E) to ensure that the strong factorization property holds.
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- In 1918 Gevrey showed that all weak solutions of the heat equation are in fact smooth and satisfy

$$\max_{x \in K} |\partial^\alpha \varphi(x)| \leq C h^{|\alpha|} (|\alpha|!)^2, \qquad \forall \alpha \in \mathbb{N}^{d+1},$$

  for all $K \Subset \mathbb{R}^{d+1}$ and some $C = C(K) > 0$, $h = h(K) > 0$. Moreover, the exponent 2 is optimal.

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Spaces of ultradifferentiable functions

- Let $(M_p)_{p\in\mathbb{N}}$ be a sequence of positive reals.
- $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ stands for the space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that for all $K \Subset \mathbb{R}^d$ there is some $h > 0$ such that

$$\sup_{\alpha\in\mathbb{N}^d} \max_{x\in K} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty. \tag{2}$$

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Conditions on M_p

- We shall impose the following conditions on M_p:

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$.

(M.2) $M_{p+q} \leq CH^p M_p M_q$, $p, q \in \mathbb{N}$, for some $C, H > 0$.

(M.5) There exist $C, q > 0$ such that M_p^q is strongly non-quasianalytic, i.e.,

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Uniform and exponentially equicontinuous representations

- Let E be a lcHs and let (π, E) be a representation of $(\mathbb{R}^d, +)$ in E.
- (π, E) is said to be uniform if for all $p \in \operatorname{csn}(E)$ there is $q \in \operatorname{csn}(E)$ such that

$$\mathbb{R}^d \times E_q \to E_p : (x, e) \to \pi(x)e$$

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Smooth vectors revisited

- Let E be a quasi-complete lcHs and let (π, E) be a representation of $(\mathbb{R}^d, +)$ in E. Denote by $\mathcal{B}(E)$ the set of absolutely convex, bounded and closed subsets of E.
- For $B \in \mathcal{B}(E)$ we define

$$E_B := \operatorname{span} B = \bigcup_{t>0} tB, \qquad q_B(e) := \inf\{t > 0 \,|\, e \in tB\}.$$

  E_B is a Banach space and continuously included in E.
- $\varphi : \mathbb{R}^d \to E$ is said to be bornologically smooth if there is $B \in \mathcal{B}(E)$ such that $\varphi \in C^\infty(\mathbb{R}^d; E_B)$.
- $E^\infty := \{e \in E \,|\, \gamma_e \text{ is bornologically smooth}\}$.
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- If E is a Fréchet space, then $E^{\{p!\}}$ coincides with the space E^ω of analytic vectors considered by G-K-L.

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Example: Weighted Beurling algebras

- For $k \geq 0$ we define the L^1_k as the Banach space of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{L^1_k} := \int_{\mathbb{R}^d} |f(x)| e^{k|x|} \, dx < \infty.$$

- Consider the representation of $(\mathbb{R}^d, +)$ in L^1_k via translation. The space $(L^1_k)^*$ of ultradifferentiable vectors of class $*$ is given by

$$\mathcal{D}^{\{M_p\}}_{L^1_k} = \bigcup_{h>0} \mathcal{D}^{M_p,h}_{L^1_k}, \qquad \mathcal{D}^{(M_p)}_{L^1_k} = \bigcap_{h>0} \mathcal{D}^{M_p,h}_{L^1_k},$$

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The induced action by π

- Let E be a quasi-complete lcHs space and let (π, E) be a uniform and exponentially equicontinuous representation of $(\mathbb{R}^d, +)$ in E.
- Let $k > \kappa$. The representation (π, E) induces an action of $(\mathcal{D}^*_{L^1_k}, *)$ on E via

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Let E be a quasi-complete lcHs space and let (π, E) be a uniform and exponentially equicontinuous representation of $(\mathbb{R}^d, +)$ in E and let $k > \kappa$. Let M_p be a weight sequence satisfying $(M.1)$, $(M.2)$, and $(M.5)$. Then, E^* has the strong factorization property, that is,

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Example: Factorization of $\mathcal{D}^*_{L^p}$ and $\dot{\mathcal{B}}^*$

- Let $E = L^p$, $1 \leq p < \infty$, or C_0.
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Gelfand-Shilov spaces (spaces of type $\mathcal{S}$)

- Let ω be a positive increasing function on $[0, \infty)$ such that $\omega(t) \to \infty$ as $t \to \infty$ and let M_p be a sequence of positive reals.
- Define $\mathcal{S}^{M_p,h}_{\omega,h}$, $h > 0$, as the space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

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- Show the strong factorization property for the space of smooth vectors of uniform exponentially equicontinuous representations. E.g.

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