A geometric characterisation of the Hjelsmlev-Moufang planes

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Abstract

Hjelmslev-Moufang planes are point-line geometries related to the exceptional algebraic groups of type E_6. More generally, point-line geometries related to spherical Tits-buildings—Lie incidence geometries—are the prominent examples of parapolar spaces: axiomatically defined geometries consisting of points, lines and symplecta (structures isomorphic to polar spaces). In this paper we classify the parapolar spaces with a similar behaviour as the Hjelmslev-Moufang planes, in the sense that their symplecta never have a non-empty intersection. Under standard assumptions, we obtain that the only such parapolar spaces are exactly given by the Hjelmslev-Moufang planes and their close relatives (arising from taking certain restrictions). On the one hand, this work complements the algebraic approach to these structures with Jordan algebras due to Faulkner in his book "The Role of Nonassociative Algebra in Projective Geometry", published by the AMS in 2014; on the other hand, it provides a new tool for classification and characterisation problems in the general theory of parapolar spaces.

Keywords: Hjelmslev-Moufang planes, Lie incidence geometries, parapolar spaces, E_6

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1 Introduction

1.1 Origin of the problem

The natural geometries of "groups of algebraic origin" (by which we mean semi-simple algebraic groups and their classical and mixed type analogues) are the (Tits-)buildings, introduces by Jacques Tits in his monumental work [14]. Special cases were considered before, especially to get a grip on the algebraic groups of exceptional type. One of these concerned the (split) groups of type E_6. The associated geometry is the so-called *Hjelmslev-Moufang plane* $\mathscr{P}$ (defined over a field k), as was formally introduced by Springer and Veldkamp [12] for the case $\mathrm{char}(k) \neq 2,3$; and studied avant-la-lettre by Tits [13] in the general case. This geometry has some remarkable and interesting properties. One is that the lines of $\mathscr{P}$, which carry the structure of a hyperbolic quadric in a projective space of dimension 9, pairwise intersect non-trivially. This is one of the defining properties of *projective remoteness planes*, which were introduced by Faulkner in [6]. He constructs such geometries using the Jordan algebras of 3×3 Hermitian matrices over composition algebras, which yields the Hjelmslev-Moufang plane and its relatives (essentially given by subplanes). Our results in particular give evidence that no other algebraic structures are likely to produce projective remoteness planes (certainly not when the lines have the structure of a polar space).

After Springer and Veldkamp introduced the Hjelmslev-Moufang planes (henceforth: HM-planes), and after Tits developed his theory of buildings, people started to define other, more general, point-line geometries to get an even better grip on the algebraic groups and the corresponding buildings. One of the central ideas was that of a *parapolar space*, introduced by Bruce Cooperstein [4] in the late 1970s. Roughly speaking, a parapolar space is a connected point-line geometry in which every quadrangle with at least one non-collinear diagonal pair of points is contained in a *convex subgeometry* isomorphic to a polar space (for the precise definition we refer to Definition 2.6). These subgeometries are usually called *symplecta* or, briefly, *symps*. Cooperstein's approach was very successful and ever since, parapolar spaces have been studied in depth, in particular by Cohen and Shult. This evolved in a rich theory, discussed at length in [1] and [10]. Although almost all known examples of parapolar spaces are related to buildings, a full classification result is not within reach.

So, in modern terminology, and referring to the definitions in Section 2.2, an HM-plane is a parapolar space of symplectic rank 5 (meaning that all its symps have rank 5), in which every two symps share at least a point (in fact, either a single point, or a maximal singular subspace, see Proposition 3.9 of [12]) and in which every pair of points is contained in a symp.

1.2 The main result

The question that we put forward in this paper can now be informally stated as: *Which parapolar spaces behave like HM-planes when the point-symp structure is considered?* In other words, can we classify all parapolar spaces with the properties that every pair of points is contained in at least one symp, and every pair of symps intersect nontrivially? We obtain the following theorem (referring to Section 2.2 for undefined notions).

Theorem 1.1. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space in which every pair of points is contained in at least one symplecton, and every pair of symplecta intersects in at least one point. Then Ω is one of the following point-line geometries.*
– *The Cartesian product of a projective line and an arbitrary projective plane;*
– *The Cartesian product of two arbitrary not necessarily isomorphic projective planes;*
– *The line Grassmannian $\mathsf{A}_{4,2}(k)$ for any skew field k;*
– *The line Grassmannian $\mathsf{A}_{5,2}(k)$ for any skew field k;*
– *The Lie incidence geometry $\mathsf{E}_{6,1}(k)$ for any field k.*

Theorem 1.1 is a special case of our main result (**Theorem 3.1**), as we can relax the assumptions strongly. Indeed, we can also carry out a classification of the above-mentioned parapolar spaces when replacing the requirement "all pairs of points are contained in a symp" by "*if* there is a symp of rank 2, then pairs points which can be joined by a shortest path of length 2, are contained in a symp". In technical terms, this means that we put no restriction on the diameter, but we do require that, in case there is a symp of rank 2, then the parapolar space should be *strong*. This classification yields the same geometries as does Theorem 1.1, except for a class of parapolar spaces with the property that all symps intersect each other in exactly a point, a situation we deal with in **Theorem 3.2**.

In some sense, the case in which the symps of the parapolar spaces all have rank at least 3 is the generic one (giving rise to $\mathsf{A}_{4,2}(k)$, $\mathsf{A}_{5,2}(k)$ and $\mathsf{E}_{6,1}(k)$). Nonetheless, the proof of the case of where there are symps of rank 2 (in which case we will prove that actually *all* symps have rank 2) is by far the most intricate (as is also reflected by the fact that strongness is required here). In that connection we quote Shult [9]: *It is not easy to live in a world with no symplecton of rank at least three in sight.*

Finally, let us explain why we could have expected the geometries, other than the $\mathsf{E}_{6,1}(k)$-geometry, occurring in Theorem 1.1–the fact that no others do is the main achievement of this paper. The line Grassmannian $\mathsf{A}_{5,2}(k)$ and the Cartesian product of two projective planes over k (also know as the Segre variety $\mathscr{S}_{2,2}(k)$) are close relatives of the $\mathsf{E}_{6,1}(k)$-geometry. Indeed, by restricting the coordinatizing algebra of the HM-plane over k (the split octonions over k) to the split quaternions over k or to $k \times k$, we exactly obtain $\mathsf{A}_{5,2}(k)$ and $\mathscr{S}_{2,2}(k)$, respectively. The latter geometries have similar incidence properties as the Hjelmslev-Moufang planes, as was also noted by Springer and Veldkamp (cf. [12], page 254). This holds true even when k is no longer commutative for $\mathsf{A}_{5,2}(k)$, or when considering the Cartesian product of any two (not necessarily isomorphic) axiomatic projective planes. The geometries $\mathsf{A}_{4,2}(k)$ and the Cartesian product of a projective line and an arbitrary projective plane are natural subgeometries of the latter, respectively.

1.3 Future perspectives

Our main theorem characterises the $\mathsf{E}_{6,1}(\mathbb{K})$-geometry and its relatives as parapolar spaces in which two symps can never have an empty intersection. It turns out that many other (exceptional) Lie incidence geometries are parapolar spaces in which there are other gaps in the spectrum of dimensions of intersections of pairs of symps. For instance, in $\mathsf{E}_{8,8}(\mathbb{K})$, whose symps have rank 7, two symps can never intersect in a k-space where $k \in \{1,3,4\}$. This then implies that in the latter's point-residue, $\mathsf{E}_{7,7}(\mathbb{K})$, two symps can never intersect in a k-space where $k \in \{0,2,3\}$. In general, letting k be any integer with $k \geq -1$, we call a parapolar space *k-lacunary* if $k \notin \{\dim(\xi_1 \cap \xi_2) \mid \xi_1, \xi_2 \text{ symps of } \Omega\}$.

The current paper can be used to classify the k-lacunary parapolar spaces Ω for $k \geq 0$, provided that each symp of Ω has rank at least $k+3$. Indeed, one can then deduce that Ω has a residue which is -1-lacunary, and these are listed in our current main result. Although it is not hard to predict the possibilities for Ω given this list, it requires non-trivial arguments to actually prove this—this will be pursued in another paper. The locally connected parapolar spaces we obtain are $\mathsf{E}_{6,2}(\mathbb{K}), \mathsf{E}_{7,1}(\mathbb{K}), \mathsf{E}_{8,8}(\mathbb{K})$ (which are *long-root geometries*) and their relatives (more precisely: residues). Surprisingly, these three Lie incidence geometries, their point-residues (namely, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$, respectively) and the latter's point-residues (namely, $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, $\mathsf{D}_{5,5}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, respectively), produce precisely the 3×3 lower south-east corner of the Freudenthal-Tits magic square.

The result mentioned above provides an additional strong tool in (classification) work related to parapolar spaces, in particular for work aiming at exceptional Lie incidence geometries. For instance, if one proves that a gap in the spectrum of intersection dimensions of symps occurs, then the problem reduces to a neat list of parapolar spaces, or, if one assumes the parapolar space does not occur in the given list, one may rely on the fact that each (sensible) dimension occurs as the dimension of the intersection of two symps.

Before stating the precise version of our main results (Theorems 3.1 and 3.2), we introduce in the next section the necessary terminology concerning parapolar spaces, including the examples relevant for this paper.

2 Parapolar spaces

We provide a gentle introduction into the theory of parapolar spaces to keep the paper self-contained. We refer to [1] and [10] for more information.

2.1 Generalities on point-line geometries

Definition 2.1. *A pair $\Omega = (X, \mathscr{L})$ is a* point-line geometry *if X is a set and $\mathscr{L}$ is a family of subsets of X of size at least 2 covering X; the elements of X are called points and those of $\mathscr{L}$ lines.*

Some terminology. • Let $\Omega = (X, \mathscr{L})$ be a point-line geometry. Two distinct points x, y of X that are contained in a common line are called *collinear*, denoted $x \perp y$. The set of points equal or collinear to a given point x is denoted $x^\perp$, and for a set $S \subseteq X$, we denote $S^\perp = \bigcap_{s \in S} s^\perp$.

• A subset $Y \subseteq X$ is called a *subspace of* Ω if for every pair of collinear points $x, y \in Y$, all lines joining x and y are entirely contained in Y; it is called *proper* if $Y \neq X$. A *geometric hyperplane* is a proper subspace which intersects every line nontrivially. A subspace $Y \subseteq X$ is called *singular* if every pair of distinct points of Y is collinear. The *generation* of a subset $A \subseteq X$ is the intersection of all subspaces of Ω containing A and is a subspace again, we denote it by $\langle A \rangle$.

•The *collinearity graph* $\Gamma(X, \mathscr{L})$ of Ω is the graph with vertex set X where adjacency is collinearity. A subspace Y is called *convex* if for every pair of points $x, y \in Y$, all points on any shortest path from x to y (in the collinearity graph) belong to Y. The intersection of all convex subspaces of Ω containing a given subset $A \subseteq X$ is called the *convex closure* of A and denoted $\mathrm{cl}(A)$. A point-line geometry is called *connected* if its collinearity graph is connected. The *diameter* of a connected point-line geometry is the diameter of its collinearity graph.

Definition 2.2. *If a point-line geometry $(X, \mathscr{L})$ is such that every pair of distinct points is contained in (at most) exactly one line, then it is a* (partial) linear space*; if it is such that every pair of lines intersect in exactly one point, then it is a* dual linear space*. The latter it is called* nontrivial *if there are at least two lines.*

Note that in the above definition, since $\mathscr{L}$ covers X, a dual linear space is automatically connected.

Definition 2.3. *A* projective plane *is a point-line geometry $(X, \mathscr{L})$ which is both a linear space and a dual linear space and which does not contain lines of size 2; a* projective space *is a point-line geometry containing at least two lines and such that every triple of points not contained in a common line generates a projective plane.*

Every projective space is either a projective plane or obtained from a vector space of dimension at least 4 by taking the 1-spaces as points and the 2-spaces as lines (with containment as incidence relation). The dimension of a projective space is 2 if it is a projective plane, and it is n is it is constructed from a vector space of dimension $n+1$ as above. In all cases the dimension is one less than the minimum size of a generating set. For convenience we will call the more or less trivial singular point-line geometry $(X, \{X\})$ a *projective space of dimension* 1 provided $|X| \geq 3$. These will also be referred to as *projective lines*. A single point will sometimes be called a *projective space of dimension* 0 and the empty set a *projective space of dimension* -1.

For each point-line geometry $\Omega = (X, \mathscr{L})$ we can study the local structure in any of its points $x \in X$ as follows. Let $\mathscr{L}_x$ be the set of lines of Ω containing x and let Π_x be the set of singular subspaces of Ω generated by two members of $\mathscr{L}_x$ (there is no guarantee that this set is nonempty), where we identify each member of Π_x with the set of lines through x it contains.

Definition 2.4. *Let $\Omega = (X, \mathscr{L})$ be a point-line geometry and let $x \in X$ be arbitrary. The point-line geometry $\Omega_x = (\mathscr{L}_x, \Pi_x)$ is called the* local geometry (at x)*, or the* point residual (at x)*. If every point residual is connected, then we say that Ω is* locally connected.

2.2 The definitions of polar and parapolar spaces

Before giving the definition of a parapolar space, we need to know that of a polar space:

Definition 2.5. A point-line geometry $\Delta = (X, \mathscr{L})$ is a *polar space* if the following axioms hold.

(PS1) Every line has at least three points.

(PS2) No point is collinear to all other points.

(PS3) Every nested sequence of singular subspaces is finite.

(PS4) For each pair $(x, L) \in X \times \mathscr{L}$ either exactly one, or all points of L are collinear to x.

Polar spaces turn out to be partial linear spaces but not linear spaces. Note that the joint axioms (PS2) and (PS4) are equivalent to "$x^\perp$ is a geometric hyperplane, for all $x \in X$". Every singular subspace of a polar space is a finite-dimensional projective space, and for each given polar space $\Delta = (X, \mathscr{L})$ there exists a natural number $r > 1$, called the *rank of* Δ, such that some singular subspace of Δ has dimension $r-1$, but no singular subspace of dimension r exists in Δ. The singular subspaces of Δ of dimension $r-1$ will—rightfully—be referred to as *maximal singular subspaces*, whereas singular subspaces of dimension $r-2$ will be referred to as *submaximal singular subspaces*. The number t of maximal singular subspaces containing a given submaximal singular subspace only depends on Δ, and not on the submaximal singular subspace. If $t > 2$, then we call Δ *thick*; otherwise $t = 2$ and Δ is *hyperbolic*. The set of maximal singular subspaces of a hyperbolic polar space can be partitioned in two subsets such that two maximal singular subspaces belong to the same subset if and only if the dimension of their intersection has the same parity as their own dimension. We often call a maximal singular subspace of a hyperbolic polar space a *generator*.

In a polar space $\Delta = (X, \mathscr{L})$ of rank $r \geq 3$, it is easy to see that the point-residual Δ_x, $x \in X$, is a polar space of rank $r-1$, which is canonically isomorphic to the subspace $x^\perp \cap y^\perp$, for every $y \in X$ not collinear to x. Also, Δ is hyperbolic if and only if Δ_x is hyperbolic.

Definition 2.6. A point-line geometry $\Omega = (X, \mathscr{L})$ is called a *parapolar space* if the following axioms hold:

(PPS1) Ω is connected and, for each pair $(x, L) \in X \times \mathscr{L}$ either none, one or all of the points of L are collinear to p, and there exists a pair $(p, L) \in X \times \mathscr{L}$ such that p is collinear to no point of L.

(PPS2) For every pair of non-collinear points p and q in X, one of the following holds:
- (a) $\mathrm{cl}(\{p, q\})$ is a polar space, called a *symplecton*, or *symp* for short;
- (b) $p^\perp \cap q^\perp$ is a single point;
- (c) $p^\perp \cap q^\perp = \emptyset$.

(PPS3) Every line is contained in at least one symplecton.

A parapolar space Ω is called *strong* if $p^\perp \cap q^\perp$ is never a single point for p not collinear to q. We say that Ω has *minimum symplectic rank* r if there is a symp of rank r but not less. We say that Ω has *at least symplectic rank* r if there is no symp of rank smaller than r. We say that Ω has *uniform symplectic rank* r if each symplecton has rank r.

Note that, contrary to what happens in polar spaces, the singular subspaces of a parapolar space need not (all) be projective spaces. However, we will prove a sufficient condition for that (see Lemma 4.1), which implies that all subspaces are projective in case the symplectic rank is at least 3; and if there are symps of rank 2, this fact will follow by our assumptions (see Lemma 5.2).

2.3 Examples of polar and parapolar spaces

Many interesting examples of parapolar spaces emerge from buildings as follows. Since a building is a numbered simplicial complex, one can take all simplices of a certain type T as point set, and then there is a well-defined mechanism that deduces a set of lines. The resulting point-line geometry is the so-called *T-Grassmannian* of the building. Now, for a certain choice of T, projective spaces and polar spaces emerge from buildings of types A_n and B_n, respectively. Other choices of T for these and for other types of buildings in general lead to parapolar spaces, and these are the ones we refer to as Lie incidence geometries. We call them *exceptional* if the corresponding building is of exceptional type.

Below, we provide specific examples of polar and parapolar spaces, whilst giving the notation used in our main theorem below. We leave the proofs that these are actual polar and parapolar spaces to the interested reader as illuminating exercises.

Example 2.7 (**Hyperbolic Polar Spaces).** Every hyperbolic polar space of rank r at least 4 is the point-line geometry naturally arising from a hyperbolic quadric in projective $(2n-1)$-space over some field (such a hyperbolic quadric has standard equation $X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0$).

Every hyperbolic polar space $(X, \mathscr{L})$ of rank 3 arises from a 4-dimensional vector space V over some skew field k by taking for point set X the set of 2-spaces of V, and as set of lines $\mathscr{L}$ the set of *pencils of* 2*-spaces*. A pencil of 2-spaces is the set of 2-spaces containing a fixed 1-space V_1 and contained in a fixed 3-space V_3, with $V_1 \subseteq V_3$. In the projective language, X is the set of lines of a projective space of dimension 3 and $\mathscr{L}$ is the set of planar line pencils.

Finally every hyperbolic polar space of rank 2 is an $(\ell_1 \times \ell_2)$-*grid*, i.e., the Cartesian product of two projective lines, see the next example.

Example 2.8 (**Product spaces).** Probably the easiest examples of parapolar spaces are the Cartesian products of two linear spaces. Let $\Lambda_i = (X_i, \mathscr{L}_i)$, $i = 1,2$, be a nonempty linear space with the property that every line has size at least 3. Assume that both Λ_1 and Λ_2 contain at least one line. Define the Cartesian product $\Lambda_1 \times \Lambda_2$ as the point-line geometry with point set $X_1 \times X_2$ and line set $\{L_1 \times \{p_2\} \mid L_1 \in \mathscr{L}_1, p_2 \in X_2\} \cup \{\{p_1\} \times L_2 \mid p_1 \in X_1, L_2 \in \mathscr{L}_2\}$. The symps here are the rank 2 hyperbolic polar spaces $L_1 \times L_2$, $L_i \in \mathscr{L}_i$, $i = 1,2$. The diameter of $\Lambda_1 \times \Lambda_2$ is 2 and the parapolar space is strong.

Note that in case Λ_1 and Λ are two projective lines. we obtain the aforementioned hyperbolic polar spaces of rank 2.

Example 2.9 (**Line Grassmannians).** Line Grassmannians are defined in exactly the same way as hyperbolic polar spaces of rank 3, using a vector space V of dimension at least 4, or a projective space of dimension at least 3. If the projective space has dimension at least 4, then we obtain strong parapolar spaces of diameter 2 with uniform symplectic rank 3, and all symps are isomorphic to a fixed hyperbolic polar space of rank 3. We denote the line Grassmannian of a projective n-space over k by $\mathsf{A}_{n,2}(k)$, using standard notation.

Example 2.10 (**Hjelmslev-Moufang Planes—Parapolar spaces of type** $\mathsf{E}_{6,1}$**).** The line Grassmannians are examples of *Lie incidence geometries*, for each projective space can be given the structure of a *spherical Tits-building* (which are the natural geometries of Lie groups and groups of Lie type). The buildings related to the groups of exceptional type have no short elementary description and we shall therefore not define these. We just content ourselves with mentioning that to every building of exceptional type E_6 (say over the field k) corresponds a Lie incidence geometry denoted $\mathsf{E}_{6,1}(k)$ which can be obtained by a standard procedure applied to the building. The point-line geometry $\mathsf{E}_{6,1}(k)$ can be defined via a trilinear form in a 27-dimensional vector space over k, or via the Zariski closure of the image of an affine Veronesean map involving a split Cayley algebra over k, or using the algebraic group of exceptional type E_6 over k. We shall not do this here since this does not yield interesting insight in the objects we defined, and it does not provide useful information for our proofs. If we denote by Ξ the family of symps of the Lie incidence geometry $\mathsf{E}_{6,1}(k) = (X, \mathscr{L})$, then the point-line geometry (X, Ξ) is the aforementioned *Hjelmslev-Moufang plane over* k.

3 Main results

3.1 The statements

We can now formulate our main results precisely. The description of the geometries occurring here can be found in Subsection 2.3 preceding this section.

Theorem 3.1. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space, assumed to be strong if the minimum symplectic rank is 2 containing no pair of disjoint symplecta and such that there is a line contained in at least two symplecta. Then Ω is one of the following point-line geometries.*
– *The Cartesian product of a projective line and an arbitrary projective plane;*
– *The Cartesian product of two arbitrary not necessarily isomorphic projective planes;*
– *The line Grassmannian $\mathsf{A}_{4,2}(k)$ for any skew field k;*
– *The line Grassmannian $\mathsf{A}_{5,2}(k)$ for any skew field k;*
– *The Lie incidence geometry $\mathsf{E}_{6,1}(k)$ for any field k.*
In particular, Ω is strong and, if the symplectic rank is at least 3, it is also locally connected.

Our result below describes what happens if each line is contained in a unique symplecton, i.e., if each pair of symplecta intersects each other in exactly a point, showing that the classification in this case is hopeless.

Theorem 3.2. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space, assumed to be strong if the minimum symplectic rank is* 2*, in which every two symplecta intersect in exactly a point. Then, all members of the family Ξ of symps have rank at least 3, and the point-line geometry (X, Ξ) is a non-trivial dual linear space with the following property: If $p_0 \in X$ belongs to two distinct members ξ_1, ξ_2 of Ξ, and $p_i \in \xi_i$, $i = 1, 2$, and $p_3 \in X$ is contained in a common member ξ_{i3} of Ξ together with p_i, $i = 1, 2$, where $p_0 \notin \{p_1, p_2, p_3\}$, then*

$$\delta_{\xi_1}(p_0, p_1) + \delta_{\xi_{13}}(p_1, p_3) + \delta_{\xi_{23}}(p_2, p_3) + \delta_{\xi_2}(p_0, p_2) \geq 5,$$

where δ_ξ is the distance in the collinearity graph of $\xi \in \Xi$, i.e., 0 *if the arguments are equal,* 1 *if they are collinear in ξ, and* 2 *otherwise.*

Conversely, let $\Upsilon = (X, \Xi)$ be a given a nontrivial dual linear space such that every line (i.e., every member of Ξ) has the structure of a polar space of rank at least 3, and satisfying the above inequality for the given restrictions on the points and symps. Let $\mathscr{L}$ be the set of lines of all these polar spaces. Then the geometry $\Omega = (X, \mathscr{L})$ is a parapolar space of symplectic rank at least 3 in which all symps intersect each other in exactly a point.

Remark: Theorem 1.1 follows immediately from Theorems 3.1 and 3.2: A parapolar space satisfying the conditions of Theorem 1.1 by definition has diameter 2, and as such the inequality of Theorem 3.2 cannot be satisfied.

3.2 Structure of the proof

In **Section 4** we start by collecting some auxiliary results which we will need in our proofs. Most of these are very minor generalizations of existing results, introducing local hypotheses instead of global, but we provide proofs for completeness' sake.

From then on, we assume that every pair of symps meets nontrivilly. In **Section 5**, we show that if some symp rank 2, then all symps have rank 2. We then classify the strong parapolar space with only symplecta of rank 2 in **Section 6**. In **Section 7**, we treat the most generic case, being the one in which the parapolar spaces have symplectic rank at least 3, under the additional assumption that there is a line contained in at least two symps.

Finally, in **Section 8**, we consider the case in which every pair of symps has exactly one point in common, and prove Theorem 3.2. Note that this situation does not occur when there are symps of rank 2, for then we assume that Ω is strong, if Ξ_1 and Ξ_2 intersect in a unique point p, then taking lines L_1 and L_2 through p in Ξ_1, Ξ_2, respectively, yields a symp through L_1 and L_2 meeting Ξ_1 and Ξ_2 in more than one point.

We want to emphasise that our proof is elementary in the sense that it only uses projective and incidence geometry. The identification of the line Grassmannians and the Hjelmslev-Moufang plane is done using a theorem of Cohen and Cooperstein [3] after having deduced the necessary conditions for using this theorem. However, we can avoid this and instead continue in an elementary way until the very end, only using the characterization of Veblen and Young of projective spaces (for the cases of the line Grassmannians) and the local characterization of buildings of type E_6 by Tits [16] for the Hjelmslev-Moufang plane. This will be explained at the end of Section 7, see Remark 7.15.

4 Some auxiliary results

Compare the next lemma with Theorem 13.4.1(2) of [10].

Lemma 4.1. *Let Ω be a parapolar space. If all points of a line L contained in a symp ξ of rank at least 3 are collinear to a point p, then p and L are contained in a symp and hence generate a projective singular plane. Consequently, if the symplectic rank is at least 3, each singular subspace is projective.*

Proof. If $p \in \xi$, we are done. If not, take a point $q \in \xi$ collinear to all points of L and not contained in the subspace $p^\perp \cap \xi$. Then p and q are at distance 2 and $L \subseteq p^\perp \cap q^\perp$, so there is a symp ξ' through p and q, which clearly contains L and p. Since ξ' is a polar space, it follows that the singular subspace generated by L and p is a projective plane. □

Lemma 4.2. *Let Ω be a parapolar space of minimum symplectic rank d. Then every singular subspace of dimension at most $d-1$ is contained in some symp.*

Proof. By Axiom (PPS3) each line is contained in a symp and by connectivity each point is contained in a line. Hence if $d = 2$ we are done. So suppose $d \geq 3$. Then Lemma 4.1 confirms that the (projective) dimension is well-defined. So let W be a singular subspace of Ω of dimension d^* with $2 \leq d^* \leq d-1$. Let $d' \leq d^*$ be the maximum number such that there exists a symp ξ with $\dim(\xi \cap W) = d'$ (well defined by the first line of this proof, which also shows that $d' \geq 1$). Suppose for a contradiction that $d' < d^*$. Then we can pick $p \in W \setminus \xi$ and $q \in \xi \setminus p^\perp$ with q collinear to all points of $W \cap \xi$. However, the symp ξ' containing p and q (well defined by the fact $d' \geq 1$) intersects W in a subspace of dimension $d'+1$, contradicting the maximality of d'. We conclude that W is contained in some symp. □

We have the following corollary.

Corollary 4.3. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space of symplectic rank at least 3. Let $x \in X$ be arbitrary. Then the point residual Ω_x is connected if and only if the graph Γ with vertex set $\mathscr{L}_x$ and two vertices adjacent if they are contained in a common symp, is connected. Consequently, a locally connected parapolar space of symplectic rank at least 3 contains at least one line which is contained in at least two symps.*

Proof. Since all symps contain planes and since every plane belongs to a symp by Lemma 4.2, we only need to show the last assertion. So suppose Ω is locally connected. Since it is not a polar space, there are at least two symps, and by connectivity some point $x \in X$ is contained in at least two symps. Hence Ω_x contains two symps and by connectivity of Ω_x and the first assertion, there is a line through x contained in at least two symps. □

Finally we need the following two elementary results for polar spaces.

Lemma 4.4. *Let Δ be a hyperbolic polar space. Given two generators, we can find a submaximal singular subspace disjoint from both generators.*

Proof. Let U and V be two generators. We proceed by induction on the rank r of Δ. If $r=2$, it is clear that we can find a point disjoint from the lines U and V. For $r \geq 3$, consider non-collinear points p_U and p_V in U and V, respectively. In $p_U^\perp \cap p_V^\perp$, U and V correspond to maximal singular subspaces, so by induction there is a singular subspace Z in $p_U^\perp \cap p_V^\perp$ of dimension $n-3$ disjoint from U and V. As the residual at Z (recursively defined as the point residual at the point corresponding to Z of the residual at a hyperplane of Z) is a rank 2 hyperbolic polar space, in which U and V correspond to lines, it contains a point disjoint from them, yielding a submaximal singular subspace of Δ disjoint from both U and V. □

We already noted that in a polar space, the points equal or collinear to a certain point form a geometric hyperplane, but we can be more precise.

Lemma 4.5. *Let $\Delta = (X, \mathscr{L})$ be a polar space and let $p \in X$ be arbitrary. Then $p^\perp$ is a geometric hyperplane of Δ which is not properly contained in another geometric hyperplane.*

Proof. Let $q \in X$ not be collinear to p and consider the subspace $H = \langle p^\perp, q\rangle$. Note that by (PS4) $q^\perp \subseteq H$. Now let $x \in X$ be arbitrary. If $x \perp x' \in q^\perp \setminus p^\perp$, then we can interchange the roles of x' and q' and obtain $x \in x'^\perp \subseteq H$. If no such x' exists, then we consider $y \in x^\perp \setminus (\{x\} \cup q^\perp)$ and observe that the previous argument now does lead to $y \in H$. Hence, if $y' = \langle x, y\rangle \cap q^\perp$, then $x \in \langle y, y'\rangle \subseteq H$. □

Standing Hypotheses. We now embark on the proof of the Main Result. In the next three sections, we let $\Omega = (X, \mathscr{L})$ be a parapolar space of minimum symplectic rank d such that every two symplecta have at least one point in common. We distinguish between the cases $d=2$ and $d \geq 3$. In the former case, we also assume that Ω is strong; in the latter case we assume that at least one line of Ω is contained in at least two symps. Such a line will be called *sympthick*.

We will also use the following notation. The family of symps of Ω is denoted by Ξ, and if two noncollinear points $x, y \in X$ are contained in a symp $\xi \in \Xi$, then we write $\xi = \xi(x,y) := cl(\{x,y\})$.

The case $d=2$ is also divided into two parts: we first show in the next section that Ω has *uniform* symplectic rank 2.

5 Minimum symplectic rank 2 implies uniform symplectic rank 2

In this section we assume that $(X, \mathscr{L})$ has minimum symplectic rank 2. The Standing Hypotheses imply that Ω is strong. The aim of this subsection is to show that all symps have rank 2.

We begin with the only two lemmas that will also be useful for the case of uniform symplectic rank 2. For convenience, we will call a symp of rank 2 a *quad* (from "quadrangle").

Lemma 5.1. *Let $(X,\mathscr{L})$ have minimum symplectic rank 2. If L_1 and L_2 are disjoint lines of any quad ξ, then either at least one of L_1,L_2 is properly contained in a singular subspace, or some line of ξ intersecting both L_1,L_2 is properly contained in a singular subspace.*

Proof. Let ξ be a quad and let L_1,L_2 be two nonintersecting lines in ξ. We claim that there exist lines M_1,M_2 not contained in ξ and meeting L_1,L_2 in points q_1,q_2, respectively. Indeed, let $i\in\{1,2\}$. By Axiom (PPS1), X does not consist of only the points of ξ, so there is a point $p\in X\setminus\xi$. Connectivity of $(X,\mathscr{L})$ yields a shortest path $(p,p_1,...,p_n,q_i)$ from p to L_i (so $q_i\in L_i$). Now if p_nq_i does not belong to ξ, then we can put $M_i=p_nq_i$. If $p_n\in\xi$, then $p_{n-1}\notin\xi$ (as otherwise we could shorten the path) and so, by strongness, p_{n-1} and q_i determine a symp ξ_i and then there is a line M_i in ξ_i through q_i not contained in ξ. The claim is proved.

Again, let $i\in\{1,2\}$. We may assume that L_i is not properly contained in a singular subspace. Consequently, since $(X,\mathscr{L})$ is strong, L_i and M_i are contained in a unique symp ξ_i and the singular subspace $\xi\cap\xi_i$ equals L_i. Hence $\xi_1\cap\xi_2$, nonempty by assumption, is not contained in ξ. For any point $q\in\xi_1\cap\xi_2$, q is collinear to a point $r_1\in L_1$ and to a point $r_2\in L_2$. Necessarily, $r_1\perp r_2$ since $q\notin\xi$. So r_1,r_2,q are contained in a singular subspace properly containing the line r_1r_2. $\square$

If there are quads we cannot invoke Lemma 4.1 to conclude that all singular subspaces are projective spaces. However, under our assumptions, we nevertheless can.

Lemma 5.2. *Let $(X,\mathscr{L})$ have minimum symplectic rank 2 and S any of its singular subspaces. Then S is projective and contains no pair of skew lines that are both contained in a quad.*

Proof. Let S be a singular subspace properly containing a line. If S does not contain two nonintersecting lines then S is a projective plane. So we may assume that two lines L_1,L_2 in S are disjoint. Suppose for a contradiction that both are contained in a quad; say $L_i\subseteq\xi_i\in\Xi$, $i=1,2$. Then $\xi_i\cap S=L_i$ and $\xi_1\cap\xi_2$ contains a point $q\notin S$. Now q is collinear to unique points p_1,p_2 on L_1,L_2, respectively. Let $r\in L_1\setminus\{p_1\}$. Then ξ_1 is determined by r and q, but since $p_2\in\{r,q\}^\perp$, we see that $p_2\in\xi_1$, a contradiction. This already shows the second part of the assertion.

Now we show Veblen's axiom. Suppose L_1 and L_2 both intersect two intersecting lines K_1,K_2 in two distinct points, and let p be the intersection of K_1 and K_2. Assume for a contradiction that L_1 and L_2 are disjoint. Then the previous paragraph implies that some symp ζ of rank at least 3 contains, say, L_1. Since p is collinear to all points of L_1, Lemma 4.1 implies that p and L_1 are contained in a projective plane, which then also contains K_1,K_2 and hence L_2. Consequently L_1 and L_2 intersect after all. Hence, by [15], S is projective.

The lemma is completely proved. $\square$

Lemma 5.3. *Let $(X,\mathscr{L})$ have minimum symplectic rank 2. Let ξ be a quad and let $L\subseteq\xi$ be a line contained in a singular plane π. Let ζ be any symp such that $\zeta\cap L=\emptyset$. Then ζ has rank 2.*

Proof. We divide the proof into two parts, based on whether or not there is a point in ζ collinear to a point $x \in \pi \setminus L$. Before heading off, note that $\zeta \cap \pi$ is empty. Indeed, suppose $\zeta \cap \pi$ is a point p (off L, by assumption). By the Standing Hypotheses, $\zeta \cap \xi$ contains a point p' (also off L). Then p' is not collinear to p, as otherwise $p \in \xi(r, p') = \xi$ for some point $r \in L$ not collinear to p', a contradiction. However, if p and p' are not collinear, $\zeta = \xi(p, p')$ contains a point on L after all, violating our assumption.

Case I: *There is a point q of ζ collinear to some point x of $\pi \setminus L$.*

Claim. Each point of ζ is collinear to at least one point of π.

Denote by Z the subset of points of ζ which are collinear to at least one point of π. We (subsequently) show that Z is a subspace containing $q^\perp \cap \zeta$ and at least one point of ζ not belonging to $q^\perp \cap \zeta$, as then Lemma 4.5 implies that $Z = \zeta$, proving the claim.

- *Z is a subspace of ζ:*
  Let q_1, q_2 be collinear points of Z. Then either they are collinear to a common point of π, in which case every point of q_1q_2 is collinear to that point, or else they are collinear to distinct points x_1, x_2, respectively, with $\delta(q_1, x_2) = \delta(q_2, x_1) = 2$. But then, in the symp $\xi(q_1, x_2) = \xi(q_2, x_1)$, every point of q_1q_2 is collinear to a unique point of the line $x_1x_2 \subseteq \pi$.

- *Z contains $q^\perp \cap \zeta$:*
  Let $r \in \zeta$ be a point collinear to q. We show that $r \in Z$. If $r \perp x$, then there is nothing to prove, so suppose $x \notin r^\perp$. Then the symp $\xi(r, x)$ intersects ξ in at least one point p^*. If $p^* \notin L$, its distance to x is 2 (like above this follows from $x \notin \xi$) and hence by convexity, $L \cap \xi(r, x)$ contains a point. Either way, $\xi(r, x) \cap \pi$ contains a line, at least one of which points is collinear to r.

- *At least one point r of ζ not belonging to $q^\perp$ belongs to Z:*
  If some point p of $\xi \cap \zeta$ is not collinear to q, then we can take $r = p$. Hence suppose $p \perp q$ for all points $p \in \xi \cap \zeta$. It suffices to find a point $r \perp q$, $q \neq r \in \zeta$, collinear to a point of $\pi \setminus L$ (because interchanging the roles of q and r will then imply $r^\perp \subseteq Z$). Assume for a contradiction that every point of $\zeta \cap q^\perp$ is collinear to some point of L. Then also q is; say $p^* \in q^\perp \cap L$. By assumption, $p^* \notin \zeta$. If some point p of $\xi \cap \zeta$ is not collinear to p^*, then $\xi = \xi(p, p^*)$ contains q (recall $p \perp q$) a contradiction. This arguments shows that $\xi \cap \zeta$ is just a point, say p, which is collinear to p^*. It also shows $q^\perp \cap L$ is exactly p^*. Consider $r \in \zeta$ with $r \in q^\perp \setminus p^\perp$. Then, since $\zeta = \xi(p, r)$ does not contain p^*, r is collinear to a unique point $p' \in L$ with $p' \neq p^*$. Whence $\xi(r, x)$ contains p' and q, and hence also $p^* \in p'^\perp \cap q^\perp$, implying $\pi \subseteq \xi(r, x)$. But then, inside $\xi(r, x)$, r is collinear to the points of a line $M \neq L$ of π as $p^* \notin r^\perp$. This shows that r is collinear to some point of $\pi \setminus L$.

As mentioned above, this shows the claim. We now show that ζ is a quad indeed.

Suppose for a contradiction that ζ has rank at least 3. Let p be a point of $\xi \cap \zeta$ and let p' be the unique point on L collinear to p. Then consider a plane α in ζ intersecting both $\xi \cap \zeta$ and $p'^\perp$ in exactly the point p. If a point $z \in \alpha, z \neq p$ were collinear with a point p^* of L, then our choice of α implies $p' \neq p^*$, but then $z \in p^\perp \cap p^{*\perp} \subseteq \xi(p, p^*) = \xi$, a contradiction. The above claim implies that each point of $\alpha \setminus \{p\}$ is collinear to a unique point of $\pi \setminus L$. A standard argument now shows that the perp correspondence restricted from α to π preserves collinearity and hence is an isomorphism of planes. Consequently some points of α different from p are collinear to points of L after all, a contradiction. This proves the lemma in Case I.

Case II: *No point of ζ is collinear to a point of $\pi \setminus L$.*

Claim. No line of π is contained in a symp of rank at least 3.

Suppose for a contradiction that some line of π were contained in a symp of rank at least 3. Lemma 4.1 then yields a symp ξ^* containing π. Let $q \in \xi^* \cap \zeta$. By assumption, no point of $\pi \setminus L$ is collinear to q. Hence all points of L are collinear to q. Let $p \in \xi \cap \zeta$ be arbitrary and set $p' = p^\perp \cap L$. Then $p' \perp q$ and, consequently, $q \perp p$ (as otherwise p' would belong to $\zeta = \xi(p,q)$, a contradiction). Hence ξ, which is defined by L and p, also contains q, a contradiction. The claim follows.

We now show that ζ is a quad, distinguishing between the following two cases.

- *Case IIa: $\zeta \cap \xi$ is a single point p.*
  Let p' be the unique point on L collinear with p. Pick an arbitrary point $y \in \pi \setminus L$ and an arbitrary point $z \in \xi \setminus L$ such that z is collinear to a point $z' \in L \setminus \{p'\}$. Then y and z are not collinear as otherwise $\xi = \xi(p', z)$ contains y. Set $\xi^* = \xi(y,z)$. Then ξ^* contains a line M of π, namely $M = yz'$. By the above claim, ξ^* is a quad and hence $\xi^* \cap \pi = M$. Noting that $p \in \zeta$ is collinear to $p' \in \pi \setminus M$, we can interchange the roles of (ξ, L) and (ξ^*, M) and then Assumption I applies again, showing that ζ is a quad.

- *Case IIb: $\zeta \cap \xi$ is a line K.*
  Select $p \in K$ arbitrarily and set $p' = p^\perp \cap L$. Select a line $M \neq K$ of ζ through p not contained in $p'^\perp$ and consider the symp ξ_1 defined by p' and M. If ξ_1 has a line in common with π, then the points of $M \setminus \{p\}$ are collinear to points of $\pi \setminus L$, contradicting our hypothesis. Hence there is a line $N \neq pp'$ of ξ_1 through p' not contained in π.

  Now either N and L are contained in a singular plane π' or they determine a symp ξ', which is in fact a quad by the above claim, since it shares the line L with π. In the first case, we replace π by π' and observe that the points of $M \setminus \{p\}$ are collinear to points of $\pi' \setminus L$; in the second case we replace ξ by ξ' and observe that the points of $K \setminus \{p\}$ are collinear to the points of $\pi \setminus L$. In both cases, these replacements imply that Case I applies again, yielding that ζ has rank 2.

This completes the proof of the lemma. □

Lemma 5.4. *Let $(X, \mathscr{L})$ have minimum symplectic rank 2. Then every symp that intersects a quad in a line is itself a quad.*

Proof. Suppose for a contradiction that a quad ξ and a symp ζ of rank at least 3 intersect in a line L. Pick $x \in \xi$ arbitrarily but not on L. As in the proof of Lemma 5.1, there is a line M through x not contained in ξ. Let M' be a line of ξ through x disjoint from L. Then

Lemma 5.3 implies that M and M' are not contained in a plane. Hence there is a symp ξ' containing M and M'. Since L is contained in some plane of ζ, Lemma 5.3 again implies that ξ' is a quad.

Claim 1: The intersection $\zeta \cap \xi'$ is a point q.
Note that our main assumption yields $\zeta \cap \xi' \neq \emptyset$. Assume for a contradiction that $\zeta \cap \xi'$ is a line K. Since $\xi \cap \xi' = M'$, the lines K and L are disjoint. For every point $z \in K$, the unique point in $z^\perp \cap M'$ and every point in $z^\perp \cap L$ (recall $L \cup K \subseteq \zeta$) are collinear as $z \notin \xi$ (implying that also $z^\perp \cap L$ is unique). It follows that each point $z \in K$ is contained in a unique plane α_z intersecting M' and L in collinear points. Since α_z contains a line of ζ, and ζ has rank at least 3, Lemma 4.1 implies the existence of a symp of rank at least 3 containing α_z and hence intersecting ξ' in the line $\alpha_z \cap \xi'$. Now, for $z \neq z' \in K$, the plane $\alpha_{z'}$ intersects ξ' in a line disjoint from $\alpha_z \cap \xi'$. This contradicts once again Lemma 5.3. The claim is proved.

Similarly as in the previous paragraph, $q^\perp \cap L = p$ and $q^\perp \cap M = q'$. Let π be any plane of ζ containing L. Then there is a point $x \in \pi \setminus L$ collinear to q and a point $r \in \xi' \cap q^\perp$ such that rq does not intersect M'.

Claim 2: r is collinear to some point of $\pi \setminus L$.
If $r \perp x$, then this is trivial. If not there is a symp $\xi(r,x)$, which intersects ξ and hence, by convexity (as in the previous proof), it has a line R in common with π. Let $x' \in R \cap r^\perp$ and suppose for a contradiction that $x' \in L$. Then the unique point x'' on M' collinear with x' is collinear to r too (since $x' \notin \xi'$) and hence $x'' \neq q'$. This also implies that $p \neq x'$ and hence $x' \notin q^\perp$. But then $\xi(r,x) = \xi(q,x') = \zeta$, a contradiction. Claim 2 is proved.

Now we replace π by another plane π^* of ζ containing L and such that π and π^* are not contained in a common 4-space. Then r is also collinear to a point x^* of $\pi^* \setminus L$. This implies that x' and x^* are collinear, contradicting our choice of π^*.

The lemma is proved. □

The main goal of this section is now within reach.

Proposition 5.5. *Let $(X, \mathscr{L})$ have minimum symplectic rank 2. Then $(X, \mathscr{L})$ has uniform symplectic rank 2.*

Proof. Assume for a contradiction that there is a symp ζ of rank at least 3. Since the minimum rank is 2, there is also a quad ξ and by the Standing Hypotheses, $\xi \cap \zeta \neq \emptyset$. Moreover, by Lemma 5.4, $\xi \cap \zeta$ is a point p. Pick lines $L \subseteq \xi$ and $M \subseteq \zeta$ both through p. If L and M are contained in a plane, then by Lemma 4.1, this plane is contained in a symp of rank at least 3 intersecting ξ in the line L, contradicting Lemma 5.4. Hence, by strongness, L and M define a symp, which has a line in common with both ξ and ζ and hence, again by Lemma 5.4, it can neither have rank at least 3 nor rank 2. This impossibility completes the proof. □

6 The case of uniform symplectic rank 2

We continue with our assumption that Ω contains at least one quad. By Proposition 5.5, Ω has uniform symplectic rank 2.

By Lemma 5.2 implies that all singular subspaces are projective. We can now easily even say more.

Lemma 6.1. *Let $(X,\mathscr{L})$ have uniform symplectic rank 2. Then every singular subspace properly containing a line is a projective plane. Moreover any two projective planes intersect in at most one point.*

Proof. By Lemma 5.2, a singular subspace does not contain disjoint lines (as there are no symps of rank at least 3). Hence as soon as it contains two lines, it is a projective plane. The second part of the Lemma follows from the first part and uniform symplectic rank 2. $\square$

The previous lemma allows us to speak about *(singular) planes* instead of "singular subspaces properly containing a line". Note also that Lemma 5.1 implies the existence of many singular planes.

Lemma 6.2. *Let $(X,\mathscr{L})$ have uniform symplectic rank 2. Then every symp and every singular plane that share a point, share a line.*

Proof. Let ξ be a symp and π a singular plane an suppose for a contradiction that $\xi \cap \pi = p$, with $p \in X$. Let L be a line in π not containing p (and hence disjoint from ξ) and let ξ_L be a symp containing L. Since ξ_L does not contain planes, $p \notin \xi \cap \xi_L$. Let q be a point of $\xi \cap \xi_L$ and denote by r the unique point of L collinear to q. Then $p \perp r \perp q$. If p and q are not collinear, then $r \in \xi$, contradicting $L \cap \xi = \emptyset$. So suppose p and q are collinear. Then p,q,r are contained in a singular plane π' which intersects π in the line pr. Let t be a point in $\pi \setminus pr$. By Lemma 6.1, t is not collinear to q; but then q and t determine a symp, which contains the planes π and π', a contradiction to the symplectic rank being 2. $\square$

Lemma 6.3. *Let $(X,\mathscr{L})$have uniform symplectic rank 2. Then every point p not contained in a singular plane π is collinear to a unique point of π.*

Proof. Let ℓ be the distance of p to π (connectivity implies that ℓ is finite). If $\ell = 1$, then it follows by an argument similar to the one used at the end of the proof of Lemma 6.2 that the point in π collinear to p is unique. Next, if $\ell = 2$, strongness implies that p is contained in a symp, which, by Lemma 6.2, shares a line L with π. But then L contains a point collinear to p, contradicting $\ell = 2$. Since by (PPS1) parapolar spaces are connected, it follows that ℓ always equals 1. Uniqueness of the point collinear with p follows from Lemma 6.1. $\square$

In case there is a singular plane intersecting every symp non-trivially, we can show that the parapolar space is a product geometry of a projective line and a projective plane. We first show, under this assumption, that each symp is non-thick.

Lemma 6.4. *Let $(X,\mathscr{L})$ have uniform symplectic rank 2. If there is a singular plane π intersecting every symp non-trivially, then each symp of $(X,\mathscr{L})$ is non-thick.*

Proof. By Lemma 6.2, π intersects each symp in a line. Let ξ be an arbitrary symp. Set $L = \pi \cap \xi$ and let q be a point in $\xi \setminus L$. Let p be the unique point on L collinear to q and take a line K in π intersecting L in p. Let L' be a line in ξ through q disjoint from L. By Lemma 6.3, p is the unique point of K collinear to q and hence, as $(X, \mathscr{L})$ is strong, there is a unique symp $\xi_{K,q}$ through K and q. Let K' be a line in $\xi_{K,q}$ through q disjoint from K (hence $K' \nsubseteq \xi$). We claim that L' and K' are contained in a singular plane π'. If not, then by strongness, L' and K' are contained in a unique symp ξ'. Since π shares a line with ξ', the latter contains a point of L. Hence ξ', containing L' and a point of L, coincides with ξ, violating $K' \nsubseteq \xi$. This shows the claim. If there would be another line qr in ξ disjoint from L, then repeating the above argument implies $r \perp K'$, contradicting the fact that r is collinear to a unique point (namely q) of π'. We conclude that ξ is non-thick. $\square$

Proposition 6.5. *Let $(X, \mathscr{L})$ have uniform symplectic rank 2. If there is a singular plane π intersecting every symp non-trivially, then $(X, \mathscr{L})$ is isomorphic to the Cartesian product of a projective line with a projective plane.*

Proof. Again, Lemma 6.2 implies that π intersects each symp in a line, and by Lemma 6.4, each symp is non-thick. Let L be an arbitrary line intersecting π in a point t (which exists since there is a symp through t).

Claim. The line L is the unique line through t not contained in π.

Indeed, suppose for a contradiction that there is a point $x \notin \pi \cup L$ with $x \perp t$. If x and L would belong to a singular plane π', we take a symp ξ' through a line L' of π' not containing t. Then $\xi' \cap \pi$ is a line L'' by assumption, and since t, if not already on L'', is collinear to two non-collinear points of L' and L'', respectively, we obtain $t \in \xi'$. This however means that $\pi' \subseteq \xi'$, a contradiction. So x is not collinear to L, and then strongness implies a symp containing x and L. By assumption this symp intersects π in a line, which contains t, implying that the symp has three lines through t, contradicting that it is non-thick. The claim is proved.

We now complete the lemma by showing that $(X, \mathscr{L})$ is isomorphic to the direct product space $\pi \times L$. Let $x \in X$ be arbitrary. If $x \in \pi \cup L$, then x can be uniquely written in $L \times \{t\} \cup \{t\} \times \pi$. So suppose $x \notin L \cup \pi$. By Lemma 6.3, x is collinear to a unique point x_π of π, which does not coincide with t by the above claim. Hence, by strongness, there is a unique symp ξ through x and t and, again by the above claim, ξ contains L as one of its two lines through t. So there is a unique point $x_L \in L$ collinear to x, and $x_L \neq t$. Just like L was the unique line through t not in π, the line xx_π is the unique line through x_π not contained in π. Therefore, since x_L is collinear with a unique point of xx_π (as x_L and x_π are not collinear), x_L and x_π determine x uniquely. Lastly, it follows from the argument in the previous proof that the lines distinct from L through any point $x' \in L \setminus \{x\}$ belong to a singular plane.

The proposition is proved. $\square$

If no plane intersects every symp, then we need to show that Ω is the Cartesian product of two projective planes. The following lemma is the crux of that proof.

Lemma 6.6. *Let $(X,\mathscr{L})$ have uniform symplectic rank 2. If some plane π is disjoint from some symp ξ, then ξ is non-thick and there exists a bijection from the point set of some line in π to one system of generators of ξ such that elements corresponding under this bijection are contained in a common singular plane.*

Proof. Let L be a line in ξ. Pick $p_1,p_2 \in L$ distinct. Let q_i be the unique respective points in π collinear to p_i, $i=1,2$. If $q_1=q_2$, then L is contained in a singular plane intersecting π in a point; if $q_1 \neq q_2$, then $\xi(p_1,q_2)$ contains L and q_1q_2 and hence collinearity is a bijection between L and q_1q_2. In the first case we say that L is π-*triangular (with centre $q_1=q_2$)*, in the second case π-*quadrangular (with axis q_1q_2)*. We show three properties.

(1) *Each pencil of lines in ξ contains at most one π-triangular line.*
Let L_1,L_2 be two intersecting lines of ξ. If both are π-triangular, the planes meet in a line, contradiction Lemma 6.3 and showing the claim.

Now let M_1 and M_2 be two disjoint π-quadrangular lines of ξ.

(2) *One or all lines meeting both M_1 and M_2 are π-triangular, according to whether the axes of M_1,M_2 are distinct or not.*
Indeed, the axes of M_1 and M_2, being contained in a projective plane, have at least one point r in common. Then r is collinear to some points s_1,s_2 on M_1,M_2, respectively. If s_1 were not collinear to s_2, then $r \in \xi$, a contradiction. Hence r,s_1,s_2 are contained in a singular plane and the line s_1s_2 of ξ is π-triangular with centre r. If the axes intersect in a unique point, there is a unique π-triangular line meeting both M_1 and M_2; if they coincide, each line meeting both M_1 and M_2 is π-triangular. The claim is proved.

It is now easy to see that the previous claim yields at least two (necessarily disjoint, by the first claim) π-triangular lines (even if ξ is non-thick), say T_1,T_2, with respective centres t_1,t_2. Let U_1,U_2,U_3 be three lines each intersecting both T_1 and T_2 non-trivially.

(3) *The lines T_1 and T_2 define a (full) grid G in ξ, one of which reguli consisting of π-triangular lines and the other of π-quadrangular lines.*
For $j \in \{1,2,3\}$, the axis B_j of U_j is a line containing t_1 and t_2 and it follows that $t_1 \neq t_2$, so $B_j=t_1t_2$. Let t be an arbitrary point on t_1t_2. Then the points on U_1,U_2,U_3 collinear to t are pairwise collinear, as above. This implies that, varying $t \in t_1t_2$, each line intersecting U_1 and U_2 non-trivially also intersects U_3 non-trivially, and, on top, is π-triangular. This shows the cliam.

By (3), it suffices to show that ξ is hyperbolic to finish the proof.

Suppose for a contradiction that ξ is thick. Let $i \in \{1,2\}$. Put $p_i = U_1 \cap T_i$ and take a line L_i through p_i distinct from U_1 and T_i. By (1), L_i is π-quadrangular with axis $A_i \ni t_i$. By (2) and the fact that U_1 is π-quadrangular, exactly one line intersecting both L_1 and L_2 is π-triangular. Consequently, there is π-quadrangular line U_1' distinct from U_1 intersecting both L_1 and L_2. Again, (2) implies a π-triangular line T' intersecting both U_1 and U_1'. However, the grid G determined by T_1 and T_2 already possessed a π-triangular line through the point $T' \cap U_1$, contradicting (1). $\square$

We can now show in general that every symp is hyperbolic.

Lemma 6.7. *Let $(X,\mathscr{L})$ have uniform symplectic rank 2. Then every symp is hyperbolic.*

Proof. Suppose for a contradiction that there is thick symp ξ. By Lemmas 5.1 and 6.1, there exists some singular plane π. By Lemmas 6.6 and 6.2, $\pi\cap\xi$ is a line L. Let M be a line in ξ disjoint from L and pick a point $p\in M$. Considering a symp through a point x of $\pi\setminus\xi$ and p (which exists since the unique point of π collinear to p is contained in L and $(X,\mathscr{L})$ is strong) we see that there exists some line $K\ni p$ not contained in ξ (and some point of K is collinear to x). Replacing M by another line through p disjoint from L (which is possible by the thickness of ξ) if necessary, we may assume that M and K are contained in a unique symp ξ'. If $\xi'\cap\pi$ contained a point q, then q, being collinear to all points of L and a unique point of M, would belong to ξ, and hence to L, a contradiction, as that point and M define $\xi\neq\xi'$. So $\xi'\cap\pi$ is empty and Lemma 6.6 implies that ξ' is non-thick.

We use the terminology of the proof of Lemma 6.6, applied to the pair (π,ξ'). Clearly, M is π-quadrangular with axis L, hence by Lemma 6.6, the line K, belonging to the other regulus, is contained in a singular plane with a unique point on L. But some point on K was collinear to x, contradicting the uniqueness assertion in Lemma 6.3. This absurdity proves the lemma. □

Theorem 6.8. *Let $(X,\mathscr{L})$ have uniform symplectic rank 2. Then $(X,\mathscr{L})$ is isomorphic to the Cartesian product of a projective plane with either another projective plane, or a projective line.*

Proof. By Proposition 6.5, we may assume that there is a singular plane disjoint from some symp. The existence of two singular planes π_1 and π_2 intersecting each other in a point p then is an easy consequence of Lemma 6.6.

Let $x\in X\setminus(\pi_1\cup\pi_2)$ be arbitrary. Then x is not collinear to p as otherwise a symp through xp has a line through x in common with both π_1 and π_2 by Lemma 6.2, contradicting hyperbolicity (cf. Lemma 6.7). Hence, using Lemma 6.3, x is collinear to unique distinct points $x_1\in\pi_1\setminus\{p\}$ and $x_2\in\pi_2\setminus\{p\}$. Conversely, given points $x_1\in\pi_1$ and $x_2\in\pi_2$ distinct from p, there is a unique symp through x_1,x_2 (again using strongness and the fact that x_1,x_2 are not collinear), which is non-thick by Lemma 6.7 and therefore contains a unique point collinear to both x_1 and x_2 and not contained in $\pi_1\cup\pi_2$. Consequently we already have that X can be written as $\pi_1\times\pi_2$ in a set-theoretic way. It remains to show that two points $x,x'\in X$ collinear to the same point $x_1\in\pi_1$ are collinear themselves. But if x and x' were not collinear, then the symp through them (note that $x\perp x_1\perp x'$) contains, by Lemma 6.2, a line in π_1, hence a third line through x', a contradiction. Similarly for $x_2\in\pi_2$.

The theorem is proved. □

7 The case of symplectic rank at least 3

From now on we may assume that $\Omega = (X, \mathscr{L})$ is a parapolar space of minimum symplectic rank d with $d \geq 3$. The Standing Hypotheses imply that we have at least one sympthick line (recall that this is a line contained in at least two symps). A symp not containing a sympthick line will be called *isolated*; in the other case *non-isolated*. Recall that every pair of symps meets non-trivially.

We aim to prove the assumptions needed in the Cooperstein-Cohen theory from [3] as updated by Shult in [10]. Hence we need to show that

(LC) Ω is locally connected,
(BD) the singular subspaces have bounded dimension,
(BR) the symps have bounded rank, and
(H) Ω satisfies the so-called *Haircut Axiom* (see Lemma 7.12).

Lemma 7.1. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Let ξ be a non-isolated symp with rank d_1. Then, for every singular subspace S of ξ of dimension $d-2$, there is a symp $\xi^* \neq \xi$ such that $S \subseteq \xi \cap \xi^*$. Furthermore, one of the following holds.*

(i) The symp ξ^* is hyperbolic, has rank d and $\dim(\xi \cap \xi^*) = d-1$.

(ii) For each singular subspace M of ξ of dimension $d-1$ through S, there is a symp ξ_M with $M \subseteq \xi \cap \xi_M$ (equality if $d_1 = d$).

Proof. By assumption, ξ contains a line L which is contained in a second symp. We first deal with singular subspaces through L; afterwards we show that this is not a restriction, by showing that each line of ξ is sympthick. So consider a singular subspace S of dimension $d-2$ with $L \subseteq S \subseteq \xi$.

Claim 1: There is a symp $\xi^* \neq \xi$ such that $S \subseteq \xi \cap \xi^*$.
Let U be a subspace of S through L, maximal with respect to the property that there exists a symp $\xi^* \in \Xi$ with $U \subseteq \xi \cap \xi^*$ (U is well defined since L satisfies this requirement). Suppose for a contradiction that $U \subsetneq S$, so there is a point $p \in S \setminus U$. The set $p^\perp \cap \xi^*$ is a singular subspace of ξ^*, clearly containing U. Also $\xi \cap \xi^*$ is a singular subspace of ξ^* containing U. Since ξ^* is a symp of rank at least d and $\dim(U) < d-2$, there is a point $q \in \xi^* \setminus \xi$ collinear to U with $q \notin p^\perp$. Then q and p are non-collinear and $U \subseteq p^\perp \cap q^\perp$. Hence there is a symp ξ' through p and q, which is distinct from ξ since $q \notin \xi$. But now $\xi \cap \xi'$ contains $\langle p, U \rangle$, contradicting the maximality of U. We conclude that there is a symp $\xi^* \neq \xi$ with $S \subseteq \xi \cap \xi^*$, showing the claim.

Now suppose that the above found symp ξ^* is either thick, or has rank at least $d+1$ or is such that $\xi \cap \xi^* = S$. Let M be any singular subspace of ξ through S of dimension $d-1$.

Claim 2: Under the above assumptions on ξ^*, there is a symp ξ_M with $M \subseteq \xi \cap \xi_M$.
Take a point $p \in M \setminus S$. We may assume that $M \nsubseteq \xi \cap \xi^*$. Our assumptions on ξ^* imply the existence of a subspace M' of dimension $d-1$ through S in ξ^* which is not contained in $p^\perp \cap \xi^*$ (which is a singular subspace of ξ^* through S) nor in $\xi \cap \xi^*$ (the latter coincides with S if ξ^* is non-thick and has rank d). Similarly as above, we take a point $q \in M' \setminus S$,

which is then symplectic to p. The unique symp ξ_M through p and q contains M. This shows the claim.

If ξ^* does not satisfy those assumptions, then ξ^* is non-thick, has rank d and $S \subsetneq \xi \cap \xi^*$. Since ξ^* has rank d and $\dim(S) = d-2$, the latter implies that $\dim(\xi \cap \xi^*) = d-1$. We now complete the lemma by showing that each line in ξ is sympthick.

Claim 3: Each line in ξ is sympthick.
Without loss of generality, we may consider a line K in ξ generating a plane π together with L. If $d_1 > 3$, π is contained in a $(d-2)$-space of ξ, so by Claim 1 we may assume that $d_1 = 3$. Likewise, by Claim 2, we may assume that a symp $\xi^* \neq \xi$ through L is non-thick, has rank 3 (since $3 = d_1 \geq d \geq 3$) and is such that $\xi \cap \xi^*$ is a plane π^* through L distinct from π. Let π' be the unique plane through L in ξ^* distinct from π^*. If $\pi \cup \pi'$ contains a pair of non-collinear points, these determine a symp containing $\pi \cup \pi'$, proving that K is sympthick. So suppose π and π' are collinear. Let q be a point of $\pi' \setminus L$ and note that $q^\perp \cap \xi = \pi$ since $d_1 = 3$. Hence a point $p \in \xi \cap K^\perp \setminus \pi$ is not collinear to q. The points p and q determine a unique symp, containing K, proving again that K is sympthick, as required. $\square$

Remark 7.2. The proof of the previous lemma did not use the assumption that every pair of symps meets nontrivially. Hence the statements are true without that assumption.

We can show that no symp is isolated, and hence the previous lemma holds for all symps of Ω.

Lemma 7.3. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then no symp is isolated.*

Proof. Suppose for a contradiction that some symp ξ is isolated, i.e., none of its lines is sympthick. Since Ω contains at least one sympthick line, there is a non-isolated symp ξ'. Then, since every two symps always intersect nontrivially, $\xi \cap \xi'$ is just a point p. Take a subspace S in ξ' of dimension $d-2$ which is not contained in $p^\perp$. By one of the two cases occurring in Lemma 7.1, there is a symp $\xi'' \neq \xi'$ through S such that $\dim(\xi' \cap \xi'') \geq d-1$. Again, our assumption on ξ implies that $\xi \cap \xi''$ is just a point p''. Then $p'' \neq p$, as $\xi' \cap \xi''$ is a singular subspace of ξ' and p is not collinear with S. Since the rank of ξ' is at least 3, the intersection $\xi' \cap \xi''$ contains at least a point q collinear to both p and p''. The point q does not belong to ξ but is collinear to the distinct points p, p'', implying p and p'' are collinear. Hence, since p'' is collinear to all points of the line pq in ξ', Lemma 4.1 says p'' and pq are contained in a symp, in particular, there is a second symp containing pp'' after all, a contradiction. $\square$

Lemma 7.4. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Let ξ be any symp of rank d. Then we have*

- (i) *for each symp ξ' with* $\dim(\xi \cap \xi') \geq d-2$, *the rank of ξ' is d and* $\dim(\xi \cap \xi') = d-1$,
- (ii) *ξ is hyperbolic of odd rank.*

Proof. (*i*) Consider opposite subspaces S_1 and S_2 of ξ of dimension $d-2$ (note that $d-2 \geq 1$). By Lemmas 7.1 and 7.3, there are symps ξ_1^* and ξ_2^* intersecting ξ in maximal singular subspaces M_1 and M_2 of ξ through S_1 and S_2, respectively. If $M_1 \cap M_2 = \emptyset$, then $\xi_1^* \cap \xi_2^*$, which contains at least a point p by the Standing Hypotheses, is disjoint from ξ. But then p is collinear to the non-collinear subspaces S_1 and S_2 of ξ, a contradiction. Hence $M_1 \cap M_2$ is a point (it cannot be more since S_1 and S_2 are opposite).

Observe that this implies that Possibility (*ii*) of Lemma 7.1 cannot occur, so any symp ξ^* with $\dim(\xi \cap \xi^*) \geq d-2$ is hyperbolic, has rank d and $\dim(\xi \cap \xi^*) = d-1$. This shows the first assertion, so we continue with the second one.

(*ii*) Firstly, suppose for a contradiction that ξ is thick. Let M_2^* be a $(d-1)$-space in ξ_2^* through S_2 distinct from M_2. Then M_2^* is collinear to at most one of the maximal singular subspaces of ξ through S_2 and, as there are at least three such subspaces, M_2^* is contained in a symp with a maximal singular subspace M_2' of ξ through S_2 which is disjoint from M_1, contradicting the first paragraph. We conclude that ξ is hyperbolic. Secondly, suppose ξ is hyperbolic of even rank d. Then M_1 and M_2, intersecting each other in a point, belong to different natural types of generators. By Lemma 4.4, there exists a subspace S_3 of ξ of dimension $d-2$ disjoint from M_1 and M_2. By Lemma 7.1, there is a symp $\xi_3^* \neq \xi$ with $S_3 \subseteq \xi \cap \xi_3^*$. By the above observation, $\xi \cap \xi_3^*$ is a maximal singular subspace M_3 of ξ through S_3. The first paragraph implies that both $M_1 \cap M_3$ and $M_2 \cap M_3$ is a point, but then the types of M_1, M_2 and M_3 should all be distinct, which is clearly impossible. □

For convenience we record a consequence of the proof of the previous lemma.

Corollary 7.5. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. If M_1 and M_2 are opposite maximal singular subspaces in a symp ξ of rank d, then at most one of them is contained in a second symp.*

Proof. This follows directly from the first paragraph of the proof of Lemma 7.4. □

Lemma 7.6. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Let ξ be any symp of rank d. Then the set Φ of maximal singular subspaces of ξ that are the intersection of ξ with another symp is precisely the set of generators belonging to one natural type.*

Proof. Suppose two generators M_1 and M_2 of ξ belong to Φ, and assume for a contradiction that they have distinct natural type. By Lemma 4.4, we can find a submaximal subspace S in ξ disjoint from M_1 and M_2. By Lemma 7.1 and 7.3, there is a symp ξ^* through S. In view of Lemma 7.4, $\xi^* \cap \xi$ is a maximal singular subspace M. By Corollary 7.5 and our choice of S, M intersects both M_1 and M_2 in exactly a point. Since M_1 and M_2 have distinct natural type, this is impossible.

We deduced that all members of Φ belong to the same natural type of generators. Conversely, to see that each generator of this type belongs to Φ, we consider any submaximal singular subspace S of ξ. As above, there is a symp ξ^* such that $\xi \cap \xi^*$ is a maximal singular subspace M of ξ containing S. The lemma follows. □

The following two lemmas are the basis to prove local connectivity and uniform rank.

Lemma 7.7. *Let $(X,\mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then a generator of some symp of rank d which is not contained in a second symp is contained in a singular d-space.*

Proof. Let ξ be an arbitrary symp of rank d and M an arbitrary generator of ξ not contained in another symp (cf. Lemma 7.6). Let M' be any generator of ξ intersecting M in a $(d-2)$-space W. Then $M' = \xi \cap \xi'$, for some $\xi' \in \Xi$. By Lemma 7.4(i), ξ' is (just as ξ) hyperbolic of odd rank d. In ξ', we consider the generator M'' containing W and distinct from M', and some point $p \in M'' \setminus M'$. If p were not collinear to all points of M, then $\{p,q\}$ is contained in a symp, for every $q \in M \setminus M''$, and that symp contains M and is different from ξ, contradicting our assumption on M. Hence p and M generate a singular subspace of dimension d. □

Lemma 7.8. *Let $\Omega = (X,\mathscr{L})$ have minimum symplectic rank $d \geq 3$. Let ξ_1 be a symp of rank d and let ξ_2 be any symp intersecting ξ_1 in exactly a point p. Then there is a singular plane through p intersecting both symps in a line.*

Proof. Consider a generator M_1 in ξ_1 through p not contained in a second symp of Ω (cf. Lemma 7.6). Then, by Lemma 7.7, there is a singular d-space W containing M_1. If W would intersect ξ_2 in more than p, the lemma follows immediately, so assume $W \cap \xi_2 = p$. We select a hyperplane H of W not containing p. Then, by Lemma 4.2, H is contained in a symp ξ. By our main hypothesis, we obtain a point x_2 contained in $\xi_2 \cap \xi$. Then $x_2 \neq p$ since otherwise ξ would contain the d-space W, whereas $\dim(\xi \cap \xi_1) \geq d-2$ implies, by Lemma 7.4(i), that ξ has rank d. Let $x_1 \in H \cap M_1$ be collinear to x_2 (x_1 exists since $\dim(H \cap M_1) \geq 1$). Since $x_2 \perp x_1 \perp p$ and both x_2 and p belong to ξ_2 we deduce that $x_2 \perp p$, and by Lemma 4.1, $\langle p, x_1, x_2 \rangle$ is a singular plane intersecting both ξ_1 and ξ_2 in the lines px_1 and px_2, respectively. □

Finally we can show that the symplectic rank is uniform.

Lemma 7.9. *Let $\Omega = (X,\mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then it has uniform symplectic rank d and therefore each symp is hyperbolic of odd rank d.*

Proof. Let ξ be any symp of rank d. By Lemma 7.4, any symp ξ' with $\dim(\xi \cap \xi') \geq d-2$ has rank d as well. Now let ξ^* be an arbitrary symp. We claim that we can find a (finite) sequence of symps between ξ^* and ξ such that successive symps in the sequence intersect each other in a subspace of dimension at least $d-2$, from which then follows that each symp in this sequence has rank d. By the Standing Hypotheses, $\xi \cap \xi^*$ is non-empty. If $\xi \cap \xi^*$ is a point, Lemma 7.8 implies the existence of a plane π intersecting both ξ and ξ^* in a line, and since $d \geq 3$, Fact 4.2 guarantees a symp through π which then shares at least a line with both ξ and ξ^*. Hence, if $d = 3$, we are done. If $d > 3$, we may already assume that $1 \leq \dim(\xi \cap \xi^*) \leq d-3$. Under this assumption we can take points p and p^* in ξ and ξ^*, respectively, collinear to $\xi \cap \xi^*$ and not collinear to each other. The symp determined by p and p^* intersects both ξ and ξ^* in a subspace strictly bigger than $\xi \cap \xi^*$.

Recursively, the claim follows and hence each symp has rank d. Lemma 7.4 now implies that each symp is hyperbolic of odd rank d. $\square$

Henceforth we could therefore drop the word "minimum" from our assumptions on Ω, but we prefer to keep it in order to remind the reader of the full context. Local connectivity now follows as a consequence of Lemma 7.8.

Lemma 7.10. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then Ω is locally connected.*

Proof. Consider two lines L_1 and L_2 through p. Let ξ_1 and ξ_2 be symps through L_1 and L_2, respectively. If $\dim(\xi_1 \cap \xi_2) \geq 1$, it is clear that L_1 and L_2 are connected via a sequence of singular planes intersecting each other in lines. If $\dim(\xi_1 \cap \xi_2) = 0$, then, as all symps have rank d now by Lemma 7.9, a link between ξ_1 and ξ_2 is provided by Lemma 7.8 (and inside the symps we are fine, as just mentioned before). $\square$

We proceed by showing boundedness of the singular rank.

Lemma 7.11. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then the dimension of a singular subspace is at most $2(d-1)$.*

Proof. Suppose there is a singular $(2d-1)$-space W in Ω. Let M_1 and M_2 be two disjoint $(d-1)$-subspaces in W. By Fact 4.2, there are symps ξ_1 and ξ_2 containing M_1 and M_2, respectively. This yields a point $p \in \xi_1 \cap \xi_2$. Since M_i is a maximal singular subspace in ξ_i, $i = 1, 2$, we know $p \notin W$. In particular $p \notin M_1 \cup M_2$ and so we can find points $q_1 \in M_1$ and $q_2 \in M_2$ with $q_1 \notin p^\perp$ and $q_2 \in p^\perp$. Then $q_2 \in p^\perp \cap q_1^\perp \subseteq \xi_1$, a contradiction. $\square$

Finally we prove the Haircut Axiom (H) [11].

Lemma 7.12. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then*

(H) *for any symp ξ and any point $p \notin \xi$, the set $p^\perp \cap \xi$ can never be a submaximal singular subspace of ξ.*

Proof. Assume for a contradiction that $p^\perp \cap \xi = H$, with H a submaximal singular subspace of ξ. Since ξ is hyperbolic, there are exactly two generators M_1, M_2 containing H. Pick $p_i \in M_i \setminus H$, $i = 1, 2$. By assumption, $p_i \notin p^\perp$, $i = 1, 2$. Then the symps $\xi(p, p_1)$ and $\xi(p, p_2)$ contain M_1 and M_2, respectively, contradicting Lemma 7.6 and the fact that M_1 and M_2 have distinct natural type. $\square$

In order to show that the uniform symplectic rank of Ω is either 3 or 5, we first show that two symps which intersect in a plane, intersect in a generator.

Lemma 7.13. *Let $(X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then two symps that have no generator in common intersect in either a point or a line.*

Proof. Recall that we know from Lemma 7.9 that each symp has rank d. The result is trivial if $d = 3$, so let $d \geq 4$. Suppose two generators ξ and ξ' intersect in a singular subspace U of dimension j, $0 \leq j \leq d-2$. Select a generator M in ξ disjoint from U

such that $M = \xi \cap \xi^*$, for some $\xi^* \in \Xi$, which is possible by Lemma 7.6. The Standing Hypotheses yield a point $p \in \xi' \cap \xi^*$. Then $p \notin \xi$ since M is disjoint from ξ'. However, p is collinear to all points of a $(d-2)$-space in M (since $p \in \xi^*$) and $\dim(p^\perp \cap U) \geq j-1$ (since $p \in \xi'$). Since $p^\perp \cap \xi$ is a singular subspace, its dimension ℓ satisfies $(d-2)+(j-1)+1 \leq \ell \leq d-1$, implying $j \leq 1$. The lemma is proved. □

Lemma 7.14. *Let $\Omega = (X, \mathscr{L})$ have minimum symplectic rank $d \geq 3$. Then Ω has uniform symplectic rank $d \in \{3,5\}$. So the symps are either hyperbolic polar spaces of rank 3, or hyperbolic polar spaces of rank 5.*

Proof. Suppose $d \geq 5$, we show that $d = 5$. Let ξ be a symp and choose two generators M, M' of ξ not contained in second symps and intersecting in a plane π. Let W and W' be d-spaces through M, M', respectively (these exist by Lemma 7.7). If all points of $W \setminus M$ were collinear to all points of $W' \setminus M'$, then all points of M would be collinear to all points of M', a contradiction. So there are points $p \in W \setminus M$ and $p' \in W' \setminus M'$ which are not collinear. Since π belongs to $p^\perp \cap p'^\perp$, p and p' determine a symp ξ^* intersecting ξ in at least the plane π, so by Lemma 7.13, $\xi \cap \xi^*$ is a generator M^*. Since $p^\perp \cap \xi = M$, we have $p^\perp \cap M^* \subseteq M$, likewise $p'^\perp \cap M^* \subseteq M'$. Both subspaces have dimension $d-2$ and are contained in M^*, and hence intersect in a $d-3$-space. On the other hand, they intersect in π only, so $d-3 \leq 2$, implying $d \leq 5$. □

End of the proof of the Main Result—Case of the existence of at least one sympthick line. Lemmas 7.10, 7.11, 7.14 and 7.12 show that conditions (LC), (BD), (BR) and (H) are satisfied. Therefore we may invoke Theorems 15.3.7 and 15.4.3 from [10], which are updates of the Main Theorem of [2] and Theorem 1 of [3]. Knowing that $d \in \{3,5\}$ (cf. Lemma 7.14), we conclude that the parapolar spaces with minimum symplectic rank $d \geq 3$, containing at least one sympthick line, and such that every two symps intersect nontrivially are precisely $\mathsf{A}_{4,2}(k)$, $\mathsf{A}_{5,2}(k)$ (and in these cases $d = 3$; k is an arbitrary skew field) and $\mathsf{E}_{6,1}(k)$ (and then $d = 5$; k is an arbitrary field).

Remark 7.15 (**Avoiding Cohen-Cooperstein theory).** With some limited additional effort, one can strengthen Lemmas 7.11 and 7.13 using direct arguments as follows. Two symps either intersect in a point, or in a generator. Also, the maximum dimension of a singular subspace is either 3 or 4 (in the case $d = 3$), or 5 (in the case of $d = 5$). This leaves us with three cases. The first two cases are dealt with in a completely elementary way identifying the elements of the projective spaces of dimension 4 and 5, respectively, from which Ω arises as line Grassmannian, as certain subspaces of Ω. A similar technique can be used for the remaining case, $d = 5$, now using a characterization of buildings of type E_6 by Jacques Tits [16]. This approach is carried out in detail in the first author's thesis [5].

8 The case of symplectic rank at least 3 where no line is sympthick

We finish the proof of the Main Result.

Let $\Omega = (X, \mathscr{L})$ be a parapolar space of symplectic rank at least 3 such that every two symps intersect nontrivally, and such that every line is contained in a unique symp. Then clearly symps intersect each other in points and the point-line geometry $\Upsilon = (X, \Xi)$ is a dual linear space.

Lemma 8.1. *Suppose $p_0 \in X$ belongs to two distinct members ξ_1, ξ_2 of Ξ. Let $p_1 \in \xi_1 \setminus \{p_0\}$ and $p_2 \in \xi_2 \setminus \{p_0\}$ be arbitrary and take any $\xi_{i3} \in \Xi$ through p_i, $i = 1,2$, and let $p_3 \in \xi_{13} \cap \xi_{23}$. Then, if $p_0 \neq p_3$, we have*

$$\delta_1(p_0, p_1) + \delta_{13}(p_1, p_3) + \delta_{23}(p_2, p_3) + \delta_2(p_0, p_2) \quad \geq \quad 5, \tag{1}$$

where $\delta_\bullet$ is the distance in the collinearity graph of $\xi_\bullet \in \Xi$, i.e., 0 if the arguments are equal, 1 if they are collinear in ξ, and 2 otherwise.

Proof. We distinguish three cases.

(i) Suppose $\xi_{13} = \xi_1$. Then $\xi_{23} \neq \xi_2$, for otherwise $p_0 = p_3$. Hence ξ_1, ξ_{23}, ξ_2 are three distinct members of Ξ, i.e., $\{p_0, p_2, p_3\}$ is a triangle in Υ. If $p_0 \perp p_2 \perp p_3 \perp p_0$ in Ω, then Lemma 4.1 implies that p_0, p_2, p_3 are contained in a common singular plane, which is, by Lemma 4.2, contained in some symp ξ. Since ξ shares a line with both ξ_1 and ξ_2, our assumption implies that $\xi_1 = \xi = \xi_2$, a contradiction. Without loss of generality, we may assume $p_2 \notin p_3^\perp$, in particular, $\delta_{23}(p_2, p_3) = 2$. Since symps are convex, we then have $p_2^\perp \cap p_3^\perp \subseteq \xi_{23}$, and so $p_0 \notin p_2^\perp \cap p_3^\perp$, implying $\delta_{13}(p_0, p_3) + \delta_2(p_0, p_2) \geq 3$. Inside the (discrete) metric space $\xi_{13} = \xi_1$, the triangle inequality now yields $\delta_1(p_0, p_1) + \delta_{13}(p_1, p_3) \geq \delta_1(p_0, p_3)$, from which (1) follows.

By symmetry we may now suppose that $\xi_{13} \neq \xi_1$ and $\xi_{23} \neq \xi_2$.

(ii) In this case, we assume that $\xi_{13} = \xi_{23} \notin \{\xi_1, \xi_2\}$. Then we can interchange the roles of p_1 and p_3 in Case (i) and conclude that (1) holds again.

(iii) Finally we assume that ξ_1, ξ_2, ξ_{13} and ξ_{23} are four distinct symps. The only way in which (1) can be violated is when $p_0 \perp p_1 \perp p_3 \perp p_2 \perp p_0$ (in Ω). But then, according to (PPS2), all four points are contained in common symp, which shares lines with the distinct symps $\xi_1, \xi_2, \xi_{13}, \xi_{23}$, contradicting the fact that lines are contained in unique symps.

□

We now show the converse. Suppose we have a nontrivial dual linear space $\Upsilon = (X, \Xi)$ such that every line (i.e., every member of Ξ) has the structure of a polar space of rank at least 3, and satisfying the inequality (1) for the given restrictions on the points and symps (we shall refer to this inequality as Condition (1)). Let $\mathscr{L}$ be the set of all lines of all these polar spaces (to avoid confusion with the lines of $\mathscr{L}$ and the symps of parapolar spaces, we now refer to them as *blocks*).

Lemma 8.2. *The geometry $\Omega = (X, \mathscr{L})$ is a parapolar space of symplectic rank at least 3, whose set of symps coincides with Ξ and in which every line is contained in a unique symp, and such that every two symps intersect each other in a unique point.*

Proof. Recall that in a dual linear space, each point is contained in a block and each two blocks intersect each other in a unique point. We now verify the axioms of a parapolar space and show that the symps of Ω are the blocks of Υ. Note that the two last assertions are satisfied if we replace "symp" by "block" (and we will show in (PPS2) that we may do so).

(PPS1) The connectivity of Ω follows from the connectivity of Υ as a geometry of points and symps, and the fact that every block is connected (being a polar space).
Now let $p_0, p_1, p_2 \in X$ be three mutually collinear points (collinearity with respect to Ω) with p_0 not on the line L joining p_1, p_2. If p_0, p_1, p_2 are contained in a common block of Υ, then p_0 is collinear to all points of L. Suppose now that the blocks ξ_{ij} of Υ containing p_i and p_j, $0 \leq i < j \leq 2$, are mutually distinct. Then Condition (1) implies (δ_{ij} is the distance in the collinearity graph of ξ_{ij})

$$3 = \delta_{01}(p_0, p_1) + \delta_{12}(p_1, p_2) + \delta_{12}(p_2, p_2) + \delta_{02}(p_0, p_2) \geq 5,$$

a contradiction. Note that $|\Xi| > 1$ by assumption, so we can find a point $p \in X$ and a line $L \in \mathscr{L}$ such that no line of $\mathscr{L}$ contains p and meets L.

(PPS2) We claim that each block of Υ is the convex closure of any pair of noncollinear points it contains (clearly, the convex closure of two such points contains the block). So suppose for a contradiction that p_1, p_2 are noncollinear points of a block ξ_{12} such that $\xi_{12} \setminus \mathrm{cl}(\{p_1, p_2\})$ contains a point p_0. By definition of closure and by the fact that lines between two points of ξ_{12} are contained in ξ_{12} (since two blocks intersect in a unique point and each line belongs to a block), we have that p_0 is collinear to two non-collinear points of $\mathrm{cl}(\{p_1, p_2\})$, so without loss, $p_1 \perp p_0 \perp p_2$. Hence, the symps ξ_{01} and ξ_{02} containing p_0p_1, and p_0p_2, respectively, are well defined and distinct. Then Condition (1) implies

$$4 = \delta_{01}(p_0, p_1) + \delta_{12}(p_1, p_2) + \delta_{12}(p_2, p_2) + \delta_{02}(p_0, p_2) \geq 5,$$

a contradiction. This shows the claim.
Suppose now that p_1, p_2 are points of X not contained in any block and suppose for a contradiction that $p_0, p_3 \in p_1^{\perp} \cap p_2^{\perp}$, with $p_0 \neq p_3$. Then there are distinct blocks ξ_{01} and ξ_{02} containing p_0, p_1 and p_0, p_2, respectively, and likewise ξ_{13} and ξ_{23} containing p_1, p_3 and p_2, p_3, respectively. With similar notation as before, Condition (1) yields

$$4 = \delta_{01}(p_0, p_1) + \delta_{13}(p_1, p_3) + \delta_{23}(p_2, p_3) + \delta_{02}(p_0, p_2) \geq 5,$$

the sought contradiction.
In particular, we showed that the blocks of Υ are precisely the symps of Ω

(PPS3) Since each line is contained in a block by definition, this follows from the above. $\square$

This completes the proof of the Main Result. Some remarks to conclude:

Remark 8.3 (**The existence of locally disconnected parapolar spaces.**). We start with a very general class of examples. Let $\Upsilon = (Y, \mathscr{B})$ be any dual linear space having at least two lines. For each $B \in \mathscr{B}$, we can select a polar space ξ_B of rank at least 3 and an injective mapping $B \to \xi_B$ such that any two elements in the image of B are non-collinear in ξ_B (this can always be achieved by choosing ξ_B "large enough"). We then identify B with its image and set $\Xi = \{\xi_B \mid B \in \mathscr{B}\}$. We also define X as the union of Y with the disjoint union of all $\xi_B \setminus B$, were B ranges over $\mathscr{B}$. Finally, let $\mathscr{L}$ be the set of all lines in all the polar spaces ξ_B, $B \in \mathscr{B}$. Then $\Omega = (X, \mathscr{L})$ is a locally disconnected parapolar space of symplectic rank at least 3 such that every two symps meet in exactly one point (it is an easy exercise to verify that Condition (1) holds).

In the previous example there are many points $x \in X$ which are contained in a unique symp. We were not able to find examples such that every point is contained in at least two symps. Particularly in the finite case this seems rather hard. In fact we conjecture that *such a finite parapolar space does not exist*.

Remark 8.4 (**On the requirement that Ω is strong when there are symps of rank 2**). One can construct several examples of *non-strong* parapolar spaces Ω in which each pair of symps has a non-trivial intersection, which are not accounted for in our main theorem (so necessarily, Ω contains symps of rank 2). Indeed, consider for instance the Cartesian product of a projective plane and a pencil of projective lines $\{L_i \mid i \in \mathscr{I}\}$, for some index set $\mathscr{I}$, such that no other relations between the lines L_i exist apart from the fact that they share a certain fixed point p. This gives us an example of a non-strong parapolar space in which every pair of symps intersects non-trivially (as they all have a line in common with π), demonstrating that one should not expect a "nice" classification of such parapolar spaces.

Nonetheless, it is hard to come up with such examples having diameter 2, or in which all lines are sympthick. We conjecture that one could obtain a neat classification of diameter 2 parapolar spaces in which all symps intersect each other non-trivially (in need adding that each line is sympthick), and we would not be surprised if all these parapolar spaces turn out to be strong.

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