A geometric connection between the split first and second row of the Freudenthal-Tits magic square

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Abstract

A projective representation G_1 of a variety of the first row of the Freudenthal-Tits magic square can be obtained as the absolute geometry of a (symplectic) polarity ρ of the projective representation G_2 of a variety one cell below. In this paper we extend this geometric connection between G_1 and G_2 by showing that any non-degenerate quadric Q of maximal Witt index containing G_2 gives rise to a variety isomorphic to G_1, in the sense that the symplecta of G_2 contained in totally isotropic subspaces of Q are the absolute symplecta of a unique (symplectic) polarity ρ of G_2. Except for the smallest case, we also show that any non-degenerate quadric containing G_2 has maximal Witt index; and in the largest case, we obtain that there are only three kinds of possibly degenerate quadrics containing the Cartan variety $\mathcal{E}_6(\mathbb{K})$.

Keywords: Veronese variety, spherical buildings, embeddings, geometric hyperplanes
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1 Introduction

The universal embedding of a point-line geometry of type $\mathsf{E}_{6,1}(\mathbb{K})$, with $\mathbb{K}$ an arbitrary (commutative) field, is given by the Cartan variety $\mathcal{E}_6(\mathbb{K})$ in 26-dimensional projective space $\mathsf{PG}(26,\mathbb{K})$ ([11, 6]). Its elements of types 1,2,3,4,5 (Bourbaki labeling) are points, 5-spaces, lines, planes and 4-spaces of $\mathsf{PG}(26,\mathbb{K})$, respectively; an element of type 6 is referred to as a *symplecton*, abstractly it is a convex substructure isomorphic to polar space of type D_5, and in $\mathsf{PG}(26,\mathbb{K})$ it appears as hyperbolic quadric $Q(9,\mathbb{K})$ in a 9-dimensional subspace. In general, there are three possibilities for the intersection of a hyperplane of $\mathsf{PG}(26,\mathbb{K})$ with $\mathcal{E}_6(\mathbb{K})$ (see [5], or Proposition 4.13 in the current paper), corresponding to the three orbits of the group $\mathsf{E}_6(\mathbb{K})$ on the points of $\mathsf{PG}(26,\mathbb{K})$ (points on the variety, points on a secant, and the remaining points). One of these types of hyperplanes intersects $\mathcal{E}_6(\mathbb{K})$ in the points of a subvariety isormorphic to $\mathcal{F}_{4,4}(\mathbb{K})$, a geometry of type $\mathsf{F}_{4,4}(\mathbb{K})$ in $\mathsf{PG}(25,\mathbb{K})$. Another way to obtain this subvariety is as the absolute geometry of a *symplectic polarity* ρ of $\mathcal{E}_6(\mathbb{K})$ ([15]). Such a polarity ρ is a duality of $\mathcal{E}_6(\mathbb{K})$ (an isomorphism from $\mathcal{E}_6(\mathbb{K})$ to its dual) with the following properties: a point x is either incident or opposite its image $\rho(x)$ and there is at least one point x with $x \in \rho(x)$. The elements of $\mathcal{E}_6(\mathbb{K})$ incident with their image are called absolute and constitute the absolute geometry. The absolute points generate a hyperplane of $\mathsf{PG}(26,\mathbb{K})$ and each point of $\mathcal{E}_6(\mathbb{K})$ in this hyperplane is absolute.

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1.1 Main results

The main achievement of this paper is the fact that we can also obtain $\mathcal{F}_{4,4}(\mathbb{K})$ by embedding $\mathcal{E}_6(\mathbb{K})$ in a non-degenerate quadric of $\mathsf{PG}(26,\mathbb{K})$. Indeed, if Q is such a quadric (the existence of which follows from Lemma 5.2), then there are two options for a symplecton Σ of $\mathcal{E}_6(\mathbb{K})$: either Σ arises as the intersection of a 9-space with Q, or Σ is contained in a totally isotropic subspace of Q of dimension 9 (cf. Lemma 3.2). In the latter case, Σ is called *projective*. An informal version of a special case of our main results (the precise statement can be found in Main Result 3.4 in Section 3) reads as follows.

Theorem 1.1 (Subcase of the main result) *Suppose $\mathcal{E}_6(\mathbb{K})$ is contained in a non-degenerate quadric Q of $\mathsf{PG}(26,\mathbb{K})$. Then there is a unique symplectic polarity ρ of $\mathcal{E}_6(\mathbb{K})$ such that the set of symplecta of $\mathcal{E}_6(\mathbb{K})$ contained in a totally isotropic subspace of Q (i.e., the projective symplecta) coincides with the set of absolute symplecta of Γ under ρ.*

To see why this is only a special case, we take a step back and go to the Freudenthal-Tits magic square (FTMS). This is a 4×4 array of Lie incidence geometries (or buildings or Lie algebras, as one prefers), where one can view these geometries abstractly or as a projective variety. We take the latter viewpoint. One finds the $\mathcal{E}_6(\mathbb{K})$ geometry in the last cell of the second row, with on its left its smaller siblings: the line Grassmannian variety $\mathcal{G}_{6,2}(\mathbb{K})$ of a projective 5-space $\mathsf{PG}(5,\mathbb{K})$, living in $\mathsf{PG}(14,\mathbb{K})$, the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ which is the direct product of two projective planes over $\mathbb{K}$, living inside $\mathsf{PG}(8,\mathbb{K})$, and the quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$, the image of $\mathsf{PG}(2,\mathbb{K})$ under the Veronese map, living in $\mathsf{PG}(5,\mathbb{K})$. To be precise, these are the varieties on the second row of the FTMS in the *split* version of the square (the non-split version consists of certain real forms of the split version). Moreover, also the variety $\mathcal{F}_{4,4}(\mathbb{K})$—the absolute geometry under the symplectic polarity ρ—can be found in the FTMS, namely in the cell just above $\mathcal{E}_6(\mathbb{K})$. A more detailed description of the geometries of the first and second row of the FTMS, split version, can be found in Section 2. The more general version of the above theorem says that, starting from a variety of the second row of the FTMS, split version, embedded in a non-degenerate quadric, the symplecta contained in totally isotropic subspaces give rise to a variety of the FTMS one cell above. There is only one slight difference, being that for the smallest case of the Veronese variety $\mathcal{V}_2(\mathbb{K})$, we have to require that the quadric of $\mathsf{PG}(5,\mathbb{K})$ containing $\mathcal{V}_2(\mathbb{K})$ has totally isotropic planes, i.e., it has maximal Witt index. For the larger cases, we can prove that a non-degenerate quadric in which they embed automatically has maximal Witt index, but there are counterexamples in the smallest case (see Remark 5.3). This is not a huge drawback though, because if Q is a non-degenerate quadric of $\mathsf{PG}(5,\mathbb{K})$ containing $\mathcal{V}_2(\mathbb{K})$, then Q has maximal Witt index if and only if there is at least one projective symp, so if the Witt index is not maximal, the geometry induced by the projective symps is trivial.

Apart from this new connection, we devote Section 6 to explaining in some more detail the common properties of the varieties of the first and second row of the FTMS, split version, and the connections using (symplectic) polarities and hyperplane sections that were already mentioned above.

Finally, for $\mathcal{E}_6(\mathbb{K})$, we also show a converse to the above theorem:

Theorem 1.2 *Given a variety $\mathcal{F}_{4,4}(\mathbb{K})$, arising from a symplectic polarity ρ of $\mathcal{E}_6(\mathbb{K})$, there is a unique polarity $\mathfrak{p}$ of $\mathsf{PG}(26,\mathbb{K})$ such that the corresponding non-degenerate quadric Q contains $\mathcal{E}_6(\mathbb{K})$ and such that its projective symplecta with respect to Q are precisely the absolute symplecta of $\mathcal{E}_6(\mathbb{K})$ with respect to ρ.*

This means that the symplectic polarities of $\mathcal{E}_6(\mathbb{K})$ are in $1-1$-correspondence with the polarities of $\mathsf{PG}(26,\mathbb{K})$ whose absolute geometries contain $\mathcal{E}_6(\mathbb{K})$. The same holds for the second and third column, but we focus on the largest and most involved case.

1.2 Consequences

We list some consequences of our main results, again focussing on the largest case of $\mathcal{E}_6(\mathbb{K})$. This time, we also restrict our attention to $\mathcal{E}_6(\mathbb{K})$ when it comes to their proofs, since this is the most involved case and the other cases could be treated analogously.

Corollary 1.3 *There are, up to projectivity, three types of (non-trivial, possibly degenerate) quadrics in* $\mathsf{PG}(26,\mathbb{K})$ *containing* $\mathcal{E}_6(\mathbb{K})$*. The radical of these quadrics has dimension* 16*,* 8 *or* -1 *and, projected from the radical, one obtains a quadric of maximal Witt index. In particular, each non-degenerate quadric of* $\mathsf{PG}(26,\mathbb{K})$ *containing* $\mathcal{E}_6(\mathbb{K})$ *is projectively equivalent to the parabolic quadric* $Q(26,\mathbb{K})$*.*

The more general situation where the quadric of $\mathsf{PG}(26,\mathbb{K})$ that contains $\mathcal{E}_6(\mathbb{K})$ is allowed to be degenerate is treated in Proposition 7.29. The three options for these quadrics correspond precisely to the three types of hyperplane intersections mentioned earlier on. Replacing 16 and 8 by $2d$ and d, the above corollary also holds for the analogs corresponding to the second and third column of the FTMS, where $d = 2$ and $d = 4$, respectively, but we do not provide a proof for this. The fact that there is only one kind of non-degenerate quadric containing $\mathcal{E}_6(\mathbb{K})$ is also true for the second and third column of the FTMS, as follows almost immediately from our other results. Together with the above corollary, this is the content of Main Result 3.6. For the first column, a similar result fails, as was mentioned in the previous subsection.

Finally, it is known that $\mathcal{E}_6(\mathbb{K})$ is projectively unique in $\mathsf{PG}(26,\mathbb{K})$. As a consequence of Theorem 1.2, we have the following more precise statement:

Corollary 1.4 *Suppose* Q *is a non-degenerate quadric in* $\mathsf{PG}(26,\mathbb{K})$ *containing* $\mathcal{E}_6(K)$*. Then* $\mathcal{E}_6(K)$ *is projectively unique on* Q*.*

Indeed, if two varieties $\mathcal{E}_6(\mathbb{K})$ are embedded in a non-degenerate quadric Q, then we show in Corollary 7.32 that there is a collineation of Q mapping one to the other. Similarly, this would also follow for the second and third column, since one could show an analogue of Theorem 1.2. For the first column, the analogue of the above corollary is proven in Proposition 7.4, as it is needed for the proof of the main theorem in this case.

1.3 Methods

For the largest case, i.e., for $\mathcal{E}_6(\mathbb{K})$, the clue is that the projective symps with respect to the non-degenerate quadric containing it, constitute a geometric hyperplane in the dual of $\mathcal{E}_6(\mathbb{K})$, meaning that for each 4-space in $\mathcal{E}_6(\mathbb{K})$, either all symps or just one symp through it, are projective. Since the dual of $\mathcal{E}_6(\mathbb{K})$ also gives an $\mathcal{E}_6(\mathbb{K})$ variety, we can then make use of the strong classification result of the geometric hyperplanes of $\mathcal{E}_6(\mathbb{K})$ due to Cooperstein and Shult (see Proposition 4.13, or [5]).

The same approach would also work for the second largest case, as there is an analogous classification result of the (dual) geometric hyperplanes, by Shult (see Proposition 4.14, or [13]). We however choose not to take this path, since we prefer to give a constructive proof where we geometrically construct the polarity of $\mathcal{G}_{6,2}(\mathbb{K})$ and its absolute geometry, which will be isomorphic to $\mathcal{C}_{3,2}(\mathbb{K})$, the line Grassmannian variety of a symplectic polar space of rank 3, living in $\mathsf{PG}(13,\mathbb{K})$. We consider this hands-on approach slightly more illuminating and, as such, a worthy complement of the more efficient approach in the largest case.

For the second smallest case of $\mathcal{S}_{2,2}(\mathbb{K})$, the approach we take is in the same vein as the second largest case. An analogous approach to the largest case fails. Indeed, there is no counterpart of the classification of the geometric hyperplanes, as there are geometric hyperplanes of $\mathcal{S}_{2,2}(\mathbb{K})$ which do not arise from hyperplanes of $\mathsf{PG}(8,\mathbb{K})$. We note however that the intersection of hyperplanes of $\mathsf{PG}(8,\mathbb{K})$ with $\mathcal{S}_{2,2}(\mathbb{K})$ still has the same behaviour as in the previous cases—more on this later.

Finally, in the smallest case of $\mathcal{V}_2(\mathbb{K})$, we have to take an entirely different approach. This is mostly due to the fact that $\mathcal{V}_2(\mathbb{K})$ contains no singular lines (no line of $\mathsf{PG}(5,\mathbb{K})$ is entirely contained in $\mathcal{V}_2(\mathbb{K})$) and the fact that the resulting geometry is not very rich in structure: there are only points.

1.4 Origin of the problem

In [8], a paper by one of the current authors, Hendrik Van Maldeghem and Narasimha Sastry, the smallest case of the problem in the current paper occured when studying the full embeddings of an abstract geometry of type $\mathsf{F}_4(\mathbb{K})$ in an abstract geometry of type $\mathsf{E}_{7,7}(\mathbb{K})$. This study entailed the embedding of a projective plane over $\mathbb{K}$ in a geometry of type $\mathsf{A}_{5,2}(\mathbb{K})$ (looking at the induced embeddings of the corresponding line residues in each other). In one of the subcases, the embedding of the projective plane in $\mathsf{A}_{5,2}(\mathbb{K})$ is such that it is actually contained in a subgeometry of type $\mathsf{A}_{3,2}(\mathbb{K})$, or equivalently, in a hyperbolic quadric $Q(5,\mathbb{K})$ in $\mathsf{PG}(5,\mathbb{K})$. This embedding was such that the points of the projective plane where points of $Q(5,\mathbb{K})$, and the lines where certain conics, which either arose as the intersection of $Q(5,\mathbb{K})$ with a non-isotropic plane, or were contained in a totally isotropic plane of $Q(5,\mathbb{K})$. This yields exactly the situation as described above: a Veronese variety $\mathcal{V}_2(\mathbb{K})$ on $Q(5,\mathbb{K})$ where some of its conics might be contained in isotropic planes and some might not. It was then proven (although, afterwards, it turned out not to be necessary for that paper) that the constellation of the lines in $\mathsf{PG}(2,\mathbb{K})$ embedded in a totally isotropic subspace of $Q(5,\mathbb{K})$ forms a dual oval. This paper contains the proof of that fact, and extends it by showing that it is actually a dual conic. Although a (dual) conic is a very generic object, Van Maldeghem linked it with the first cell of the first row of the FTMS, since the first cell of the second row of the FTMS contains the Veronese variety $\mathcal{V}_2(\mathbb{K})$. Based on this single observation, he then conjectured that also the other geometries of the second row of the FTMS should exhibit similar behaviour. In this paper, we studied this phenomenon for the entire second row of the FTMS (split version). The definition of the resulting geometry—which was not clear at the time—can be found in Definition 3.3, preceeding our main theorem. We thank Van Maldeghem for the interesting suggestion and his helpful remarks on our findings.

2 Preliminaries

In this section we provide the necessary background and introduce the varieties that we encounter.

2.1 Point-line geometries, parapolar spaces

Most of the geometries that we will encounter, are instances of parapolar spaces. Parapolar spaces are point-line geometries which are locally polar spaces: points at distance 2 either are on a unique shortest path or their convex closure is isomorphic to a polar space. The formal definition hence requires the notions "polar space", "distance", "connectedness", and "convexity", which we provide for clarity.

A *point-line geometry* Γ is a pair $\Gamma = (Y, \mathcal{M})$ where Y is a set of points and $\mathcal{M}$ a non-empty set of lines, each of which is a subset of Y. Some standard terminology:

- A *subspace* S of Γ is a subset of Y with the property that each line not contained in S intersects S in at most one point.
- *Collinearity* between points corresponds to being contained in a common line (not necessarily unique), and we denote this by the symbol $\perp$. Note that a point is collinear to itself if there is at least one line through it.
- A subspace is called *singular* if each pair of its points is collinear. Two singular subspaces S_1, S_2 are collinear to each other if each point of S_1 is collinear to each point of S_2. If $S_1, \ldots, S_n$ is a collection of pairwise collinear singular subspaces, then we denote the subspace generated by them by $\langle S_1, \ldots, S_n \rangle$. In the special case that two distinct collinear points x, y determine a unique line, we also denote this line by xy.
- The *collinearity graph* of Γ is the graph on Y with collinearity as adjacency relation. The *distance* $\delta(x, y)$ between two points $x, y \in Y$ is the distance between x and y in the collinearity graph (possibly $\delta(x, y) = \infty$ if there is no path between them). A path between x and y of length $\delta(x, y)$ is called a *shortest path.* The diameter of Γ is the diameter of its collinearity graph. We say that Γ is *connected* if for every two points x, y of Y, $\delta(x, y) < \infty$.
- A subspace $S \subseteq Y$ is called *convex* if all shortest paths between points $x, y \in S$ are contained in S. The *convex subspace closure* of a set $S \subseteq Y$ is the intersection of all convex subspaces containing S (this is well defined since Y is a convex subspace itself).
- A *geometric hyperplane* of Γ is a subspace S of Γ with the property that each line of Γ shares at least one point with S; it is called proper if it does not coincide with Y.

Definition 2.1 A point-line geometry $\Delta = (Y, \mathcal{M})$ is a *polar space* if the Buekenhout-Shult axioms are satisfied:

(BS1) every line has at least three points;
(BS2) no point is collinear to all other points;
(BS3) every nested sequence of singular subspaces is finite;
(BS4) for each pair $(y, M) \in Y \times \mathcal{M}$ either one or all points of M are collinear to y.

The last axiom is called the *1-or-all axiom.* One can show that every singular subspace of Δ is a projective space, finite-dimensional by (BS3), and that there is a natural number $r \geq 2$ (called the *rank* of Δ) such that the maximal singular subspaces of Δ all have dimension $r - 1$. These axioms where introduced by Buekenhout and Shult in the '70s and are equivalent to the original definition given by Tits in the '60s.

Definition 2.2 A connected point-line geometry $\Gamma = (Y, \mathcal{M})$ is a *parapolar space* if

(PPS1) for every pair of non-collinear points p and q in Y, with $|p^\perp \cap q^\perp| > 1$, the convex subspace closure of $\{p, q\}$ is a polar space, called a *symplecton* (a *symp* for short);
(PPS2) each line of $\mathcal{M}$ is contained in a symplecton;
(PPS3) no symplecton contains all points of Y.

The parapolar space is called *strong* if there are no pairs of points $p, q \in Y$ with $|p^\perp \cap q^\perp| = 1$.

In general, a parapolar space is not necessarily embedded in a projective space, yet the parapolar spaces we will encounter in the current paper are. Consequently, its singular subspaces will be projective. The symps of a parapolar space need not be isomorphic, and can even have different ranks. The symplectic rank of a parapolar space is hence the set of natural numbers occurring as the rank of a symp.

2.2 Quadrics, ovoids and conics

Henceforth, let $\mathbb{K}$ be a (commutative) field. For a non-zero cardinal number n, we denote by $\mathsf{PG}(n, \mathbb{K})$ the n-dimensional projective space over $\mathbb{K}$.

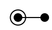
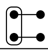
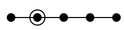
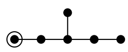
A *quadric* Q in $\mathsf{PG}(n, \mathbb{K})$, n finite, is the null set of a quadratic homogeneous polynomial in the (homogeneous) coordinates of points of $\mathsf{PG}(n, \mathbb{K})$.

- Two points x_1, x_2 on Q are called *collinear* if all points on the line x_1x_2 are contained in Q (notation: $x_1 \perp x_2$) (we also say that the line x_1x_2 is *singular*); if x_1 and x_2 are not collinear, then $x_1x_2 \cap Q = \{x_1, x_2\}$.
- A *tangent line to Q (at a point $x \in Q$)* is a line which has either only x or all its points in Q. The union of the set of tangent lines to Q at one of its points x is a subspace, denoted by $T_x(Q)$, of dimension at least $n - 1$.
- The set of points $\{x \in Q \mid T_x(Q) = \mathsf{PG}(n, \mathbb{K})\}$ forms a subspace of $\mathsf{PG}(n, \mathbb{K})$ and is called the *radical* of Q. If the radical is empty, then we say that Q is non-degenerate. Projecting from the radical yields a non-degenerate quadric.
- The *projective index* of Q is the (common) dimension of the maximal singular subspaces of $\mathsf{PG}(n, \mathbb{K})$ entirely contained in Q, the *Witt index* of Q is the projective index plus one. A non-degenerate quadric of Witt index w with $w > 1$ is in particular a polar space of rank w. We say that a non-degenerate quadric Q has *maximal Witt index* if it has Witt index $\lfloor \frac{n+1}{2} \rfloor$; these are also known as *parabolic* quadrics (if n is even) or as *hyperbolic* quadrics (if n is odd). We denote the non-degenerate quadric of maximal Witt index by $Q(n, \mathbb{K})$. If $n = 2$, then $Q(2, \mathbb{K})$ is also called a *conic*.

An *ovoid* O of $\mathsf{PG}(n, \mathbb{K})$ is a set of points which behaves like a non-degenerate quadric of Witt index 1: no three points of O are collinear, and the union of the set of tangent lines (defined as above) at each point is a hyperplane of $\mathsf{PG}(n, \mathbb{K})$. If $n = 2$, an ovoid is referred to as an *oval*.

2.3 The varieties of the second row of the split FTMS

The projective varieties in the table below

$\mathcal{V}_2(\mathbb{K})$	$\mathcal{S}_{2,2}(\mathbb{K})$	$\mathcal{G}_{6,2}(\mathbb{K})$	$\mathcal{E}_6(\mathbb{K})$
			

form the split version of the second row of the FTMS (the encircled node is the one corresponding to the point set). Their absolute types are, respectively, $\mathsf{A}_{2,1}(\mathbb{K})$, $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, $\mathsf{A}_{5,2}(\mathbb{K})$ and $\mathsf{E}_{6,1}(\mathbb{K})$. We make a distinction between the variety, i.e., the geometry embedded in a certain way in projective space, and the abstract geometry, by using curly letters for the former.

We first describe each of these geometries separately, and then give their common properties in Section 4.1.

Quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$ in $\mathsf{PG}(5, \mathbb{K})$ — The *quadric Veronese variety* $\mathcal{V}_2(\mathbb{K})$ is the set of points in $\mathsf{PG}(5, \mathbb{K})$ obtained by taking the images of all points of $\mathsf{PG}(2, \mathbb{K})$ under the Veronese map ν, which maps the point (x, y, z) of $\mathsf{PG}(2, \mathbb{K})$ to the point $(x^2, y^2, z^2; yz, zx, xy)$. The set of points of $\mathsf{PG}(5, \mathbb{K})$ that are in the image of ν is denoted by X. The set of planes of $\mathsf{PG}(5, \mathbb{K})$ which contain the image of a line of $\mathsf{PG}(2, \mathbb{K})$ is denoted by Ξ. One can verify that for each $\xi \in \Xi$, the intersection $\xi \cap X$ is a conic in the plane ξ.

Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ in $\mathsf{PG}(8, \mathbb{K})$ — The *Segre variety* $\mathcal{S}_{2,2}(\mathbb{K})$ is the direct product of two projective planes π_1 and π_2 over $\mathbb{K}$, its set of points in $\mathsf{PG}(8, \mathbb{K})$ being given by taking the images of all pairs of points (x_0, x_1, x_2) in π_1 and (y_0, y_1, y_2) in π_2, under the Segre map

$$\sigma((x_0, x_1, x_2), (y_0, y_1, y_2)) = (x_iy_j)_{0 \leq i \leq 2; 0 \leq j \leq 2}.$$

Note that the image of two lines of π_1 and π_2 respectively under σ is isomorphic to the direct product of two lines and gives hyperbolic quadric of rank 2 in a 3-space of $\mathsf{PG}(8, \mathbb{K})$.

Line Grassmannian variety $\mathcal{G}_{6,2}$ in $\mathsf{PG}(14, \mathbb{K})$ — The *line Grassmannian variety* $\mathcal{G}_{6,2}(\mathbb{K})$ of $\mathsf{PG}(5, \mathbb{K})$ is the set of points of $\mathsf{PG}(14, \mathbb{K})$ obtained by taking the images of all lines of $\mathsf{PG}(5, \mathbb{K})$ under the Plücker map

$$\tau(\langle(x_0, x_1, \ldots, x_5), (y_0, y_1, \ldots, y_5)\rangle) = \left(\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \right)_{0 \leq i < j \leq 5}.$$

In this case, the image of a 3-space is, by the Klein correspondence, a hyperbolic quadric of rank 3 in a 5-space of $\mathsf{PG}(14, \mathbb{K})$.

The Cartan variety $\mathcal{E}_6(\mathbb{K})$ in $\mathsf{PG}(26, \mathbb{K})$ — Since we will not need the precise definition of the variety $\mathcal{E}_6(\mathbb{K})$, which is the projective version of the well known 27-dimensional module of the (split) exceptional group of Lie type E_6, we simply refer to the literature here. Aschbacher [1] provides an algebraic description, Cohen [4] provides a construction using intersections of quadrics, a version of which can also be found in Section 10 of [10]. We only note that, just as in the previous cases, $\mathcal{E}_6(\mathbb{K})$ consists of a point set and a set of hyperbolic quadrics, this time of rank 5 and living in 9-dimensional subspaces of $\mathsf{PG}(26, \mathbb{K})$.

The above geometries can all be described as the image of the Veronese map, applied to certain point-line geometries, namely ring projective planes over a split composition algebra $\mathbb{A}$ over $\mathbb{K}$, more precisely: either over $\mathbb{K}$ itself, over $\mathbb{K} \times \mathbb{K}$, over the split quaternions $\mathbb{H}'$ over $\mathbb{K}$, or over the split octonions $\mathbb{O}'$ over $\mathbb{K}$ (see Section 10 of [10]). Consequently, these geometries live in $\mathsf{PG}(3d + 2, \mathbb{K})$, where $d = \dim_{\mathbb{K}}(\mathbb{A}) \in \{1, 2, 4, 8\}$. Hence, in accordance with the smallest case $\mathcal{V}_2(\mathbb{K})$, we will denote the point set of the above varieties with X_d (corresponding to the encircled node) and their family of $(d+1)$-subspaces containing the images of the lines with Ξ_d, for $d \in \{1, 2, 4, 8\}$ (corresponding to the node symmetric to the encircled node). One of the properties of Veronese varieties is that, for each $\xi \in \Xi_d$, the intersection $X_d \cap \xi$ is precisely the image of a line, i.e., it contains no additional points of X_d. If $d > 1$, the image of a line is isomorphic to a hyperbolic polar space, i.e., a polar space of rank $2, 3, 5$ living in dimension $3, 5, 9$, respectively. We discuss the common features in more detail in Section 4.1.

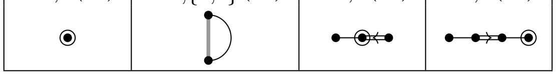
Associated point-line geometries We define $\mathcal{M}_d$ as the set of lines M of $\mathsf{PG}(3d+2, \mathbb{K})$ with $M \subseteq X_d$, i.e., the singular lines of X_d. Note that $\mathcal{M}_d = \emptyset$ precisely if $d = 1$. With slight abuse of notation in case $d = 1$, the geometry $\Gamma_d := (X_d, \mathcal{M}_d)$ is referred to as the *point-line geometry associated to* (X_d, Ξ_d).

Next, we describe the *dual point-line geometry* associated to (X_d, Ξ_d), which has Ξ_d as its point set (whence "dual"). Two members ξ_1, ξ_2 of Ξ_d will be collinear when their intersection is more than a point, or equivalently (cf. Fact 4.1), a subspace of dimension $\lceil \frac{d}{2} \rceil$ (so two members of Ξ_1 are never collinear). The line determined by the collinear quadrics ξ_1 and ξ_2 consists of the members of Ξ_d containing $\xi_1 \cap \xi_2$ which moreover, if $d = 4$, share a plane with a singular 3-space (note that this 3-space then automatically intersects the plane $\xi_1 \cap \xi_2$ in a line). For each d, we denote the line determined by ξ_1, ξ_2 by $S(\xi_1, \xi_2)$ or by $S(X(\xi_1), X(\xi_2))$ (the notation $\langle \xi_1, \xi_2 \rangle$ would be ambiguous), and set of all such lines by $\mathcal{S}_d$. The resulting dual point-line geometry, again with abuse of notation if $d = 1$, is denoted by $\Gamma_d^* := (\Xi_d, \mathcal{S}_d)$. For the geometries we are concerned with, $\Gamma_d \cong \Gamma_d^*$ (as can be seen from the symmetry in the diagram). All of this can be deduced from the properties of these geometries (see also Fact 4.9).

2.4 The varieties of the first row of the split FTMS

Below we give the first row of the FTMS (split version), as Dynkin diagrams and their corresponding relative type as a building.

$\mathsf{A}_{1,1}(\mathbb{K})$	$\mathsf{A}_{1,\{1,2\}}(\mathbb{K})$	$\mathsf{C}_{3,2}(\mathbb{K})$	$\mathsf{F}_{4,4}(\mathbb{K})$



Conic in $\mathsf{PG}(4,\mathbb{K})$ — With this, we mean the image under the Veronese map ν of a (non-degenerate) conic C in the projective plane $\mathsf{PG}(2,\mathbb{K})$.

Thin generalized hexagon in $\mathsf{PG}(7,\mathbb{K})$ — A generalized hexagon (6-gon) is a point-line geometry $(\mathcal{P},\mathcal{L})$ which has no ordinary m-gons for $2 \le m < 6$ (in particular, it is a partial linear space) but for which each pair $X, Y \in \mathcal{P} \cup \mathcal{L}$ is contained in an ordinary 6-gon. The generalized hexagon that we encounter is thin: each point is contained in exactly two lines.

Polar line Grassmannian variety $\mathcal{C}_{3,2}(\mathbb{K})$ **in** $\mathsf{PG}(13,\mathbb{K})$ — The polar space of type $\mathsf{C}_{3,1}(\mathbb{K})$ is the symplectic polar space in $\mathsf{PG}(5,\mathbb{K})$, i.e., it arises as the absolute geometry of a symplectic polarity p of $\mathsf{PG}(5,\mathbb{K})$. By the variety $\mathcal{C}_{3,2}(\mathbb{K})$ we mean the subvariety of $\mathcal{G}_{6,2}(\mathbb{K})$ obtained by considering the image under the Plücker map of the absolute lines of $\mathsf{PG}(5,\mathbb{K})$ under p.

Metasymplectic space $\mathcal{F}_{4,4}(\mathbb{K})$ **in** $\mathsf{PG}(25,\mathbb{K})$ — The metasymplectic space under consideration is the geometry associated to a split thick building of type $\mathsf{F}_{4,4}(\mathbb{K})$. It has points, lines and planes as singular subspaces and its symps are isomorphic to symplectic polar spaces of rank 3. For more details on its geometric properties, we refer to [3]. The variety $\mathcal{F}_{4,4}(\mathbb{K})$ that we will encounter is the absolute geometry of a certain polarity of a split building of type $\mathsf{E}_{6,1}(\mathbb{K})$.

3 Main result

As mentioned above, the geometries $\mathcal{V}_2(\mathbb{K})$, $\mathcal{S}_{2,2}(\mathbb{K})$, $\mathcal{G}_{6,2}(\mathbb{K})$ and $\mathcal{E}_6(\mathbb{K})$ of the second row of the split FTMS are point-quadric varieties (X_d, Ξ_d) with $d = 1, 2, 4, 8$, respectively. The associated point-line geometry is $\Gamma_d = (X_d, \mathcal{M}_d)$, and $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$ is its dual. We will need the following notion.

Definition 3.1 A point-line geometry $(Y, \mathcal{M})$ is *fully embedded* in a point-line geometry $(Y', \mathcal{M}')$ if $Y \subseteq Y'$ and for each $M \in \mathcal{M}$ there is a line $M' \in \mathcal{M}'$ such that $M = M'$ (viewed as subsets of Y and Y').

The first (full) embedding that we consider is that of Γ_d in a quadric Q_d of $\mathsf{PG}(3d+2,\mathbb{K})$ (the existence of which is proven in Section 5), where Γ_d is embedded in $\mathsf{PG}(3d+2,\mathbb{K})$ in the usual way (as described above), and where Q_d is considered as a point-line geometry in the natural way: using its set of points and its set of singular lines.

Terminology. Two points of X_d which are on a line of $\mathcal{M}_d$ are called Γ_d-collinear, and since the embedding is full, these points are also on a line of Q_d and hence Q_d-collinear. If the context is clear we will sometimes just call them collinear. Two points of X_d that are Q_d-collinear are not necessarily Γ_d-collinear.

Before stating the main result, we have a look at the embedding of the members of Ξ_d in Q_d, as this will be a key element.

Lemma 3.2 *Suppose $\Gamma_d = (X_d, \mathcal{M}_d)$, with $d \in \{1,2,4,8\}$ and a quadric Q_d of $\mathsf{PG}(3d+2,\mathbb{K})$ (hyperbolic if $d=1$) are embedded in $\mathsf{PG}(3d+2,\mathbb{K})$ as usual, and such that Γ_d is fully embedded in Q_d. Let ξ be a member of Ξ_d. Then ξ is either a singular subspace of Q_d, or $X_d \cap \xi$ embeds isometrically in Q_d (i.e., non-collinear points of $X_d \cap \xi$ are non-collinear in Q_d) and arises as the intersection of ξ with Q_d.*

Proof If $d > 1$, this follows from Lemma 3.19 of [8]. So suppose $d = 1$. Take any $\xi \in \Xi_1$ and recall that ξ is a plane. If ξ is a singular plane of Q_1, there is nothing to show, so suppose that ξ is not contained in a singular subspace of Q_1. By assumption $\xi \cap X_1$ is a non-degenerate conic, which is contained in $\xi \cap Q_1$, and the latter is a (possibly degenerate) conic. We claim that $\xi \cap X_1$ coincides with $\xi \cap Q_1$. Since five points determine a unique conic (whether it is degenerate or not), the claim is trivally true if $|\xi \cap X_1| = |\mathbb{K}| + 1 \geq 5$. If $|\mathbb{K}| < 4$, we consider a quadratic field extension $\mathbb{L}$ of $\mathbb{K}$, and the image of $\mathsf{PG}(2,\mathbb{L})$ under the Veronese map then lies on a hyperbolic quadric in $\mathsf{PG}(5,\mathbb{L})$ with the same equation as Q_1. Since $|\mathbb{L}| \geq 4$, the claim also follows in these smaller cases. □

Our object of interest is given by the members of Ξ_d which embed in a singular subspace of Q_d, with which we define a subgeometry of the dual geometry $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$ associated to $\Gamma_d = (X_d, \mathcal{M}_d)$ (recall its definition given at the end of Subsection 2.3). More precisely we define:

Definition 3.3 – Let $\mathcal{P}_d$ denote the set $\{\xi \cap X_d \mid \xi \in \Xi_d \text{ with } \xi \text{ in a singular subspace of } Q_d\}$, whose members will be referred to as *projective* conics (if $d=1$) or symps (if $d>1$).
- Let $\mathcal{L}_d$ be the subset of $\mathcal{M}_d$, where a line L belongs to $\mathcal{L}_d$ if for each $\xi \in \Xi_d$ with $L \subseteq \xi$, the conic/symp $X_d \cap \xi$ belongs to $\mathcal{P}_d$. Note that $\mathcal{L}_1$ is empty.
- We say that distinct members Σ_1, Σ_2 of $\mathcal{P}_d$ are collinear (denoted $\Sigma_1 \perp_d \Sigma_2$) if $\Sigma_1 \cap \Sigma_2$ is covered by lines of $\mathcal{L}_d$ fully contained in $\Sigma_1 \cap \Sigma_2$, i.e.,

$$\Sigma_1 \perp_d \Sigma_2 \Leftrightarrow \Sigma_1 \cap \Sigma_2 = \bigcup \{L \in \mathcal{L}_d : L \subseteq \Sigma_1 \cap \Sigma_2\}.$$

  The line determined by collinear Σ_1, Σ_2 is then the line $S(\Sigma_1, \Sigma_2)$ of the dual point-line geometry $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$ associated to (X_d, Ξ_d) determined by Σ_1 and Σ_2. We denote the point-line geometry with point set $\mathcal{P}_d$ and the described induced line set by $\widetilde{\Gamma_d^*}$.
- If $d > 1$, let $\mathcal{X}_d$ denote the subset of X_d of points contained in some line of $\mathcal{L}_d$, i.e., $\mathcal{X}_d = \bigcup \mathcal{L}_d$.
- Let $\mathcal{X}_1$ denote the subset of X_1 of points contained in a unique member of $\mathcal{P}_d$.

With this definition at hand, we can now formulate our first main result.

Main Result 3.4 *Let $\Gamma_d = (X_d, \mathcal{M}_d)$ be the point-line geometry associated to the geometry $\mathcal{V}_2(\mathbb{K})$ (if $d=1$), $\mathcal{S}_{2,2}(\mathbb{K})$ (if $d=2$), $\mathcal{G}_{6,2}(\mathbb{K})$ (if $d=4$) or $\mathcal{E}_6(\mathbb{K})$ (if $d=8$), living inside $\mathbb{P}_d = \mathsf{PG}(3d+2,\mathbb{K})$. Then Γ_d is fully embedded in a non-degenerate quadric Q_d of $\mathbb{P}_d$ (of maximal Witt index, if $d=1$), and $\widetilde{\Gamma_d^*}$, as defined above, is a full subgeometry of the dual geometry $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$ and*

- *$\widetilde{\Gamma_1^*}$ is isomorphic to a conic;*
- *$\widetilde{\Gamma_2^*}$ is isomorphic to a thin generalized hexagon;*
- *$\widetilde{\Gamma_4^*}$ is isomorphic to a polar line Grassmannian $\mathcal{C}_{3,2}(\mathbb{K})$;*
- *$\widetilde{\Gamma_8^*}$ is isomorphic to a metasymplectic space $\mathcal{F}_{4,4}(\mathbb{K})$.*

Moreover, there is a canonical bijection between $\mathcal{P}_d$ and $\mathcal{X}_d$, and if $d>1$, this extends to an isomorphism between $\widetilde{\Gamma_d^*}$ and the subgeometry $(\mathcal{X}_d, \mathcal{L}_d)$ of Γ_d. Finally, the points of $\mathcal{X}_d$ arise as a hyperplane section, i.e., $\mathcal{X}_d = X_d \cap H_d$ with H_d a hyperplane of $\mathbb{P}_d$.

The definition of collinearity in $\widetilde{\Gamma_d^*}$ might seem strange at first, hence we add a case-by-case characterisation, which we will encounter throughout the paper.

Lemma 3.5 *Two distinct members of $\mathcal{P}_1$ are never collinear; two distinct members of $\mathcal{P}_d$ with $d > 1$ are collinear if and only if:*

($d = 2$) $\Sigma_1 \cap \Sigma_2$ is a line of $\mathcal{L}_2$;
($d = 4$) $\Sigma_1 \cap \Sigma_2$ is a plane each line of which belongs to of $\mathcal{L}_4$;
($d = 8$) $\Sigma_1 \cap \Sigma_2$ is a 4-space and there is a unique line L such that the set of lines of $\mathcal{L}_8$ inside $\Sigma_1 \cap \Sigma_2$ is exactly the set of lines of $\Sigma_1 \cap \Sigma_2$ which meet L in at least a point.

Proof For $d = 1$ this is clear, for $d \in \{2, 4, 8\}$, this is shown in Lemmas 7.6, 7.17 and 7.30, respectively. □

As a side result, we obtain the following theorem concerning the quadrics containing a variety of the second row.

Main Result 3.6 *Let $\Gamma_d = (X_d, \mathcal{M}_d)$ be the point-line geometry associated to the geometry $\mathcal{S}_{2,2}(\mathbb{K})$ (if $d = 2$), $\mathcal{G}_{6,2}(\mathbb{K})$ (if $d = 4$) or $\mathcal{E}_6(\mathbb{K})$ (if $d = 8$), living inside $\mathbb{P}_d = \mathsf{PG}(3d + 2, \mathbb{K})$. Then each non-degenerate quadric Q_d in $\mathsf{PG}(3d + 2, \mathbb{K})$ containing Γ_d has maximal Witt index. Moreover, if $d = 8$, then the only non-trivial quadrics of $\mathsf{PG}(26, \mathbb{K})$ containing $\mathcal{E}_6(\mathbb{K})$, are the following:*

- *a non-degenerate parabolic quadric $Q(26, \mathbb{K})$;*
- *a degenerate quadric whose radical is a 16-space (which is the tangent space $T_p(X_8)$ of a point p of Γ_d) and whose base is a hyperbolic quadric of rank 4 in dimension 9;*
- *a degenerate quadric whose radical is an 8-space (a subquadric of a symp, of type $Q(8, \mathbb{K})$) and whose base is a hyperbolic quadric of rank 9 in dimension 17.*

Proof This is shown in Lemma 6.10 for $d = 2, 4$ and in Proposition 7.29 for $d = 8$. □

Structure of the proof

In Section 4.1, we first discuss the common featues of the varieties of the first and second row of the FTMS, and describe their connections (in terms of hyperplane sections and as absolute geometry of a polarity). In Section 5, we show that the varieties of the second row can be embedded in a non-degenerate quadric, with for the first cell the extra condition that it has maximal Witt index, by giving the equations explicitly. The different behaviour between the first cell ($d = 1$) and ($d = 2, 4, 8$) becomes even more pronounced in Section 6, where we start with a general treatment of the cases with $d \in \{2, 4, 8\}$. After that, in the final Section 7, a case-by-case proof is given, including the $d = 1$ case. Crucial to the approach taken for $d = 8$ is the classification of the geometric hyperplanes of $\mathcal{G}_{6,2}(\mathbb{K})$ and $\mathcal{E}_6(\mathbb{K})$, as given in Propositions 4.13. It is possible to treat the cases $d = 4$ and $d = 8$ analogously, however, we choose to give a constructive proof for the $d = 4$ case instead—constructive in the sense that we construct a polarity of $\mathcal{G}_{6,2}(\mathbb{K})$ having $\mathcal{P}_4$ as set of absolute symps, and whose absolute geometry is isomorphic to $\mathcal{C}_{3,2}(\mathbb{K})$. This complements the directness of the proof for the $d = 8$ case and provides additional insight in these structures. Since the nature of the geometric hyperplanes of $\mathcal{S}_{2,2}(\mathbb{K})$ is different, as mentioned before, the case $d = 2$ has to be dealt with differently, with common elements to the constructive proof for the $d = 4$ case.

4 Properties of and connections between the varieties of the first and second row of the FTMS, split version

In this section we give an overview of the geometric properties of the varieties on the first and second row of the FTMS, split version. Our interest goes out mostly to, on the one hand, the geometric properties of the varieties on the second row, split version, and on the other hand, the way that a variety of the first row can be seen 'inside' the one on the row below it.

4.1 Common properties of the varieties of the second row

Let (X_d, Ξ_d) be the point-quadric system associated to $\mathcal{V}_2(\mathbb{K})$ ($d = 1$), $\mathcal{S}_{2,2}(\mathbb{K})$ ($d = 2$), $\mathcal{G}_{6,2}(\mathbb{K})$ ($d = 4$) or $\mathcal{E}_6(\mathbb{K})$ ($d = 8$). Recall that $\mathcal{M}_d$ denotes the set of lines of the ambient projective space whose points are completely contained in X_d. We use the symbol $\perp$ to denote collinearity with respect to the line set $\mathcal{M}_d$, or sometimes we use $\perp_{\Gamma_d}$.

We list some well-known properties satisfied by these geometries. Many of them can be verified by using the diagram, or, for $d = \{1, 2\}$, the properties of the Veronese and Segre map, or for $d = 4$, by using the correspondence with the projective 5-space $\mathsf{PG}(5, \mathbb{K})$. We refer to [2] especially for the largest case $d = 8$.

In [12], the geometries under consideration have been characterised by a much weaker version of the following properties.

Fact 4.1 *For each $d \in \{1, 2, 4, 8\}$, the geometry (X_d, Ξ_d) satisfies the following properties:*

(P1) *Any pair of non-collinear points x and y of X_d lies in a unique element of Ξ_d, and each singular subspace of $(X_d, \mathcal{M}_d)$ of dimension at most $\lfloor \frac{d}{2} \rfloor$ is contained in a member of Ξ_d.*
(P2) *If $\xi_1, \xi_2 \in \Xi_d$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subseteq X_d$. Moreover, $\dim(\xi_1 \cap \xi_2) \in \{0, \lfloor \frac{d}{2} \rfloor\}$.*
(P3) *If $x \in X_d$, then $T_x(X_d) := \langle T_x(\xi) \mid x \in \xi \in \Xi_d \rangle$ has dimension $2d$.*

Notation. If $x, y \in X_d$ are non-collinear points, then we denote the unique element of Ξ_d containing them by $[x, y]$ and we denote the conic or symp $X_d \cap [x, y]$ by $\Sigma_{x,y}$.

Remark 4.2 *For each $d \in \{2, 4, 8\}$, $\Gamma_d = (X_d, \mathcal{M}_d)$ is a strong parapolar space of diameter 2 with hyperbolic symplecta of rank $\frac{d}{2} + 1$. In particular, for each $\xi \in \Xi_d$, the symp $\xi \cap X_d$ is the convex closure of any two of its non-collinear points.*

Just like $\xi_1 \cap \xi_2$ is either a point or a maximal singular subspace for distinct $\xi_1, \xi_2 \in \Xi_d$, there are only two options for the set of X_d-points of some $\xi \in \Xi_d$ collinear to a point $x \in X_d \setminus \xi$:

Fact 4.3 *Let $x \in X_d$ and $\xi \in \Xi_d$ be such that $x \notin \Xi_d$, with $d \in \{1, 2, 4, 8\}$. Then either $x^\perp \cap \xi_d$ is empty or $x^\perp \cap \xi_d$ is a maximal singular subspace of $X_d \cap \xi$. Moreover, the first case always occurs, and the latter case does not occur when $d = 1$.*

A consequence that we will use a couple of times when $d > 1$ is the following:

Corollary 4.4 *Suppose $d \in \{2, 4, 8\}$. Let ξ_1, ξ_2 be distinct members of Ξ_d and suppose $x_1 \in X_d \cap \xi_1 \setminus \xi_2$ and $x_2 \in X_d \cap \xi_2 \setminus \xi_1$ are collinear. Then:*

(i) If $\xi_1 \cap \xi_2$ is a unique point p, then $x_1 \perp_{\Gamma_d} p \perp_{\Gamma_d} x_2$;

(*ii*) *If* $\xi_1 \cap \xi_2$ *is a maximal singular subspace* S*, then* $x_1^{\perp_{\Gamma_d}} \cap S = x_2^{\perp_{\Gamma_d}} \cap S$ *and this is a hyperplane of* S*.*

Proof Since $x_1 \perp_{\Gamma_d} x_2$, it follows from Fact 4.3 that x_1 is collinear to a maximal singular subspace M_2 of $X_d \cap \xi_2$, and $\dim M_2 = \frac{d}{2} \geq 1$. By Property (P2), there are two options indeed.

(*i*) Suppose first that $\xi_1 \cap \xi_2$ is a unique point p. If $p \notin M_2$, then p is collinear to a point p_2 of M_2 and hence $p_2 \in x_1^{\perp_{\Gamma_d}} \cap p^{\perp_{\Gamma_d}} \subseteq \xi_1 = [p, x_1]$ by Remark 4.2, a contradiction.

(*ii*) Next, suppose $\xi_1 \cap \xi_2$ has dimension $\frac{d}{2}$. Inside the symp $X_d \cap \xi_1$, the point x_1 is collinear to a hyperplane H of S (note that $x_1 \notin S$). Clearly, $M_2 = \langle H, x_2 \rangle$. Inside the symp $X_d \cap \xi_2$, it follows from $x_2 \in M_2$, that also x_2 is collinear to H. The statement follows. □

Definition 4.5 For a point $x \in X_d$ and a member $\xi \in \Xi_d$, we say that x and ξ are opposite if $x^\perp \cap \xi = \emptyset$. If $x \notin \xi$ but x and ξ are not opposite, we say that they are *close*.

The above notion of opposition corresponds to opposition in the related buildings. For each point x, there exists at least one $\xi \in \Xi_d$ opposite x; likewise, for each $\xi \in \Xi_d$, such a point exists. For $d = 1$ this is trivial, since each point p and each $\xi \in \Xi_1$ with $p \notin \xi$ are opposite (cf. Fact 4.3).

The following follows from the study of the (geometric) hyperplanes [5, 13] for $d = 4$ and $d = 8$ (see also Facts 4.14 and 4.13), and for the two smallest cases it can be verified too:

Fact 4.6 *For each symp* $\xi \in \Xi_d$*, the points of* X_d *not opposite* ξ *generate a hyperplane* H_ξ *of* $\mathsf{PG}(3d + 2, \mathbb{K})$ *intersecting* X_d *in precisely the set of points of* X_d *which are not opposite* ξ*.*

In Fact 4.11 we will see that the map $\xi \mapsto H_\xi$ induces an embedding of the dual geometry $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$ in the dual of $\mathsf{PG}(3d+2, \mathbb{K})$, for $d = 8$ (it is true for the other values of d too, but we provide a proof for the $d = 8$ case).

A symp $\xi \in \Xi_d$ together with a point of X_d opposite ξ, generates the entire space. More precisely, it follows from Lemma 4.14 in [12] (which says that $T_x(X_d)$ contains no points non-collinear to x) and a dimension argument, that:

Fact 4.7 *For each symp* $\xi \in \Xi_d$ *and each point* $x \in X_d$ *opposite* ξ*, the subspaces* ξ *and* $T_x(X_d)$ *are complementary, that is, they are disjoint and generate* $\mathsf{PG}(3d + 2, \mathbb{K})$*.*

Maximal singular subspaces. For $d = 2$, i.e., for the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ it follows from its definition that its maximal singular subspaces are planes; the maximal singular subspaces of its symps are lines, and each such line is contained in a singular plane. This behaviour is different when d grows larger. Indeed, for $\mathcal{G}_{6,2}(\mathbb{K})$, the maximal singular subspaces are certain singular planes (contained in symplecta) and singular 4-spaces. For $\mathcal{E}_6(\mathbb{K})$, the maximal singular subspaces have dimension 4 or 5. A 5-space and a symp are called *incident* when they share a 4-dimensional subspace. This intersection is, although 4-dimensional, clearly not maximal. To make the distinction, we refer to maximal singular subspaces of dimension 4 as 4-spaces, and to the other kind as 4′=spaces. The following fact tells us more about the maximal singular subspaces of a symp.

Fact 4.8 *Suppose* $d > 2$ *and let* $\xi \in \Xi_d$ *be arbitrary. Then the two families of maximal singular subspaces of the symp* $X_d \cap \xi$ *can be distinguished as follows: the members of one family occur as the intersection with other members of* Ξ_d*, the members of the other family are strictly contained in singular subspaces of* $(X_d, \mathcal{M}_d)$*.*

If $d > 2$, certain maximal singular subspaces of $\xi \in \Xi_d$ are hence actual maximal singular subspaces in the geometry $(X_d, \mathcal{M}_d)$ too, and others are not.

The following fact can be deduced from the previous facts. It gives a correspondence between the point-residue and the symps.

Fact 4.9 *Let $x \in X_d$ and $\xi \in \Xi_d$ be such that $x^\perp \cap \xi$ is empty. Then the map taking a point $y \in X_d \cap \xi$ to $\Sigma_{x,y}$ is a bijection between the points of $X_d \cap \xi$ and the conics/symps containing x. Moreover, if $d > 1$, this correspondence is an isomorphism. Indeed, if $y, z \in X_d \cap \xi$, then:*

- *(i) $y \perp z$ if and only if $\xi_y \cap \xi_z$ is a maximal singular subspace in both $\Sigma_{x,y}$ and $\Sigma_{x,z}$; y and z are at distance 2 if and only if $\Sigma_{x,y} \cap \Sigma_{x,z} = \{x\}$;*
- *(ii) If $y \perp z$, then the points on the line yz correspond to the members of the line $S(\Sigma_{x,y}, \Sigma_{x,z})$ of the dual geometry. Moreover, if $d = 4$, the set of all members of Ξ_d containing $\Sigma_{x,y} \cap \Sigma_{x,z}$ corresponds to the set of points of a (non-maximal) singular plane of $X \cap \xi$.*

For $\mathcal{E}_6(\mathbb{K})$, we will also need the mutual position between two 5-spaces.

Fact 4.10 *Two 5-spaces U, V of $\mathcal{E}_6(\mathbb{K})$ intersect each other in either the empty subspace, a point or a plane. In case they share a unique point, then there is a unique symp of $\mathcal{E}_6(\mathbb{K})$ incident with both U and V.*

Proof This can be deduced from a study of the point-residual $\mathcal{D}_{5,5}(\mathbb{K})$, who's properties can be determined in terms of the corresponding polar space of type D_5. □

Universal embeddings. The (Veronese) varieties $\mathcal{S}_{2,2}(\mathbb{K})$ in $\mathsf{PG}(8, \mathbb{K})$ $(d = 2)$, $\mathcal{G}_{6,2}(\mathbb{K})$ in $\mathsf{PG}(14, \mathbb{K})$ $(d = 4)$ and $\mathcal{E}_6(\mathbb{K})$ in $\mathsf{PG}(26, \mathbb{K})$ $(d = 8)$ are the universal embeddings of the abstract geometries of type $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, $\mathsf{A}_{5,2}(\mathbb{K})$ and $\mathsf{E}_{6,1}(\mathbb{K})$, respectively. This follows from result of Zanella [19] for $d = 2$, from Wells [18] for $d = 4$, and from Kasikova, Shult, and Cooperstein for $d = 8$ (combining the main result of [11], showing in particular that a geometry of type $\mathsf{E}_{6,1}(\mathbb{K})$ has a universal embedding), and [6], which bounds the dimension of this universal embedding).

There is also a notion of the dual Veronese variety, by associating to a symp $\xi \in \Xi_d$ of the dual geometry $\Gamma^* = (\Xi_d, \mathcal{S}_d)$ the unique hyperplane H_ξ, which is a point of the dual of $\mathsf{PG}(3d+2, \mathbb{K})$. For each $d \in \{1, 2, 4, 8\}$, this gives a variety in (the dual of) $\mathsf{PG}(3d + 2, \mathbb{K})$ which is isomorphic to the original Veronese variety, i.e., the universal embedding. Since we did not find a proof of this fact in the literature, and since we will rely on this fact for $d = 8$, we provide a proof for the case $d = 8$.

Proposition 4.11 *The map taking a symp $\xi \in \Xi_8$ to the hyperplane H_ξ of $\mathsf{PG}(26, \mathbb{K})$ gives the universal embedding of Γ_8^* in (the dual of) $\mathsf{PG}(26, \mathbb{K})$ (i.e., as the Cartan variety $\mathcal{E}_6(\mathbb{K})$).*

Proof As mentioned before, the dual geometry $\Gamma_8^* = (\Xi_d, \mathcal{S}_8)$ is abstractly isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. We first show that the map $\xi \mapsto H_\xi$ is a (full) embedding in the dual of $\mathsf{PG}(26, \mathbb{K})$. Let ξ_1, ξ_2 be distinct symps of Γ_8. The existence of a point of Γ close to ξ_1 and opposite ξ_2 is straightforward, and hence injectivity follows (in particular, we know that $\dim(H_{\xi_1} \cap H_{\xi_2}) = 24$).

Claim 1: If $\xi_1 \cap \xi_2$ is a 4-space S, then $H_{\xi_1} \cap H_{\xi_2} \cap \Gamma_8$ is the set of points of Γ_8 that are equal or collinear to at least a point of S.
If p is equal or collinear to a point of S, then by definition, $p \in H_{\xi_1} \cap H_{\xi_2}$. For the converse statement, suppose first that $p \in \Gamma_8$ belongs to $H_{\xi_1} \cap H_{\xi_2}$. If $p \in \xi_1 \cup \xi_2$, then p is collinear to at least a point of S. So suppose $p \notin \xi_1 \cup \xi_2$ and let U_1 and U_2 be the respective $4'$-spaces of ξ_1 and ξ_2 collinear to p. Suppose for a contradiction that U_1 and U_2 are disjoint from S. The 5-spaces

$\langle p, U_1\rangle$ and $\langle p, U_2\rangle$ then violate the relations between 5-spaces as listed in Fact 4.10 since there are several symps meeting both of them in a $4'$-space. So $S \cap U_1$ is non-empty, hence the claim holds.

If $\xi_1 \cap \xi_2$ is a 4-space S, then we denote by H_S the subspace generated by all points of Γ that are equal or collinear to a point of S.

Claim 2: If $\xi_1 \cap \xi_2$ is a 4-space S, then H_S equals $H_{\xi_1} \cap H_{\xi_2}$ and has dimension 24.
By Claim 1, we know that $H_S \subseteq H_{\xi_1} \cap H_{\xi_2}$. We show that $\dim H_S = 24$. Consider 5-spaces V_1 and V_2 meeting ξ_1 and ξ_2 in respective $4'$-spaces V_1' and V_2' that are disjoint from S. It is easily verified that V_1 and V_2 are opposite 5-spaces in Γ_8 (noting that each point of V_1' is collinear to a unique point of V_2'). Next, let $E(V_1, V_2)$ denote the subset of points of Γ_8 which are simultanously collinear to a 3-space of V_1 and a 3-space of V_2. Equipped with the lines of Γ_8 that $E(V_1, V_2)$ contains, $E(V_1, V_2)$ is a geometry of type $\mathsf{A}_{5,2}(\mathbb{K})$. This geometry is called an equator geometry (with poles V_1 and V_2), for more information we refer to [16]. Now, since Γ_8 is embedded in $\mathsf{PG}(26, \mathbb{K})$, we moreover have that $E(V_1, V_2)$ is embedded in a 14-space complementary to the 11-space $\langle V_1, V_2\rangle$ as the line Grassmannian variety $\mathcal{G}_{6,2}(\mathbb{K})$. Clearly, S is a 4-space of $E(V_1, V_2)$. Since in $\mathcal{G}_{6,2}(\mathbb{K})$, each point is collinear to a point of a 4-space, we have that $E(V_1, V_2) \subseteq H_S$. Since also $V_1' \cup V_2' \subseteq H_S$, we obtain $\langle E(V_1, V_2), V_1, V_2\rangle \subseteq H_S \subseteq H_{\xi_1} \cap H_{\xi_2}$. Noting that a point q which is collinear to a $4'$-space of ξ_1 disjoint from S does not belong to H_{ξ_2}, it follows that $\dim(H_{\xi_1} \cap H_{\xi_2}) = 24$. Since also $\dim\langle E(V_1, V_2), V_1, V_2\rangle = 24$, the claim is proven.

Claim 3: The set of symps containing a 4-space S of Γ^* corresponds bijectively to the set of hyperplanes of $\mathsf{PG}(26, \mathbb{K})$ containing H_S.
Let ξ be any member of Ξ_8 containing S. By definition, each point p of H_S is collinear to a point of $S \subseteq \xi$, so $p \in H_\xi$ and $H_S \subseteq H_\xi$. Conversely, consider a hyperplane H of $\mathsf{PG}(26, \mathbb{K})$ containing H_S. Let ξ_1 and ξ_2 be distinct members of Ξ_8 containing S. Take a point q_1 in Γ_8 close to ξ_1 but not to ξ_2, and consider a singular line L containing q_1 with $L \nsubseteq H_{\xi_1}$. Then L contains a unique point $q_2 \in H_{\xi_2}$, and a unique point $q \in H$. A line L_1 containing q_1 and meeting ξ_1 in a point q_1', determines a unique symp of Γ_8 together with L, and this symp meets the 5-space $\langle q_2, q_2^\perp \cap \xi_2\rangle$ in a line q_2q_2' for some point $q_2' \in \xi_2$. Then q is collinear to a unique point q' on the line $q_1'q_2'$, and q' is collinear to the 3-space ${q'}_1^\perp \cap S = {q'}_2^\perp \cap S$ (cf. Corollary 4.4). Let ξ be the unique member of Ξ_8 containing q' and S. By construction, $q \in H_\xi$ (because $q \perp q'$) and $H_S \subseteq H_\xi$ (because $S \subseteq \xi$), so $H_\xi = \langle q, H_S\rangle = H$. The claim follows.

At this point, we conclude that $\xi \mapsto H_\xi$ is a full embedding.

Claim 4: $\bigcap_{\xi\in\Xi_8} H_\xi = \emptyset$.
As an intermediate step, we show for each point $p \in \Gamma_8$ that $\bigcap_{p\in\xi} H_\xi = T_p(X_8)$. Take $\xi' \in \Xi$ opposite p and let q be an arbitrary point of $X_8(\xi')$. Consider a point q' in $X_8(\xi')$ non-collinear to q. Then q is opposite the unique symp $\xi_{p,q'}$ and hence $q \notin H_{\xi_{p,q'}}$. Since $q \in X_8(\xi')$ was arbitrary, ξ' is disjoint from $\bigcap_{p\in\xi} H_\xi$. On the other hand, $T_p(X_8)$ is contained in $\bigcap_{p\in\xi} H_\xi$ because $T_p(X_8)$ is generated by the singular lines through p, which are close to or contained in the symps through p. Since $T_p(X_8)$ and ξ' are complementary subspaces of $\mathsf{PG}(26, \mathbb{K})$ (cf. Fact 4.7), we obtain that $\bigcap_{p\in\xi} H_\xi = T_p$ indeed. Therefore $\bigcap_{p\in\xi} H_\xi \subseteq \bigcap_{p\in X_8} T_p = \emptyset$. The claim follows.

We conclude that $\xi \mapsto H_\xi$ yields a full embedding of the geometry Γ_8^* of type $\mathsf{E}_{6,1}(\mathbb{K})$ in a projective space $\mathsf{PG}(26, \mathbb{K})$, where the points corresponding to the hyperplanes H_ξ generate the entire projective space. We conclude that this embedding is the universal embedding as Cartan variety $\mathcal{E}_6(\mathbb{K})$. □

We continue with the connections between the first and second row.

4.2 (Symplectic) polarities of the varieties of the second row of the FTMS

One way to obtain the varieties of the first row from the ones below them is via (symplectic) polarities or related maps. We start with the third and fourth cell, as these provide clearer examples.

For the polar line Grassmannian, the polar space of type $\mathsf{C}_{3,1}(\mathbb{K})$ arises as the geometry of absolute elements of a symplectic polarity p of $\mathsf{PG}(5,\mathbb{K})$. This polarity extends to a polarity $\tilde{p}$ of $\mathcal{G}_{6,2}(\mathbb{K})$, mapping points (i.e., a line of $\mathsf{PG}(5,\mathbb{K})$) to symps (i.e., a 3-space of $\mathsf{PG}(5,\mathbb{K})$). The absolute points of $\mathcal{G}_{6,2}(\mathbb{K})$ under $\tilde{p}$ correspond precisely to the absolute lines of $\mathsf{PG}(5,\mathbb{K})$ under p. This means that the absolute geometry of $\tilde{p}$ is of type $\mathsf{C}_{3,2}(\mathbb{K})$ indeed.

By definition, a *symplectic polarity* of $\mathsf{E}_{6,1}(\mathbb{K})$ is such that the geometry of its absolute elements is of type $\mathsf{F}_{4,4}(\mathbb{K})$. We note that the polarity $\tilde{p}$ on $\mathcal{G}_{6,2}(\mathbb{K})$ has the property that a non-absolute point is mapped to an opposite symp (as can be seen from the action of p on $\mathsf{PG}(5,\mathbb{K})$). Also the symplectic polarities of $\mathsf{E}_{6,1}(\mathbb{K})$ do, and it even characterises them (provided that the absolute geometry is non-trivial), according to Main Result 2.1 of [15]:

Fact 4.12 *A (non-anisotropic) symplectic polarity of* $\mathsf{E}_{6,1}(\mathbb{K})$ *is a duality of* $\mathsf{E}_{6,1}(\mathbb{K})$ *with the property that there is at least one absolute point and the non-absolute points are mapped to opposite symps.*

We continue with the second cell. Recall that we introduced $\mathcal{S}_{2,2}(\mathbb{K})$ as the image under the Segre map σ of two projective planes π_1 and π_2. Consider the map sending a point $\sigma((x_0,x_1,x_2),(y_0,y_1,y_2))$ to the symp given by the image under σ of the lines in π_1 and π_2 with respective equations $y_0X_0+y_1X_1+y_2X_2=0$ and $x_0X_0+x_1X_1+x_2X_2=0$. Then the point is absolute if and only if $y_0x_0+y_1x_1+y_2x_2=0$, which clearly is a hyperplane section of $\mathcal{S}_{2,2}(\mathbb{K})$ in $\mathsf{PG}(8,\mathbb{K})$. Its geometry of absolute elements is isomorphic to the flag geometry of π_1: consider all pairs (p,L) where p is a point of π_1 and L a line of π_1 with $p\in L$, where (p,L) and (q,M) are collinear if either $p=q$ or $L=M$, and hence through (p,L) there are exactly two lines: $\{(q,L)\mid q\in L\}$ and $\{(p,M)\mid M\ni p\}$. This yields a thin generalised hexagon. This map can be consider as a polarity of the Hjelmslev-Moufang plane over $\mathbb{K}\times\mathbb{K}$, which is isomorphic to $\mathcal{S}_{2,2}(\mathbb{K})$ (its points and lines are the points and symps of $\mathcal{S}_{2,2}(\mathbb{K})$. It also has the property that it sends a non-absolute point to an opposite symp.

Finally, for the smallest case, $\mathcal{V}_2(\mathbb{K})$, the situation is slightly special. Indeed, by its very definition, the conic of the first row is exactly the image of a (non-degenerate) conic C in $\mathsf{PG}(2,\mathbb{K})$ under the Veronese map ν. Consider the map ρ taking a point of C to its tangent line. If $\operatorname{char}\mathbb{K}\neq 2$, then ρ induces a polarity of $\mathsf{PG}(2,\mathbb{K})$, which extends to a polarity of $\mathsf{PG}(5,\mathbb{K})$ stabilising $\mathcal{V}_2(\mathbb{K})$ and with $\nu(C)$ as absolute geometry. However, if $\operatorname{char}\mathbb{K}=2$ then all tangent lines to C go through a common point (the nucleus of C) and hence the map induced by ρ on $\mathsf{PG}(2,\mathbb{K})$ is degenerate.

So in each of the four cases, except in the smallest case if $\operatorname{char}\mathbb{K}=2$, there is a polarity whose absolute geometry is isomorphic to the geometry of the cell above it in the FTMS.

4.3 Geometric hyperplanes of the varieties of the second row of the FTMS

The geometries of the first row, as obtained in the previous section as a subvariety of those in the cell below, have in common that they all lie in a hyperplane of the ambient projective space, even stronger, they arise as a hyperplane section. We discuss this common feature, occurring as a hyperplane section, or more generally speaking as a geometric hyperplane, in this subsection.

The geometric hyperplanes of the (abstract) point-line geometry $\mathsf{E}_{6,1}(\mathbb{K})$ and of Grassmannians of projective spaces, in particular, $\mathsf{A}_{5,2}(\mathbb{K})$, have been studied in a more general setting by Cooperstein and Shult ([5]) and by Shult ([13]), respectively. In the cases of $\mathsf{E}_{6,1}(\mathbb{K})$ and $\mathsf{A}_{5,2}(\mathbb{K})$ it turns out that there are three types of geometric hyperplanes, all of which can be realised as a hyperplane section of their respective universal embeddings $\mathcal{E}_6(\mathbb{K})$ in $\mathsf{PG}(26,\mathbb{K})$ and $\mathcal{G}_{6,2}(\mathbb{K})$ in $\mathsf{PG}(14,\mathbb{K})$.

The following proposition contains the description of the three types of geometric hyperplanes of $\mathcal{E}_6(\mathbb{K})$.

Proposition 4.13 *Let H be a proper geometric hyperplane of $\mathcal{E}_6(\mathbb{K})$ in $\mathsf{PG}(26,\mathbb{K})$. Then H arises as a hyperplane section and is of one of the following types:*

- *The points of $\mathcal{E}_6(\mathbb{K})$ in H are the points collinear to at least one point of a given symp $\xi \in \Xi$ (H is called a white hyperplane);*
- *The points of $\mathcal{E}_6(\mathbb{K})$ in H are the union of a set of symps Σ through a point $p \in X$ such that, in $p^\perp$ the set of symps corresponding to the members of Σ is the point set of a quadric of type $\mathsf{B}_{4,1}(\mathbb{K})$ (H is called a grey hyperplane);*
- *The points of $\mathcal{E}_6(\mathbb{K})$ in H are the absolute points of a symplectic polarity ρ of $\mathcal{E}_6(\mathbb{K})$, and the geometry of absolute elements under ρ is isomorphic to the variety $\mathcal{F}_{4,4}(\mathbb{K})$ (H is called a black hyperplane).*

Proof This is proven (somewhat implicitly) by Cooperstein and Cohen in [5]. See also 3.2 of [9] for a more detailed, geometric account. □

A similar situation occurs in the case of the line Grassmannians. The proposition is a special case of a result of Shult, who treats geometric hyperplanes of Grassmannians of projective spaces in [13].

Proposition 4.14 *Let H be a proper geometric hyperplane of $\mathcal{G}_{6,2}(\mathbb{K})$ in $\mathsf{PG}(15,\mathbb{K})$. Then H arises as a hyperplane section and can be obtained by the absolute elements of a possibly degenerate alternating bilinear form f on $\mathsf{PG}(5,\mathbb{K})$, f not the null form. In case H contains no symp of $\mathcal{G}_{6,2}(\mathbb{K})$, f is non-degenerate and the absolute elements of f in $\mathcal{G}_{6,2}(\mathbb{K})$ give a polar line Grassmannian variety $\mathcal{C}_{3,2}(\mathbb{K})$.*

Proof The first statement is a special case of Main Results 1 and 2 of [13]. Observing that the only non-trivial alternating bilinear forms f on $\mathsf{PG}(5,\mathbb{K})$ are those whose radical is either empty, a line or a 3-space; there are only three types of geometric hyperplanes. Note that the absolute lines of $\mathsf{PG}(5,\mathbb{K})$ under f are precisely the points of H. In case f is non-degenerate (i.e., the radical is empty), the geometry of absolute elements of $\mathsf{PG}(5,\mathbb{K})$ under f is a symplectic polar space (i.e., of type $\mathsf{C}_{3,1}(\mathbb{K})$); therefore, H is isomorphic to $\mathsf{C}_{3,2}(\mathbb{K})$. Since $\mathsf{C}_{3,1}(\mathbb{K})$ has no singular 3-spaces, there are no symps of $\mathcal{G}_{6,2}(\mathbb{K})$ which are contained in H. Next, suppose that f has a non-trivial radical. In this case, it is easily seen that the absolute geometry of $\mathsf{PG}(5,\mathbb{K})$ under f contains 3-spaces, and hence H contains symps. □

The situation is different for the second column. There are geometric hyperplanes of $\mathcal{S}_{2,2}(\mathbb{K})$ that give rise to a thin generalized hexagon and arise as the intersection of $\mathcal{S}_{2,2}(\mathbb{K})$ with a hyperplane of $\mathsf{PG}(8,\mathbb{K})$, but some geometric hyperplanes of $\mathcal{S}_{2,2}(\mathbb{K})$ generate $\mathsf{PG}(8,\mathbb{K})$. For an example, see Thas and Van Maldeghem in Section 2 of [14].

For the first column, the variety is given by the image of a conic C in $\mathsf{PG}(2,\mathbb{K})$ under the Veronese map ν. Since C is given by a quadratic equation in x, y, z, the points of $\nu(C)$ satisfy a linear equation in $x^2, y^2, z^2, yz, zx, xy$ and hence $\nu(C)$ is contained in a hyperplane of $\mathsf{PG}(5,\mathbb{K})$, more precisely, one can verify that $\nu(C)$ is the intersection of the Veronese variety with a hyperplane of $\mathsf{PG}(5,\mathbb{K})$.

4.4 Parapolar spaces

Finally, another way to see the intimite relation between the varieties on the same row of the split version of the FTMS is in the setting of parapolar spaces (see Remark 4.2).

The point-line geometries $(X_d, \mathcal{M}_d)$, $d \in \{2,4,8\}$, associatied to the varieties $\mathcal{S}_{2,2}(\mathbb{K})$, $\mathcal{G}_{6,2}(\mathbb{K})$, $\mathcal{E}_6(\mathbb{K})$ (second row, columns 2 to 4) are strong parapolar spaces of diameter 2 (see Remark 4.2). Each symp of this parapolar space is a hyperbolic quadric living in a subspace of dimension $d+1$, and the set of the thus obtained $(d+1)$-spaces coincides with Ξ: each $\xi \in \Xi_d$ can be obtained by taking the subspace generated by the convex subspace closure of two non-collinear points of X_d.

For the first row, the last two columns give non-strong parapolar spaces of diameter 3 whose symps are symplectic. From the foregoing, we know that these are contained in the parapolar spaces corresponding to the row below it in two ways (simultanously): as the absolute geometry of a polarity p and as the intersection with a hyperplane H of the ambient projective space. For $\mathcal{G}_{6,2}(\mathbb{K})$, a maximal singular 4-space V shares a 3-space with H and the absolute geometry of $V \cap H$ under the restriction of p to $V \cap H$ gives a symplectic quadrangle in $V \cap H$. These are the symps of the polar line Grassmannian variety $\mathcal{C}_{3,2}(\mathbb{K})$. Likewise, in $\mathcal{E}_6(\mathbb{K})$, a 5-space V which is fixed under p is fully contained in H, and the absolute geometry of V under the restriction of p to V induces a symplectic polar space of rank 3. These are the symps of $\mathcal{F}_{4,4}(\mathbb{K})$.

By extension, if one would allow the set of symps to be empty, also the second column of the first row (i.e., the thin generalised hexagon) can be thought of as a non-strong parapolar space of diameter 3.

5 Embeddings on quadrics

We first show, for each $d \in \{1,2,4,8\}$, the existence of a non-degenerate quadric Q_d in which $\Gamma_d = (X_d, \Xi_d)$ is fully embedded, where for $d = 1$ we moreover show the existence of a hyperbolic such quadric.

Lemma 5.1 *The quadric Veronese variety $\mathcal{V}_2(\mathbb{K}) = (X_1, \Xi_1)$, in its standard embedding in $\mathsf{PG}(5,\mathbb{K})$, is fully embedded in a non-degenerate hyperbolic quadric Q_1.*

Proof The point set of $\mathcal{V}_2(\mathbb{K})$ is given by points $(x^2, y^2, z^2, yz, zx, xy)$. We label the coordinates of $\mathsf{PG}(5,\mathbb{K})$ as $(x_{-1}, x_{-2}, x_{-3}, x_1, x_2, x_3)$ and put $y_1 = x_{-1} - x_3$, $y_2 = x_{-2} - x_1$ and $y_3 = x_{-3} - x_2$. Then the coordinates of the points in the image of the Veronese map all satisfy $x_1y_1 + x_2y_2 + x_3y_3 = 0$, and hence $\mathcal{V}_2(\mathbb{K})$ is contained in a hyperbolic quadric Q_1. Since there are no lines on $\mathcal{V}_2(\mathbb{K})$, there is nothing else to show. □

Lemma 5.2 *Let $\Gamma_d = (X_d, \mathcal{L}_d)$ be the point-line geometry associated to the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ (if $d = 2$), the Grassmannian variety $\mathcal{G}_{6,2}(\mathbb{K})$ (if $d = 4$) and the Cartan variety $\mathcal{E}_{6,1}(\mathbb{K})$ (if $d = 8$). Then Γ_d, in its standard embedding in $\mathsf{PG}(3d+2,\mathbb{K})$, is fully embedded in a non-degenerate quadric Q_d in $\mathsf{PG}(3d+2,\mathbb{K})$.*

Proof Following the construction of the standard embeddings of $\Gamma_d = (X_d, \mathcal{L}_d)$ in $\mathsf{PG}(3d+2,\mathbb{K})$ as done in Section 10 of [10], X_d can be given by 6-tuples $(x_1, x_2, x_3, X_1, X_2, X_3)$, with $x_i \in \mathbb{K}$ and $X_i \in \mathbb{A}$, where $\mathbb{A}$ is the unique split composition algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}} \mathbb{A} = d$

(and hence X_i is actually a d-tuple in $\mathbb{K}$), that satisfy the following set of $3d+3$ equations (where $i \in \{1,2,3\}$ and the indices are to be read modulo 3):

$$x_{i+1}x_{i+2} - X_i\overline{X}_i = 0,$$
$$x_i\overline{X}_i - X_{i+1}X_{i+2} = 0.$$

Clearly, every quadric in $\mathsf{PG}(3d+2,\mathbb{K})$ with equation a linear combination of these $3d+3$ equations, contains X_d. On the other hand, it is shown in [17] that also the converse is true: every quadric in $\mathsf{PG}(3d+2,\mathbb{K})$ that contains X_d is a linear combination of these $3d+3$ equations. Consider such a linear combination of these $3d+3$ quadrics, which then has equation:

$$\sum_{i\in\{1,2,3\}} a_i(x_{i+1}x_{i+2} - X_i\overline{X}_i) + A_i(x_i\overline{X}_i - X_{i+1}X_{i+2}) = 0,$$

with $a_i \in \mathbb{K}$ and $A_i \in \mathbb{A}$. A standard calculation yields that this quadric is non-degenerate if and only if

$$a_1a_2a_3 - a_1A_1\overline{A}_1 - a_2A_2\overline{A}_2 - a_3A_3\overline{A}_3 - A_1A_2A_3 \neq 0.$$

Choosing $a_1 = a_2 = a_3 = 1$ and $A_1 = A_2 = A_3 = 0$, we hence obtain that

$$x_1x_2 + x_2x_3 + x_3x_1 = X_1\overline{X}_1 = X_2\overline{X}_2 + X_3\overline{X}_3$$

is the equation of a non-degenerate quadric in $\mathsf{PG}(3d+2,\mathbb{K})$ containing X_d. Recalling that a line of $\mathcal{L}_d$ is a line of $\mathsf{PG}(3d+2,\mathbb{K})$ which is entirely contained in X_d, the embedding of $(X_d,\mathcal{L}_d)$ in the quadric is automatically full. □

Remark 5.3 Note that the above proof actually gives the equation of each quadric in $\mathsf{PG}(3d+2,\mathbb{K})$ containing $\Gamma_d = (X_d,\mathcal{L}_d)$ with $d \in \{2,4,8\}$. Later on, we will show in Lemma 6.10 and Proposition 7.29 that, for $d \in \{2,4,8\}$, each of the above quadrics, i.e., each non-degenerate quadric containing $\Gamma_d = (X_d,\mathcal{M}_d)$ has maximal Witt index (they are parabolic). On the other hand, when $d = 1$, there also exist non-degenerate quadrics in $\mathsf{PG}(5,\mathbb{K})$ containing $\mathcal{V}_2(\mathbb{K})$ which are *not* hyperbolic, for instance the quadric given by the following equation (using the same notation as in the proof of Lemma 5.1):

$$x_{-1}x_{-2} + x_{-2}x_{-3} + x_{-3}x_{-1} = x_1^2 + x_2^2 + x_3^2.$$

Indeed, one can verify that for instance of $\mathbb{R}$, this quadric has Witt index 1, meaning that there are no lines on it.

6 The general case for $d > 1$

Let $d > 1$. We consider the embedding of $\Gamma_d = (X_d,\mathcal{M}_d)$ in a non-degenerate quadric Q_d (cf. Lemma 5.2). Only when mentioned explicitly, we also allow Q_d to be degenerate.

Notation. Since $d > 1$, the intersection of a $\xi \in \Xi_d$ with X_d is always a symp (not a conic) and hence we will speak of a symp Σ (instead of $\xi \cap X_d$) and its ambient $(d+1)$-space $\langle\Sigma\rangle$ (instead of ξ).

Recall that $\mathcal{L}_d$ denotes the subset of $\mathcal{M}_d$ consisting of the lines L such that each symp containing L is projective, i.e., belongs to $\mathcal{P}_d$. We will show that $\mathcal{P}_d$ is a geometric hyperplane of the dual geometry $\Gamma_d^* = (\Xi_d,\mathcal{S}_d)$, meaning that each line of the dual geometry either is incident with a unique projective symp, or all symps incident with it are projective.

6.1 $\mathcal{P}_d$ as geometric hyperplane of the dual geometry $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$

Our first goal is to show that $\mathcal{P}_d$ is a geometric hyperplane of the dual geometry $\Gamma_d^* = (\Xi_d, \mathcal{S}_d)$. In this subsection, we also include the possibility that Q_d is degenerate.

We start with an observation, which we mention explicitly because of its frequent use in the sequel.

Lemma 6.1 *Let Γ_d be embedded in a possibly degenerate quadric Q_d. Let x, y be points of X_d which are not collinear in Γ_d. The following are equivalent:*

(i) the symp $\Sigma_{x,y}$ containing x, y belongs to $\mathcal{P}_d$;
(ii) x and y are Q_d-collinear;
(iii) $y \in T_x(Q_d)$ or $x \in T_y(Q_d)$.

Proof Note that, by Property (P1) from Fact 4.1, there is indeed a unique symp $\Sigma_{x,y}$ containing x, y. The equivalence of (i) and (ii) follows immediately from Lemma 3.2: if x and y are Q_d-collinear, then $\Sigma_{x,y}$ cannot be embedded isometrically in Q_d and hence $\Sigma_{x,y} \in \mathcal{P}_d$; the other direction is by definition. The equivalence of (ii) and (iii) is trivial, recalling that $T_x(Q_d)$ of Q_d at x is generated by the singular lines of Q_d through x. □

This observation allows us to deduce valuable information about the projective symps through a point.

Lemma 6.2 *Let Γ_d be embedded in a possibly degenerate quadric Q_d. Let $x \in X_d$ be arbitrary and take any symp Σ opposite x. Then there are three possibilities:*

(i) $T_x(Q_d) \cap \Sigma$ is a degenerate hyperplane of Σ, i.e., of the form $T_q(\Sigma) \cap \Sigma$ for some $q \in \Sigma$.
(ii) $T_x(Q_d) \cap \Sigma$ is a non-degenerate hyperplane of Σ, i.e., isomorphic to $Q(d, \mathbb{K})$.
(iii) $T_x(Q_d) \cap \Sigma = \Sigma$. In this case, $T_x(Q_d) = \mathsf{PG}(3d+2, \mathbb{K})$ and is generated by the projective symps containing x.

Proof Recall that Σ opposite x means that $x^{\perp_{\Gamma_d}} \cap \Sigma = \emptyset$. Since $T_x(Q_d)$ has dimension at least $3d+1$, it meets the $(d+1)$-space $\langle \Sigma \rangle$ in at least a d-space. Suppose first that $T_x(Q_d)$ contains Σ. Then, since each symp containing x intersects Σ in a point by Fact 4.9, Lemma 6.1 implies that each symp containing x belongs to $\mathcal{P}_d$. But then all symps through x are contained in $T_x(Q_d)$, and hence $T_x(Q_d)$ has dimension $3d+2$ indeed.

So suppose $T_x(Q_d) \cap \langle \Sigma \rangle$ is a proper hyperplane of $\langle \Sigma \rangle$. If $T_x(Q_d) \cap \langle \Sigma \rangle$ contains a maximal singular subspace of Σ, then the unique option is that $T_x(Q_d) \cap \Sigma$ is a cone with vertex q over a hyperbolic quadric of rank one less than the rank of Σ, and then $T_x(Q_d) \cap \langle \Sigma \rangle = T_q(\Sigma)$. If $T_x(Q_d) \cap \Sigma$ contains no maximal singular subspace of Σ, then it contains opposite submaximal singular subspaces and hence it is isomorphic to a parabolic quadric in dimension d. □

Notation. If H is a degenerate hyperplane of Σ of the form $T_q(\Sigma)$ for some point $q \in \Sigma$, then we say that H has vertex q.

The following lemma allows us to immediately finish the proof of the case $d = 8$ in Proposition 7.29, using the fact that the dual geometry $\Gamma_8^* = (\Xi_8, \mathcal{S}_8)$ has a natural embedding in $\mathsf{PG}(26, \mathbb{K})$ by Lemma 4.11, enabling us to use Proposition 4.13 on the geometric hyperplanes. As alluded to before, a highly similar approach would also work for the case where $d = 4$, but we prefer to give an elementary proof, not relying on the classification of the geometric hyperplanes of $\mathcal{G}_{6,2}(\mathbb{K})$, since this gives more intuition. For $d = 2$, we need a different approach anyway since the geometric hyperplanes of $\mathcal{S}_{2,2}(\mathbb{K})$ behave differently.

Lemma 6.3 *Let Γ_d be embedded in a possibly degenerate quadric Q_d. The set $\mathcal{P}_d$ forms a geometric hyperplane of the dual geometry $(\Xi_d, \mathcal{S}_d)$.*

Proof We consider an arbitrary line of $\mathcal{S}_d$, determined by two collinear members Σ_1, Σ_2 of Ξ_d, intersecting each other in a maximal singular subspace S of Γ_d. We claim that one or all members of Ξ_d on the line $S(\Sigma_1, \Sigma_2)$ belong to $\mathcal{P}_d$. To that end, we take a point $x \in S$ and a symp Σ opposite x. By Lemma 4.9(ii), the members of Ξ_d on $S(\Sigma_1, \Sigma_2)$ correspond bijectively to the points of a line L of Σ. Since L either has a unique point in $T_x(Q_d)$ or is contained in it, the claim follows from Lemma 6.1. □

In the next sections, we again assume that Q_d is non-degenerate, unless explicitly mentioned otherwise.

6.2 $\mathcal{X}_d$ as geometric hyperplane of the geometry $\Gamma_d = (X_d, \mathcal{M}_d)$, $d \in \{2, 4\}$

Henceforth, assume $d \in \{2, 4\}$ (the below also holds for $d = 8$, but we provide no proofs for that case since it has already been settled). We proceed by attaching a point to a projective symp, more precisely, we will introduce a bijection between the set $\mathcal{X}_d$ (the union of the lines $\mathcal{L}_d$) and the set $\mathcal{P}_d$. Note that for now we only know that $\mathcal{P}_d$ is non-empty. It will then follow that $\mathcal{X}_d$ is a geometric hyperplane of $(X_d, \mathcal{M}_d)$.

If $x \in \mathcal{X}_d$, then the hyperplane $T_x(Q_d) \cap \Sigma$ is always degenerate, as we show next.

Lemma 6.4 *Let x be any point in X_d. Then for any symp Σ opposite x, $T_x(Q_d) \cap \langle\Sigma\rangle$ is a degenerate hyperplane if and only if $x \in \mathcal{X}_d$.*

Proof By Fact 4.9, the symps through a singular line $L \ni x$ are in bijective correspondence with the points of a maximal singular subspace Π_L of Σ. By Lemma 6.1, L belongs to $\mathcal{L}_d$ if and only if $\Pi_L \subseteq T_x(Q_d)$. Moreover, by Lemma 6.2, $T_x(Q_d) \cap \Sigma$ is a hyperplane of Σ (since Q_d is non-degenerate), which is degenerate if and only if it contains a maximal singular subspace of Σ. So, if $x \in \mathcal{X}_d$, then by definition there is a line L of $\mathcal{L}_d$ through it and the corresponding subspace Π_L is a maximal singular subspace of Σ contained in $T_x(Q_d)$, showing that $T_x(Q_d) \cap \langle\Sigma\rangle$ is degenerate indeed. Conversely, if Π is a maximal singular subspace of Σ contained in $T_x(Q_d)$, then the corresponding line through x belongs to $\mathcal{L}_d$ and hence $x \in \mathcal{X}_d$. □

Suppose $x \in \mathcal{X}_d$ and Σ are opposite. By the previous lemma, $T_x(Q_d) \cap \Sigma$ is a degenerate hyperplane, i.e., it is given by $T_q(\Sigma)$ for some point $q \in \Sigma$. Then the symp $\Sigma_{x,q}$ will play a special role with respect to x.

Lemma 6.5 *Let x be any point in $\mathcal{X}_d$. Then there is a unique symp $\Sigma(x)$ through x such that the set of all singular lines of $\Sigma(x)$ through x coincides with the subset of $\mathcal{L}$ of lines containing x. Furthermore, each projective symp through x shares a maximal singular subspace with $\Sigma(x)$.*

Proof Since distinct symps intersect each other in a singular subspace, there can be at most one symp with the desired properties, so it suffices to show existence. To that end, let Σ be any symp opposite x. By Lemma 6.4, $T_x(Q_d) \cap \langle\Sigma\rangle$ coincides with $T_q(\Sigma)$ for some point $q \in \Sigma$. We show that $\Sigma(x) := \Sigma_{x,q}$ is as required.

Take any line $L \in \mathcal{L}_d$ containing x. By Fact 4.9 and Lemma 6.2, the symps through L (which are projective) correspond bijectively to the points of a maximal singular subspace Π_L of $T_q(\Sigma)$. In particular, $q \in \Pi_L$ and therefore the symp $\Sigma_{x,q}$ contains L. Since $L \in \mathcal{L}_d$ was arbitrary, $\Sigma(x)$

contains all lines of $\mathcal{L}_d$ through x. For the converse inclusion, take any singular line M in $\Sigma_{x,q}$ through x. Then, again by Fact 4.9, the symps through M correspond to a maximal singular subspace Π_M containing q, which hence belongs to $T_q(\Sigma) = T_x(Q_d) \cap \Sigma$. Therefore, all symps through M are projective, so $M \in \mathcal{L}_d$ indeed.

For the final statement, consider a projective symp Σ' through x. Then Σ' meets Σ in a point collinear to q, and hence $\Sigma' \cap \Sigma(x)$ is a maximal singular subspace, according to Fact 4.9. $\square$

Notation. For each point $x \in \mathcal{X}_d$, we keep the notation introduced in the previous lemma: $\Sigma(x)$ denotes the unique symp whose set of singular lines through x coincides with the set of lines of $\mathcal{L}$ containing x. Necessarily, $\Sigma(x)$ is a projective symp.

We record a consequence of the previous proof.

Corollary 6.6 *For any $x \in \mathcal{X}_d$ and any symp Σ opposite x, the point $\Sigma(x) \cap \Sigma$ is the vertex of the degenerate hyperplane $T_x(Q_d) \cap \Sigma$.*

Our goal is to show that the correspondence $\varphi_d : x \mapsto \Sigma(x)$ between $\mathcal{X}_d$ and $\mathcal{P}_d$ is bijective. This requires some work and our method depends on $d \in \{2, 4\}$. We will provide the proofs in separate sections, but record the statement here so that we can continue our general treatment. For $d = 8$, it can be checked that it follows from Proposition 7.29 that the below proposition is also true and that $\Sigma(x)$ coincides with $\rho(x)$ where ρ is the symplectic polarity of Γ_8 with $\mathcal{P}_8$ as set of absolute symps (and $\mathcal{X}_8$ as set of absolute points).

Proposition 6.7 *For $d \in \{2, 4\}$, the map $\varphi_d : \mathcal{X}_d \to \mathcal{P}_d : x \mapsto \Sigma(x)$ is a bijection.*

Proof For $d = 2$, this is proven in Lemma 7.8 (injectivity) and Lemma 7.9 (surjectivity), for $d = 4$, in Lemma 7.8 (injectivity) and Lemma 7.19 (surjectivity). $\square$

We could make the previous proposition stronger by also showing that φ_d preserves collinearity, and then it would follow from Lemma 6.3 that $\mathcal{X}_d$ is a geometric hyperplane of $\Gamma_d = (X_d, \mathcal{M}_d)$, yet the following method is shorter.

Lemma 6.8 *For any $x \in \mathcal{X}_d$, we have $\Sigma(x) \cap \mathcal{X}_d = x^\perp \cap \Sigma(x)$.*

Proof By Lemma 6.5, all points of $\Sigma(x)$ collinear to x belong to $\mathcal{X}_d$. Suppose for a contradiction that y is a point of $\Sigma(x) \cap \mathcal{X}_d$ not collinear to x. Since $\Sigma(x)$ is a projective symp containing y, it meets $\Sigma(y)$ in a maximal singular subspace S of $\Sigma(y)$ by Lemma 6.5. Consider a point p on $S \cap x^\perp$. Then px and py are non-collinear lines of $\mathcal{L}$ in $\Sigma(x)$ and, again by Lemma 6.5, this implies $\Sigma(p) = \Sigma(x)$, contradicting the injectivity of φ_d (cf. Proposition 6.7). $\square$

And then indeed:

Lemma 6.9 *The set $\mathcal{X}_d$ is a proper geometric hyperplane of Γ_d.*

Proof Take any line $M \in \mathcal{M}_d$. As a consequence of Lemma 6.3, there is a projective symp Σ containing M. Let $x \in \Sigma$ be the unique point with $\Sigma(x) = \Sigma$ (cf. Proposition 6.7). Then $x^\perp \cap \Sigma$ is contained in $\mathcal{X}_d$, and hence M either contains a unique point of $\mathcal{X}_d$ or is contained in $\mathcal{X}_d$. So $\mathcal{X}_d$ is indeed a geometric hyperplane of Γ_d. The fact that it is a proper subspace follows from Lemma 6.8. $\square$

Since $\mathcal{X}_d$ in particular is not empty, it also follows that Q_d has maximal Witt index:

Lemma 6.10 *If $d \in \{2, 4\}$, then a non-degenerate quadric Q_d containing Γ_d has maximal Witt index.*

Proof If $d = 2$, then we are in dimension 8 and hence have to show that Q_2 has 3-spaces; if $d = 4$ then we are in dimension 14 and hence Q_4 needs to have 6-space. If $d = 2$ this actually already follows from the fact that $\mathcal{P}_2$ is non-empty: any projective symp lives in a singular 3-space of Q_2. If $d = 4$, consider a point $x \in \mathcal{X}$ and its associated projective symp $\Sigma(x)$. Consider a singular 4-space V of Γ_4 meeting $\Sigma(x)$ in a line L containing x. Take any line K through x in V and any singular line M through x in $\Sigma(x)$, with $K \neq L \neq M$. Then K and M are contained in a symp, and since $M \in \mathcal{L}$ by Lemma 6.5, this symp is projective, so K and M are Q_4-collinear. We conclude that $\langle V, x^{\perp_{\Gamma_4}} \cap \Sigma(x)\rangle$ is a singular 6-space of Q_4. □

7 Cell-by-cell approach

Having treated the general properties in the previous sections, we now focus on the differences and treat the cases separately.

7.1 First cell: $\mathcal{V}_2(\mathbb{K})$

In this section we deal with the case where $d = 1$, i.e., we consider the quadric Veronese variety $\mathcal{V} := \mathcal{V}_2(\mathbb{K})$ in $\mathsf{PG}(5, \mathbb{K})$, embedded in a non-degenerate hyperbolic quadric Q_1 (the existence of such a quadric follows from Lemma 5.1). We will make use of the Klein correspondence between the points of Q_1 and the lines of $\mathsf{PG}(5, \mathbb{K})$, and briefly recall some of its basic properties.

Klein correspondence — Firstly, collinear points of Q_1 correspond to intersecting lines in $\mathsf{PG}(3, \mathbb{K})$, and the two types of singular planes of Q_1 correspond to the sets of lines through either a common point or a common plane. If Q^- is an elliptic quadric arising as the intersection of a 3-space with Q_1, then it corresponds to a regular line spread of $\mathsf{PG}(3, \mathbb{K})$; in particular a conic of $\mathsf{PG}(5, \mathbb{K})$ arising as the intersection of a (non-singular) plane with Q_1, corresponds to a regulus in $\mathsf{PG}(3, \mathbb{K})$, that is, one system of lines of a hyperbolic quadric in 3 dimensions.

So, since $\mathcal{V} \subseteq Q_1$, its points correspond under the Klein correspondence to lines of $\mathsf{PG}(3, \mathbb{K})$, a projective conic corresponds either to the set of lines forming a cone with vertex a point and base a conic in a plane, or to a set of lines in a plane forming a dual conic; a polar conic corresponds to a set of lines which forms a regulus. By definition of $\mathcal{V}$, these lines of $\mathsf{PG}(3, \mathbb{K})$ equipped with these line sets, form a projective plane.

From the definition of oval it follows that a *dual oval* O^* is a set of lines in $\mathsf{PG}(2, \mathbb{K})$, no three of which are concurrent, and each line of $\mathsf{PG}(2, \mathbb{K})$ has a unique point which is not contained in O^*.

Proposition 7.1 *The lines of $\mathsf{PG}(2, \mathbb{K})$ corresponding to the projective conics of $\mathcal{V}$ form a dual oval. In particular, each $C \in \mathcal{P}_1$ contains a unique point p_C not contained in a second member of $\mathcal{P}_1$, and all other points of C are contained in exactly two members of $\mathcal{P}_1$.*

Proof According to Lemma 7.7 of [7], no projective plane consisting of lines of $\mathsf{PG}(3, \mathbb{K})$ and reguli of $\mathsf{PG}(3, \mathbb{K})$ exists. Applying the Klein correspondence, this implies that there exists at least one projective conic C. Now assume for a contradiction that this is the unique projective conic. Let D be any other conic. By Property (P2) of Fact 4.1, $C \cap D = \langle C\rangle \cap \langle D\rangle$ is a unique point p. Take any point $x \in C$. Our assumption implies that no point of $D \setminus \{p\}$ is collinear to x, and hence the hyperplane $x^\perp$ of $\mathsf{PG}(5, \mathbb{K})$ intersects the plane $\langle D\rangle$ in the tangent line L to

D at p. Therefore, as $x \in C$ was arbitrary, $L \subseteq \langle C\rangle^\perp$. Since $\langle C\rangle$ is a singular plane of Q_1, we know $\langle C\rangle^\perp = \langle C\rangle$. Hence $L \subseteq \langle C\rangle \cap \langle D\rangle$, contradicting $\langle C\rangle \cap \langle D\rangle = \{p\}$. Hence there are at least two projective conics.

Now let C be an arbitrary projective conic. We show that C contains a unique point p_C not contained in a second projective conic, and that all other points of C are contained in a unique second projective conic. By the first paragraph, there is a second projective conic C', and again by Property (P2), $C \cap C' = \langle C\rangle \cap \langle C'\rangle = \{p\}$ for a unique point p. Let L and L' be the tangent lines at p to C and C', respectively. By Property (P3), $\pi := \langle L, L'\rangle$ is the tangent plane at p to $\mathcal{V}$. Recalling that $\langle C\rangle \cap \langle C'\rangle = \{p\}$, collinearity in Q_1 gives a bijection between the set of lines through p in $\langle C\rangle$ and the set of lines through p in $\langle C'\rangle$. Suppose for a contradiction that $L \perp L'$. Consider any conic D not through p and any conic D' through p. Since $\langle D'\rangle$ shares a singular line with π, the conic D' is projective. By Property (P2) of Fact 4.1, D and D' share a point. Since D' was arbitrary, p is collinear to each point of D, leading to a singular 3-space $\langle p, D\rangle$ on Q_1, a contradiction. Hence the line $L'^\perp \cap \langle C\rangle$ meets $C \setminus \{p\}$ in a unique point p_C. Since p_C is not collinear to any point of $C' \setminus \{p\}$, C is the unique projective conic through p_C. Likewise, for any point $q \in C \setminus \{p, p_C\}$, the line $q^\perp \cap \langle C'\rangle$ meets $C' \setminus \{p\}$ in a unique point q' and $[q, q']$ is the unique projective conic through q other than C. We conclude that the projective conics of $\mathcal{V}$ form a dual oval indeed. $\square$

We now show that the constellation of projective conics $\mathcal{P}$ is projectively unique, more precisely we show that its image under the Klein correspondence is, up to duality, a *normal rational curve*, which is projectively unique. A normal rational curve in $\mathsf{PG}(3,\mathbb{K})$ (also known as a twisted cubic) is a set of points which can be parametrised as follows,

$$\{(1, t, t^2, f(t)) \mid t \in \mathbb{K}^\times\} \cup \{(0,0,0,1)\},$$

where f is a polynomial of degree 3 with coefficients in $\mathbb{K}$.

The following property of normal rational curves is well-known, yet we provide a proof for completeness.

Lemma 7.2 *A set of points $\mathcal{C}$ in $\mathsf{PG}(3,\mathbb{K})$, no four of which are contained in a plane, with the property that for each point $x \in \mathcal{C}$, the projection of $\mathcal{C} \setminus \{x\}$ from x is the set of all but one points of a conic, is a normal rational curve.*

Proof For small fields with $|\mathbb{K}| \le 4$ this is trivial (because every set of $|\mathbb{K}|+1$ points no four in a plane is a normal rational curve). So we may assume $|\mathbb{K}| \ge 5$.

Take any point $p_0 \in \mathcal{C}$ and a plane α_0 in $\mathsf{PG}(3,\mathbb{K})$ disjoint from p_0 containing at least one point p_1 from $\mathcal{C}$. By assumption, the projection from p_0 onto α_0 of the points of $\mathcal{C} \setminus \{p_0\}$ gives all but one points of a conic $\mathcal{K}_0$ in α. Let p_2 denote the unique point of $\mathcal{K}_0$ such that $p_0p_2 \cap \mathcal{C} = \{p_0\}$.

We choose coordinates (x_1, x_2, x_2, x_4) in $\mathsf{PG}(3,\mathbb{K})$ as follows. Firstly, we assign $p_0 := (0,0,0,1)$, $p_1 := (1,0,0,0)$, $p_2 := (0,0,1,0)$; secondly, we let $p_3 := (0,1,0,0)$ be the intersection point of the two tangent lines at $\mathcal{K}_0$ in the points p_0 and p_2; finally, $p_4 := (1,1,1,1)$ is a point in $\mathcal{C} \setminus \{p_0\}$ not on α. Hence we obtain that $\mathcal{K}_0$ has equations $X_1X_3 = X_2^2$ and $X_4 = 0$ (recall that $|\mathcal{C}| - 1 \ge 5$ and that five points in a plane determine a unique conic). Hence each point of $\mathcal{C} \setminus \{p_0\}$ is of the form $(1, t, t^2, f(t))$, where $t \in \mathbb{K}$ and $f : \mathbb{K} \to \mathbb{K}$ a mapping. Since $p_1 \in \mathcal{C}$, we have $f(0) = 0$.

We determine $f(t)$ for $t \ne 0$. To that end, we consider the projection of $\mathcal{C} \setminus \{p_1\}$ from p_1 onto the plane with equation $X_1 = 0$, which gives us a set of points $\{(0, t, t^2, f(t)) \mid t \in \mathbb{K}^\times\} \cup \{(0,0,0,1)\}$. Again, this set of points is contained in a conic $\mathcal{K}_1$, say with equations $X_1 = 0$ and $a_2X_2^2 + a_3X_3^2 + a_4X_4^2 + b_2X_3X_4 + b_3X_2X_4 + b_4X_2X_3 = 0$. The condition $p_0 \in \mathcal{K}_1$ readily implies $a_4 = 0$.

Observe that the planes $\langle p_0, p_1, p_3\rangle$ and $\langle p_0, p_1, p_2\rangle$ do not contain points of $\mathcal{C}$ other than p_0 and p_1. Therefore, exactly one of the lines $\langle p_0, p_2\rangle$ (with equation $X_1 = X_2 = 0$) and $\langle p_0, p_3\rangle$ (with equation $X_1 = X_3 = 0$) is a tangent to $\mathcal{K}_1$ at the point p_0. Since the tangent line at p_0 to $\mathcal{K}_1$ has equations $X_1 = 0$ and $b_2X_3 + b_3X_2 = 0$, we obtain that exactly one of b_2, b_3 is 0.

As $(0, t, t^2, f(t))$ with $t \in \mathbb{K} \setminus \{0\}$ belongs to $\mathcal{K}_1$, we have

$$a_2t^2 + a_3t^4 + b_2t^2 f(t) + b_3tf(t) + b_4t^3 = 0,$$

so since not both b_2 and b_3 are 0, we have for $t \neq 0$:

$$f(t) = -\frac{a_2t + b_4t^2 + a_3t^3}{b_3 + b_2t}.$$

If $b_3 = 0$, then all points of $\mathcal{C} \setminus \{p_0, p_1\}$ are contained in the plane with equation $a_2X_1 + b_4X_2 + a_3X_3 + b_2X_4 = 0$, a contradiction. Hence $b_2 = 0$. This implies $a_3 \neq 0$, as otherwise $\mathcal{K}_1$ is degenerate. As also $b_3 \neq 0$, we can set $b_3 = -1$. Then the set $\mathcal{C}$ is given by the point p_0 together with the points $(1, t, t^2, a_2t + b_4t^2 + a_3t^3)$, for $t \in \mathbb{K}$. This is indeed a normal rational curve, as $a_3 \neq 0$. □

Let Π be the set of singular planes of Q_1 supporting projective conics of $\mathcal{P}$.

Proposition 7.3 *The set Π belongs to one class of generators of Q_1. Let $\mathcal{C}$ be the image in $\mathsf{PG}(3, \mathbb{K})$ of Π under the Klein correspondence. Then $\mathcal{C}$ is either a normal rational curve, or the dual of one.*

Proof Each two projective conics of $\mathcal{P}$ meet in a point, and by (P2), the corresponding planes of Q_1 also intersect in exactly that point, and hence belong to the same class of generators of Q_1. Using the Klein correspondence and the self-duality of $\mathsf{PG}(3, \mathbb{K})$, we may assume that $\mathcal{C}$ is a set of points of $\mathsf{PG}(3, \mathbb{K})$. We show that $\mathcal{C}$ is a normal rational curve.

Let C be an arbitrary projective conic of $\mathcal{P}$. Under the Klein correspondence, these points of C correspond to generators of a quadratic cone Q_C in $\mathsf{PG}(3, \mathbb{K})$ with vertex the unique point x_C of $\mathcal{C}$ corresponding to C. By Lemma 7.1, each generator of Q_C but one contains exactly one other point of $\mathcal{C}$, and this way we obtained all points of $\mathcal{C}$ (because each projective conic meets C). This also shows that no three points of $\mathcal{C} \setminus x_C$ are contained in a plane through x_C. Since C was arbitrary, the proposition follows from Lemma 7.2. □

Given Π, we show that we can reconstruct Γ.

Proposition 7.4 *The set Π determines $\Gamma = \mathcal{V}_2(\mathbb{K})$ completely. Consequently, the inclusion $\Gamma \subseteq Q_1$ is projectively unique.*

Proof Since a normal rational curve in $\mathsf{PG}(3, \mathbb{K})$ is projectively unique, the set Π is projectively unique on Q_1. It suffices to show that Π uniquely and unambiguously defines Γ.

Suppose first that $|\mathbb{K}| \geq 5$. Consider any plane $\pi \in \Pi$. By Lemma 7.1, all but one points of $\pi \cap \Gamma$ are given as $\pi \cap \pi'$, where π' varies over $\Pi \setminus \{\pi\}$. We refer to these as the *points of the first generation*. Since $|\Pi \setminus \{\pi\}| = |\mathbb{K}| \geq 5$ (cf. Proposition 7.3), these intersection points define a unique conic in π and hence also the remaining point of $\pi \cap \Gamma$ is determined. We refer to these points as the *points of the second generation*. This yields all projective conics of Γ. Now let C be a non-projective conic of Γ. From the fact that (Π, X) is an oval by Lemma 7.1 and $\langle C\rangle \notin \Pi$, one can deduce that there are at least $\lceil |\mathbb{K}|/2\rceil + 1 \geq 4$ points on C which are contained in a member of Π. The other points of C can be obtained by considering $\langle C\rangle \cap Q_1$.

Conversely, in Γ it is known that no four points not on a conic of Γ (i.e., in Ξ_1) are coplanar, and hence each set of four coplanar points that lie on members of Π, no three of which are on a line of $\mathsf{PG}(5,\mathbb{K})$, determine a unique non-projective conic of Γ (this well-known fact can also be deduced from (P2)). This yields all conics. Since the conics of Γ cover the point set of Γ, we have unambiguously recovered Γ from Π in case $|\mathbb{K}| > 4$. We treat the remaining cases separately.

So suppose $|\mathbb{K}| = 4$. Take $C \in \mathcal{P}$ and let c_1, c_2, c_3, c_4 be its 4 first generation points. There are exactly two points c_5, c_6 in $\langle C\rangle$ that complete $\{c_1, c_2, c_3, c_4\}$ to a conic, namely the two points such that $\{c_1, ..., c_6\}$ is a hyperoval. We show that there is a projectivity preserving Π and interchanging c_5 and c_6. Indeed, applying the Klein correspondence, Π corresponds to a normal rational curve $\mathcal{C}$—which is just a frame—in $\mathsf{PG}(3,4)$. Let us consider the frame $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,1,1)\}$, and suppose $\langle C\rangle$ corresponds to $(0,0,0,1)$. The points c_1, c_2, c_3, c_4 correspond to the lines containing $(0,0,0,1)$ and the other points of the frame. Let L_5 and L_6 denote the respective lines through $(0,0,0,1)$ corresponding to the points c_5 and c_6. Projecting these six lines from $(0,0,0,1)$ on the plane π with equation $X_4 = 0$, we obtain that $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,1,0), L_5\cap\pi, L_6\cap\pi\}$ is a hyperoval. So, if $\epsilon \in \mathbb{F}_4 \setminus \{0,1\}$, then we may assume $L_5 \cap \pi = (1,\epsilon,\epsilon^2)$ and $L_6 \cap \pi = (1,\epsilon^2,\epsilon)$. We now see that the linear mapping interchanging the second and third coordinate preserves $\mathcal{C}$ and interchanges L_5 and L_6. Hence we may assume that C is known. Let C' be any other projective conic. Then its unique point c_5' of second generation is uniquely determined as the unique point extending $C' \setminus \{c_5'\}$ to a conic which is not Q_1-collinear to c_5 Hence also C' is determined, and we conclude that all 15 points of Γ on projective conics are known (10 of which are of first generation).

Let C_1, C_2 be two projective conics, meeting in a point c. Then $C_1 \cup C_2$ contains 7 points of the first generation, and 2 points of the second generation. Hence there are 3 points a_1, a_2, a_3 of the first generation and 3 points b_1, b_2, b_3 of the second generation not Q_1-collinear to c. The point b_1 lies on a unique projective conic and hence is Q_1-collinear to exactly 4 points of the first generation, 2 of which are contained in $C_1 \cup C_2$, the remaining 2 belonging to $\{a_1, a_2, a_3\}$; likewise for b_2 and b_3. We may choose the labelling so that b_i and a_i are not Q_1-collinear, $i = 1,2,3$. Let D_1, D_2, D_3 the three non-projective conics of Γ through c. By the foregoing, a_1, a_2, a_3 are pairwise Q_1-collinear, and hence we may assume that $a_i \in D_i$ for $i = 1,2,3$. Likewise, since b_1 is Q_1-collinear to a_2, a_3, it follows that $b_i \in D_i$. Hence the points of D_i are given by the intersection $Q_1 \cap \langle c, a_i, b_i\rangle$. This determines the point set of Γ, and hence Γ, completely.

Next, suppose $|\mathbb{K}| = 3$. This time we have 4 projective conics with 6 points of the first generation. With a similar argument as in the previous case, we may assume that one projective conic C is fully known. Let c_1, c_2, c_3 denote its points of first generation and c_4 its point of second generation. Note that there are in theory 4 points in $\langle C\rangle$ completing $\{c_1, c_2, c_3\}$ to a conic. Take a second projective conic $C' = \{c_1, c_2', c_3', c_4'\}$, with c_1, c_2', c_3' points of the first generation. Observe that the line c_1c_4' is the unique line in $\langle C'\rangle$ through c_1 not containing any of c_2', c_3' and not collinear to c_4. Let d_4' be the unique point on the line c_1c_4' distinct from c_4' and extending $\{c_1, c_2', c_3'\}$ to a conic. Then there exists a projectivity of Q_1 mapping d_4' to c_4' while stabilizing Π, C and $\{c_1, c_2', c_3'\}$. So we may assume that $C \cup C'$ is known. In fact, the pointwise stabilizer G of Π in Q_1 has order 8, and it is easily seen that this implies that $\mathcal{C}$ has 8 different images under G. These are precisely determined by the 8 choices we had so far, as we will now show. Indeed, if C'' is any other projective conic, then its point of second generation is the unique point of $\langle C''\rangle$ extending the three points of the first generation of C'' to a conic and collinear to neither c_4 nor c_4'. Hence all points of the second generation are determined. Recall that, using the (inverse) Veronese map, Π corresponds to a set of lines of $\mathsf{PG}(2,3)$ which form a dual oval, and since $\mathbb{K} = \mathbb{F}_3$, this is actually a dual conic. Since $\mathbb{K} = \mathbb{F}_3$ has odd characteristic, it also follows that the points of second generation actually correspond to a conic, say $\overline{\mathcal{C}}$, in $\mathsf{PG}(2,3)$, being the tangent points to the dual conic.

Now let $p \in \Gamma$ be a point not on any projective conic. Then p is an internal point of the set of points of $\overline{\mathcal{C}}$. Then p is the intersection of two conics of Γ each containing two points of the second generation and one point of the first generation. These points are well defined: given two points b_1, b_2 of the second generation, the only point of the first generation that can be on a conic with b_1, b_2 is the intersection of the two projective conics distinct from those containing b_1 or b_2. Again, this determines (the point set of) Γ completely.

At last suppose $|\mathbb{K}| = 2$. Completely similar to the previous case, we may assume that all points of the 3 projective conics are known (even the analogue of the group G above has also order 8). This determines 6 of the 7 points of Γ. The last point of Γ is the unique point of Q_1 not collinear to any of these 6 points. This completes the proof of the theorem. □

The above proposition allows us to deduce that the dual oval of Proposition 7.1 is actually a dual conic. Moreover, the correspondence taking a conic $C \in \mathcal{P}_1$ to the unique point $p_C \in C$ contained in a unique member of $\mathcal{P}_1$ (see Proposition 7.1) is clearly a bijection. We also show that the point set $\{p_C \mid C \in \mathcal{P}_1\}$ arises as the intersection of a hyperplane with $\Gamma \cong \mathcal{V}_2(\mathbb{K})$.

Proposition 7.5 *The lines of* $\mathsf{PG}(2,\mathbb{K})$ *corresponding to the projective conics of* Γ *form a dual conic. Moreover, the points* $\mathcal{C} := \{p_C \mid C \in \mathcal{P}_1\}$ *are given by* $H \cap X$*, where* H *is a hyperplane of* $\mathsf{PG}(5,\mathbb{K})$ *and* X *is the point set of* Γ*.*

Proof Let Q_1^* be the Klein quadric with equation as given in Lemma 5.1, containing $\Gamma^* \cong \mathcal{V}_2(\mathbb{K})$, the quadric Veronese variety in its standard embedding in $\mathsf{PG}(5,\mathbb{K})$. We determine the corresponding set $\mathcal{P}_1^*$ of projective symps. In order to do this, we look at the corresponding set of lines of $\mathsf{PG}(2,\mathbb{K})$ under the Veronese correspondence. A general line of $\mathsf{PG}(2,\mathbb{K})$ has equation $aX + bY + cZ = 0$ for some $a, b, c \in \mathbb{K}$. One can verify that, under the Veronese map, this line corresponds to the conic of Γ that lies in the plane of $\mathsf{PG}(5,\mathbb{K})$ determined by the following three hyperplanes of $\mathsf{PG}(5,\mathbb{K})$ (where the coordinates are given by $(x_{-1}, x_{-2}, x_{-3}, x_1, x_2, x_3)$):

$$\begin{cases} cx_{-3} + bx_1 + ax_2 = 0, \\ bx_{-2} + cx_1 + ax_3 = 0, \\ ax_{-1} + cx_2 + bx_3 = 0. \end{cases}$$

Another calculation shows that this gives a singular plane of Q_1^* if and only if $ab + ac + bc = 0$. So the set of lines of $\mathsf{PG}(2,\mathbb{K})$ which corresponds to $\mathcal{P}_1^*$ under the Veronese correspondence is, in the dual of $\mathsf{PG}(2,\mathbb{K})$, given by the equation $x_0x_1 + x_1x_2 + x_0x_2 = 0$, which is a non-degenerate conic of $\mathsf{PG}(2,\mathbb{K})$ indeed. In other words, it forms a dual conic, as required. The set of points in $\mathsf{PG}(2,\mathbb{K})$ corresponding to $\mathcal{C}^* := \{p_C \mid C \in \mathcal{P}_1^*\}$ are then precisely the points of a conic if $\mathrm{char}\,(\mathbb{K}) \neq 2$ (the dual of the dual conic), or the set of points of a line if $\mathrm{char}\,(\mathbb{K}) = 2$ (the nucleus line of the dual conic). When viewing the latter as a degenerate conic (two coinciding lines), one sees that the quadratic equation in $\mathsf{PG}(2,\mathbb{K})$ transforms to a linear equation in $\mathsf{PG}(5,\mathbb{K})$ under the Veronese map, giving a hyperplane which meets Γ in exactly the points of the (degenerate) conic, i.e., the points of $\mathcal{C}^*$.

Since all Klein quadrics in $\mathsf{PG}(5,\mathbb{K})$ are projectively equivalent, there is a projectivity p mapping Q_1^* to Q_1, which of course preserves the singular subspaces of Q_1^* and hence maps projective conics of Γ^* to projective conics of $p(\Gamma^*) \cong \mathcal{V}_2(\mathbb{K})$. Hence, we may assume that Γ^* and Γ are both contained in Q_1. Since $\mathcal{P}_1$ and $\mathcal{P}_1^*$ are projectively equivalent on Q_1 as a consequence of Proposition 7.3 (normal rational curves being projectively unique in $\mathsf{PG}(3,\mathbb{K})$), we may also assume that $\mathcal{P}_1 = \mathcal{P}_1^*$. Finally, by Proposition 7.4, also the Veronese varieties Γ and Γ^* coincide. Therefore we conclude that also $\mathcal{P}_1$ is a dual conic, and that $\mathcal{C} := \{p_C \mid C \in \mathcal{P}_1\}$ is the intersection of Γ with a hyperplane. □

Conclusion. Proposition 7.5 shows the first part ($d = 1$) of Theorem 3.4.

7.2 Second cell: $\mathcal{S}_{2,2}(\mathbb{K})$

In this section we treat the case where $d = 2$, i.e., we consider the geometry $\Gamma_2 = (X_2, \Xi_2)$ which is a Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$, fully embedded in a non-degenerate quadric Q_2 in $\mathsf{PG}(8,\mathbb{K})$, cf. Lemma 5.2. Since we work with a fixed value of d, we will omit the index 2 when using Γ_2, X_2, Ξ_2, etc.

In this particular case, the symps of Γ are hyperbolic quadrangles in $\mathsf{PG}(3,\mathbb{K})$, whose maximal singular subspaces are lines. Observe that Definition 3.3 implies that two members Σ_1, Σ_2 of $\mathcal{P}$ are collinear if $\Sigma_1 \cap \Sigma_2$ is a line of $\mathcal{L}$. Conversely, for a line of $\mathcal{M}$ to belong to $\mathcal{L}$, it suffices to be contained in two members of $\mathcal{P}$:

Lemma 7.6 *Let $M \in \mathcal{M}$ be a singular line of Γ. Then $M \in \mathcal{L}$ if and only if there are at least two members of $\mathcal{P}$ containing M.*

Proof If $M \in \mathcal{L}$ then by definition, all symps through it (and there are at least three such symps) are projective. The converse statement follows from Lemma 6.3, since the symps through a line M correspond to a line of the dual geometry. □

The mutual position of two lines of $\mathcal{L}$ is limited, as we show in the next lemma. Note however that, at this point, we do not yet know that $\mathcal{L}$ is non-empty (for $\mathcal{P}$ we do know this, it is a consequence of Lemma 6.2). Recall that, for a point $x \in \mathcal{X}$, $\Sigma(x)$ is the unique symp of Γ such that the lines of $\mathcal{L}$ through x are precisely the singular lines of Σ containing x (cf. Lemma 6.5). The following lemma will allow us to deduce injectivity of $\varphi(x)$, but it is also of more general use.

Lemma 7.7 *Suppose L_1 and L_2 are distinct lines of $\mathcal{L}$. Then either L_1 and L_2 are disjoint and are not contained in a symp of Γ, or L_1 and L_2 intersect in a point x and belong to $\Sigma(x)$ and are the only lines of $\mathcal{L}$ containing x. In particular, a singular plane of Γ contains at most one line of $\mathcal{L}$.*

Proof Suppose first that L_1 and L_2 share a point x, which belongs to $\mathcal{X}$ by definition. Then by Lemma 6.5, L_1 and L_2 belong to $\Sigma(x)$ and are hence not contained in a singular plane. Moreover, by the same lemma, there are no other lines of $\mathcal{L}$ through x than L_1 and L_2. Since two lines in a singular plane of Γ intersect in a point, the last statement follows too.

Now suppose L_1 and L_2 are disjoint. Suppose for a contradiction that there is a symp Σ containing $L_1 \cup L_2$. Let Σ_i be a symp distinct from Σ containing L_i, $i = 1, 2$. Then Σ_1 and Σ_2 share at least a point p by Property (P2) of Fact 4.1. Now $L_1, L_2 \in \mathcal{L}$ and hence $\Sigma_1, \Sigma_2 \in \mathcal{P}$, which means that p is Q-collinear to $\langle L_1, L_2 \rangle = \langle \Sigma \rangle$. Since $\langle \Sigma \rangle$ is a maximal singular subspace of Q (as the latter is non-degenerate), we obtain $p \in \langle \Sigma \rangle \cap X = \Sigma$. However, this implies $p \in \Sigma \cap \Sigma_i = L_i$ for $i \in \{1, 2\}$, contradicting the fact that $L_1 \cap L_2$ is empty. We conclude that L_1 and L_2 are not contained in a common symp of Γ. □

This already leads us to injectivity of $\varphi(x)$:

Lemma 7.8 *Let $x, y \in \mathcal{X}$ be such that $\Sigma(x) = \Sigma(y)$. Then $x = y$.*

Proof Suppose for a contradiction that x, y are distinct points of $\mathcal{X}$ with $\Sigma(x) = \Sigma(y)$. Then $\Sigma(x)$ contains two lines of $\mathcal{L}$ containing x and two lines of $\mathcal{L}$ containing y. Among these lines, at least two are either disjoint or are contained in a singular plane, contradicting Lemma 7.7. □

Next, we show surjectivity.

Lemma 7.9 *For each $\Sigma \in \mathcal{P}$, there is a $x \in \mathcal{X}$ such that $\Sigma = \Sigma(x)$, i.e., Σ contains two intersecting lines of $\mathcal{L}$. In particular, $\mathcal{L}$ is not empty.*

Proof Let Σ be any symp of $\mathcal{P}$. Take a singular plane π in Γ meeting Σ in a line M and let p be a point in $\pi \setminus M$. Consider the tangent space $T_p(Q)$, which does not contain Σ, for otherwise $\langle p, \Sigma\rangle$ would be a singular subspace of Q of dimension 4, a contradiction (to Q being non-degenerate). Hence T_p intersects the 3-space $\langle\Sigma\rangle$ in a plane α containing the line M. Therefore, $\alpha \cap \Sigma$ is the union of two lines M and L. Take a point q on $L \setminus M$. In Γ, the points p and q are not collinear, and hence they determine a unique symp $\Sigma_{p,q}$ of Γ. By Lemma 6.1, $\Sigma_{p,q} \in \mathcal{P}$. Then $L = \Sigma \cap \Sigma_{p,q}$, thus $L \in \mathcal{L}$. Note that L and M are lines of different types in Σ.

Repeating the argument with a plane π' meeting Σ in a line M' of the other type, i.e., a line M' meeting M in a unique point, we obtain a second line $L' \in \mathcal{L}$ (L and L' are distinct since they have different types). □

We have shown Proposition 6.7 in the case that $d = 2$. We now extend the bijection given by φ to an isomorphism.

Lemma 7.10 *The point-line geometry $\widetilde{\Gamma^*}$, with point set $\mathcal{P}$ and induced line set, is isomorphic to $(\mathcal{X}, \mathcal{L})$. More precisely, if x, y, z are three points of $\mathcal{X}$ on a line L of $\mathcal{L}$, then $\Sigma(x), \Sigma(y), \Sigma(z)$ are three projective symps containing L; and conversely, if $\Sigma(x), \Sigma(y), \Sigma(z)$ are pairwise collinear and on a line of the dual geometry, then x, y, z are on a line of $\mathcal{L}$.*

Proof Let x, y, z be three points on a line L of $\mathcal{L}$ (note that $L \subseteq \mathcal{X}$). Then $\Sigma(x)$, $\Sigma(y)$ and $\Sigma(z)$ contain L and hence $\Sigma(x) \cap \Sigma(y) = \Sigma(y) \cap \Sigma(z) = \Sigma(z) \cap \Sigma(x) = L$. Conversely, if $\Sigma(x), \Sigma(y), \Sigma(z)$ are pairwise collinear projective symps on a line of the dual geometry, then by definition they share a line L which belongs to $\mathcal{L}$. Since lines are maximal singular subspaces of symps, we have $x, y, z \in L$ by Lemma 6.8. □

Moreover, we observe the following property, which will turn out not be true for higher values of d; indeed, if $d > 2$, there are Γ-collinear points in $\mathcal{X}$ which are not collinear (they are symplectic instead) in the geometry $(\mathcal{X}, \mathcal{L})$, see Fact 7.28.

Lemma 7.11 *If $x, y \in \mathcal{X}$ are distinct Γ-collinear points, then $xy \in \mathcal{L}$.*

Proof Suppose for a contradiction that $xy \notin \mathcal{L}$. By Lemma 7.10 and Definition 3.3, this is equivalent with $\Sigma(x) \cap \Sigma(y)$ not being a line of $\mathcal{L}$. By Lemma 7.6, this at its turn is equivalent with $\Sigma(x) \cap \Sigma(y)$ being just a point, say p. Since $x \perp_\Gamma y$, Corollary 4.4(i) implies that $\langle x, p, y\rangle$ is a singular plane. By Lemma 6.8, the lines xp and yp belong to $\mathcal{L}$. As such, xp and py are two lines of $\mathcal{L}$ in a singular plane of Γ, contradicting Lemma 7.7. We conclude that $xy \in \mathcal{L}$. □

Our next goal is to show that the point-line geometry $(\mathcal{X}, \mathcal{L})$ is a (thin) generalized hexagon. To that end we show that $(\mathcal{X}, \mathcal{L})$ contains no ordinary m-gons for $2 \leq m < 6$ and that every point-line pair is contained in an ordinary 6-gon.

Lemma 7.12 *The geometry $(\mathcal{X}, \mathcal{L})$ does not contain an ordinary m-gon for $2 \leq m < 6$.*

Proof Clearly, two points of $\mathcal{X}$ are on at most one line of $\mathcal{M}$ and hence also on at most one line of $\mathcal{L}$, so there are no digons. In particular, any m-gon with $m > 2$ is determined by its points or lines. By Lemma 7.7 there are no triangles (this would yield two lines of $\mathcal{L}$ in a common singular plane), nor quadrangles (this would yield a symp containing two disjoint lines of $\mathcal{L}$). Now suppose there is a pentagon in $(\mathcal{P}, \mathcal{L})$, with points $p_0, \dots, p_4$. By Lemma 7.7, the lines p_0p_1 and p_0p_4 determine a unique projective symp, and therefore p_1 is Q-collinear to p_4. Obviously, p_0 is also Q-collinear to p_1 and p_4. We conclude that all pairs of points of $\{p_0, \dots, p_4\}$ are Q-collinear. If the planes $\langle p_0, p_1, p_2\rangle$ and $\langle p_0, p_3, p_4\rangle$ (which are singular planes of Q and non-singular planes of Γ) share more than just $\{p_0\}$, then this leads to a contradiction to Property (P2) of Fact 4.1. Therefore $\langle p_0, \dots, p_4\rangle$ generates a 4-dimensional singular space of Q, a contradiction. □

Lemma 7.13 *Let $L_0, \dots, L_4$ be five distinct lines of $\mathcal{L}$, such that L_i and L_{i+1} intersect in a point p_i for $i = 0, \dots, 3$. Then there is an ordinary hexagon in $(\mathcal{X}, \mathcal{L})$ containing $L_0, \dots, L_4$.*

Proof Note that the points $p_0, \dots, p_3$ are the only points that lie on more than one of these lines: if there would be other intersection points, there would be an ordinary m-gon for $2 \le m < 6$, a contradiction to Lemma 7.12. We claim that p_0 is opposite the (projective) symp $\Sigma(p_3)$, the unique symp containing L_3 and L_4. Clearly, $p_0 \notin \Sigma(p_3)$ since $L_3 \cup L_4 = \Sigma(p_3) \cap \mathcal{X}$ by Lemma 6.8. So suppose for a contradiction that p_0 is collinear to a line M of $\Sigma(p_3)$. Then p_2 is collinear to a point $p \in M$ and hence the symp $\Sigma(p_1)$ meets $\Sigma(p_3)$ in the line p_2p. Since $\Sigma(p_1)$ and $\Sigma(p_3)$ are projective symps containing p_2p, it follows from Lemma 7.6 that $p_2p \in \mathcal{L}$. However, as noted above, $L_3 \cup L_4 = \Sigma(p_3) \cap \mathcal{X}$, and hence $M = L_3$, leading to three lines of $\mathcal{L}$ in $\Sigma(p_2)$, a contradiction to Lemma 6.8. So p_0 is far from $\Sigma(p_3)$ indeed. As such, the line L_0 contains a unique point p_5 (distinct from p_0) collinear to a line N of $\Sigma(p_3)$ (cf. Fact 4.6). The two lines of $\mathcal{L}$ in the symp $\Sigma(p_5)$ are L_0 and a unique line L_5 in the singular plane $\langle p_0, N\rangle$. Let p_4 be the point $L_5 \cap N$. Then $p_4 \in \mathcal{X}$ and hence by Lemma 6.8, the line $p_4p_3 \in \mathcal{L}$ (more precisely, it is L_4 or L_3). By Lemma 7.12, $p_4p_3 = L_4$ and $L_0, \dots, L_5$ is an ordinary hexagon. □

Lemma 7.14 *Every two elements of $\mathcal{X} \cup \mathcal{L}$ are contained in an ordinary hexagon.*

Proof First, let x, y be two elements of $\mathcal{X}$. If $x = y$ or $xy \in \mathcal{L}$, this follows from Lemma 7.13: each point of $\mathcal{X}$ lies on two lines of $\mathcal{L}$ and hence by Lemma 7.12 we can make a path of length 5. So, by Lemma 7.11, we may suppose that x and y are not Γ-collinear. If $\Sigma(x) \cap \Sigma(y)$ is a line M, then $x^\perp \cap M = M \cap \mathcal{X} = y^\perp \cap M$ and the latter is a unique point p, so $xp, py \in \mathcal{L}$ and again, the statement follows from Lemma 7.13. So suppose $\Sigma(x) \cap \Sigma(y)$ is a point p. By the above, we may assume that $p \notin \mathcal{X}$ and hence p is not collinear to x nor to y (cf. Lemma 6.8). Let L_i^x and L_i^y be the respective lines of $\Sigma(x)$ and $\Sigma(y)$ ($i = 1, 2$) contained in $\mathcal{L}$. Denote by q_i^x the unique point on L_i^x collinear to p. Then q_i^x is collinear to a line N_i of $\Sigma(y)$ containing p. Renumbering if necessarily, N_i meets L_i^y in a point q_i^y. By Lemma 7.11, $q_i^x q_i^y \in \mathcal{L}$ and hence we obtained a hexagon contanining x and y.

Next, consider a point x and a line $L \in \mathcal{L}$. Take any point $y \in L$. By the above, there is a hexagon containing x and y, and since y is contained in exactly two lines of $\mathcal{L}$ by Lemma 6.5, the hexagon contains L. Likewise, if we start from two lines of $\mathcal{L}$. □

In the next lemma, we show that $\mathcal{X}$ arises as a hyperplane section of $\mathcal{S}_{2,2}(\mathbb{K})$.

Lemma 7.15 *There is a hyperplane H of $\mathsf{PG}(8, \mathbb{K})$ such that $X \cap H$ is precisely $\mathcal{X}$.*

Proof By Lemma 5.2, the points of Q are given by the equation

$$a(X_1X_2 - X_3X_4) + b(X_0X_2 - X_5X_6) + c(X_0X_1 - X_7X_8) + d(X_0X_4 - X_6X_8) + e(X_0X_3 - X_5X_7) \\ + f(X_1X_6 - X_4X_7) + g(X_1X_5 - X_3X_8) + h(X_2X_8 - X_4X_5) + i(X_2X_7 - X_3X_6) = 0,$$

with $a,b,c,d,e,f,g,h,i\in\mathbb{K}$ and $abc-ade-bfg-chi-dgi-efh\neq 0$.

By Proposition 6.7 and Lemma 7.10, the correspondence $\varphi:\mathcal{X}\to\mathcal{P}$ is an isomorphism between the point-line geometry $(\mathcal{X},\mathcal{L})$ and the point-line geometry $\widetilde{\Gamma^*}$, with point set $\mathcal{P}$ and induced collinearity.

By Lemma 6.4, a point $x\in X$ belongs to $\mathcal{X}$ if and only if, for any symp Σ opposite x holds that $T_x(Q)\cap\Sigma$ is a degenerate hyperplane of Σ (i.e., two intersecting lines).

Let $(x_1x_2, y_1y_2, z_1z_2, y_1z_2, y_2z_1, x_1z_2, x_2z_1, x_2y_1, x_1y_2)$ be an arbitrary point x of Γ, without loss of generality we may suppose $x_1=x_2=1$, so we get $(1, y_1y_2, z_1z_2, y_1z_2, y_2z_1, z_2, z_1, y_1, y_2)$. The tangent space to Q in this point has equation

$$\begin{aligned}&(cy_1y_2+bz_1z_2-ey_1z_2-dy_2z_1)X_0+(c+az_1z_2-gz_2-fz_1)X_1+(b+ay_1y_2-iy_1-hy_2)X_2\\&+(-e-ay_2z_1+iz_1+gy_2)X_3+(-d-ay_1z_2+hz_2+fy_1)X_4+(-gy_1y_2+hy_2z_1-bz_1+ey_1)X_5\\&+(-fy_1y_2+iy_1z_2-bz_2+dy_2)X_6+(-iz_1z_2+fy_2z_1+ez_2-cy_2)X_7+(-hz_1z_2+dz_1-cy_1+gy_1z_2)X_8=0.\end{aligned}$$

This point lies opposite the symp Σ determined by $X_0=X_5=X_6=X_7=X_8=0$ and $X_1X_2=X_3X_4$. We determine the intersection of the tangent space and this symp.

If $c+az_1z_2-gz_2-fz_1=b+ay_1y_2-iy_1-hy_2=-e-ay_2z_1+iz_1+gy_2=-d-ay_1z_2+hz_2+fy_1=0$, then $T_x(Q)\cap\Sigma=\Sigma$, contradicting Lemma 6.2 and the fact that Q is non-degenerate. Without loss of generalization, we may hence suppose that $c+az_1z_2-gz_2-fz_1\neq 0$. Then $T_x(Q)\cap\Sigma$ is given by

$$\begin{cases}X_0=X_5=X_6=X_7=X_8=0,\\X_1X_2=X_3X_4\\(c+az_1z_2-gz_2-fz_1)X_1+(b+ay_1y_2-iy_1-hy_2)X_2+\\\qquad(-e-ay_2z_1+iz_1+gy_2)X_3+(-d-ay_1z_2+hz_2+fy_1)X_4=0\end{cases}$$

This is a conic in a plane and it is easily verified that it is degenerate if and only if

$$(b+ay_1y_2-iy_1-hy_2)(c+az_1z_2-gz_2-fz_1)-(-e-ay_2z_1+iz_1+gy_2)(-d-ay_1z_2+hz_2+fy_1)=0.$$

This is at its turn equivalent to

$$\begin{aligned}&(ed-bc)+(fg-ac)y_1y_2+(hi-ab)z_1z_2+(ae-gi)y_1z_2+\\&\qquad(ad-fh)y_2z_1+(bg-eh)z_2+(bf-di)z_1+(ci-ef)y_1+(ch-dg)y_2=0.\end{aligned}$$

It follows that $x\in X$ belongs to $\mathcal{X}$ if and only if it lies in the hyperplane with equation

$$\begin{aligned}&(ed-bc)X_0+(fg-ac)X_1+(hi-ab)X_2+(ae-gi)X_3+(ad-fh)X_4+\\&\qquad(bg-eh)X_5+(bf-di)X_6+(ci-ef)X_7+(ch-dg)X_8=0.\end{aligned}$$

□

Conclusion. By Proposition 6.7 and Lemma 7.10, the geometry $\widetilde{\Gamma^*}$ (see Definition 3.3) is isomorphic to the subgeometry $(\mathcal{X},\mathcal{L})$ of $\Gamma=(X,\mathcal{M})$, and hence by Lemma 7.14, $\widetilde{\Gamma^*}$ is a full subgeometry of Γ^* isomorphic to a thin generalised hexagon, and $\mathcal{X}=H\cap X$ for a hyperplane H of $\mathsf{PG}(8,\mathbb{K})$ by Lemma 7.15. We have proven the second case of Theorem 3.4.

7.3 Third cell: $\mathcal{G}_{6,2}(\mathbb{K})$

In this section, $d = 4$ and $\Gamma_4 = (X_4, \mathcal{M}_4)$ is the line Grassmannian variety $\mathcal{G}_{6,2}(\mathbb{K})$ embedded in a non-degenerate quadric Q_4 of $\mathsf{PG}(14, \mathbb{K})$ (abstractly, Γ_4 is of type $\mathsf{A}_{5,2}(\mathbb{K})$). As in the previous section, we will omit the index 4.

Recall that the maximal singular subspaces of Γ are either planes or 4-dimensional subspaces. These maximal planes occur as the intersections of symps, moreover, the maximal singular subspaces of symps come into two natural families as the symps are hyperbolic quadrics $Q(5, \mathbb{K})$. The members of one family are exactly those that occur as the intersection with other symps, the members of the other family are exactly those that are contained in a singular 4-space (cf. Fact 4.8). We will refer to these two types as the *maximal type* and the *non-maximal type*. About the planes of maximal type, we can say the following:

Lemma 7.16 *Suppose $\Sigma_1 \cap \Sigma_2$ is a plane π. Then each symp containing a line M of π, contains π.*

Proof Suppose Σ contains a line M of π. Then Σ meets Σ_1 in a (maximal) plane π_1 and since there is only one plane of maximal type through M in Σ_1 by Fact 4.8, $\pi_1 = \pi$. □

Lemma 7.17 *Let Σ_1, Σ_2 be distinct members of $\mathcal{P}$. The following are equivalent.*

- *(i) Σ_1 and Σ_2 are collinear (w.r.t. Definition 3.3);*
- *(ii) $\Sigma_1 \cap \Sigma_2$ is a plane at least one line of which belongs to of $\mathcal{L}$;*
- *(iii) $\Sigma_1 \cap \Sigma_2$ is a plane each line of which belongs to of $\mathcal{L}$.*

Proof The implications (*i*) ⇒ (*ii*) and (*iii*) ⇒ (*i*) are trivial, so we show (*ii*) ⇒ (*iii*). Take any line L in the plane $\Sigma_1 \cap \Sigma_2$. Then each symp containing L also contains π by Lemma 7.16, in particular, it contains a line of $\mathcal{L}$, and hence it is projective. So $L \in \mathcal{L}$ indeed, and (*iii*) follows. □

In view of the above, it is useful to introduce a distinguished set of planes.

Definition 7.18 Let Π denote the set of planes of maximal type, each line of which belongs to $\mathcal{L}$, or equivalently, the set of planes of maximal type containing at least one line of $\mathcal{L}$.

As mentioned in Section 6, our aim is to show that $\varphi : \mathcal{X} \to \mathcal{P} : x \mapsto \Sigma(x)$ is bijective. We start with surjectivity. Note that this implies that the sets $\mathcal{L}$ and $\mathcal{X}$ are non-empty (the set of projective symps is non-empty by Lemma 6.2).

Lemma 7.19 *Let Σ be a projective symp. Then Σ contains a point $x \in \mathcal{X}$ such that $\Sigma(x) = \Sigma$.*

Proof Let π be a singular plane of Σ which is contained in a singular 4-space V, and let M be a line in V disjoint from π. Then $T_M(Q) \cap \langle \Sigma \rangle$ contains π and is either a 3-space or a 4-space of Σ (if $M \perp_q \langle \Sigma \rangle$ then this would yield a 7-space of Q, a contradiction). In the first case, $T_M(Q) \cap \Sigma$ contains a unique singular plane π' of the other type, intersecting π in a line; in the second case, $T_M(Q) \cap \Sigma$ is a degenerate hyperplane $T_q(\Sigma)$ for some $q \in \Sigma$. In the latter case, let π' be any singular plane of Σ intersecting π in a line containing q.

Put $M' := \pi \cap \pi'$. Take a point $p \in \pi' \setminus M'$. Consider a plane α in V containing M, disjoint from M'. Since $p^{\perp_\Gamma} \cap V = M'$, the point p is not Γ-collinear to any point of α. Now let Σ' be any symp containing α (cf. Property (P1) of Fact 4.1). If p were Γ-collinear to a point of Σ', then p is Γ-collinear to a singular plane α' of Σ' by Fact 4.3. The planes α and α' are clearly non-maximal, so by Fact 4.8, they share a point, contradicting that p is not Γ-collinear to any point of α. So p and Σ' are opposite.

We claim that the singular plane of Σ' whose points correspond to the symps through π' is exactly α (cf. Fact 4.9). Put differently, we claim that for each point r of α, the symp $\Sigma_{p,r}$ contains π'. Indeed, r is Γ-collinear to the line M' in π', so $\Sigma_{p,r}$ contains π'. The claim follows. By construction, α belongs to $T_p(Q) \cap \Sigma'$, because π' and V are collinear on Q: π' is Q-collinear to M by choice of π' and to π since Σ is projective. The symps $\Sigma_{p,r}$ with $r \in \alpha$ are hence projective, i.e., all symps through π' are projective. By Lemma 7.16, each line of π' belongs to $\mathcal{L}$ and so $\pi' \in \Pi$.

Since π was an arbitrary plane in Σ, it follows that there are at least two planes of Π in Σ. These planes intersect each other in a point $x \in \mathcal{X}$, and $\Sigma(x) = \Sigma$ by Lemma 6.5. □

Next, we work towards injectivity.

Notation. Define $X(\Sigma)$ as the set of points x of $\Sigma \cap \mathcal{X}$ with $\Sigma(x) = \Sigma$.

Lemma 7.20 *Let Σ be a projective symp. Then $X(\Sigma)$ is either a unique point, the set of points of a singular plane of Σ of non-maximal type, or the entire point-set of Σ.*

Proof By Lemma 7.19, $X(\Sigma)$ is non-empty. We first show that $X(\Sigma)$ is a subspace. So suppose that x and y are distinct, collinear points of $X(\Sigma)$. Let α be the unique plane in Σ of non-maximal type containing the line xy. Consider any point $p \in \pi \setminus xy$. Then Σ contains a unique plane π_{xp} of Π containing the line xp and a unique plane π_{yp} of Π containing the line yp by Lemma 6.5. By the same lemma, $\Sigma(p) = \Sigma$ since it contains the (distinct) planes π_{xp} and π_{yp} of Π. Therefore, $p \in X(\Sigma)$. Switching the roles of p and y, the same holds for the points on the line xy and $\pi \subseteq X(\Sigma)$. We conclude that, if $X(\Sigma)$ contains no pair of non-collinear points, then it is either a unique point or a plane of non-maximal type (namely α).

Finally, suppose that $X(\Sigma)$ contains two points x, y which are non-collinear. Take any point p in Σ collinear to both x and y. The same argument as in the previous paragraph shows that $\Sigma(p) = \Sigma$. It hence follows that $X(\Sigma)$ is a non-singular convex subspace of Σ, i.e., Σ itself. The lemma follows. □

Lemma 7.21 *Let Σ be a projective symp. The set $X(\Sigma)$ is a unique point.*

Proof Consider a point $p \in X(\Sigma)$, which is possible by Lemma 7.19. Suppose for a contradiction that $X(\Sigma)$ is not just $\{p\}$. Then Lemma 7.20 implies that either $X(\Sigma)$ is Σ itself (Case (i)) or a plane α of non-maximal type (Case (ii)). Take a point $z \in \Sigma$ not collinear to p (in particular, $z \notin \alpha$ in Case (ii)). We claim that we can take a symp Σ' as follows, depending on the case:

- (i) Let Σ' be any symp meeting Σ in precisely z.
- (ii) Let Σ' be a projective symp meeting Σ in precisely z.

The point-residue of $(X, \mathcal{M})$ at z is isomorphic to $\mathcal{S}_{1,3}(\mathbb{K})$, in which there clearly is a symp which is disjoint from the one corresponding to Σ. So in Case (i), there is a symp Σ' as required. In Case (ii) we additionally require that Σ' is projective. To that end, recall that $\Sigma(z) \neq \Sigma$ and hence $\Sigma(z)$ meets the projective symp Σ in a plane $\pi \in \Pi$ by Lemma 6.5. Now take a plane π' in $\Sigma(z)$, meeting Σ in $\{z\}$ only. Again considering the point-residue at z, it follows that there is a symp Σ' containing π' which meets Σ in z only, and since Σ' contains $\pi' \in \Pi$, Σ' is projective.

In both cases, $p^\perp \cap \Sigma' = \emptyset$: if not, then p is Γ-collinear to a plane of Σ', and by Corollary 4.4(i), this implies $p \perp z$, a contradiction. In other words, p is opposite Σ'. Since $\Sigma(p) = \Sigma$, Corollary 6.6 implies that $T_p(Q) \cap \Sigma'$ coincides with the singular hyperplane $T_z(\Sigma')$. Take a line L with $\langle z, L\rangle$ a plane of Σ' of non-maximal type. By Lemma 6.5, there is a plane of Π through p such that the symps through it correspond to the points of the plane $\langle z, L\rangle$, in particular, the point p is Q-collinear with $\langle z, L\rangle$. We now distinguish between the two cases.

In Case (i), p can be replaced by any point of Σ non-collinear to z since $X(\Sigma) = \Sigma$. From this we deduce that L is Q-collinear to all points of Σ opposite z, and since these points generate $\langle\Sigma\rangle$, we obtain that $\langle L, \Sigma\rangle$ is a singular subspace of Q. However, $\langle \Sigma, L\rangle$ has dimension 7 since $L \cap \langle\Sigma\rangle = \emptyset$, and the maximal singular subspaces of Q have dimension 6, a contradiction.

In Case (ii), we first vary L to obtain that p is Q-collinear to all points of $z^\perp \cap \Sigma'$, which generates a singular 4-space of Q because Σ' is projective. Next, we vary the point p among the points of the plane α that are not Σ-collinear to z. From this we deduce that the entire plane α is Q-collinear to the 4-space $\langle z^\perp \cap \Sigma'\rangle$, again leading to a 7-dimensional subspace on Q, a contradiction. □

We have now shown Proposition 6.7 for $d = 4$. This allowed us to deduce that $\mathcal{X}$ is a geometric hyperplane of $\Gamma = (X, \mathcal{M})$ (cf. Lemma 6.9). As mentioned earlier, we do not rely on the classification of the geometric hyperplanes of Γ (see Proposition 4.14) for this case, and give constructive proof instead, complementing the approach used for $d = 8$.

Our first goal is to show that the singular 4-spaces of $\mathcal{G}_{6,2}(\mathbb{K})$ give rise to symps of $\mathcal{C}_{3,2}(\mathbb{K})$. So let V be a singular 4-space and define $\mathcal{X}_V = V \cap \mathcal{X}$ and $\mathcal{L}_V = \{L \in \mathcal{L} \mid L \subseteq V\}$.

Lemma 7.22 *Let V be a singular 4-space of Γ. Then $\mathcal{X}_V$ is a 3-dimensional subspace of $\mathcal{X}$ and for each point $x \in \mathcal{X}_V$, the lines of $\mathcal{L}_V$ through x in V are precisely the lines through x in the plane $\pi(x) := \Sigma(x) \cap V$. Moreover, the map $x \mapsto \pi(x)$ is injective.*

Proof By Lemma 6.9, it suffices to show that V is not fully contained in $\mathcal{X}$. Suppose for a contradiction that $V \subseteq \mathcal{X}$. Take any point $x \in V$. Then $\Sigma(x) \cap V$ is a plane π (of non-maximal type since $\pi \subseteq V$). Consider a point p in $\Sigma(x) \cap x^\perp$, with $p \notin V$. Then $px \in \mathcal{L}$ by Lemma 6.5 and hence $\Sigma(p)$ contains x and therefore shares a plane π' with V. Now $\pi' \subseteq V \subseteq \mathcal{X}$ and hence Lemma 6.8 implies that $\pi' \subseteq p^\perp$. As the maximal singular subspaces of $\Sigma(p)$ are planes, this implies $p \in \pi'$, contradicting our choice of $p \notin V$. This shows the first statement.

Take any point $x \in \mathcal{X}_V$ and consider the plane $\pi(x) := \Sigma(x) \cap \mathcal{X}_V$. Note that Lemma 6.5 implies that $\pi(x)$ can also be defined as the unique plane in V containing all lines of $\mathcal{L}$ in V containing x. We claim that the correspondence $x \mapsto \pi(x)$ is injective. Suppose that $\pi(x) = \pi(y)$. Then $\Sigma(x) \cap \Sigma(y)$ contains the non-maximal plane $\pi(x) = \pi(y)$ and hence $\Sigma(x) = \Sigma(y)$. We conclude from Lemma 7.21 that $x = y$. This shows the claim. □

Corollary 7.23 *Let V be a singular 4-space of Γ. Then three points x_1, x_2, x_3 of $\mathcal{X}_V$ which pairwise determine a line of $\mathcal{L}_V$, lie on one line.*

Proof Suppose for a contradiction that $\langle x_1, x_2, x_3\rangle$ generate a plane. Let L_k denote the line $x_i x_j \in \mathcal{L}$, with $\{i, j, k\} = \{1, 2, 3\}$. By Lemma 7.22, we see that $\pi(x_1) = \langle x_1, x_2, x_3\rangle$, since it is generated by the lines x_1x_2 and x_1x_3. Likewise, $\pi(x_2) = \langle x_1, x_2, x_3\rangle$ and hence, also by Lemma 7.22. we obtain that $x_1 = x_2$, a contradiction. The corollary follows. □

Remark 7.24 Take any 4-space V of Γ. Then the correspondence $x \mapsto \pi(x)$ for $x \in \mathcal{X}_V$ induces a symplectic polarity in the projective 3-space $\mathcal{X}_V$. We will show indirectly in the next proposition that $(\mathcal{X}_V, \mathcal{L}_V)$ is a symplectic quadrangle indeed.

Next, we also show that each maximal plane which is contained in $\mathcal{X}$, belongs to Π.

Lemma 7.25 *Suppose α is a maximal singular plane, with $\alpha \subseteq \mathcal{X}$. Then every symp through α is projective, i.e., $\alpha \in \Pi$. Moreover, if $\alpha \subseteq \Sigma(x)$, then $x \in \alpha$.*

Proof Take any point $p \in \alpha$ and consider a symp Σ opposite p. By Lemma 6.2, there is at least one projective symp Σ' containing α. By Proposition 6.7, Σ' contains a unique point x with $\Sigma' = \Sigma(x)$. According to Lemma 6.8, the points of $\mathcal{X}$ in Σ' are precisely those collinear to x, and hence $x \in \alpha$. So each line of α through x belongs to $\mathcal{L}$. But then every symp through α contains a line of $\mathcal{L}$ and hence is projective. The lemma follows. □

We reach our goal:

Proposition 7.26 *The point-line geometry $\Gamma' = (\mathcal{X}, \mathcal{L})$ is isomorphic to the point-line geometry associated to a geometry of type $\mathsf{C}_{3,2}(\mathbb{K})$ and each symp of Γ' corresponds to a 4-space V of Γ in the sense that $(\mathcal{X}_V, \mathcal{L}_V)$ is a symplectic quadrangle in the projective 3-space $\mathcal{X}_V$ and vice versa.*

Proof We define yet another point-line geometry, namely $\Gamma'' := (\mathcal{V}, \mathcal{X})$ (with natural incidence), where $\mathcal{V}$ is the set of 4-spaces of Γ and a 'line' is the set of singular 4-spaces through a point of $\mathcal{X}$. We show that Γ'' is a symplectic polar space of rank 3, which already leads us to a geometry of type $\mathsf{C}_{3,1}(\mathbb{K})$ with line set $\mathcal{X}$.

Observe that any two 4-spaces of Γ intersect each other in a unique point of Γ (since they correspond to distinct points of a projective 5-space, which are on a unique line). We verify the axioms of polar spaces for Γ''.

(BS1) There are at least three 4-spaces through any point of $\mathcal{X}$ (noting that the set of 4-spaces through a point of Γ corresponds to the set of points on a line in $\mathsf{PG}(5, \mathbb{K})$).

(BS2) Let V be any 4-space and let p be a point in $V \setminus \mathcal{X}_V$ (which exists by Lemma 7.22). Then any 4-space V' through p distinct from V is non-collinear to V in Γ'' since $V \cap V' = \{p\} \notin \mathcal{X}$.

(BS3) We claim that the set of maximal singular subspaces of Γ'' is precisely Π. Firstly, if $\pi \in \Pi$, then any 4-space meeting π in a point, meets π in a line and hence two 4-spaces containing π intersect each other in a point of $\pi \subseteq \mathcal{X}$ and are therefore collinear in Γ''. We now show that, conversely, for any maximal set of pairwise collinear 4-spaces of Γ'', there is a plane of Π meeting each of these 4-spaces in a line.

Put $\{i, j, k\} = \{1, 2, 3\}$. Let V_1, V_2, V_3 be three 4-spaces of $\mathcal{V}$ with three pairwise distinct intersection points $x_k := V_i \cap V_j \in \mathcal{X}$. Then x_1, x_2, x_3 are pairwise collinear points in Γ, which generate a singular plane α of Γ (they do not lie on a line, for this line would then be contained in $V_1 \cap V_2 \cap V_3$, a contradiction). Since two 4-spaces meet in a point, π is a maximal singular plane (and π meets each V_i in a line). Moreover, $\pi \subseteq \mathcal{X}$ since x_1, x_2 and x_3 belong to $\mathcal{X}$ and the latter is a subspace (cf. Lemma 6.9). Also, by Lemma 7.25, each symp through π is projective and hence $\pi \in \Pi$. Therefore $x_i x_j$ is a line of $\mathcal{L}_{V_k}$.

Next, suppose $V_4 \in \mathcal{V}$ meets V_i in a point x'_i of $\mathcal{X}$. Then V_4 together with any two members of $\{V_1, V_2, V_3\}$ play the same role as do V_1, V_2, V_3 and hence $x'_i x_j$ and $x'_i x_k$ are lines of $\mathcal{L}$ in V_i. By Lemma 7.23, $x'_i \in x_j x_k$. Since $i \in \{1, 2, 3\}$ was arbitrary, we obtain that V_4 meets π in a line too.

Finally, any 4-space meeting the plane π in a point, meets π in a line and hence also meets V_i in a point on the line $x_j x_k$. We conclude that the plane of Γ'' generated by V_1, V_2, V_3 is precisely the set of 4-spaces meeting π in a line. The claim follows. This also implies that the rank of Γ'' will be 3.

(BS4) Take any $V \in \mathcal{V}$ and $x \in \mathcal{X}$ with $x \notin V$. Then x is Γ-collinear to a unique line L of V. A 4-space V' through x meets V in a point of L, and V' is Γ''-collinear to V precisely if this point belongs to $\mathcal{X}$. By Lemma 6.9, L either has a unique point in $\mathcal{X}$ or is entirely contained in it, so the one-or-all axiom follows.

We conclude that $\Gamma'' = (\mathcal{V}, \mathcal{X})$ is a polar space of rank 3, whose set of singular planes is Π. For each point V of Γ'', we deduce that its point-residual is isomorphic to $(\mathcal{X}_V, \mathcal{L}_V)$ (since each plane of Π incident with V meets V in a line of $\mathcal{L}$). So $(\mathcal{X}_V, \mathcal{L}_V)$ is a polar space of rank 2. According to Lemma 7.22, its points are precisely the points of the projective 3-space $\mathcal{X}_V = \mathcal{X} \cap V$ and its line set is a subset of the line set of $\mathcal{X}_V$, and it is a well-known fact that this means that the polar space is symplectic. This at its turn implies that Γ'' is symplectic.

The points of the line Grassmannian of Γ'' coincide with $\mathcal{X}$ by very definition. Now consider a line L of the line Grassmannian of Γ''. Then L consists of the points of $\mathcal{X}$ that are simultanously incident with a plane $\pi \in \Pi$ and a 4-space V, where V and π are incident. Since V and π are incident, their intersection is a line, which belongs to $\mathcal{L}$ as it belongs to π. Therefore, (the set of points on) L coincides with the (set of points on) the line $\pi \cap V$. Finally, note that the symps of the line Grassmannian of Γ'' correspond with the points of Γ'', and hence the bijective-correspondence between the 4-spaces and the symps also follows. The proposition is proven. $\square$

For the desired conclusion, we still need to show that the bijection φ between $\mathcal{X}$ and $\mathcal{P}$ induces an isomorphism between $(\mathcal{X}, \mathcal{L})$ and $\widetilde{\Gamma^*_d}$.

Lemma 7.27 *Three points x, y, z of $\mathcal{X}$ lie on a line of $\mathcal{L}$ if and only if $\Sigma(x), \Sigma(y)$ and $\Sigma(z)$ are pairwise collinear and lie on a line of the dual geometry Γ^*.*

Proof Suppose that x, y, z lie on a line $L \in \mathcal{L}$. Then $\Sigma(x)$, $\Sigma(y)$ and $\Sigma(z)$ contain L and by Lemma 7.16, they intersect each other in a common plane π. By Lemma 7.17, each line of π belongs to $\mathcal{L}$ and the symps are pairwise collinear w.r.t. Definition 3.3. We claim that they are on a line of the dual geometry, meaning that each singular 3-space meeting $\Sigma(x)$ and $\Sigma(y)$ in a plane, also meets $\Sigma(z)$ in a plane. Let Π be such a 3-space and put $\alpha = \Pi \cap \Sigma(x)$ and $\beta = \Pi \cap \Sigma(y)$, and note that $\alpha \cap \beta$ is a line $L \subseteq \pi$. Then Π is contained in a unique 4-space V, which also meets $\Sigma(z)$ in a plane $\gamma \supseteq L$. Clearly, $\alpha \subseteq \mathcal{X}$ since $\alpha \subseteq x^\perp \cap \Sigma(x)$; likewise also β and γ are contained in $\mathcal{X}$. By Lemma 7.22, $\alpha \cup \beta \cup \gamma$ is contained in the 3-space $\mathcal{X}_V = \mathcal{X} \cap V = \Pi$, and hence $\gamma \subseteq \langle \alpha, \beta \rangle = \Pi$. The claim follows.

Conversely, suppose $\Sigma(x)$, $\Sigma(y)$ and $\Sigma(z)$ are pairwise collinear projective symps on a line of Γ^*. Let π denote the common plane they contain, which is covered by lines of $\mathcal{L}$ by Definition 3.3. Again by Lemma 7.17, this means that each line of π belongs to $\mathcal{L}$. Since $\Sigma(x) \cap \mathcal{X} = x^\perp \cap \Sigma(x)$, likewise for y, z, we also know that $x, y, z \in \pi$. We claim that x, y, z are on a line of π (which belongs to $\mathcal{L}$ indeed). Let Π be a 3-space which meets $\Sigma(x)$ and $\Sigma(y)$ in planes through xy. Since the three symps are on a line of the dual geometry, Π also meets $\Sigma(z)$ in a plane through xy. Let V be the unique 4-space containing Π. By Lemma 7.22, $\mathcal{X}_V = V \cap \mathcal{X} = \Pi$ (since $\Pi \cap \Sigma(x) \subseteq x^\perp \cap \Sigma(x) \subseteq \mathcal{X}$, likewise for y). Hence also $\Pi \cap \Sigma(z) \subseteq \mathcal{X}$, meaning that this plane belongs to $z^\perp$, which is only possible if $z \in xy$. The claim follows. $\square$

Conclusion. The geometry $\widetilde{\Gamma^*}$ (see Definition 3.3), is isomorphic to the geometry $(\mathcal{X}, \mathcal{L})$ by Lemma 7.27, and by Proposition 7.26, the latter is abstractly isomorphic to a geometry of type $\mathsf{C}_{3,2}(\mathbb{K})$. Since $\mathcal{X}$ is a geometric hyperplane of $\Gamma = (X, \mathcal{M})$, we also know by Proposition 4.14 that $\mathcal{X}$ arises as a hyperplane section of Γ and that $(\mathcal{X}, \mathcal{L})$ is, as a variety, isomorphic to the line Grassmannian variety $\mathcal{C}_{3,2}(\mathbb{K})$.

To make the overview complete, we describe (without proof) the mutual relations between points of $(\mathcal{X}, \mathcal{L})$ (namely collinear, symplectic, special, opposite) in terms of the corresponding projective symps, to make the overview complete.

Fact 7.28 Let x, y be distinct points of $\mathcal{X}$.

- x and y are *collinear* if and only if $\Sigma(x) \cap \Sigma(y)$ is a plane containing xy. In this case, $xy \in \mathcal{L}$. We note that the singular planes of $(\mathcal{X}, \mathcal{L})$ are precisely those of Π.
- x and y are *symplectic* if and only if $\Sigma(x) \cap \Sigma(y)$ is a plane α not containing x nor y. In this case, x and y are collinear in Γ but $xy \notin \mathcal{L}$ (and hence $\alpha \notin \Pi$). Moreover, the unique symp of $(\mathcal{X}, \mathcal{L})$ containing x and y is given by the subgeometry of $(\mathcal{X}, \mathcal{L})$ induced on the 3-space of $(X, \mathcal{M})$ determined by xy and the line $x^\perp \cap \alpha = y^\perp \cap \alpha$.
- x and y are *special* if and only if $\Sigma(x)$ and $\Sigma(y)$ share a unique point p which is Γ-collinear to both x and y (equivalently, $p \in \mathcal{X}$). In this case, px and $py \in \mathcal{L}$. Also, x and y are not collinear in Γ, and the Γ-symp containing them is the projective symp $\Sigma(p)$ (which hence shares a plane with each of $\Sigma(x)$ and $\Sigma(y)$).
- x and y are *opposite* if and only if $\Sigma(x) \cap \Sigma(y)$ is a unique point p not Γ-collinear to x nor y (equivalently, $p \notin \mathcal{X}$). In this case, x and y are not collinear in Γ and the unique Γ-symp containing them is not projective and hence meets $\Sigma(x)$ and $\Sigma(y)$ only in the respective points x and y.

7.4 Fourth cell: $\mathcal{E}_6(\mathbb{K})$

In this section, $d = 8$ and $\Gamma_8 = (X_8, \mathcal{M}_8)$ is the point-line geometry associated to the Cartan variety $\mathcal{E}_6(\mathbb{K})$ in $\mathsf{PG}(26, \mathbb{K})$ (abstractly, Γ_8 is of type $\mathsf{E}_{6,1}(\mathbb{K})$), contained in a quadric Q_8, which in this section is possibly degenerate. Once more, we will omit the index 8.

Proposition 7.29 *Let Q be a (possibly degenerate, but non-trivial) quadric of $\mathsf{PG}(26, \mathbb{K})$ containing a point-line geometry $\Gamma = (X, \mathcal{M})$ associated to $\mathcal{E}_6(\mathbb{K})$. Then there are three options for Q, and accordingly, for the corresponding set of projective symps $\mathcal{P}$:*

- (*i*) *Q is a degenerate quadric whose radical is a 16-space which is the tangent space $T_p(X)$ of a point p of Γ and whose base is a hyperbolic quadric of rank 4 in dimension 9. In this case, $\mathcal{P}$ is a white geometric hyperplane of $(\Xi, \mathcal{S})$.*
- (*ii*) *Q is a degenerate quadric whose radical is an 8-space which meets a certain symp Σ of Γ in a quadric of type $\mathsf{B}_{4,1}(\mathbb{K})$, and whose base is a hyperbolic quadric of rank 9 in dimension 17. In this case, $\mathcal{P}$ is a grey geometric hyperplane of $(\Xi, \mathcal{S})$.*
- (*iii*) *Q is a non-degenerate, parabolic quadric (of rank 13 in dimension 26). In this case, $\mathcal{P}$ is a black geometric hyperplane of $(\Xi, \mathcal{S})$, meaning that $\mathcal{P}$ is the set of absolute symps of a unique symplectic polarity ρ of Γ. The set of absolute points of ρ is $\mathcal{X}$, the set of absolute lines is $\mathcal{L}$.*

Moreover, in all cases, Q is the unique quadric in $\mathsf{PG}(26, \mathbb{K})$ with $\mathcal{P}$ as set of singular symps.

Proof The last statement essentially follows from Lemma 6.2, since for each point $x \in \Gamma$, we know $T_x(Q_d)$ (it is a straightforward verification that the projective symps through a point generate $T_p(Q_d)$). Since the points of Γ generate $\mathsf{PG}(3d + 2, \mathbb{K})$, the quadric Q is uniquely determined.

By Lemma 6.3, $\mathcal{P}$ is a geometric hyperplane of the dual geometry $\Gamma^* = (\Xi, \mathcal{S})$. We consider the embedding of $(\Xi, \mathcal{S})$ in $\mathsf{PG}(26, \mathbb{K})$ induced by the map $\xi \mapsto H_\xi$ (cf. Facts 4.6 and 4.11) and view the hyperplane H_ξ as a point. Then Proposition 4.13 lists the three possibilities for $\mathcal{P}$ (white, grey or black hyperplane). We describe the possibilities below, in terms of the original geometry $\Gamma = (X, \mathcal{M})$.

(white) The symps of $\mathcal{P}$ are the symps which are not opposite a given point $p \in X$. In this case, each point q in $T_p(X) \cap X$ is Q-collinear to every point of X: take any point $x \in X$ and consider a symp Σ through q and x. Since q is collinear to p, the symp Σ is not opposite p and hence belongs to $\mathcal{P}$. It follows that q and x are Q-collinear. Since the radical of Q is a subspace of $\mathsf{PG}(26, \mathbb{K})$, we have that the 16-space $\langle T_p(X) \cap X \rangle = T_p(X)$ is contained in the radical. Now let Σ be a symp opposite p. Then $\langle T_p(X), \Sigma \rangle = \mathsf{PG}(26, \mathbb{K})$ by Fact 4.7. Moreover, Σ is opposite p and hence embeds isometrically, with $\langle \Sigma \rangle \cap Q = \Sigma$ by Lemma 3.2. Hence Q indeed has $T_p(X)$ as its radical, and base isomorphic to Σ, so to a hyperbolic quadric of rank 5.

(grey) Consider a quadric Q' of type $\mathsf{B}_{4,1}(\mathbb{K})$ (i.e., $Q(8, \mathbb{K})$) which is a subquadric of a given symp Σ of Γ. In this case, the symps of $\mathcal{P}$ are the symps which meet Q' non-trivially. We claim that the radical of Q is precisely the 8-space $\langle Q' \rangle$. Let q be a point of Q' and x any point of X. As in the previous case, any symp through q and x is projective and hence q and x are Q-collinear. On the other hand, if q is any point of $\Gamma \setminus Q'$, then there exists a symp through q which meets Σ in a unique point which does not belong to Q' and hence q is not Q-collinear to all of X, in other words, q does not belong to the radical. The claim follows. Next, we claim that the projection of Q from the radical $\langle Q' \rangle$ is a hyperbolic quadric of rank 9. Take a point $x \in \Sigma \setminus Q'$ and a point $y \in X$ not Q-collinear to x (note that this implies that y is opposite Σ, for otherwise there would be a projective symp through x and y). Consider two symps Σ_1, Σ_2 through y meeting Q' in points x_1, x_2 with x_1 and x_2 not Γ-collinear, so that $\Sigma_1 \cap \Sigma_2 = \{y\}$. Also $\Sigma_i \cap \Sigma = \{x_i\}$ for $i \in \{1, 2\}$. So, $\langle \Sigma_1, Q' \rangle$ and $\langle \Sigma_2, Q' \rangle$ correspond to singular 8-spaces (say U_1 and U_2, respectively) of the residue $\mathrm{Res}_Q(Q')$, intersecting each other in the point (say y') corresponding to $\langle Q', y \rangle$. The point (say x') corresponding to $\langle x, Q' \rangle = \langle \Sigma \rangle$ is not collinear to y' and is hence collinear to a 7-space of U_2. The subspaces U_1 and $\langle y', y'^{\perp} \cap U_2 \rangle$ are then disjoint 8-spaces of $\mathrm{Res}_Q(Q')$ in the 17-space $\mathrm{Res}_{\mathbb{P}}(Q')$, where $\mathbb{P} = \mathsf{PG}(26, \mathbb{K})$. We conclude that $\mathrm{Res}_Q(Q')$ is a hyperbolic quadric of rank 9 indeed.

(black) This time, there is a symplectic polarity ρ of Γ so that the set of absolute symps is precisely $\mathcal{P}$. Recall that the geometry Γ' of absolute elements of Γ under ρ is of type $\mathsf{F}_{4,4}(\mathbb{K})$. A standard property of ρ is that the set of absolute lines of ρ is precisely the set of lines of Γ through which each symp is absolute, i.e., $\mathcal{L}$. It is also well known that each point of an absolute line is absolute, and that there is an absolute line through each point, meaning that the set of absolute points is the union of $\mathcal{L}$, i.e., $\mathcal{X}$. We first verify that ρ is the unique symplectic polarity of Γ with $\mathcal{P}$ as set of absolute symps. Indeed, as mentioned above, we can deduce the set of absolute lines in terms of $\mathcal{P}$ and the set of absolute points in terms in terms of the absolute lines. Now, for an absolute point x, there is a unique symp containing all absolute lines through x, and this is precisely the image of x under ρ. Hence ρ is determined. Next, we show that Q is a non-degenerate parabolic quadric. It is a straightforward verification that the absolute symps through a point x in Γ generate a white hyperplane if x is absolute and a grey hyperplane if x is not absolute. As this determines Q already (as mentioned in the beginning of this proof), we already see that Q is non-degenerate. Now take an absolute point x and consider a 5-space U meeting the symp $\rho(x)$ in a line $L \ni x$. Consider a line $M \ni x$ in U and a line $K \ni x$ in $\rho(x)$, with $K \neq L$. Then K and M are contained in a symp, which is absolute because it contains the absolute line L (the absolute lines through x are precisely the lines through x in $\rho(x)$, moreover, each symp containing an absolute line is absolute). Therefore, U is Q-collinear to the 8-space $x^{\perp_\Gamma} \cap \rho(x)$, yielding a singular 12-space on Q. We conclude that Q is a non-degenerate parabolic quadric of rank 13.

□

In order to prove case $d = 8$ of the main result, we only need to verify that, in case Q is non-degenerate, the collinearity we defined in Definition 3.3 matches the collinearity in the $\mathcal{F}_{4,4}(\mathbb{K})$ geometry determined by the symplectic polarity ρ, i.e., two symps $\Sigma_1 = \rho(x_1)$ and $\Sigma_2 = \rho(x_2)$ of $\mathcal{P}$ should be collinear precisely if x_1 and x_2 are collinear in $\mathcal{F}_{4,4}(\mathbb{K})$, meaning that $x_1x_2 \in \mathcal{L}$. We show slightly more, to also show the characterisation given in Lemma 3.5:

Lemma 7.30 *Let Q be a non-degenerate quadric containing Γ and let ρ be the unique symplectic polarity with $\mathcal{P}$ as set of absolute symps. Let $\Sigma_1 = \rho(x_1)$ and $\Sigma_2 = \rho(x_2)$ be two members of $\mathcal{P}$. Then the following are equivalent:*

(i) *Σ_1 and Σ_2 are collinear (w.r.t. Definition 3.3);*
(ii) *$x_1x_2 \in \mathcal{L}$;*
(iii) *$\Sigma_1 \cap \Sigma_2$ is a 4-space containing a unique line L such that the set of lines of $\mathcal{L}$ in $\Sigma_1 \cap \Sigma_2$ is exactly the set of lines meeting L in at least a point.*

Moreover, suppose $\Sigma_3 = \rho(x_3)$ belongs to $\mathcal{P}$, then $\Sigma_3 \in S(\Sigma_1, \Sigma_2)$ if and only if $x_3 \in x_1x_2$.

Proof We show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. We use the following property of ρ: for each point $x \in \mathcal{X}$, the set of lines of $\mathcal{L}$ containing x is the set of lines of $\rho(x)$ containing x; in particular, $\rho(x) \cap \mathcal{X} = x^\perp \cap \rho(x)$.

$(i) \Rightarrow (ii)$: By definition, $\Sigma_1 \perp \Sigma_2$ means that $\Sigma_1 \cap \Sigma_2$ is the union of lines in $\mathcal{L}$. Therefore, $\Sigma_1 \cap \Sigma_2$ is a 4-space contained in $\mathcal{X}$. By the above property and the fact that $\Sigma_1 \cap \Sigma_2$ is a maximal singular subspace in both Σ_1 and Σ_2, we obtain that $x_1, x_2 \in \Sigma_1 \cap \Sigma_2$, and hence x_1x_2 is a line through x_1 in $\rho(x_1)$, so $x_1x_2 \in \mathcal{L}$ indeed.

$(ii) \Rightarrow (iii)$: Again using the above property, we have that x_1x_2 belongs to $\Sigma_1 \cap \Sigma_2$. Moreover, each line in $\Sigma_1 \cap \Sigma_2$ containing x_1 or x_2 also belongs to $\mathcal{L}$. Now let M be any line of $\Sigma_1 \cap \Sigma_2$, meeting x_1x_2 in a point $x_3 \neq x_1, x_2$, and consider lines L_1 and L_2 containing x_1 and x_2 respectively, which meet M in a point $y \neq x_3$. Since $L_1, L_2 \in \mathcal{L}$, the point y belongs to $\mathcal{X}$, and by the above property, $\langle L_1, L_2 \rangle \subseteq \rho(y)$. Therefore also M is a line containing y and belonging to $\rho(y)$, so $M \in \mathcal{L}$ indeed. Next, suppose for a contradiction that there is a line K in $\rho(x) \cap \rho(y)$ disjoint from x_1x_2 which belongs to $\mathcal{L}$. By the foregoing, each point of x_1x_2 is on a line of $\mathcal{L}$ with each point of x_1x_2 and hence $\langle K, x_1, x_2 \rangle$ generates a singular 3-space in $(\mathcal{X}, \mathcal{L})$. Since the latter is the point-line geometry of a geometry of type $\mathsf{F}_{4,4}(\mathbb{K})$, which contains no singular 3-spaces, this is a contradiction.

$(iii) \Rightarrow (i)$: This is trivial.

For the final statement, suppose $\Sigma_1 \perp \Sigma_2$, or equivalently, $x_1x_2 \in \mathcal{L}$. Then $x_1x_2 = \rho(\Sigma_1 \cap \Sigma_2)$ and hence, Σ_3 contains $\Sigma_1 \cap \Sigma_2$ if and only if $x_3 \in x_1x_2$. □

Conclusion. If Q is a non-degenerate quadric in $\mathsf{PG}(26, \mathbb{K})$ containing Γ, then it follows from Proposition 7.29 and Lemma 7.30 that Q is parabolic and that the subgeometry $\widetilde{\Gamma^*}$ of $\Gamma^* = (\Xi, \mathcal{S})$ with point set $\mathcal{P}$ (as given in Definition 3.3) arises from a symplectic polarity ρ of Γ, whose absolute points are $\mathcal{X}$ and whose absolute lines are $\mathcal{L}$. In particular, as a point-line geometry, $\widetilde{\Gamma^*}$ is isomorphic to $(\mathcal{X}, \mathcal{L})$, and the latter is, as a variety, isomorphic to $\mathcal{F}_{4,4}(\mathbb{K})$. Moreover, one of the properties of ρ is that the set $\mathcal{X}$ arises as the intersection of Γ with a hyperplane of $\mathsf{PG}(26, \mathbb{K})$ (cf. Proposition 4.13).

This completes the proof of the main theorem. We have one final result to end with, relating the symplectic polarity of Γ to the polarity of $\mathsf{PG}(26, \mathbb{K})$ defining the quadric Q containing Γ.

Indeed, provided that Q is non-degenerate, the embedding of Γ on Q yields a geometry isomorphic to $\mathcal{F}_{4,4}(\mathbb{K})$ on Γ coming from a symplectic polarity ρ on Γ, whose absolute symps are precisely the projective symps of Γ. So the polarity of $\mathsf{PG}(26,\mathbb{K})$ defining Q gives a unique symplectic polarity of Γ. Conversely, given a symplectic polarity ρ of Γ, or equivalently, a subgeometry of Γ isomorphic to $\mathcal{F}_{4,4}(\mathbb{K})$, we show in the next proposition that there is a unique non-degenerate quadric of $\mathsf{PG}(26,\mathbb{K})$ containing Γ such that the projective symps of Γ w.r.t. that quadric are exactly the absolute symps of ρ. So also the symplectic polarity of Γ determines a unique polarity of $\mathsf{PG}(26,\mathbb{K})$.

Proposition 7.31 *Given a subgeometry $\widetilde{\Gamma} \cong \mathcal{F}_{4,4}(\mathbb{K})$ of $\Gamma \cong \mathcal{E}_6(\mathbb{K})$ in $\mathsf{PG}(26,\mathbb{K})$ arising as the absolute geometry of a symplectic polarity ρ of Γ, there is a unique non-degenerate quadric Q of $\mathsf{PG}(26,\mathbb{K})$ containing Γ such that the projective symps of Γ w.r.t. that quadric are exactly the absolute symps of ρ.*

Proof By Lemma 5.2, there is a non-degenerate quadric Q^* in which Γ fully embeds. Proposition 7.29 yields a symplectic polarity ρ^* whose fixed point geometry $\widetilde{\Gamma^*}$ is isomorphic to $\mathcal{F}_{4,4}(\mathbb{K})$, and whose absolute symps are precisely the symps of Γ that are singular with respect to Q^*. Consider an isomorphism τ of $\Gamma \cong \mathcal{E}_6(\mathbb{K})$ mapping $\widetilde{\Gamma^*}$ to $\widetilde{\Gamma}$ (cf. [7]) and let $\overline{\tau}$ be the unique collineation of $\mathsf{PG}(26,\mathbb{K})$ that τ extends to. Then $Q := \overline{\tau}(Q^*)$ is a non-degenerate quadric in $\mathsf{PG}(26,\mathbb{K})$ in which $\overline{\tau}(\Gamma) = \Gamma$ fully embeds. Since $\overline{\tau}$ preserves singular subspaces, the absolute symps of Γ with respect to ρ^* are precisely the singular symps of Γ with respect to Q. We show uniqueness of Q by determining the polarity p of $\mathsf{PG}(26,\mathbb{K})$ whose absolute geometry is Q, in terms of ρ.

Let x be any point of Γ. By Lemma 6.1, $T_x(Q)$ contains all singular symps of Γ through x, or equivalently, all absolute symps of Γ through x. It follows from Lemma 6.2 and the description of the geometric hyperplanes of Γ in Proposition 4.13 that:

- If x is absolute, then the absolute symps of Γ containing x are precisely the symps sharing a 4-space with $\rho^*(x)$. It follows that their union is the set of points of Γ close to $\rho^*(x)$. We obtain that $T_x(Q)$ coincides with the hyperplane H generated by the (white) geometric hyperplane of points not opposite $\rho^*(x)$.
- If x is not absolute, then the absolute symps of Γ containing x are precisely the symps through x that meet the symp $\rho^*(x)$, which is opposite x, in the points of a subquadric of type $\mathsf{B}_{4,1}(\mathbb{K})$ (namely, the quadric formed by the absolute points contained in $\rho^*(x)$). It follows that their union is the set of points of a (grey) geometric hyperplane of Γ, and the unique hyperplane of $\mathsf{PG}(26,\mathbb{K})$ generated by these points coincides with $T_x(Q)$.

Taking 27 points of Γ that form a frame of $\mathsf{PG}(26,\mathbb{K})$, we let p be the unique polarity of $\mathsf{PG}(26,\mathbb{K})$ that maps a point z to $T_z(Q)$ with the latter determined in terms of absolute symps as above. Clearly, p then coincides with the polarity associated to Q, and hence Q is unique indeed. □

As a consequence of this, Γ is projectively unique on Q.

Corollary 7.32 *Let $\Gamma \cong \mathcal{E}_6(\mathbb{K})$ be embedded in a non-degenerate quadric Q. Then Γ is projectively unique in Q, i.e., if $\Gamma' \cong \mathcal{E}_6(\mathbb{K})$ is also embedded in Q, then there is collineation of $\mathsf{PG}(26,\mathbb{K})$ stabilising Q which maps Γ to Γ'.*

Proof Let $\widetilde{\Gamma}$ be the $\mathcal{F}_{4,4}(\mathbb{K})$ subvariety of Γ, arising from a symplectic polarity ρ of Γ whose absolute symps are the singular symps of Γ w.r.t. Q. Likewise, we define $\widetilde{\Gamma'}$. Using the fact that $\mathcal{F}_{4,4}(\mathbb{K})$ is projectively unique in $\mathcal{E}_6(\mathbb{K})$, with a projectivity that extends to $\mathsf{PG}(26,\mathbb{K})$, and the fact that $\mathcal{E}_6(\mathbb{K})$ is also projectively unique in $\mathsf{PG}(26,\mathbb{K})$, there is a projectivity τ of $\mathsf{PG}(26,\mathbb{K})$ which maps $(\widetilde{\Gamma},\Gamma)$ to $(\widetilde{\Gamma'},\Gamma')$. According to Proposition 7.31, τ stabilises Q. □

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