Sequence space representations for Gelfand-Shilov spaces

Andreas Debrouwere

Ghent University

GF2020

September 4, 2020

Introduction

- $\mathcal{S} \cong s$ (Hermite expansions).
- More generally, Vogt (1983) obtained sequence space representations for the Fréchet spaces

$$\mathcal{K}(\eta_p) := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} \max_{n\leq p} |f^{(n)}(x)|\eta_p(x) < \infty, \forall p \in \mathbb{N}\}.$$

- Pelczinsky-Vogt decomposition method (E complemented in $s\widehat{\otimes}F$ and vice versa $\Rightarrow E \cong s\widehat{\otimes}F$).

Main problem

Obtain sequence space representations for spaces of ultradifferentiable functions with rapid decay (= Gelfand-Shilov spaces).

Introduction

- $\mathcal{S} \cong s$ (Hermite expansions).
- More generally, Vogt (1983) obtained sequence space representations for the Fréchet spaces

$$\mathcal{K}(\eta_p) := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} \max_{n\leq p} |f^{(n)}(x)|\eta_p(x) < \infty, \forall p \in \mathbb{N}\}.$$

- Pelczinsky-Vogt decomposition method (E complemented in $s\widehat{\otimes}F$ and vice versa $\Rightarrow E \cong s\widehat{\otimes}F$).

Main problem

Obtain sequence space representations for spaces of ultradifferentiable functions with rapid decay ($=$ Gelfand-Shilov spaces).

Introduction

- $\mathcal{S} \cong s$ (Hermite expansions).
- More generally, Vogt (1983) obtained sequence space representations for the Fréchet spaces

$$\mathcal{K}(\eta_p) := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} \max_{n\leq p} |f^{(n)}(x)|\eta_p(x) < \infty, \ \forall p \in \mathbb{N}\}.$$

- Pelczinsky-Vogt decomposition method (E complemented in $s\widehat{\otimes}F$ and vice versa $\Rightarrow E \cong s\widehat{\otimes}F$).

Main problem

Obtain sequence space representations for spaces of ultradifferentiable functions with rapid decay (= Gelfand-Shilov spaces).

Introduction

- $\mathcal{S} \cong s$ (Hermite expansions).
- More generally, Vogt (1983) obtained sequence space representations for the Fréchet spaces

$$\mathcal{K}(\eta_p) := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} \max_{n\leq p} |f^{(n)}(x)|\eta_p(x) < \infty, \forall p \in \mathbb{N}\}.$$

- Pelczinsky-Vogt decomposition method (E complemented in $s\widehat{\otimes}F$ and vice versa $\Rightarrow E \cong s\widehat{\otimes}F$).

Main problem

Obtain sequence space representations for spaces of ultradifferentiable functions with rapid decay (= Gelfand-Shilov spaces).

- $\mathcal{S} \cong s$ (Hermite expansions).
- More generally, Vogt (1983) obtained sequence space representations for the Fréchet spaces

$$\mathcal{K}(\eta_p) := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} \max_{n\leq p} |f^{(n)}(x)|\eta_p(x) < \infty, \forall p \in \mathbb{N}\}.$$

- Pelczinsky-Vogt decomposition method (E complemented in $s\widehat{\otimes}F$ and vice versa $\Rightarrow E \cong s\widehat{\otimes}F$).

Main problem

Obtain sequence space representations for spaces of ultradifferentiable functions with rapid decay ($=$ Gelfand-Shilov spaces).

Gelfand-Shilov spaces

- A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is called a weight function if $\omega(0) = 0$, $\log t = o(\omega(t))$ and there is $C > 0$ such that
$$\omega(2t) \leq C\omega(t) + C, \qquad \forall t \geq 0.$$

- $\omega(t) = t^{\frac{1}{\alpha}} \log(1 + t)^{\beta}$ $(\alpha > 0, \beta \in \mathbb{R})$; $\omega(t) = e^{(\log t)^{\alpha}}$ $(0 < \alpha < 1)$.
- Given two weight functions ω and η, we define $\mathcal{S}^{\omega,\lambda}_{\eta,\lambda}$, $\lambda > 0$, as the Banach space consisting of all $f \in \mathcal{S}(\mathbb{R})$ such that
$$\sup_{x \in \mathbb{R}} |f(x)| e^{\lambda \eta(|x|)} < \infty \qquad \text{and} \qquad \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| e^{\lambda \omega(|\xi|)} < \infty.$$
- We set
$$\mathcal{S}^{(\omega)}_{(\eta)} := \bigcap_{\lambda > 0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}, \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} := \bigcup_{\lambda > 0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}.$$

Gelfand-Shilov spaces

- A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is called a weight function if $\omega(0) = 0$, $\log t = o(\omega(t))$ and there is $C > 0$ such that
$$\omega(2t) \leq C\omega(t) + C, \qquad \forall t \geq 0.$$

- $\omega(t) = t^{\frac{1}{\alpha}} \log(1 + t)^{\beta}$ $(\alpha > 0, \beta \in \mathbb{R})$; $\omega(t) = e^{(\log t)^{\alpha}}$ $(0 < \alpha < 1)$.
- Given two weight functions ω and η, we define $\mathcal{S}^{\omega,\lambda}_{\eta,\lambda}$, $\lambda > 0$, as the Banach space consisting of all $f \in \mathcal{S}(\mathbb{R})$ such that
$$\sup_{x\in\mathbb{R}} |f(x)|e^{\lambda\eta(|x|)} < \infty \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}} |\widehat{f}(\xi)|e^{\lambda\omega(|\xi|)} < \infty.$$

- We set
$$\mathcal{S}^{(\omega)}_{(\eta)} := \bigcap_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}, \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} := \bigcup_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}.$$

Gelfand-Shilov spaces

- A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is called a weight function if $\omega(0) = 0$, $\log t = o(\omega(t))$ and there is $C > 0$ such that
$$\omega(2t) \leq C\omega(t) + C, \qquad \forall t \geq 0.$$
- $\omega(t) = t^{\frac{1}{\alpha}} \log(1 + t)^{\beta}$ $(\alpha > 0, \beta \in \mathbb{R})$; $\omega(t) = e^{(\log t)^{\alpha}}$ $(0 < \alpha < 1)$.
- Given two weight functions ω and η, we define $\mathcal{S}^{\omega,\lambda}_{\eta,\lambda}$, $\lambda > 0$, as the Banach space consisting of all $f \in \mathcal{S}(\mathbb{R})$ such that
$$\sup_{x\in\mathbb{R}} |f(x)|e^{\lambda\eta(|x|)} < \infty \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}} |\widehat{f}(\xi)|e^{\lambda\omega(|\xi|)} < \infty.$$
- We set
$$\mathcal{S}^{(\omega)}_{(\eta)} := \bigcap_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}, \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} := \bigcup_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}.$$

Gelfand-Shilov spaces

- A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is called a weight function if $\omega(0) = 0$, $\log t = o(\omega(t))$ and there is $C > 0$ such that
$$\omega(2t) \leq C\omega(t) + C, \qquad \forall t \geq 0.$$

- $\omega(t) = t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$ $(\alpha > 0, \beta \in \mathbb{R})$; $\omega(t) = e^{(\log t)^{\alpha}}$ $(0 < \alpha < 1)$.
- Given two weight functions ω and η, we define $\mathcal{S}^{\omega,\lambda}_{\eta,\lambda}$,$\lambda > 0$, as the Banach space consisting of all $f \in \mathcal{S}(\mathbb{R})$ such that
$$\sup_{x\in\mathbb{R}} |f(x)|e^{\lambda\eta(|x|)} < \infty \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}} |\widehat{f}(\xi)|e^{\lambda\omega(|\xi|)} < \infty.$$

- We set
$$\mathcal{S}^{(\omega)}_{(\eta)} := \bigcap_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}, \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} := \bigcup_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}.$$

Gelfand-Shilov spaces

- A continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ is called a weight function if $\omega(0) = 0$, $\log t = o(\omega(t))$ and there is $C > 0$ such that
$$\omega(2t) \leq C\omega(t) + C, \qquad \forall t \geq 0.$$

- $\omega(t) = t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$ $(\alpha > 0, \beta \in \mathbb{R})$; $\omega(t) = e^{(\log t)^{\alpha}}$ $(0 < \alpha < 1)$.
- Given two weight functions ω and η, we define $\mathcal{S}^{\omega,\lambda}_{\eta,\lambda}$,$\lambda > 0$, as the Banach space consisting of all $f \in \mathcal{S}(\mathbb{R})$ such that
$$\sup_{x\in\mathbb{R}} |f(x)|e^{\lambda\eta(|x|)} < \infty \qquad \text{and} \qquad \sup_{\xi\in\mathbb{R}} |\widehat{f}(\xi)|e^{\lambda\omega(|\xi|)} < \infty.$$

- We set
$$\mathcal{S}^{(\omega)}_{(\eta)} := \bigcap_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}, \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} := \bigcup_{\lambda>0} \mathcal{S}^{\omega,\lambda}_{\eta,\lambda}.$$

The classical Gelfand-Shilov spaces

- Let $\alpha, \beta > 0$. We define Σ^{α}_{β} ($\mathcal{S}^{\alpha}_{\beta}$) as the space consisting of all $f \in C^{\infty}(\mathbb{R})$ such that for all $\lambda > 0$ (for some $\lambda > 0$)

$$\sup_{x \in \mathbb{R}} \sup_{p,q \in \mathbb{N}} \frac{|x^p f^{(q)}(x)|}{\lambda^{p+q} p!^{\beta} q!^{\alpha}} < \infty.$$

- We have that

$$\Sigma^{\alpha}_{\beta} = \mathcal{S}^{(t^{1/\alpha})}_{(t^{1/\beta})}, \qquad \mathcal{S}^{\alpha}_{\beta} = \mathcal{S}^{\{t^{1/\alpha}\}}_{\{t^{1/\beta}\}}.$$

- The spaces $\mathcal{S}^{\alpha}_{\beta}$ were introduced by Gelfand and Shilov in 1968. They showed that $\mathcal{S}^{\alpha}_{\beta} \neq \{0\}$ if and only if $\alpha + \beta \geq 1$.

The classical Gelfand-Shilov spaces

- Let $\alpha, \beta > 0$. We define Σ^{α}_{β} ($\mathcal{S}^{\alpha}_{\beta}$) as the space consisting of all $f \in C^{\infty}(\mathbb{R})$ such that for all $\lambda > 0$ (for some $\lambda > 0$)

$$\sup_{x \in \mathbb{R}} \sup_{p,q \in \mathbb{N}} \frac{|x^p f^{(q)}(x)|}{\lambda^{p+q} p!^{\beta} q!^{\alpha}} < \infty.$$

- We have that

$$\Sigma^{\alpha}_{\beta} = \mathcal{S}^{(t^{1/\alpha})}_{(t^{1/\beta})}, \qquad \mathcal{S}^{\alpha}_{\beta} = \mathcal{S}^{\{t^{1/\alpha}\}}_{\{t^{1/\beta}\}}.$$

- The spaces $\mathcal{S}^{\alpha}_{\beta}$ were introduced by Gelfand and Shilov in 1968. They showed that $\mathcal{S}^{\alpha}_{\beta} \neq \{0\}$ if and only if $\alpha + \beta \geq 1$.

The classical Gelfand-Shilov spaces

- Let $\alpha, \beta > 0$. We define Σ^α_β ($\mathcal{S}^\alpha_\beta$) as the space consisting of all $f \in C^\infty(\mathbb{R})$ such that for all $\lambda > 0$ (for some $\lambda > 0$)

$$\sup_{x \in \mathbb{R}} \sup_{p,q \in \mathbb{N}} \frac{|x^p f^{(q)}(x)|}{\lambda^{p+q} p!^\beta q!^\alpha} < \infty.$$

- We have that

$$\Sigma^\alpha_\beta = \mathcal{S}^{(t^{1/\alpha})}_{(t^{1/\beta})}, \qquad \mathcal{S}^\alpha_\beta = \mathcal{S}^{\{t^{1/\alpha}\}}_{\{t^{1/\beta}\}}.$$

- The spaces $\mathcal{S}^\alpha_\beta$ were introduced by Gelfand and Shilov in 1968. They showed that $\mathcal{S}^\alpha_\beta \neq \{0\}$ if and only if $\alpha + \beta \geq 1$.

The classical Gelfand-Shilov spaces

- Let $\alpha, \beta > 0$. We define Σ^{α}_{β} ($\mathcal{S}^{\alpha}_{\beta}$) as the space consisting of all $f \in C^{\infty}(\mathbb{R})$ such that for all $\lambda > 0$ (for some $\lambda > 0$)

$$\sup_{x \in \mathbb{R}} \sup_{p,q \in \mathbb{N}} \frac{|x^p f^{(q)}(x)|}{\lambda^{p+q} p!^{\beta} q!^{\alpha}} < \infty.$$

- We have that

$$\Sigma^{\alpha}_{\beta} = \mathcal{S}^{(t^{1/\alpha})}_{(t^{1/\beta})}, \qquad \mathcal{S}^{\alpha}_{\beta} = \mathcal{S}^{\{t^{1/\alpha}\}}_{\{t^{1/\beta}\}}.$$

- The spaces $\mathcal{S}^{\alpha}_{\beta}$ were introduced by Gelfand and Shilov in 1968. They showed that $\mathcal{S}^{\alpha}_{\beta} \neq \{0\}$ if and only if $\alpha + \beta \geq 1$.

Power series spaces

- Let $\nu = (\nu_n)_{n\in\mathbb{N}}$ be a positive increasing sequence.
- We define $\Lambda^\lambda(\nu)$, $\lambda \in \mathbb{R}$, as the space consisting of all $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}} |c_n| e^{\lambda\nu_n} < \infty.$$

- We set

$$\Lambda_\infty(\nu) := \bigcap_{\lambda>0} \Lambda^\lambda(\nu), \qquad \Lambda_0(\nu) := \bigcap_{\lambda>0} \Lambda^{-\lambda}(\nu).$$

- If $\log(n) = o(\nu_n)$, we have that

$$\Lambda'_\infty(\nu) := \bigcup_{\lambda>0} \Lambda^{-\lambda}(\nu), \qquad \Lambda'_0(\nu) := \bigcup_{\lambda>0} \Lambda^{\lambda}(\nu).$$

Power series spaces

- Let $\nu = (\nu_n)_{n\in\mathbb{N}}$ be a positive increasing sequence.
- We define $\Lambda^\lambda(\nu)$, $\lambda \in \mathbb{R}$, as the space consisting of all $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |c_n| e^{\lambda\nu_n} < \infty.$$

- We set

$$\Lambda_\infty(\nu) := \bigcap_{\lambda>0} \Lambda^\lambda(\nu), \qquad \Lambda_0(\nu) := \bigcap_{\lambda>0} \Lambda^{-\lambda}(\nu).$$

- If $\log(n) = o(\nu_n)$, we have that

$$\Lambda'_\infty(\nu) := \bigcup_{\lambda>0} \Lambda^{-\lambda}(\nu), \qquad \Lambda'_0(\nu) := \bigcup_{\lambda>0} \Lambda^{\lambda}(\nu).$$

Power series spaces

- Let $\nu = (\nu_n)_{n\in\mathbb{N}}$ be a positive increasing sequence.
- We define $\Lambda^\lambda(\nu)$, $\lambda \in \mathbb{R}$, as the space consisting of all $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |c_n| e^{\lambda\nu_n} < \infty.$$

- We set

$$\Lambda_\infty(\nu) := \bigcap_{\lambda>0} \Lambda^\lambda(\nu), \qquad \Lambda_0(\nu) := \bigcap_{\lambda>0} \Lambda^{-\lambda}(\nu).$$

- If $\log(n) = o(\nu_n)$, we have that

$$\Lambda'_\infty(\nu) := \bigcup_{\lambda>0} \Lambda^{-\lambda}(\nu), \qquad \Lambda'_0(\nu) := \bigcup_{\lambda>0} \Lambda^{\lambda}(\nu).$$

Power series spaces

- Let $\nu = (\nu_n)_{n\in\mathbb{N}}$ be a positive increasing sequence.
- We define $\Lambda^\lambda(\nu)$, $\lambda \in \mathbb{R}$, as the space consisting of all $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |c_n| e^{\lambda\nu_n} < \infty.$$

- We set

$$\Lambda_\infty(\nu) := \bigcap_{\lambda>0} \Lambda^\lambda(\nu), \qquad \Lambda_0(\nu) := \bigcap_{\lambda>0} \Lambda^{-\lambda}(\nu).$$

- If $\log(n) = o(\nu_n)$, we have that

$$\Lambda'_\infty(\nu) := \bigcup_{\lambda>0} \Lambda^{-\lambda}(\nu), \qquad \Lambda'_0(\nu) := \bigcup_{\lambda>0} \Lambda^{\lambda}(\nu).$$

Known results (1)

Langenbruch, 2006

Let $\alpha > 1/2$ ($\alpha \geq 1/2$). Then,

$$\Sigma_\alpha^\alpha \cong \Lambda_\infty(n^{\frac{1}{2\alpha}}), \qquad \mathcal{S}_\beta^\alpha \cong \Lambda_0'(n^{\frac{1}{2\alpha}}).$$

- Hermite expansions, or more generally, eigenfunction expansions with respect to certain elliptic PDO (Gramchev, Rodino, Pilipovic (2011); Vindas and Vuckovic (2016)).
- If $\omega = \omega_M$ is the associated function of a weight sequence M subject to some standard conditions, we have that

$$\mathcal{S}_{(M)}^{(M)} = \mathcal{S}_{(\omega_M)}^{(\omega_M)} = \Lambda_\infty(\omega_M(n^{\frac{1}{2}})),$$
$$\mathcal{S}_{\{M\}}^{\{M\}} = \mathcal{S}_{\{\omega_M\}}^{\{\omega_M\}} = \Lambda_0'(\omega_M(n^{\frac{1}{2}})).$$

- The above spaces are invariant under the Fourier transform!

Known results (1)

Langenbruch, 2006

Let $\alpha > 1/2$ ($\alpha \geq 1/2$). Then,

$$\Sigma_{\alpha}^{\alpha} \cong \Lambda_{\infty}(n^{\frac{1}{2\alpha}}), \qquad \mathcal{S}_{\beta}^{\alpha} \cong \Lambda_0'(n^{\frac{1}{2\alpha}}).$$

- Hermite expansions, or more generally, eigenfunction expansions with respect to certain elliptic PDO (Gramchev, Rodino, Pilipovic (2011); Vindas and Vuckovic (2016)).
- If $\omega = \omega_M$ is the associated function of a weight sequence M subject to some standard conditions, we have that

$$\mathcal{S}_{(M)}^{(M)} = \mathcal{S}_{(\omega_M)}^{(\omega_M)} = \Lambda_{\infty}(\omega_M(n^{\frac{1}{2}})),$$
$$\mathcal{S}_{\{M\}}^{\{M\}} = \mathcal{S}_{\{\omega_M\}}^{\{\omega_M\}} = \Lambda_0'(\omega_M(n^{\frac{1}{2}})).$$

- The above spaces are invariant under the Fourier transform!

Known results (1)

Langenbruch, 2006

Let $\alpha > 1/2$ ($\alpha \geq 1/2$). Then,

$$\Sigma_{\alpha}^{\alpha} \cong \Lambda_{\infty}(n^{\frac{1}{2\alpha}}), \qquad \mathcal{S}_{\beta}^{\alpha} \cong \Lambda_0'(n^{\frac{1}{2\alpha}}).$$

- Hermite expansions, or more generally, eigenfunction expansions with respect to certain elliptic PDO (Gramchev, Rodino, Pilipovic (2011); Vindas and Vuckovic (2016)).
- If $\omega = \omega_M$ is the associated function of a weight sequence M subject to some standard conditions, we have that

$$\mathcal{S}_{(M)}^{(M)} = \mathcal{S}_{(\omega_M)}^{(\omega_M)} = \Lambda_{\infty}(\omega_M(n^{\frac{1}{2}})),$$
$$\mathcal{S}_{\{M\}}^{\{M\}} = \mathcal{S}_{\{\omega_M\}}^{\{\omega_M\}} = \Lambda_0'(\omega_M(n^{\frac{1}{2}})).$$

- The above spaces are invariant under the Fourier transform!

Known results (1)

Langenbruch, 2006

Let $\alpha > 1/2$ ($\alpha \geq 1/2$). Then,

$$\Sigma_\alpha^\alpha \cong \Lambda_\infty(n^{\frac{1}{2\alpha}}), \qquad \mathcal{S}_\beta^\alpha \cong \Lambda_0'(n^{\frac{1}{2\alpha}}).$$

- Hermite expansions, or more generally, eigenfunction expansions with respect to certain elliptic PDO (Gramchev, Rodino, Pilipovic (2011); Vindas and Vuckovic (2016)).
- If $\omega = \omega_M$ is the associated function of a weight sequence M subject to some standard conditions, we have that

$$\mathcal{S}_{(M)}^{(M)} = \mathcal{S}_{(\omega_M)}^{(\omega_M)} = \Lambda_\infty(\omega_M(n^{\frac{1}{2}})),$$
$$\mathcal{S}_{\{M\}}^{\{M\}} = \mathcal{S}_{\{\omega_M\}}^{\{\omega_M\}} = \Lambda_0'(\omega_M(n^{\frac{1}{2}})).$$

- The above spaces are invariant under the Fourier transform!

Known results (2)

Cappiello, Gramchev, Pilipovic, Rodino, 2019

Let $\alpha+\beta>1$ ($\alpha+\beta\geq 1$) be such that $\alpha/\beta\in\mathbb{Q}$. Then,

$$\Sigma_\beta^\alpha \cong \Lambda_\infty(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}_\alpha^\alpha \cong \Lambda_0'(n^{\frac{1}{\alpha+\beta}}).$$

- Eigenfunction expansions with respect to certain elliptic PDO, e.g.,

$$(-\Delta)^m + x^{2k}$$

for suitable $k, m \in \mathbb{N}$.

Question

Does a similar result hold for the spaces $\mathcal{S}_{(N)}^{(M)}$ and $\mathcal{S}_{\{N\}}^{\{M\}}$? What is the correct generalization of the condition $\alpha/\beta\in\mathbb{Q}$?

Known results (2)

Cappiello, Gramchev, Pilipovic, Rodino, 2019

Let $\alpha + \beta > 1$ ($\alpha + \beta \geq 1$) be such that $\alpha/\beta \in \mathbb{Q}$. Then,

$$\Sigma_\beta^\alpha \cong \Lambda_\infty(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}_\alpha^\alpha \cong \Lambda_0'(n^{\frac{1}{\alpha+\beta}}).$$

- Eigenfunction expansions with respect to certain elliptic PDO, e.g.,

$$(-\Delta)^m + x^{2k}$$

for suitable $k, m \in \mathbb{N}$.

Question

Does a similar result hold for the spaces $\mathcal{S}_{(N)}^{(M)}$ and $\mathcal{S}_{\{N\}}^{\{M\}}$? What is the correct generalization of the condition $\alpha/\beta \in \mathbb{Q}$?

Known results (2)

Cappiello, Gramchev, Pilipovic, Rodino, 2019

Let $\alpha+\beta>1$ ($\alpha+\beta\geq 1$) be such that $\alpha/\beta\in\mathbb{Q}$. Then,

$$\Sigma_\beta^\alpha \cong \Lambda_\infty(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}_\alpha^\alpha \cong \Lambda_0'(n^{\frac{1}{\alpha+\beta}}).$$

- Eigenfunction expansions with respect to certain elliptic PDO, e.g.,

$$(-\Delta)^m + x^{2k}$$

for suitable $k, m \in \mathbb{N}$.

Question

Does a similar result hold for the spaces $\mathcal{S}_{(N)}^{(M)}$ and $\mathcal{S}_{\{N\}}^{\{M\}}$? What is the correct generalization of the condition $\alpha/\beta\in\mathbb{Q}$?

Known results (2)

Cappiello, Gramchev, Pilipovic, Rodino, 2019

Let $\alpha+\beta>1$ ($\alpha+\beta\geq 1$) be such that $\alpha/\beta\in\mathbb{Q}$. Then,

$$\Sigma_{\beta}^{\alpha}\cong\Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}),\qquad \mathcal{S}_{\alpha}^{\alpha}\cong\Lambda_{0}'(n^{\frac{1}{\alpha+\beta}}).$$

- Eigenfunction expansions with respect to certain elliptic PDO, e.g.,

$$(-\Delta)^{m}+x^{2k}$$

for suitable $k,m\in\mathbb{N}$.

Question

Does a similar result hold for the spaces $\mathcal{S}_{(N)}^{(M)}$ and $\mathcal{S}_{\{N\}}^{\{M\}}$? What is the correct generalization of the condition $\alpha/\beta\in\mathbb{Q}$?

The case $\omega(t) = t$

- For $\lambda > 0$ we set $V_\lambda = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \lambda\}$.
- Given a weight function η, we define $\mathcal{H}_{\eta,\lambda}(V_\lambda)$ as the Banach space consisting of all $f \in \mathcal{O}(V_\lambda)$ such that

$$\sup_{z \in V_\lambda} |f(z)| e^{\lambda\omega(|\operatorname{Re} z|)} < \infty.$$

- We set

$$\mathcal{H}_{(\eta)} := \bigcap_{\lambda > 0} \mathcal{H}_{\eta,\lambda}(V_\lambda), \qquad \mathcal{H}_{\{\eta\}} := \bigcup_{\lambda > 0} \mathcal{H}_{\eta,\lambda}(V_\lambda).$$

- We have that

$$\mathcal{H}_{(\eta)} = \mathcal{S}^{(t)}_{(\eta)}, \qquad \mathcal{H}_{\{\eta\}} = \mathcal{S}^{\{t\}}_{\{\eta\}}.$$

The case $\omega(t) = t$

- For $\lambda > 0$ we set $V_\lambda = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \lambda\}$.
- Given a weight function η, we define $\mathcal{H}_{\eta,\lambda}(V_\lambda)$ as the Banach space consisting of all $f \in \mathcal{O}(V_\lambda)$ such that

$$\sup_{z \in V_\lambda} |f(z)| e^{\lambda\omega(|\operatorname{Re} z|)} < \infty.$$

- We set

$$\mathcal{H}_{(\eta)} := \bigcap_{\lambda>0} \mathcal{H}_{\eta,\lambda}(V_\lambda), \qquad \mathcal{H}_{\{\eta\}} := \bigcup_{\lambda>0} \mathcal{H}_{\eta,\lambda}(V_\lambda).$$

- We have that

$$\mathcal{H}_{(\eta)} = \mathcal{S}^{(t)}_{(\eta)}, \qquad \mathcal{H}_{\{\eta\}} = \mathcal{S}^{\{t\}}_{\{\eta\}}.$$

The case $\omega(t) = t$

- For $\lambda > 0$ we set $V_\lambda = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \lambda\}$.
- Given a weight function η, we define $\mathcal{H}_{\eta,\lambda}(V_\lambda)$ as the Banach space consisting of all $f \in \mathcal{O}(V_\lambda)$ such that

$$\sup_{z \in V_\lambda} |f(z)| e^{\lambda\omega(|\operatorname{Re} z|)} < \infty.$$

- We set

$$\mathcal{H}_{(\eta)} := \bigcap_{\lambda>0} \mathcal{H}_{\eta,\lambda}(V_\lambda), \qquad \mathcal{H}_{\{\eta\}} := \bigcup_{\lambda>0} \mathcal{H}_{\eta,\lambda}(V_\lambda).$$

- We have that

$$\mathcal{H}_{(\eta)} = \mathcal{S}^{(t)}_{(\eta)}, \qquad \mathcal{H}_{\{\eta\}} = \mathcal{S}^{\{t\}}_{\{\eta\}}.$$

The case $\omega(t) = t$

- For $\lambda > 0$ we set $V_\lambda = \{z \in \mathbb{C} \,|\, |\operatorname{Im} z| < \lambda\}$.
- Given a weight function η, we define $\mathcal{H}_{\eta,\lambda}(V_\lambda)$ as the Banach space consisting of all $f \in \mathcal{O}(V_\lambda)$ such that

$$\sup_{z \in V_\lambda} |f(z)| e^{\lambda\omega(|\operatorname{Re} z|)} < \infty.$$

- We set

$$\mathcal{H}_{(\eta)} := \bigcap_{\lambda > 0} \mathcal{H}_{\eta,\lambda}(V_\lambda), \qquad \mathcal{H}_{\{\eta\}} := \bigcup_{\lambda > 0} \mathcal{H}_{\eta,\lambda}(V_\lambda).$$

- We have that

$$\mathcal{H}_{(\eta)} = \mathcal{S}^{(t)}_{(\eta)}, \qquad \mathcal{H}_{\{\eta\}} = \mathcal{S}^{\{t\}}_{\{\eta\}}.$$

Sequence space representations for $\mathcal{H}_{\{\eta\}}$

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

Sequence space representations for $\mathcal{H}_{\{\eta\}}$

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

Sequence space representations for $\mathcal{H}_{\{\eta\}}$

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

Sequence space representations for $\mathcal{H}_{\{\eta\}}$

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

Sequence space representations for $\mathcal{H}_{\{\eta\}}$

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

Sequence space representations for $\mathcal{H}_{\{\eta\}}$

- Let η be a weight function. We define $\eta^* = (t\eta^{-1}(t))^{-1}$.

Langenbruch, 2012 and 2016

$$\mathcal{H}_{\{\eta\}} \cong \Lambda_0'(\eta^*(n)) \cong \Lambda_0'(n)\widehat{\otimes}\Lambda_0'(\eta(n)).$$

Vogt, 1982

Let E be an infinite-dimensional nuclear Fréchet space satisfying $(\underline{DN})$ and $(\overline{\Omega})$. Then, $E \cong \Lambda_0(\nu)$ for some positive increasing sequence ν.

- $(\underline{DN})$: Quantified decomposition theorem for holomorphic functions on strips with rapid decay.
- $(\overline{\Omega})$: Weighted version of Hadamard's three-lines theorem.
- Determine diametral dimension of $\mathcal{H}'_{\{\eta\}}$ to show that $\nu = (\eta^*(n))_{n\in\mathbb{N}}$.

- Up to O-equivalence, we have that

η	η^*
$t^{\frac{1}{\alpha}}$	$t^{\frac{1}{\alpha+1}}$
$t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$	$t^{\frac{1}{\alpha+1}} \log(1+t)^{\frac{\alpha\beta}{\alpha+1}}$
$e^{(\log t)^{\alpha}}$	$e^{(\log t)^{\alpha}}$ if $0 < \alpha < 1/2$
	$e^{(\log t)^{\alpha} - \alpha(\log t)^{2\alpha-1}}$ if $1/2 \le \alpha < 2/3$
	$\ldots$

Corollary

$$\mathcal{S}_{\alpha}^{1} \cong \mathcal{S}_{1}^{\alpha} \cong \Lambda_0'(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

- Up to O-equivalence, we have that

η	η^*
$t^{\frac{1}{\alpha}}$	$t^{\frac{1}{\alpha+1}}$
$t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$	$t^{\frac{1}{\alpha+1}} \log(1+t)^{\frac{\alpha\beta}{\alpha+1}}$
$e^{(\log t)^{\alpha}}$	$e^{(\log t)^{\alpha}}$ if $0 < \alpha < 1/2$
	$e^{(\log t)^{\alpha} - \alpha(\log t)^{2\alpha-1}}$ if $1/2 \leq \alpha < 2/3$
	$\cdots$

Corollary

$$\mathcal{S}^1_{\alpha} \cong \mathcal{S}^{\alpha}_1 \cong \Lambda'_0(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Examples

- Up to O-equivalence, we have that

η	η^*
$t^{\frac{1}{\alpha}}$	$t^{\frac{1}{\alpha+1}}$
$t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$	$t^{\frac{1}{\alpha+1}} \log(1+t)^{\frac{\alpha\beta}{\alpha+1}}$
$e^{(\log t)^{\alpha}}$	$e^{(\log t)^{\alpha}}$ if $0 < \alpha < 1/2$
	$e^{(\log t)^{\alpha} - \alpha(\log t)^{2\alpha-1}}$ if $1/2 \le \alpha < 2/3$
	$\cdots$

Corollary

$$\mathcal{S}_{\alpha}^{1} \cong \mathcal{S}_{1}^{\alpha} \cong \Lambda_0'(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Examples

- Up to O-equivalence, we have that

η	η^*
$t^{\frac{1}{\alpha}}$	$t^{\frac{1}{\alpha+1}}$
$t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$	$t^{\frac{1}{\alpha+1}} \log(1+t)^{\frac{\alpha\beta}{\alpha+1}}$
$e^{(\log t)^{\alpha}}$	$e^{(\log t)^{\alpha}}$ if $0 < \alpha < 1/2$
	$e^{(\log t)^{\alpha} - \alpha(\log t)^{2\alpha-1}}$ if $1/2 \leq \alpha < 2/3$
	$\cdots$

Corollary

$$\mathcal{S}^1_{\alpha} \cong \mathcal{S}^{\alpha}_1 \cong \Lambda'_0(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Examples

- Up to O-equivalence, we have that

η	η^*
$t^{\frac{1}{\alpha}}$	$t^{\frac{1}{\alpha+1}}$
$t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$	$t^{\frac{1}{\alpha+1}} \log(1+t)^{\frac{\alpha\beta}{\alpha+1}}$
$e^{(\log t)^{\alpha}}$	$e^{(\log t)^{\alpha}}$ if $0 < \alpha < 1/2$
	$e^{(\log t)^{\alpha} - \alpha(\log t)^{2\alpha-1}}$ if $1/2 \leq \alpha < 2/3$
	$\dots$

Corollary

$$\mathcal{S}_{\alpha}^{1} \cong \mathcal{S}_{1}^{\alpha} \cong \Lambda_0'(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Examples

- Up to O-equivalence, we have that

η	η^*
$t^{\frac{1}{\alpha}}$	$t^{\frac{1}{\alpha+1}}$
$t^{\frac{1}{\alpha}} \log(1+t)^{\beta}$	$t^{\frac{1}{\alpha+1}} \log(1+t)^{\frac{\alpha\beta}{\alpha+1}}$
$e^{(\log t)^{\alpha}}$	$e^{(\log t)^{\alpha}}$ if $0 < \alpha < 1/2$
	$e^{(\log t)^{\alpha} - \alpha(\log t)^{2\alpha-1}}$ if $1/2 \leq \alpha < 2/3$
	$\ldots$

Corollary

$$\mathcal{S}^{1}_{\alpha} \cong \mathcal{S}^{\alpha}_{1} \cong \Lambda'_0(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Sequence space representations for $\mathcal{H}_{(\eta)}$

D., 2020

$$\mathcal{H}_{(\eta)} \cong \Lambda_\infty(\eta^*(n)).$$

In particular,

$$\Sigma_\alpha^1 \cong \Sigma_1^\alpha \cong \Lambda_\infty(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Vogt, 1982; Aytuna, Krone, Terzioglu, 1989

Let E be a nuclear Fréchet space satisfying (DN) and (Ω) with $\Delta(E) = \Delta(\Lambda_\infty(\nu)) = \Lambda'_\infty(\nu)$ for some positive sequence ν with $\nu_{2n} = O(\nu_n)$. Then, $E \cong \Lambda_\infty(\nu)$.

- (DN): Weighted version of Hadamard's three-lines theorem.
- (Ω): Mapping properties of the STFT on $\mathcal{H}_{(\eta)}$.
- $\Delta(\mathcal{H}_{(\eta)}) = \Lambda'_\infty(\eta^*(\eta))$: STFT + result of Langenbruch.

Sequence space representations for $\mathcal{H}_{(\eta)}$

D., 2020

$$\mathcal{H}_{(\eta)} \cong \Lambda_\infty(\eta^*(n)).$$

In particular,

$$\Sigma_\alpha^1 \cong \Sigma_1^\alpha \cong \Lambda_\infty(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Vogt, 1982; Aytuna, Krone, Terzioglu, 1989

Let E be a nuclear Fréchet space satisfying (DN) and (Ω) with $\Delta(E) = \Delta(\Lambda_\infty(\nu)) = \Lambda'_\infty(\nu)$ for some positive sequence ν with $\nu_{2n} = O(\nu_n)$. Then, $E \cong \Lambda_\infty(\nu)$.

- (DN): Weighted version of Hadamard's three-lines theorem.
- (Ω): Mapping properties of the STFT on $\mathcal{H}_{(\eta)}$.
- $\Delta(\mathcal{H}_{(\eta)}) = \Lambda'_\infty(\eta^*(\eta))$: STFT + result of Langenbruch.

D., 2020

$$\mathcal{H}_{(\eta)} \cong \Lambda_\infty(\eta^*(n)).$$

In particular,

$$\Sigma^1_\alpha \cong \Sigma^\alpha_1 \cong \Lambda_\infty(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Vogt, 1982; Aytuna, Krone, Terzioglu, 1989

Let E be a nuclear Fréchet space satisfying (DN) and (Ω) with $\Delta(E) = \Delta(\Lambda_\infty(\nu)) = \Lambda'_\infty(\nu)$ for some positive sequence ν with $\nu_{2n} = O(\nu_n)$. Then, $E \cong \Lambda_\infty(\nu)$.

- (DN): Weighted version of Hadamard's three-lines theorem.
- (Ω): Mapping properties of the STFT on $\mathcal{H}_{(\eta)}$.
- $\Delta(\mathcal{H}_{(\eta)}) = \Lambda'_\infty(\eta^*(\eta))$: STFT + result of Langenbruch.

Sequence space representations for $\mathcal{H}_{(\eta)}$

D., 2020

$$\mathcal{H}_{(\eta)} \cong \Lambda_\infty(\eta^*(n)).$$

In particular,

$$\Sigma^1_\alpha \cong \Sigma^\alpha_1 \cong \Lambda_\infty(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Vogt, 1982; Aytuna, Krone, Terzioglu, 1989

Let E be a nuclear Fréchet space satisfying (DN) and (Ω) with $\Delta(E) = \Delta(\Lambda_\infty(\nu)) = \Lambda'_\infty(\nu)$ for some positive sequence ν with $\nu_{2n} = O(\nu_n)$. Then, $E \cong \Lambda_\infty(\nu)$.

- (DN): Weighted version of Hadamard's three-lines theorem.
- (Ω): Mapping properties of the STFT on $\mathcal{H}_{(\eta)}$.
- $\Delta(\mathcal{H}_{(\eta)}) = \Lambda'_\infty(\eta^*(\eta))$: STFT + result of Langenbruch.

Sequence space representations for $\mathcal{H}_{(\eta)}$

D., 2020

$$\mathcal{H}_{(\eta)} \cong \Lambda_\infty(\eta^*(n)).$$

In particular,

$$\Sigma_\alpha^1 \cong \Sigma_1^\alpha \cong \Lambda_\infty(n^{\frac{1}{\alpha+1}}), \qquad \alpha > 0.$$

Vogt, 1982; Aytuna, Krone, Terzioglu, 1989

Let E be a nuclear Fréchet space satisfying (DN) and (Ω) with $\Delta(E) = \Delta(\Lambda_\infty(\nu)) = \Lambda_\infty'(\nu)$ for some positive sequence ν with $\nu_{2n} = O(\nu_n)$. Then, $E \cong \Lambda_\infty(\nu)$.

- (DN): Weighted version of Hadamard's three-lines theorem.
- (Ω): Mapping properties of the STFT on $\mathcal{H}_{(\eta)}$.
- $\Delta(\mathcal{H}_{(\eta)}) = \Lambda_\infty'(\eta^*(\eta))$: STFT + result of Langenbruch.

Sequence space representations for $\mathcal{S}_{(\eta)}^{(\omega)}$ and $\mathcal{S}_{\{\eta\}}^{\{\omega\}}$

- Let ω and η be weight functions. We define $\omega \sharp \eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}_{(\eta)}^{(\omega)}$ ($\psi, \gamma \in \mathcal{S}_{\{\eta\}}^{\{\omega\}}$) such that

$$\sum_{j \in \mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}_{(\eta)}^{(\omega)} \cong \Lambda_\infty(\omega \sharp \eta(n)) \cong \Lambda_\infty(\omega(n)) \widehat{\otimes} \Lambda_\infty(\eta(n)),$$

$$\mathcal{S}_{\{\eta\}}^{\{\omega\}} \cong \Lambda_0'(\omega \sharp \eta(n)) \cong \Lambda_0'(\omega(n)) \widehat{\otimes} \Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

Sequence space representations for $\mathcal{S}^{(\omega)}_{(\eta)}$ and $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$

- Let ω and η be weight functions. We define $\omega \sharp \eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}^{(\omega)}_{(\eta)}$ ($\psi, \gamma \in \mathcal{S}^{\{\omega\}}_{\{\eta\}}$) such that

$$\sum_{j \in \mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega \sharp \eta(n)) \cong \Lambda_\infty(\omega(n)) \widehat{\otimes} \Lambda_\infty(\eta(n)),$$

$$\mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega \sharp \eta(n)) \cong \Lambda_0'(\omega(n)) \widehat{\otimes} \Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

Sequence space representations for $\mathcal{S}^{(\omega)}_{(\eta)}$ and $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$

- Let ω and η be weight functions. We define $\omega\sharp\eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}^{(\omega)}_{(\eta)}$ ($\psi, \gamma \in \mathcal{S}^{\{\omega\}}_{\{\eta\}}$) such that

$$\sum_{j \in \mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)) \cong \Lambda_\infty(\omega(n))\widehat{\otimes}\Lambda_\infty(\eta(n)),$$

$$\mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n)) \cong \Lambda_0'(\omega(n))\widehat{\otimes}\Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

Sequence space representations for $\mathcal{S}^{(\omega)}_{(\eta)}$ and $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$

- Let ω and η be weight functions. We define $\omega\sharp\eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}^{(\omega)}_{(\eta)}$ ($\psi, \gamma \in \mathcal{S}^{\{\omega\}}_{\{\eta\}}$) such that

$$\sum_{j \in \mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)) \cong \Lambda_\infty(\omega(n))\widehat{\otimes}\Lambda_\infty(\eta(n)),$$

$$\mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n)) \cong \Lambda_0'(\omega(n))\widehat{\otimes}\Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

Sequence space representations for $\mathcal{S}^{(\omega)}_{(\eta)}$ and $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$

- Let ω and η be weight functions. We define $\omega\sharp\eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}^{(\omega)}_{(\eta)}$ ($\psi, \gamma \in \mathcal{S}^{\{\omega\}}_{\{\eta\}}$) such that

$$\sum_{j\in\mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)) \cong \Lambda_\infty(\omega(n))\widehat{\otimes}\Lambda_\infty(\eta(n)),$$

$$\mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n)) \cong \Lambda_0'(\omega(n))\widehat{\otimes}\Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

Sequence space representations for $\mathcal{S}^{(\omega)}_{(\eta)}$ and $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$

- Let ω and η be weight functions. We define $\omega\sharp\eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}^{(\omega)}_{(\eta)}$ $(\psi, \gamma \in \mathcal{S}^{\{\omega\}}_{\{\eta\}})$ such that

$$\sum_{j\in\mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)) \cong \Lambda_\infty(\omega(n))\widehat{\otimes}\Lambda_\infty(\eta(n)),$$

$$\mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n)) \cong \Lambda_0'(\omega(n))\widehat{\otimes}\Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

Sequence space representations for $\mathcal{S}^{(\omega)}_{(\eta)}$ and $\mathcal{S}^{\{\omega\}}_{\{\eta\}}$

- Let ω and η be weight functions. We define $\omega\sharp\eta = (\omega^{-1}(t)\eta^{-1}(t))^{-1}$.

D., 2020

Suppose that there are $a, b > 0$ and $\psi, \gamma \in \mathcal{S}^{(\omega)}_{(\eta)}$ ($\psi, \gamma \in \mathcal{S}^{\{\omega\}}_{\{\eta\}}$) such that

$$\sum_{j\in\mathbb{Z}^d} \psi(x - aj - bk)\gamma(x - aj) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{1}$$

Then,

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)) \cong \Lambda_\infty(\omega(n))\widehat{\otimes}\Lambda_\infty(\eta(n)),$$

$$\mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n)) \cong \Lambda_0'(\omega(n))\widehat{\otimes}\Lambda_0'(\eta(n)).$$

- Pelczinsky-Vogt decomposition method (E complemented in $\Lambda_\infty(\nu)$ ($\Lambda_0(\nu)$) and vice versa $\Rightarrow E \cong \Lambda_\infty(\nu)$ ($E \cong \Lambda_0(\nu)$) (Vogt, 1982).

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda_0'(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2 \, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda'_{0}(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2 \, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda'_0(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2 \, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda'_{0}(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2 \, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda'_0(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2 \, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_\infty(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda'_0(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2 \, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

On the condition (1)

- Condition (1) is satisfied in the following two cases:
  1. ω is non-quasianalytic: Existence of cut-off functions.
  2. $\omega = o(t^2)$ and $\eta = o(t^2)$ ($\omega = O(t^2)$ and $\eta = O(t^2)$): Condition (1) is equivalent to the existence of a dual pair of Gabor frame windows in $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) (Wexler-Raz biorthogonality relations). Bölcskei and Janssen (2000) showed that there exist a dual pair of Gabor frame windows in $\mathcal{S}^{1/2}_{1/2}$.

Corollary

$$\Sigma^{\alpha}_{\beta} \cong \Lambda_{\infty}(n^{\frac{1}{\alpha+\beta}}), \qquad \mathcal{S}^{\alpha}_{\beta} \cong \Lambda'_{0}(n^{\frac{1}{\alpha+\beta}}),$$

provided that either

$$\alpha \geq 1 \text{ or } \beta \geq 1$$

or

$$\alpha > 1/2 \text{ and } \beta > 1/2\, (\alpha \geq 1/2 \text{ and } \beta \geq 1/2).$$

Open problems

Conjecture

$$\mathcal{S}_{(\eta)}^{(\omega)} \cong \Lambda_\infty(\omega \sharp \eta(n)), \qquad \mathcal{S}_{\{\eta\}}^{\{\omega\}} \cong \Lambda_0'(\omega \sharp \eta(n))$$

for all non-trivial spaces $\mathcal{S}_{(\eta)}^{(\omega)}$ ($\mathcal{S}_{\{\eta\}}^{\{\omega\}}$)

- By using the STFT one can show that $\mathcal{S}_{(\eta)}^{(\omega)}$ and $(\mathcal{S}_{\{\eta\}}^{\{\omega\}})'$ satisfy the suitable (DN) and (Ω) type conditions. How to determine the diametral dimension of these spaces?
- Does each non-trivial space $\mathcal{S}_{(\eta)}^{(\omega)}$ ($\mathcal{S}_{\{\eta\}}^{\{\omega\}}$) contain a pair of functions satisfying (1)? Equivalently, does it contain a dual pair of Gabor frame windows? Does $\mathcal{S}_\beta^\alpha$, $\alpha + \beta \geq 1$, contain a dual pair of Gabor frame windows?

Open problems

Conjecture

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)), \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n))$$

for all non-trivial spaces $\mathcal{S}^{(\omega)}_{(\eta)}$ $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})$

- By using the STFT one can show that $\mathcal{S}^{(\omega)}_{(\eta)}$ and $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})'$ satisfy the suitable (DN) and (Ω) type conditions. How to determine the diametral dimension of these spaces?
- Does each non-trivial space $\mathcal{S}^{(\omega)}_{(\eta)}$ $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})$ contain a pair of functions satisfying (1)? Equivalently, does it contain a dual pair of Gabor frame windows? Does $\mathcal{S}^{\alpha}_{\beta}$, $\alpha + \beta \geq 1$, contain a dual pair of Gabor frame windows?

Open problems

Conjecture

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)), \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n))$$

for all non-trivial spaces $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$)

- By using the STFT one can show that $\mathcal{S}^{(\omega)}_{(\eta)}$ and $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})'$ satisfy the suitable (DN) and (Ω) type conditions. How to determine the diametral dimension of these spaces?
- Does each non-trivial space $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) contain a pair of functions satisfying (1)? Equivalently, does it contain a dual pair of Gabor frame windows? Does $\mathcal{S}^{\alpha}_{\beta}$, $\alpha + \beta \geq 1$, contain a dual pair of Gabor frame windows?

Open problems

Conjecture

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega\sharp\eta(n)), \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega\sharp\eta(n))$$

for all non-trivial spaces $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$)

- By using the STFT one can show that $\mathcal{S}^{(\omega)}_{(\eta)}$ and $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})'$ satisfy the suitable (DN) and (Ω) type conditions. How to determine the diametral dimension of these spaces?
- Does each non-trivial space $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) contain a pair of functions satisfying (1)? Equivalently, does it contain a dual pair of Gabor frame windows? Does $\mathcal{S}^{\alpha}_{\beta}$, $\alpha + \beta \geq 1$, contain a dual pair of Gabor frame windows?

Open problems

Conjecture

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega \sharp \eta(n)), \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega \sharp \eta(n))$$

for all non-trivial spaces $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$)

- By using the STFT one can show that $\mathcal{S}^{(\omega)}_{(\eta)}$ and $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})'$ satisfy the suitable (DN) and (Ω) type conditions. How to determine the diametral dimension of these spaces?
- Does each non-trivial space $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) contain a pair of functions satisfying (1)? Equivalently, does it contain a dual pair of Gabor frame windows? Does $\mathcal{S}^{\alpha}_{\beta}$, $\alpha + \beta \geq 1$, contain a dual pair of Gabor frame windows?

Open problems

Conjecture

$$\mathcal{S}^{(\omega)}_{(\eta)} \cong \Lambda_\infty(\omega \sharp \eta(n)), \qquad \mathcal{S}^{\{\omega\}}_{\{\eta\}} \cong \Lambda_0'(\omega \sharp \eta(n))$$

for all non-trivial spaces $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$)

- By using the STFT one can show that $\mathcal{S}^{(\omega)}_{(\eta)}$ and $(\mathcal{S}^{\{\omega\}}_{\{\eta\}})'$ satisfy the suitable (DN) and (Ω) type conditions. How to determine the diametral dimension of these spaces?
- Does each non-trivial space $\mathcal{S}^{(\omega)}_{(\eta)}$ ($\mathcal{S}^{\{\omega\}}_{\{\eta\}}$) contain a pair of functions satisfying (1)? Equivalently, does it contain a dual pair of Gabor frame windows? Does $\mathcal{S}^{\alpha}_{\beta}$, $\alpha + \beta \geq 1$, contain a dual pair of Gabor frame windows?