On Exceptional Lie Geometries

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Dedicated to the memory of Ernie Shult

Abstract
Parapolar spaces are point-line geometries introduced as a geometric approach to (exceptional) algebraic groups. We provide a characterization of a wide class of Lie geometries as parapolar spaces satisfying a simple intersection property. In particular many of the exceptional Lie geometries occur. In fact, our approach unifies and extends several earlier characterizations of (exceptional) Lie geometries arising from spherical Tits-buildings.

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1 Introduction

1.1 General context

Buildings, sometimes more specifically called Tits-buildings, were introduced by Jacques Tits [22] in order to have a geometric interpretation of semi-simple groups of algebraic origin (semi-simple algebraic groups, classical groups, groups of mixed type, (twisted) Chevalley groups). The definition of a building is, however, somewhat involved, and it does not immediately provide a good intuition. On the other hand, projective spaces and polar spaces—which are natural point-line geometries—can be given the structure of Tits-buildings and provide excellent permutation representations for the classical groups. In fact, projective and polar spaces are Grassmannians of certain Tits-buildings. More exactly, since a Tits-building is a numbered simplicial complex, one can take all simplices of a certain type $T$ as point set, and then there is a well-defined mechanism that deduces a set of lines. The resulting point-line geometry
is the so-called $T$-Grassmannian of the Tits-building. For a certain choice of $T$, projective spaces and polar spaces emerge from spherical Tits-buildings of types $A_n$ and $B_n$, respectively. Other choices of $T$ for these and for other types of spherical Tits-buildings in general lead to parapolar spaces, which were first studied by Cooperstein [6]. For a precise definition of these, see Section 3.

In fact, parapolar spaces are point-line geometries introduced in the literature to approach in a geometrical way the spherical Tits-buildings and Chevalley groups mainly of exceptional types. The pure definition of a parapolar space is yet general enough to also capture many other geometries, in particular many Grassmannian geometries related to classical groups and (spherical) Tits-buildings. In general, the Grassmannian geometries related to a spherical Tits-building (or, as soon as the rank is at least 3, related to a semi-simple algebraic, classical or mixed group), and by extension also to a non-spherical one, are called Lie (incidence) geometries. A lot of work in the past went into characterizing, using additional properties, certain classes of parapolar spaces, preferably containing as many of exceptional type as possible. Perhaps the most far reaching in that respect is a relatively recent result by Shult [16] characterizing many Lie incidence geometries with basically only one additional axiom, which he called the “Haircut Axiom”. The latter axiom expresses a gap in the spectrum of dimensions of singular subspaces of symplecta arising from intersecting the latter with the perp of a point. More exactly, the axiom states that the set of points collinear to a given point and belonging to a given symplecton can never be a submaximal singular subspace (i.e., a singular subspace of dimension two less than the rank of the symplecton; maximal singular subspaces have dimension one less than the rank).

In the present paper we start with the observation that in the most popular exceptional Lie geometries, gaps also appear in another type of spectrum, namely, in the spectrum of the dimensions of the singular subspaces that occur as intersections of two symplecta. A parapolar space with at least one gap in that spectrum will be called here lacunary, or, more exactly, $k$-lacunary, if $k$-dimensional singular subspaces never appear as the intersection of two symplecta, and all symplecta really possess $k$-dimensional singular subspaces. The exceptionality of this behavior is then proved by our Main Theorem, which completelyclassifies all lacunary parapolar spaces of symplectic rank at least 3. Moreover, we also include rank 2 symplecta using a harmless additional condition, which we explain in more detail below (Subsection 1.3).

Finally, we would like to mention Cohen and Ivanyos’s approach to the exceptional geometries via root filtration spaces [5]. There is a close link with parapolar spaces, but they manage to also include the geometries which can be seen as degenerate parapolar spaces of diameter 3—namely, they do not contain symplecta and all pairs of points at mutual distance 2 are special. Their approach is specifically aimed at the so-called long root geometries, and they find all of them, whereas in our approach, the ones of type $A$ and $G$ are missing (and also those of type $C$, which play a somewhat isolated role anyway). We do not go into details here.

1.2 Connection with the Freudenthal-Tits Magic Square

In the 1960s Hans Freudenthal [7] and Jacques Tits [19, 21] provided a remarkable uniform construction of some semi-simple Lie algebras, among which many of exceptional type. The types of these Lie algebras can be arranged in a symmetric $4 \times 4$ square. In his habilitation thesis [20], Tits describes a class of geometries related to Lie groups whose types are arranged in the same $4 \times 4$ square. In fact, the geometries described by Tits are Grassmanians of certain buildings, most of which are parapolar spaces avant-la-lettre. With the modern notation, the table of types looks as follows.

Our approach allows to elegantly characterize many geometries (and their subgeometries) related to this so-called Freudenthal-Tits Magic Square as the $k$-lacunary parapolar spaces with symplectic rank at least $k + 3$ (strong if $k = -1$); all geometries of the South-East $3 \times 3$ subsquare are thus captured.
This is in fact our main motivation: single out the properties of the parapolar spaces in the Magic Square. Moreover these results will be of use to the second and third author in their investigation of the projective varieties associated to the square \([11, 12, 13]\), in particular for the study of the Lagrangian Grassmannians (third row) and the adjoint varieties (fourth row).

Zooming in on the second row of the Magic Square, one obvious property is that every pair of symplecta intersects non-trivially. In fact, this was already observed by Freudenthal [7] and Springer and Veldkamp [17], who considered these geometries as “projective planes over rings”, and by Tits [18], who viewed the buildings of type \(E_6\) as “projective planes over split octonions”. The corresponding geometry is also called “the Hjelmslev-Moufang plane” in the literature. Its structure resembles that of an ordinary projective plane, except that each line has the structure of a polar space and two lines can meet in more than one point (namely, in a maximal singular subspace of the corresponding polar spaces). So, more generally, one can ask whether there are other geometries (which are necessarily parapolar spaces) behaving like projective planes, and where the structure of the lines is a convex polar space. We consider it as a major achievement to have been able to classify all such parapolar spaces in the present paper. This completely answers the following natural question:

What are the geometries consisting of points and polar spaces, such that each polar space is a convex subspace in the induced point-line geometry and such that each pair of points is contained in at least one polar space, each pair of polar spaces intersects in a nonempty singular subspace of both?

This corresponds to (a slightly stronger form of) our previously introduced notion of \((-1)\)-lacunarity, since every pair of symplecta intersects non-trivially, and hence the empty intersection (dimension \(-1\)) cannot occur. Our methods are elementary and can be understood without deep knowledge of the theory of parapolar spaces. That result, together with the classification of 0-lacunary parapolar spaces admitting symplecta of rank 2, is then taken as the first step of an inductive process to determine all lacunary parapolar spaces. To carry out this classification, it is convenient and very efficient to rely on other, more deep, characterization theorems in the literature, although our induction hypothesis provides enough information for a further elementary proof (also in the \((-1)\)-case we use a deeper theorem, namely, Shult’s Haircut Theorem itself, but we indicate how it can be avoided).

A comment on the case of 0-lacunarity is in order here. This notion expresses that in a parapolar space, symplecta that meet in a point, automatically share a line. In a private conversation to the second author, Shult mentioned that he would have liked to classify these parapolar spaces. This is also indicated by Exercise 13.26 in his book [15], which deals exactly with the 0-lacunary hypothesis. However, what Shult was missing was the structure of the point-residuals; hence he was missing the classification of \((-1)\)-lacunary parapolar spaces, in particular the ones with symplecta of rank 2. About the latter, though, he writes in [14] “It is not easy to live in a world with no symplecton of rank at least three in sight.” Nevertheless, in our context, we manage to deal with parapolar spaces whose symplecta are all of rank 2, provided, as mentioned above, we add a harmless condition, explained in the next subsection.
1.3 When some symplecta have rank 2

When there are symplecta of rank 2 around, then we add the following additional restriction: We assume that the parapolar space is strong (though for 1-lacunary parapolar spaces it is possible to drop that assumption, see Remark 10.9), i.e., every pair of intersecting lines is contained in either a singular subspace, or in a symplecton. The reason is that we consider the case that all symplecta have rank at least 3 as the main case, but the rank 2 case is needed in the induction process, where it only turns up under the strongness assumption. Hence, in this view, it is a harmless restriction. Note, however, that some important parapolar spaces escape in this way (e.g., the point-plane geometry arising from a projective space of dimension 3), but also a lot of dull examples are excluded this way (e.g., the direct product of an arbitrary number of polar spaces of rank 2).

The fact that we require a parapolar space with symplecta of rank 2 to be strong allows us also to omit any restriction on the connectivity of the point-residuals. Indeed, if the symplectic rank is at least 3, then we develop a theory to deduce a classification of the locally disconnected lacunary parapolar spaces from the classification of the locally connected ones. And if there are symplecta of rank 2, then strongness allows a complete classification without any assumption on the local connectivity. So our result holds also for not necessarily locally connected parapolar spaces. We deal with these explicitly in the last section, where it should also become clear how to handle that situation in other circumstances (e.g., the local connectivity could be left out in Shult’s Haircut Theorem, provided in the conclusion certain additional parapolar spaces, as described in our last section, are added).

1.4 Structure of the paper

The paper is structured as follows. In Section 2 we present our main results, and some corollaries. In Section 3, we provide all necessary definitions, and we review some known results in the theory of parapolar spaces, which we will use in our proof. In Section 4 we describe the examples and tabulate the geometries in the conclusions of our main results. Section 5 describes the structure of our proof, and the actual proofs of our main results are given in the remaining sections.

2 Main Results

In this section we collect our main results. For the precise definitions of the notions we refer to Section 3; for the definitions and descriptions of the geometries in the conclusions, we refer to Section 4. We content ourselves to informally define a \( k \)-lacunary parapolar space as a parapolar with symplectic rank at least \( k + 1 \) (i.e., all symplecta (henceforth sometimes called symps) have rank at least \( k + 1 \) as polar spaces) such that symps never intersect at precisely a \( k \)-dimensional singular subspace. If \( k \) is not specified, we just say that the parapolar space is lacunary. We say that a parapolar space has minimum symplectic rank \( d \) if it has symplectic rank at least \( d \) and there exists a symp of rank \( d \). We also note that we do not consider polar spaces of infinite rank; these could be included by the interested reader with any definition. But there does not seem to be a general agreement on what such polar spaces should precisely be, so we leave them out and consider inclusion of them as a minor additional effort that can be easily performed (they will arise in (B) and (D) below, but the rest remains unchanged).

The heart of our Main Result is as follows, an extended version is provided further in this section.

**Main Theorem**  Let \( \Omega = (X, \mathcal{L}) \) be a locally connected lacunary parapolar space with symplectic rank at least 3. Then \( \Omega \) is one of the following Lie incidence geometries (\( \mathbb{K} \) is any commutative field, \( \mathbb{L} \) is an arbitrary skew field):

(A) \( A_{5,3}(\mathbb{L}) \) or the line Grassmannian of a not necessarily finite-dimensional projective space of dimension at least 4;
Then, \( \Omega \) arises from one of the following Lie incidence geometries:

- The Cartesian product of a thick line and an arbitrary projective plane (i.e., \( A_{1,1}(\ast) \times A_{2,1}(\ast) \));
- The Cartesian product of two arbitrary not necessarily isomorphic projective planes (i.e., \( A_{2,1}(\ast) \times A_{2,1}(\ast) \));
- The line-Grassmannian of any projective space of dimension 4 (i.e., \( A_{4,2}(\mathbb{L}) \));
- The line-Grassmannian of any projective space of dimension 5 (i.e., \( A_{5,2}(\mathbb{L}) \));
- The Lie incidence geometry \( E_{6,1}(\mathbb{K}) \), \( \mathbb{K} \) any field.

Moreover, if we discard the third requirement when \( \Xi \) does not contain members of rank 2, then the same conclusion holds, except that there is one further possibility:

- \((X, \Xi)\) is a non-trivial partial linear space with the following property: If \( p_0 \in X \) belongs to two distinct members \( \xi_1, \xi_2 \) of \( \Xi \), and \( p_i \in \xi_i, i = 1, 2, \) and \( p_3 \in X \) is contained in a common member \( \xi_3 \) of \( \Xi \) together with \( p_i, i = 1, 2, \) where \( p_0 \notin \{p_1, p_2, p_3\} \), then

\[
\delta_{\xi_1}(p_0, p_1) + \delta_{\xi_2}(p_1, p_3) + \delta_{\xi_3}(p_2, p_3) + \delta_{\xi_3}(p_0, p_2) \geq 5,
\]

where \( \delta_{\xi} \) is the distance in \( \xi \in \Xi \), i.e., 0 if the arguments are equal, 1 if they are collinear in \( \xi \), and 2 otherwise.

The last geometry mentioned in the previous result is in fact not locally connected. Hence, in the locally connected case, and if \( \Xi \) does not contain polar spaces of rank 2, then the third requirement follows from the others. This is the serendipity that makes the classification of all lacunary parapolar spaces work, see Lemma 7.2, which appears to be crucial for the classification of 0-lacunary parapolar spaces of symplectic rank at least 3. There is no direct proof of that fact, it just happily follows from the classification.

We now state the most general form of the main result of this paper.

**Main Result—Extended Version** Let \( \Omega = (X, \mathcal{L}) \) be a lacunary parapolar space with minimum symplectic rank \( d \geq 2 \), which is also assumed to be strong if \( d = 2 \). Let \( \Xi \) be the family of symps and let \( \Sigma \) be the family of maximal singular subspaces (not assumed to be projective or finite-dimensional).

Then either \( \Omega \) is locally disconnected and we have:
(i) $d \geq 3$ and there exists at least one point $p \in X$ for which the point-residual in $p$ is not connected. Then $\Omega$ arises from a family of locally connected $k$-lacunary parapolar spaces and polar spaces of rank at least $k + 1$ as described in Construction 10.4 and called a $k$-buttoned parapolar space, $k \geq -1, k \neq 0$.

or $\Omega$ is locally connected and one of the following holds (where $\mathbb{L}$ denotes any skew field and $\mathbb{K}$ any commutative field).

(ii) $d = 2$ and $\Omega$ is $-1$-lacunary. Then $\Omega$ is one of the following: $A_{1,1}(\ast) \times A_{2,1}(\ast), A_{2,1}(\ast) \times A_{2,1}(\ast)$.

(iii) $d \geq 3$ and $\Omega$ is $-1$-lacunary. Then $\Omega$ is one of the following: $A_{d,2}(\mathbb{L}), A_{d,2}(\mathbb{L}), E_{d,1}(\mathbb{K})$

(iv) $d = 2$, $\Omega$ is $0$-lacunary, $\text{Diam} \Omega = 2$ and at least one symp is thick. Then $(X, \Sigma)$ is a generalized quadrangle, each member of $\Xi$ is an ideal subquadrangle of $(X, \Sigma)$, and no maximal singular subspace is a projective space.

(v) $d = 2$, $\Omega$ is $0$-lacunary, $\text{Diam} \Omega = 2$ and all symps are grids. Then $\Omega$ is the Cartesian product of a thick line with an arbitrary but non-trivial linear space (with thick lines).

(vi) $d = 2$, $\Omega$ is $0$-lacunary and $\text{Diam} \Omega = 3$. Then $\Omega$ is either the Cartesian product of a line and an arbitrary (non-degenerate) polar space of rank at least 2, or is a dual polar space of rank 3.

(vii) $d \geq 3$, $\Omega$ is $0$-lacunary and $\text{Diam} \Omega = 2$. Then $\Omega$ is one of the following: $A_{d,2}(\mathbb{L}), D_{d,5}(\mathbb{K})$.

(viii) $d \geq 3$, $\Omega$ is $0$-lacunary and $\text{Diam} \Omega = 3$. Then $\Omega$ is one of the following: $A_{d,3}(\mathbb{L}), D_{d,6}(\mathbb{K}), E_{d,7}(\mathbb{K})$.

(ix) $d \geq 3$ and $\Omega$ is $1$-lacunary. Then $\Omega$ is one of the following: $A_{n,2}(\mathbb{L}) (n \geq 4)$, the line grassmannian of a (thick or non-thick) polar space of rank at least 4, $D_{d,5}(\mathbb{K}), E_{d,6}(\mathbb{K}), E_{d,7}(\mathbb{K}), E_{7,1}(\mathbb{K}), E_{8,8}(\mathbb{K})$, a metasymplesctic space.

(x) $d \geq 4$ and $\Omega$ is $2$-lacunary. Then $\Omega$ is one of the following: a homomorphic image of $D_{n,9}(\mathbb{K}), n \geq 5$ (isomorphic if $n \leq 9$), $E_{d,1}(\mathbb{K}), E_{7,7}(\mathbb{K})$.

(xi) $d \geq 5$ and $\Omega$ is $3$-lacunary. Then $\Omega$ is one of the following: $E_{d,1}(\mathbb{K}), E_{d,1}(\mathbb{K}), E_{d,7}(\mathbb{K}), E_{8,8}(\mathbb{K})$, a homomorphic image of $E_{8,1}(\mathbb{K}), a$ homomorphic image of $E_{n,1}(\ast), n \geq 9$.

(xii) $d \geq 6$ and $\Omega$ is $4$-lacunary. Then $\Omega$ is one of the following: $E_{7,7}(\mathbb{K}), E_{8,8}(\mathbb{K})$.

(xiii) $d = 7$ and $\Omega$ is $5$-lacunary. Then $\Omega$ is isomorphic to $E_{8,8}(\mathbb{K})$.

In particular

(xiv) no locally connected parapolar space of minimum symplectic rank $d \geq 3$ is $(d - 1)$-lacunary, and

(xv) no locally connected parapolar space of minimum symplectic rank $d \geq 8$ is lacunary.
3 Preliminaries

3.1 Definitions and basic facts

Parapolar spaces were introduced to capture the spherical buildings of exceptional type (spherical buildings comprise projective spaces, polar spaces and the geometries of exceptional type). Parapolar spaces are, as the name suggests, full of polar spaces. Hence we first provide a definition of polar spaces.

Polar spaces have been introduced by Veldkamp [24], later on included in the theory of buildings by Tits [22], and around the same time the axioms have been simplified by Buekenhout & Shult [3]. It is the latter point of view we take here.

Let \( \Gamma = (X, \mathcal{L}, *) \) be a point-line geometry (\( X \) is the set of points, \( \mathcal{L} \) the set of lines, and * a symmetric incidence relation). We will not consider geometries with repeated lines, so henceforth we view \( \mathcal{L} \) as a subset of the power set of \( X \), and *, which is then just inclusion made symmetric, is not mentioned explicitly. The incidence graph is the bipartite graph on \( X \cup \mathcal{L} \) with incidence as adjacency relation. The dual of \((X, \mathcal{L}, *)\) is the point-line geometry \((\mathcal{L}, X, *)\).

A subspace of \( \Gamma \) is a subset \( S \) of the point set such that, if two points \( a, b \) belong to \( S \), then all lines containing both \( a \) and \( b \) are contained in \( S \). A subspace \( H \) is called a geometric hyperplane if each line of \( \Gamma \) has either one or all its points contained in \( H \). Points contained in a common line will be called collinear; dually, lines sharing at least one point are called concurrent. A singular subspace is a subspace every two points of which are collinear. Note that the empty set and a single point are legible singular subspaces. If \( S \) is a set of pairwise collinear points, then \( \langle S \rangle \) denotes the singular subspace generated by the points of \( S \) (this is the intersection of all singular subspaces containing \( S \)); if \( S \) consists of two collinear points \( p \) and \( q \), then the line through them is denoted by \( pq \).

The collinearity graph is the graph on \( X \) with collinearity as adjacency relation. The distance \( \delta \) between two points \( p, q \in X \) (denoted \( \delta(p,q) \)) is the distance between \( p \) and \( q \) in the collinearity graph, possibly, if there is no path between \( p \) and \( q \), then \( \delta(p,q) = \infty \). The distance between a point \( p \) and a line \( L \) is given by the minimum distance, i.e., \( \delta(p,L) = \min\{\delta(p,q) \mid q \in L\} \). If \( \delta := \delta(p,q) \) is finite, then a geodesic path or a shortest path between \( p \) and \( q \) is a path in the collinearity graph of length \( \delta \). The diameter of \( \Gamma \) (denoted \( \text{Diam} \Gamma \)) is the diameter of the collinearity graph.

**Definition 3.1.** A point-line geometry \( \Delta = (X, \mathcal{L}) \) is called a polar space if the following axioms holds.

- (PS1) Every line contains at least three points.
- (PS2) No point is collinear to all other points.
- (PS3) Every nested sequence of singular subspaces is finite.
- (PS4) For any point \( x \) and any line \( L \), either one or all points on \( L \) are collinear to \( x \).

**Some basic properties** — Let \( \Delta \) be a polar space. Then every one of its singular subspaces is a projective space, and its dimension can hence be defined as the dimension of the projective space. There exists an integer \( r \geq 2 \) such that each nested sequence of singular subspaces has length \( r + 1 \). We call \( r \) the rank of \( \Delta \). Consequently, the maximal singular subspaces of \( \Delta \) have dimension \( r - 1 \). Note that axiom (PS3) implies that the rank is finite, which is not strictly necessary for polar spaces. Yet we will only consider polar spaces of finite rank, hence our preference for the above set of axioms.

Two points which are not collinear are called opposite. Two singular subspaces \( S_1 \) and \( S_2 \) are opposite if for each point \( p_1 \) of \( S_1 \) there is a point \( p_2 \in S_2 \) opposite to it, and vice versa. Opposite singular subspaces always have equal dimension and are always disjoint. Two maximal singular subspaces are opposite if
and only if they are disjoint. For any singular subspace, there is a subspace opposite to it. If two singular subspaces $S_1$ and $S_2$ are opposite, then $S_1 \perp S_2$ forms a polar space of rank $n - \dim(S_1) - 1$.

For a point $p \in X$, the point-residual at $p$ of $\Delta$, denoted $\Delta_p$, is defined by taking as points all lines through $p$, and as lines all planar line-pencil with vertex $p$ and contained in a singular plane of $\Delta$. This gives a polar space of rank one less than $\Delta$. For a singular subspace $K$ of $\Delta$, we can define the residual $\Delta_K$ by successively taking point-residuals for a basis $\{p_1, \ldots, p_{k+1}\}$ of $K$. Then $\Delta_K$ is a polar space of rank $n - \dim(K) - 1$.

We say that $\Delta$ is hyperbolic if for each singular subspace of dimension $r - 2$ (in general, also called a submaximal singular subspace) there are exactly two maximal singular subspaces through it; if $\Delta$ is not hyperbolic this number always exceeds 2. For this reason, we call a hyperbolic polar space non-thick and a non-hyperbolic polar space thick. If $\Delta$ is hyperbolic, there are two natural types of maximal singular subspaces—also called generators in this context—by stating that two generators sharing a submaximal singular subspace have different types. This is well defined and implies that two generators $M$ and $M'$ intersecting each other in a subspace $S$ of odd codimension (i.e., $\dim(M) - \dim(S)$ is odd) have different types. The set of generators belonging to the same type form a natural system of generators. We will need the following fact concerning hyperbolic polar spaces.

**Lemma 3.2.** Let $\Delta$ be a hyperbolic polar space. Given two generators, we can find a submaximal singular subspace disjoint from both generators.

**Proof.** Let $U$ and $V$ be two generators. We proceed by induction. If $n = 2$, it is clear that we can find a point disjoint from the lines $U$ and $V$. For general $n$, consider non-collinear points $p_U$ and $p_V$ in $U$ and $V$, respectively. In $p_U^\perp \cap p_V^\perp$, $U$ and $V$ correspond to maximal singular subspaces, so by induction there is a singular subspace $Z$ in $p_U^\perp \cap p_V^\perp$ of dimension $n - 3$ disjoint from $U$ and $V$. As the residual of $Z$ is a hyperbolic quadric, in which $U$ and $V$ correspond to lines, it contains a point disjoint from them, yielding a submaximal singular subspace of $\Delta$ disjoint from both $U$ and $V$. □

We will also need the following well-known fact.

**Fact 3.3.** Let $\Delta = (X, \mathcal{L})$ be a polar space and let $p \in X$ be arbitrary. Then $p^\perp$ is a geometric hyperplane of $\Delta$ and is not properly contained in another geometric hyperplane.

We need some more notions before introducing parapolar spaces.

Let $\Gamma = (X, \mathcal{L})$ again be a point-line geometry. A subspace $S$ of $\Gamma$ is called convex if, for any pair of points $\{p, q\} \subseteq S$, every point incident with a line occurring in a shortest path between $p$ and $q$ is contained in $S$. Also, $\Gamma$ is called connected if its incidence graph is connected. If $p$ and $q$ are collinear we write $p \parallel q$ and the set of all points collinear to $p$ is denoted by $p^\parallel$, i.e., $p^\parallel = \{q \in X \mid p \parallel q\}$. If $p, q \in X$ are non-collinear, let $S_{p,q}$ denote the set $p^\perp \cap q^\perp$; if $p, q \in X$ are collinear, we use the notation $S_{p,q}$ for the points on $pq$.

**Fact 3.4.** Let $\Delta = (X, \mathcal{L})$ be a polar space and let $p, q \in X$ be non-collinear. Then the convex closure of $p$ and $q$ consists of all points in $S_{p,q} \cup \{S_{p',q'} \mid p', q' \in S_{p,q} \cup \{p, q\}\}$, and coincides with $\Delta$.

**Definition 3.5.** A point-line geometry $\Omega = (X, \mathcal{L})$ is called a parapolar space if the following axioms hold:

1. (PPS1) $\Omega$ is connected and, for each line $L$ and each point $p \notin L$, $p$ is collinear to either none, one or all of the points of $L$ and there exists a pair $(p, L) \in X \times \mathcal{L}$ with $p \notin L$ such that $p$ is collinear to no point of $L$. 

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We conclude that \( d \) by the first line of this proof). Suppose for a contradiction that \( d \) is a single point; therefore, if \( d \) is a single point.

Lemma 3.6. Let \( \Omega \) be a parapolar space. Given a symp \( \xi \) and a point \( p \notin \xi \), \( p^+ \cap \xi \) is a singular subspace of \( \xi \).

\begin{proof}
Suppose \( p_1 \) and \( p_2 \) are two non-collinear points in \( \xi \) collinear to \( p \). Then \( p \) belongs to the symp determined by \( p_1 \) and \( p_2 \), contradicting \( p \notin \xi \).
\end{proof}

Lemma 3.7. Let \( \Omega \) be a parapolar space. If all points of a line \( L \) contained in a symp \( \xi \) of rank at least 3 are collinear to a point \( p \), then \( p \) and \( L \) are contained in a symp and hence generate a projective singular plane. Consequently, if the symplectic rank is at least 3, each singular subspace is projective.

\begin{proof}
If \( p \in \xi \), we are done. If not, take a point \( q \in \xi \) collinear to all points of \( L \) and not contained in the subspace \( p^+ \cap \xi \). Then \( p \) and \( q \) are at distance 2 and \( L \subseteq p^+ \cap q^+ \), so there is a symp \( \xi' \) through \( p \) and \( q \), which clearly contains \( L \) and \( p \). Since \( \xi' \) is a polar space, it follows that the singular subspace generated by \( L \) and \( p \) is a projective plane.
\end{proof}

Lemma 3.8. Let \( \Omega \) be a parapolar space of uniform symplectic rank \( d \). Then every singular subspace of dimension at most \( d - 1 \) is contained in some symp.

\begin{proof}
By Axiom (PPS3) each line is contained in a symp and by connectivity each point is contained in a line. Hence if \( d = 2 \) we are done. So suppose \( d \geq 3 \). Then Lemma 3.7 confirms that the (projective) dimension is well-defined. So let \( W \) be a singular subspace of \( \Omega \) of dimension \( d^* \) with \( 2 \leq d^* \leq d - 1 \). Let \( d' \leq d^* \) be the maximum number such that there exists a symp \( \xi \) with \( \dim(\xi \cap W) = d' \) (well defined by the first line of this proof). Suppose for a contradiction that \( d' < d^* \). Then we can pick \( p \in W \setminus \xi \) and \( q \in \xi \setminus p^+ \) with \( q \) collinear to all points of \( W \cap \xi \). However, \( \xi(p,q) \) contradicts the maximality of \( d' \). We conclude that \( W \) is contained in some symp.
\end{proof}
For each point of our parapolar space, we introduce the following graph.

**Definition 3.9.** Let $\Omega = (X, \mathcal{L})$ be a parapolar space and $p$ one of its points. We call $\Omega$ *locally connected at* $p$ if each two lines through $p$ are contained in a finite sequence of singular planes consecutively intersecting in lines through $p$.

**Definition 3.10.** Let $\Omega = (X, \mathcal{L})$ be a parapolar space and let $p$ be one of its points. We define the *point-residual at* $p$, denoted $\Omega_p = (X_p, \mathcal{L}_p)$, as follows:

- $X_p$ is the set of lines through $p$;
- $\mathcal{L}_p$ is the set of planar line-pencils with vertex $p$ contained in singular planes through $p$ which are contained in a symplecton of $\Omega$.

Note that, if the symplectic rank is at least 3, then Lemma 3.7 implies that each pair of collinear lines through $p$ are contained in a singular plane through $p$ which is contained in a sym.

The following fact is based on Theorem 13.4.1 of [15].

**Fact 3.11.** Let $\Omega = (X, \mathcal{L})$ be a parapolar space, assumed to be strong if its minimum symplectic rank is 2, and let $p$ be any of its points. Let $C$ be a connected component of $\Omega_p$. We have the following possibilities.

- $C$ is a single element of $\mathcal{L}$ (which then corresponds to a line of $\Omega$ through $p$ only contained in symps of rank 2);
- $C$ corresponds to the lines through a point in a symp $\xi$ of $\Omega$ of rank at least 3 (which happens if no line of $\xi$ is contained in a second symp of $\Omega$ of rank at least 3);
- $C$ is a strong parapolar space. There is a bijective correspondence between the singular subspaces of $\Omega$ through $p$ and the singular subspaces of $\Omega_p$ and between the symps of $\Omega$ through $p$ of rank at least 3 and the symps of $\Omega_p$ (by taking the point-residual at $p$ of each of those symps).

In case $\Omega_p$ is a parapolar space, we can again consider its point-residual at some point $p'$ of $\Omega_p$, and $(\Omega_p)_{p'}$ is then denoted by $\Omega_{pp'}$. This inductively defines $\Omega_k$ for any singular $k$-space $K$ with $k \in \mathbb{N}$, provided that the subsequent point-residuals yield parapolar spaces.

We list some properties of the point-residual $\Omega_p$.

**Fact 3.12.** Let $\Omega = (X, \mathcal{L})$ be a parapolar space with symplectic rank at least 3. Then $\Omega$ is locally connected if and only if $\Omega_p$ is connected for all $p \in X$.

**Fact 3.13.** Let $\Omega = (X, \mathcal{L})$ be a parapolar space with symplectic rank at least $d$. Then $\Omega$ is strong and $d \geq 3$ if and only if $\text{Diam} \Omega_p = 2$ for all points $p$.

**Proof.** Suppose first that $\Omega$ is strong and $d \geq 3$. Let $L_1$ and $L_2$ be two lines through any point $p$ which are not contained in a singular plane. Then there is a symplecton $\xi$ through $L_1$ and $L_2$, yielding the existence of a line $L$ through $p$ contained in singular planes in $\xi$ with both $L_1$ and $L_2$, respectively, showing $\text{Diam} \Omega_p = 2$ for any point $p$.

Conversely, suppose $\text{Diam} \Omega_p = 2$ for all points $p$. Let $L_1$ and $L_2$ be two lines through any point $p$ at distance 2 in $\Omega_p$. Then there exists a line $L$ such that, for $i = 1, 2$, $\langle L, L_i \rangle$ is a singular plane contained in a symp $\xi_i$. If $p_i$ are points in $L_i \setminus p$, then $p_1$ and $p_2$ are not collinear (otherwise Lemma 3.7 implies that they are at distance 1) and hence, since $L \subseteq p_1^+ \cap p_2^+$, there is a symp of rank at least 3 in $\Omega$ containing $L_1$ and $L_2$. Since $p$ was arbitrary, $\Omega$ contains no special pairs indeed and there are no symps of rank 2. \qed
3.2 Important theorems

For easy reference, we list existing theorems on parapolar spaces that we will use later on.


Theorem 3.14. Suppose \( \Omega = (X, \mathcal{L}) \) is a strong parapolar space with these three properties:

- For every point-symplecton pair \((p, \xi)\), we have \( p^\perp \cap \xi \neq \emptyset \).
- For each point \( p \), the set of points at distance at most 2 from \( p \) is a subspace of \( \Omega \).
- If the symplectic rank is at least 3, every maximal singular subspace has finite dimension.

Then \( \Omega \) is one of the following:

(i) \( D_{6,6}(\mathbb{K}), A_{5,3}(\mathbb{L}) \) or \( E_{7,7}(\mathbb{K}) \), for \( \mathbb{K} \) any commutative field and \( \mathbb{L} \) any skew field.
(ii) A dual polar space of rank 3.
(iii) A Cartesian product geometry \( L \times \Delta \), where \( L \) is a thick line, and \( \Delta \) is a polar space of rank at least 2.

We now recall Theorem 15.4.5 of [15], which is an updated version of a result of Cohen & Cooperstein [4]. Consider the following property in a parapolar space \( \Omega \).

\((\text{CC})_{d-2}\) If, for any point \( p \) and any sym \( \xi \) with \( p \not\in \xi \), the intersection \( p^\perp \cap \xi \) has dimension at least \( d - 2 \), then it has dimension \( d - 1 \).

Theorem 3.15. Let \( \Omega \) be a locally connected parapolar space of uniform symplectic rank \( d \geq 3 \). If \( d \geq 4 \), it is also assumed that \( \Omega \) is a strong parapolar space of which all singular subspaces have finite dimension. Then, if \( \Omega \) satisfies property \((\text{CC})_{d-2}\), then \( \Omega \) is one of the following (where \( \mathbb{K} \) denotes a commutative field):

(i) If \( d = 3 \), then \( \Omega \) is either:
   (a) The Grassmannian of \( \ell \)-spaces distinct from the hyperplanes of a vector space \( V \) over \( \mathbb{L} \) of dimension \( m \) (possibly infinite), with \( \dim V \geq 4 \) and \( \ell \in \mathbb{N}_{\geq 2} \), with, if \( m \) is finite, \( \ell \leq \lceil \frac{m-1}{2} \rceil \), or
   (b) The quotient \( A_{2n-1,6}(\mathbb{L})/\langle \sigma \rangle \), where \( \sigma \) is a polarity of \( V \) of Witt index at most \( n-5 \), \( n \geq 5 \).
(ii) If \( d = 4 \), \( \Omega \) is a homomorphic (isomorphic if \( n \leq 9 \)) image of \( D_{n,6}(\mathbb{K}) \), \( n \geq 5 \).
(iii) If \( d = 5 \), then \( \Omega \) is the Lie incidence geometry \( E_{6,1}(\mathbb{K}) \).
(iv) If \( d = 6 \), then \( \Omega \) is the Lie incidence geometry \( E_{7,7}(\mathbb{K}) \).

The following is Lemma 3.2 in [16].

Lemma 3.16. Suppose \( \Omega = A_{2n-1,6}(\mathbb{L})/\langle \sigma \rangle \), where \( \sigma \) is a polarity of \( V \) of Witt index at most \( n-5 \) and \( \mathbb{L} \) a skew field. Then \( \text{Diam} \Omega \geq 5 \).

We also use some results about imbrex geometries, see [9].

Proposition 3.17. Let \( \Omega = (X, \mathcal{L}) \) be a strong parapolar space of diameter 2, symplectic rank 2 and set of maximal singular subspaces \( \Sigma \), such that every pair of symps which share a point \( p \) and intersect a common line at points not collinear to \( p \) share a line. Then either \( \Omega \) is isomorphic to a product space \( Y \times Z \), where \( Y \) and \( Z \) are arbitrary linear spaces with thick lines, or \( (X, \Sigma) \) is a thick generalized quadrangle and each element of \( \Sigma \) is an ideal subquadrangle of \( (X, \Sigma) \); moreover no maximal singular subspace is a projective space.

\( \text{---} \)

1 The book contains the misprint \( E_{7,1} \) instead of \( E_{7,7} \)
We also use the following Theorem from [14], which itself strengthened [10].

**Theorem 3.18.** Suppose $\Omega$ is a parapolar space of symplectic rank $d$ with $d \geq 3$ satisfying these axioms:

1. **(NP)** Given a point $p$ not incident with a symp $\xi$, the intersection $p^{\perp} \cap \xi$ is never just a point.
2. **(F)** If $d \geq 4$, every maximal singular subspace has finite singular rank.

Then $\Omega$ is one of the following$^2$ (where $\mathbb{K}$ is an arbitrary commutative field):

1. $E_{6,2}(\mathbb{K})$, $E_{7,1}(\mathbb{K})$ or $E_{8,8}(\mathbb{K})$;
2. a metasymplectic space;
3. $B_{n,2}(\ast)$ or $D_{n,2}(\mathbb{K})$, $n \geq 4$;
4. a strong parapolar space of diameter 2.

Finally we use Shult’s Haircut Theorem, see [16]. This uses the following property (called the “Haircut Axiom”) in a parapolar space of minimum symplectic rank $d \geq 3$ (if the rank is uniform then this coincides with property $(CC)_{d-2}$).

1. **(H)** For any point $p$ and any symp $\xi$ with $p \notin \xi$, the intersection $p^{\perp} \cap \xi$ is never a submaximal singular subspace of $\xi$.

The above is a residual property (cf. Lemma 2.4 of [16]):

**Lemma 3.19.** Suppose $\Omega$ is a parapolar space of symplectic rank at least 3. Then $\Omega$ satisfies property (H) if and only if $\Omega_p$ also has the property (H) for each point $p$.

Shult’s Haircut Theorem then goes as follows.

**Theorem 3.20.** Suppose $\Omega$ is a locally connected parapolar space of symplectic rank at least 3, satisfying the following:

- each singular space possesses a finite projective dimension; moreover, there exists an upper bound to the rank of a symplecton.
- the Haircut Axiom (H).

Then $\Omega$ has a uniform symplectic rank $d \geq 3$ and one of the following occurs ($\mathbb{K}$ is any commutative field, $\mathbb{L}$ is any skew field):

1. $d = 3$ and $\Omega$ is either the $d$-Grassmannian $A_{n,d}(\mathbb{L})$ or a homomorphic image $A_{2n-1,n}(\mathbb{L})/\langle \sigma \rangle$ where $\sigma$ is a polarity of $A_{2n-1}(\mathbb{L})$ of Witt index at most $n-5$, $n \geq 5$;
2. $d = 4$ and $\Omega$ is a $Y_1$ geometry or a twisted version thereof (these include $E_{6,2}(\mathbb{K})$, $D_{n,n}(\mathbb{K})$, $n \geq 5$);
3. $d = 5$ and $\Omega$ is a homomorphic image of a building geometry $E_{m+4,1}(\ast)$ with $m \geq 2$ (this includes $E_{6,1}(\mathbb{K})$, $E_{7,1}(\mathbb{K})$, $E_{8,1}(\mathbb{K})$);
4. $d = 6$ and $\Omega$ is $E_{7,7}(\mathbb{K})$;
5. $d = 7$ and $\Omega$ is $E_{8,8}(\mathbb{K})$.

Conversely, all of the listed geometries satisfy the hypotheses.

The object mentioned in (ii) and named “a $Y_1$ geometry or a twisted version thereof” will be of little importance to us (and hence we will not define or discuss them). The reason is that such parapolar space has hyperbolic symplecta of rank 4, and the point-residuals are parapolar spaces of type $A_{n,i}$, with $n \geq 4$ and $i \in \{2, \ldots, n-1\}$. Only the cases $A_{5,3}$ and $A_{n,2}$ will be of interest to us, but then the corresponding $Y_1$ geometry has type $E_{6,2}$ and $D_{n+1,n+1}$, respectively.

$^2[14]$ uses a different convention for the numbering of the nodes, and allows infinite symplectic rank in (iii)
4 Description of the geometries

4.1 Important examples of parapolar spaces—Lie incidence geometries

Most of the geometries in the conclusion of our Main Result—Extended Version are Lie incidence geometries, i.e., they arise from Tits-buildings or their quotients by selecting a node $i$ of the corresponding Coxeter diagram and considering the so-called $i$-Grassmannian of the (quotient of the) building. Since the construction of the examples is not essential in our arguments, we only present some general information, in particular, we do not define buildings rigorously, but content ourselves with mentioning that most of them arise from simple algebraic groups and close relatives like classical groups, (twisted) Chevalley groups, groups of mixed type as the geometry of the Borel subgroup and the parabolic subgroups. More exactly, we can think of the building as an $n$-partite graph where each partition class consists of the cosets of a parabolic subgroup (with respect to a fixed chosen Borel subgroup) and where adjacency is given by “intersecting non-trivially”. Then $n$ is called the rank of the building. The $n$-cliques are called chambers and each clique is contained in some chamber. When each clique of size $n-1$ is contained in at least three chambers, then the building is called thick. The induced bipartite graph obtained from an $(n-2)$-clique $C$ by considering all vertices $v$ such that $C \cup \{v\}$ is an $(n-1)$-clique is a building of rank 2. These graphs have diameter $\ell$ and girth $2\ell$, for some natural number $\ell \geq 2$. If $\ell = 2$, this is a complete bipartite graph, if $\ell = 3$, then this is the incidence graph of a projective plane, if $\ell = 4$, then we have the incidence graph of a generalized quadrangle (in the thick case a thick polar space of rank 2). We will not need $\ell \geq 5$. It so happens that $\ell$ only depends on the $n-2$ partition classes containing a vertex of the $(n-2)$-clique $C$. So we can build a diagram with set of nodes the partition classes and no edge, a single edge or a double edge between two nodes if $\ell = 2, 3, 4$, respectively, where $C$ is any $(n-2)$-clique not containing any member of the partition classes corresponding to the two nodes under consideration. This is the Coxeter diagram of the building. There is an enhanced notion of Dynkin diagram, but we will not need this. The nodes of a Coxeter diagram have a standard numbering, following Bourbaki [1]. We also adopt this numbering. For a certain node $\ell$, the $\ell$-Grassmannian geometry is then the point-line geometry with set of points the partition class corresponding with the node $\ell$, and where a typical line is the set of vertices of type $\ell$ contained in a chamber with a given clique of size $n-1$ (not containing a vertex of type $\ell$). Thickness then exactly implies that the lines are thick, i.e., have at least three points. There are certain choices for $\ell$ for which the corresponding $\ell$-Grassmannian geometry is called a long root geometry. Although this is not essential in this paper, we pass along this information below. Long root automatically means that the parapolar space has diameter 3 and is non-strong, hence cannot appear as a point-residual in another parapolar space (and that is relevant for the present paper).

We now review some examples and relate parapolar spaces to these.

(A) Grassmannians of vector spaces—Projective spaces. Let $V$ be a vector space of dimension $n+1$ over some skew field $L$. The dimension need not be finite. Let $\ell$ be a natural number at most $(n+1)/2$. Let $V_\ell$ be the set of all $\ell$-dimensional subspaces of $V$, and let $\mathcal{L}_\ell$ be the family of $\ell$-pencils, i.e., the family of subsets $L(W,U)$ of $V_\ell$ consisting of all $\ell$-spaces containing a given $(\ell-1)$-space $W$ and being contained in a given $(\ell+1)$-space $U$, with $W \subseteq U$. Then the point-line geometry $(V_\ell, \mathcal{L}_\ell)$ is called the $\ell$-Grassmann geometry and it is denoted by $A_{n,\ell}(L)$. If $\ell = 1$, we obtain a projective space of dimension $n$. For $\ell > 1$, we can view $A_{n,\ell}$ also as the Grassmannian of $(\ell-1)$-dimensional subspaces of a projective space over $L$ of dimension $n$, defined similarly. If $n \geq 5$, it is a strong parapolar space of uniform symplectic rank 3; the symps are of hyperbolic type, namely, isomorphic to $A_{3,2}(L)$. If $n$ is finite, then the singular rank is $\{\ell, n+1-\ell\}$; otherwise there are infinite-dimensional singular subspaces. The condition $\ell \leq (n+1)/2$ is only there to avoid isomorphism between Grassmannians; the definition makes perfect sense also for $\ell > (n+1)/2$, with $\ell \leq n$. But then $A_{n,\ell}(L)$ is isomorphic to $A_{n,n+1-\ell}(L)$. If $n = 2$, then $A_{2,1}(L)$ is just an
Remark 4.1. The above examples are mostly related to algebraic groups of relative types $D$ related to any (interesting) algebraic objects. However, as soon as the diagram branches, like in the algebraic group. The rank 2 examples, finally, could have trivial automorphism groups and are not quadratic form in an infinite dimensional vector space, then we have a classical group behaving like. However, when there is a skew field around which is infinite-dimensional over its centre, or a pseudo-field), then we write ordinary projective plane over $L$. If we allow any projective plane (not necessarily over a skew field), then we write $A_{2,1}(\ast)$. Likewise, for $n = 1$ the symbol $A_{1,1}(\ast)$ will refer to any set of cardinality at least 3, which we call a thick line. The corresponding long root geometries are \{1, n\}-Grassmannians, which we shall not introduce.

(B) **Thick polar spaces and polar Grassmannians.** Let $\Delta = (X, L)$ be a polar space of rank $n$, not of hyperbolic type. Recall that polar spaces by definition have finite singular rank $n - 1$. If $2 \leq \ell \leq n - 1$, then we can consider the following $(\ell - 1)$-Grassmannian of $\Delta$. The point set is the set $\Delta_{\ell-1}$ of all $(\ell - 1)$-dimensional singular subspaces of $\Delta$: a typical line is a pencil, i.e., the set of $(\ell - 1)$-dimensional singular subspaces containing a given $(\ell - 2)$-dimensional subspace $W$ and being contained in a given $\ell$-dimensional singular subspace $U$, with $W \subseteq U$. This geometry is called a polar $\ell$-Grassmannian. Since a polar space is usually not uniquely defined by the underlying skew field of the singular projective spaces, we refer to a polar $\ell$-Grassmannian of a rank $n$ polar space as $B_{n,\ell}(\ast)$. Likewise, $B_{n,1}(\ast)$ refers to the ordinary polar space. Now, $B_{n,n}(\ast)$ is a dual polar space, i.e., the points are the maximal singular subspaces of $\Delta$ and the lines are the sets of maximal singular subspaces containing a fixed given submaximal singular subspace. Note that for $n = 2$, the geometries $B_{2,1}(\ast)$ are the same as $B_{2,2}(\ast)$ (by considering the dual generalized quadrangles). The Lie incidence geometries $B_{n,\ell}(\ast)$, with $2 \leq \ell \leq n$, are parapolar spaces with symps of ranks 3 and $n + 1 - \ell$ and maximal singular subspaces of dimensions $\ell$ and $n - \ell$, if $\ell \leq n - 2$, and symps of rank 2 if $\ell \in \{n - 1, n\}$ (and maximal singular subspaces of dimension $n - 1$ if $\ell = n - 1$ and 1 if $\ell = n$).

The case $\ell = 2$ corresponds to the long root geometries.

**Remark 4.1.** The above examples are mostly related to algebraic groups of relative types $A_n$ and $BC_n$. However, when there is a skew field around which is infinite-dimensional over its centre, or a pseudo-quadratic form in an infinite dimensional vector space, then we have a classical group behaving like an algebraic group. The rank 2 examples, finally, could have trivial automorphism groups and are not related to any (interesting) algebraic objects. However, as soon as the diagram branches, like in the D and E cases which follow, then everything is determined by a unique field and an algebraic group.

(D) **Non-thick polar spaces and their Grassmannians.** Let $\Delta = (X, \mathcal{L})$ be a hyperbolic polar space of rank $n$. Then, similarly as in Case B above, we can define the polar Grassmannian $D_{n,\ell}(K)$, 1 $\leq \ell \leq n$, where $K$ is any field. The symmetry of the diagram implies that $D_{n,n-1}(K)$ is isomorphic to $D_{n,n}(K)$, and we will always use the latter notation. This Lie incidence geometry is often called a half spin geometry. If $n \neq 4$, it is a strong parapolar space with uniform symplectic rank 4, where the symps are hyperbolic polar spaces. The maximal singular subspaces have dimension $n - 1$. If $2 \leq \ell \leq n - 3$, then $D_{n,\ell}(K)$ is a parapolar space with symps of ranks 3 and $n + 1 - \ell$, and maximal singular subspaces of dimensions $\ell$ and $n - \ell$. If $\ell = n - 2$, then we have uniform symplectic rank 3 and singular rank $n - 2$ (with also maximal singular subspaces of dimension 2); if $\ell = n$, then we have uniform symplectic rank 4 and singular rank $n - 1$. Finally, for $\ell = 1$, we have an ordinary hyperbolic polar space of course.

Here again, the case $\ell = 2$ corresponds to the long root geometries.

(E) **Exceptional parapolar spaces of type $E_i$, $i = 6, 7, 8$.** Let $\Delta$ be a spherical building of type $E_i$, $i \in \{6, 7, 8\}$, defined over the field $K$. Then there are certain choices for $\ell$ such that the $\ell$-Grassmannian of $\Delta$, denoted $E_{i,\ell}(K)$, has small diameter and constant symplectic rank. We list these, together with their diameter, symplectic rank $d$ and set $S$ of dimensions of the maximal singular subspaces. We include also the next case (metasymplectic spaces) in the table.
Diam

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>Diam$\Omega$</th>
<th>$d$</th>
<th>$S$</th>
<th>strong</th>
<th>long root</th>
</tr>
</thead>
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<tr>
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<td>5</td>
<td>${4, 5}$</td>
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</tr>
<tr>
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<td>4</td>
<td>${4}$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$E_{7,7}(K)$</td>
<td>3</td>
<td>6</td>
<td>${5, 6}$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$E_{7,1}(K)$</td>
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<td>5</td>
<td>${4, 6}$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$E_{8,8}(K)$</td>
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<td>7</td>
<td>${6, 7}$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$E_{8,1}(K)$</td>
<td>5</td>
<td>5</td>
<td>${4, 7}$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$F_{4,1}(\ast)$</td>
<td>3</td>
<td>3</td>
<td>${2}$</td>
<td>✓</td>
<td></td>
</tr>
</tbody>
</table>

(F) **Metasymplectic spaces.** Traditionally, the natural geometries corresponding to buildings of type $F_4$ are called *metasymplectic spaces*, after Freudenthal [8]. Since a building of type $F_4$ is not determined by a field or skew field, but by a certain pair of division rings, which is of no importance to us, we will denote metasymplectic spaces by $F_{4,1}(\ast)$. This class coincides with the family of geometries $F_{4,4}(\ast)$ by the symmetry of the diagram (but here, for a given building $\Delta$ of type $F_4$, these two geometries need not be isomorphic). Metasymplectic spaces are long root geometries with uniform symplectic rank 3 and singular rank 2. Their diameter is 3. We will not need geometries $F_{4,2}(\ast)$.

(Y) **Parapolar spaces from non-spherical buildings.** Let $\Delta$ be a building corresponding to the diagram $E_n$ obtained from $E_8$ by extending it at the long arm with a simple path of length $n - 8$, $n \geq 9$ (the case $n = 9$ is exactly $E_8$). Extend the Bourbaki numbering in the obvious way. Then $E_{n,1}(\ast)$ (again, these buildings need not be unique for a given field, hence the notation with a star) is a parapolar space of uniform symplectic rank 3 and the dimensions of the maximal singular subspaces are 4 and $n - 1$. It is never strong (for $n \geq 9$) and the diameter is unbounded.

Sometimes, the diagram $E_n$ is called $Y_{1,2,n-3}$, where the letter $Y$ indicates the shape of the diagram, and the subscripts are the lengths of the various arms of the $Y$.

If the diameter of the parapolar space in one of the examples above is at least 5, then there exist homomorphic images that are again parapolar spaces. Since in the last section, we construct in a universal way all relevant parapolar spaces which are not locally connected, we may assume that the homomorphic images are locally connected. That such things exist is shown for example by the case $A_{2n-1,n}(\mathbb{L})$, $n \geq 5$, where an identification of $n$-subspaces that correspond under a given polarity of Witt index at most $n - 5$, produces a homomorphic image that is again a locally connected parapolar space with the same polar rank and singular rank, and the same diameter. We write a superscript $h$ to indicate a homomorphic image (but it could also be an isomorphic image). In the above examples that are relevant for us, homomorphic images are possible in the geometries $A_{2n-1,n}(\mathbb{L})$, $n \geq 5$, $D_{n,n}(K)$, $n \geq 9$, and $E_{n,1}(\ast)$, $n \geq 8$.

4.2 **Some further examples— Cartesian product spaces**

Let $\Omega_i = (X_i, \mathcal{L}_i)$, $i = 1, 2$, be two parapolar spaces. Define the Cartesian product space $\Omega := \Omega_1 \times \Omega_2$ as the point-line geometry with point set the Cartesian product $X_1 \times X_2$ and set of lines

$$\{(p_1) \times L_2 : p_1 \in X_1, L_2 \in \mathcal{L}_2\} \cup \{L_1 \times \{p_2\} : L_1 \in \mathcal{L}_1, p_2 \in X_2\}.$$

This is again a parapolar space with symps $\ast \{p_1\} \times \xi_2, p_1 \in X_1, \xi_2 \in \mathcal{E}_2$.  

16
* $\xi_1 \times \{p_2\}$, $\xi_1 \in \Xi_1$, $p_2 \in X_2$, and
* $L_1 \times L_2$, $L_1 \in \mathcal{L}_1$, $L_2 \in \mathcal{L}_2$.

If both $\Omega_1$ and $\Omega_2$ are strong, then also $\Omega$ is strong. Its diameter equals $\text{Diam } \Omega_1 + \text{Diam } \Omega_2$, and it has always minimum symplectic rank 2. In fact, the above definition of product space makes sense for any pair of point-line geometries, and as soon as both components are members of the class of thick lines, linear spaces with only thick lines, polar spaces and parapolar spaces, and not both are thick lines, the product space is a parapolar space. If one uses linear spaces, then it is clear that the maximal singular subspaces of the parapolar space are not necessarily projective spaces.

Let us also mention that the product space of two projective spaces of respective dimensions $n, m$ but over the same field, admits a representation in projective space that is often called a Segre variety of type $(n, m)$. Since we are dealing with geometries rather than with their representations, we shall call the Cartesian product of a pair of projective spaces of respective dimensions $n, m$ a Segre geometry of type $(n, m)$. Since in our approach the algebraic properties of the projective spaces will not matter, we include abstract projective planes and projective lines of any cardinality at least 3.

As mentioned before, a Lie (incidence) geometry is a $J$-Grassmannian of a building of some type $X_n$. We then call $X_{n,J}$ the Coxeter type of the parapolar space.

### 4.3 The Main Result—Extended Version in tabular form

We now tabulate all $k$-lacunary locally connected parapolar spaces $\Omega$, together with the following parameters: the rank and thickness of the symplecta, structure and dimensions of the maximal singular subspaces, diameter and strongness of $\Omega$. As the tables below will show, these parameters suffice to distinguish all $k$-lacunary parapolar spaces with projective singular subspaces, for a given $k$. This is crucial to our inductive approach, as we need to know that for any two points $x, x' \in \Omega$, the point-residuals $\Omega_x$ and $\Omega_{x'}$ are isomorphic. We start by dividing them in two classes: symplectic rank $d \geq k + 3$ and $d = k + 2$.

#### 4.3.1 The $k$-lacunary parapolar spaces with (minimum) symplectic rank $d \geq k + 3$

Table 1 contains the locally connected $k$-lacunary parapolar spaces $\Omega$ with minimum symplectic rank $d \geq k + 3$, strong if $d = 2$. It turns out that in fact their rank is uniform. Moreover, these are such that, for any $k$-dimensional singular subspace $K$ of $\Omega$, the residual $\Omega_K$ is a $(-1)$-lacunary parapolar space. The table is such that, for any cell in any row, going one cell to the left in that row yields its point-residual. We denote the set of dimensions of the maximal singular subspaces of $\Omega$ by $S$. White cells contain strong parapolar spaces of diameter 2, grey cells contain the ones of diameter 3. If the parapolar space appears in white, it means that it is non-strong. Since taking a point-residual yields a strong parapolar space, only the rightmost parapolar spaces can be non-strong. For each of the parapolar spaces in Table 1, the symplecta are non-thick.

#### 4.3.2 The $k$-lacunary parapolar spaces with minimum symplectic rank $k + 2$

Table 2 contains the $k$-lacunary parapolar spaces with minimum symplectic rank $d = k + 2$ (note that $d < k + 2$ is excluded by the definition of lacunarity). These are such that, for any $(k - 1)$-dimensional singular subspace $K$ of $\Omega$, the residual $\Omega_K$ is a 0-lacunary parapolar space of (minimum) symplectic rank 2. Opposed to the previous table, this time the symplecta can be thick. Only in the second last row of Table 2, both thick and non-thick symplecta occur at the same time, and also only in this row and the row before, the rank is not uniform. For both reasons, we call the symplecta “mixed” in the second last row. We need some abbreviations.
Table 1: The $k$-lacunary parapolar spaces with symplectic rank $d \geq k + 3$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$S$</th>
<th>$k = -1$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k + 3$</td>
<td>${k + 2, k + 3}$</td>
<td>$A_{1,1}(\ast) \times A_{2,1}(\ast)$</td>
<td>$A_{4,2}(L)$</td>
<td>$D_{5,5}(K)$</td>
<td>$E_{6,1}(K)$</td>
<td>$E_{7,7}(K)$</td>
<td>$E_{8,8}(K)$</td>
</tr>
<tr>
<td></td>
<td>${k + 3}$</td>
<td>$A_{2,1}(\ast) \times A_{2,1}(\ast)$</td>
<td>$A_{5,3}(L)$</td>
<td>$E_{6,2}(K)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k + 4$</td>
<td>${k + 3, k + 4}$</td>
<td>$A_{4,2}(L)$</td>
<td>$D_{5,5}(K)$</td>
<td>$E_{6,1}(K)$</td>
<td>$E_{7,7}(K)$</td>
<td>$E_{8,8}(K)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>${k + 3, k + 5}$</td>
<td>$A_{5,2}(L)$</td>
<td>$D_{6,6}(K)$</td>
<td>$E_{7,1}(K)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k + 6$</td>
<td>${k + 5, k + 6}$</td>
<td>$E_{6,1}(K)$</td>
<td>$E_{7,7}(K)$</td>
<td></td>
<td></td>
<td></td>
<td>$E_{8,8}(K)$</td>
</tr>
</tbody>
</table>

(GQ) This refers to the case where $\Omega$ is an imbrex geometry and $(X, \Sigma)$ is a generalized quadrangle (hence GQ) and each member of $\Xi$ is an ideal subquadrangle of $(X, \Sigma)$.

(LS) Stands for a linear space with only thick lines and which contains at least two lines.

($h$) Recall that, if $Z$ is some locally connected parapolar space, then $Z^h$ denotes a locally connected homomorphic image of it, which is still a parapolar space (fibers only contain points at distance at least 5 from each other).

(A1) Shorthand for $A_{1,1}(\ast)$.

Note also that the various $A_1$ (thick lines) appearing in the same cell need not be isomorphic, i.e., they can have different sizes.

Again, $S$ denotes the set of dimensions of the singular subspaces of $\Omega$. If a class of parapolar spaces is such that its maximal singular subspaces can never be projective, then we say that its singular rank is empty (“∅”). If a class of parapolar spaces is such that its maximal singular subspaces are in some, but not all, cases projective, we write “−”, because in case the maximal singular subspaces are not projective, their dimension is undefined (by definition, this notation implies that, in case the maximal singular subspace happens to be projective, its dimension is at least 2). The number $n$ denotes the rank of the corresponding building. The shades of grey reflect the diameter: the darker the cell, the higher the diameter; the white color indicates that $\Omega$ is non-strong. Also here, reading from right to left corresponds to taking point-residuals; so non-strong parapolar spaces can only occur at the rightmost position in a row.

It is not a coincidence that there are never three grey cells in one row, and that a cell written with white letters is always preceded by a grey cell. This follows from the property that $\Omega$ is strong if and only if its point-residual has diameter 2.

4.4 Counterexamples

In this subsection we motivate our requirement that $k$-lacunary parapolar spaces only contain syms of rank at least $k + 1$. It is obvious that at least two syms have rank $k + 1$, since otherwise the $k$-lacunarity is no restriction at all. Let us call a parapolar space weakly $k$-lacunary if no pair of syms intersects in a $k$-space, there are at least two syms with rank at least $k + 1$, but syms of rank at most $k$ are
allowed. Fix \( k \geq 2 \). Then the Cartesian product of an arbitrary number of \( k \)-lacunary parapolar spaces, parapolar spaces and linear spaces, is a weakly \( k \)-lacunary parapolar space. If the direct factors are strong, then the whole parapolar space is strong and admittedly, it appears as the point-residual of a weakly \( (k + 1) \)-lacunary parapolar space, which also could be strong, etc. For example, \( E_{6,1}(\mathbb{K}) \) is \( 2 \)-lacunary, and so the direct product with a thick line is weakly \( 2 \)-lacunary; hence \( E_{8,7}(\mathbb{K}) \), or any homomorphic image, is weakly \( 3 \)-lacunary. The same argument with \( D_{5,5}(\mathbb{K}) \) leads to \( E_{7,6}(\mathbb{K}) \). In fact, by taking the direct product with a projective space over \( \mathbb{K} \), the geometries \( E_{n,6}(\mathbb{K}) \) and \( E_{n,7}(\mathbb{K}) \), \( n \geq 8 \), can be seen as weakly \( 3 \)-lacunary parapolar spaces. And there is more: We can consider a multiple direct product of \( E_{6,1}(\mathbb{K}) \) and/or \( D_{5,5}(\mathbb{K}) \) with several thick lines and projective spaces, and then the number of weakly \( 3 \)-lacunary parapolar spaces arising from non-spherical, non-Euclidean and non-hyperbolic Tits-buildings with a connected Coxeter diagram becomes insurmountable. This motivates our present definition of \( k \)-lacunarity.

### 5 Structure of the proof

Basically, the proof of our Main Result—Extended Version is by induction on the lacunary index \( k \). The inclusion of not necessarily locally connected parapolar spaces, however, forces us to treat the cases \( k \in \{-1, 0\} \) a little differently than the others, concerning assumptions. Also, the cases \( k \in \{-1, 0, 1\} \) are the only cases in which symps can have rank 2. This also distinguishes these cases from the rest. What we do is the following.

For \( k = -1 \), we first prove in Section 6 that, if there is a symp of rank 2 (and hence by assumption the parapolar space is strong), then all symps have rank 2 (Subsection 6.1). Then, in Subsection 6.2, we classify strong parapolar spaces of uniform symplectic rank 2 with lacunary index \(-1\). From then on, we may assume the symplectic rank is at least 3. In order to be able to also treat the locally disconnected case, we need to explicitly consider a slightly more general class of locally connected parapolar spaces with symplectic rank at least 3 and lacunary index \(-1\), namely, parapolar spaces with symplectic rank at least 3, lacunary index \(-1\), and containing at least one line which is contained in at least two different
symps. The last condition enables us to show that such parapolar spaces are in fact locally connected, and later on, in Section 10, we can use this to classify the general case \( k = -1 \). So, in Subsection 6.3, we classify parapolar spaces with symplectic rank at least 3, lacunary index \(-1\), and containing at least one line which is contained in at least two different symps. For efficiency reasons we use Theorem 3.20, but an independent more elementary proof could be given, as we already deduce the precise isomorphism class of the symps and the uniform symplectic rank.

The case \( k = 0 \), handled in Section 7, is also a special case, since we can show that in this case the parapolar space is automatically locally connected. Also the diameter is bounded from the beginning, namely, these spaces have diameter 2 or 3. Moreover, such parapolar spaces are always strong. All these properties are proved in Subsection 7.1. This allows for a case distinction, reducing three cases to well-known theorems. If the diameter is 2, then we show in Paragraph 7.2.1 that minimum symplectic rank 2 implies uniform symplectic rank 2 and we then can use Proposition 3.17. If the diameter is 2 and the symplectic rank is at least 3, then we use Theorem 3.15 (see Paragraph 7.2.2), whereas if the diameter is 3, then we use Theorem 3.14 regardless of the symplectic rank (see Subsection 7.3).

The case \( k = 1 \), handled in Section 8, is the last case where symps of rank 2 are involved. However, it is very easy to prove that the symplectic rank is at least 3 (this is due to the fact that we assume strongness if the minimum symplectic rank is 2; but we can also handle that case if we do not assume strongness, see Remark 10.9). Then we show that all point-residuals have the same Coxeter type. This allows to distinguish the case of diameters 2 and 3 in the point-residuals. If the point-residuals have diameter 3, then we use Theorem 3.18 to conclude, and if the point-residuals have diameter 2, then we invoke Theorem 3.15.

The remaining locally connected cases are handled in Section 9. These can be treated together using an induction argument and Shult’s Haircut Theorem 3.20.

So far for the locally connected case. Note that, each time we prove an item of the Main Result—Extended Version, we put it in a framed box, so that the reader can easily verify that all cases of the statement are covered. This should improve the readability of the paper.

In Section 10 we treat the locally disconnected case. We show a general theorem how locally disconnected parapolar spaces are built from locally connected ones, and then apply this to lacunary parapolar spaces with symplectic rank a least 3 to arrive at a universal construction for locally disconnected parapolar spaces, using as building blocks the locally connected ones classified in the previous sections. Here, the case \( k = 0 \) does not show up since 0-lacunary parapolar spaces are automatically locally connected.

The case \( k = -1 \) is special, since the property of \((-1\)-lacunarity of the building blocks is not preserved under the construction of the global, locally disconnected, parapolar space, unlike the cases \( k \geq 1 \).

### 6 Parapolar spaces with lacunary index \(-1\)

Let \( \Omega = (X, \mathcal{L}) \) be a parapolar space of minimum symplectic rank \( d \) with lacunary index \(-1\). We distinguish between the cases \( d = 2 \) and \( d \geq 3 \). In the former case, we assume—as usual when the minimum symplectic rank is 2—that \( \Omega \) is strong; in the latter case we assume that at least one line of \( \Omega \) is contained in at least two symps. Such a line will be called **sympthick**. The existence of a sympthick line will allow us to show that \( \Omega \) is in fact locally connected.

The case \( d = 2 \) is also divided into two parts: we first show that \( \Omega \) has uniform symplectic rank 2.

#### 6.1 Strong parapolar spaces with minimum symplectic rank 2 and with lacunary index \(-1\)

In this subsection \((X, \mathcal{L})\) is a strong parapolar space of minimum symplectic rank 2. As usual we denote the set of symps with \( \Xi \). Our main hypothesis, lacunary index \(-1\), means that every pair of symps meets
non-trivially. The aim of this subsection is to show that all symps have rank 2.

We begin with the only two lemmas that will also be useful for the case of uniform symplectic rank 2. For convenience, we will call a symp of rank 2 a quad (from “quadrangle”).

**Lemma 6.1.** Let \((X,\mathcal{L})\) be a strong parapolar space of minimum symplectic rank 2 with lacunary index \(-1\). Let \(\xi\) be an arbitrary quad. Let \(L_1, L_2\) be disjoint lines of \(\xi\). Then at least one of \(L_1, L_2\) is properly contained in a singular subspace, or some line of \(\xi\) intersecting both \(L_1, L_2\) is properly contained in a singular subspace.

*Proof.* Let \(\xi\) be a quad and let \(L_1, L_2\) be two nonintersecting lines in \(\xi\). We claim that there exist lines \(M_1, M_2\) not contained in \(\xi\) and meeting \(L_1, L_2\) in points \(q_1, q_2\), respectively. Indeed, let \(i \in \{1, 2\}\). Since, by Axiom (PPS1), \(X\) does not consist of only the points of \(\xi\), there is a point \(p \in X \setminus \xi\). Connectivity of \((X,\mathcal{L})\) yields a shortest path \((p, p_1, \ldots, p_n, q_i)\) from \(p\) to \(L_i\) (so \(q_i \in L_i\)). Now if \(p_nq_i\) does not belong to \(\xi\), then we can put \(M_i = p_nq_i\). If \(p_n \in \xi\), then \(p_{n-1} \notin \xi\) (as otherwise we could shorten the path) and so, by strongness, \(p_{n-1}\) and \(q_i\) determine a symp \(\xi\) and then there is a line \(M_i\) in \(\xi\) through \(q_i\) not contained in \(\xi\). The claim is proved.

Again, let \(i \in \{1, 2\}\). We may assume that \(L_i\) is not properly contained in a singular subspace. Consequently, since \((X,\mathcal{L})\) is strong, \(L_i, M_i\) are contained in a unique symp \(\xi\) and the singular subspace \(\xi \cap \xi\) equals \(L_i\). Hence \(\xi_1 \cap \xi_2\), nonempty by assumption, is not contained in \(\xi\). For any point \(q \in \xi_1 \cap \xi_2\), \(q\) is collinear to a point \(r_1 \in L_1\) and to a point \(r_2 \in L_2\). Necessarily, \(r_1 \perp r_2\) since \(q \notin \xi\). So \(r_1, r_2, q\) are contained in a singular subspace properly containing the line \(r_1r_2\).

*Lemma 6.2.* Let \((X,\mathcal{L})\) be a strong parapolar space of minimum symplectic rank 2 with lacunary index \(-1\). Then every singular subspace \(S\) is projective. Moreover, if \(S\) has dimension at least 3, then for every pair of nonintersecting lines of \(S\) at least one is not contained in a quad.

*Proof.* Let \(S\) be a singular subspace properly containing a line. If \(S\) does not contain two nonintersecting lines then \(S\) is a projective plane. So we may assume that two lines \(L_1, L_2\) in \(S\) are disjoint. Suppose for a contradiction that both are contained in a quad; say \(L_i \subseteq \xi_i \in \mathcal{L}, i = 1, 2\). Then \(\xi_1 \cap S = L_1\) and \(\xi_1 \cap \xi_2\) contains a point \(q \notin S\). Now \(q\) is collinear to unique points \(p_1, p_2\) on \(L_1, L_2\), respectively. Let \(r \in L_1 \setminus \{p_1\}\). Then \(\xi_i\) is determined by \(r\) and \(q\), but since \(p_2 \in \{r, q\}^\perp\), we see that \(p_2 \notin \xi_1\), a contradiction.

Now we show Veblen’s axiom. Suppose \(L_1\) and \(L_2\) both intersect two intersecting lines \(K_1, K_2\) in two distinct points, and let \(p\) be the intersection of \(K_1\) and \(K_2\). Assume for a contradiction that \(L_1\) and \(L_2\) are disjoint. Then the previous paragraph implies that some symp \(\xi\) of rank at least 3 contains, say, \(L_1\). Since \(p\) is collinear to all points of \(L_1\), Lemma 3.7 implies that \(p\) and \(L_1\) are contained in a projective plane, which then also contains \(K_1, K_2\) and hence \(L_2\). Consequently \(L_1\) and \(L_2\) intersect after all. Hence, by [23], \(S\) is projective.

The lemma is completely proved.

*Lemma 6.3.* Let \((X,\mathcal{L})\) be a strong parapolar space of minimum symplectic rank 2 with lacunary index \(-1\). Let \(\xi\) be a quad and let \(L \subseteq \xi\) be a line contained in a singular plane \(\pi\). Let \(\zeta\) be any symp such that \(\xi \cap L = \emptyset\). Then \(\zeta\) has rank 2.

*Proof.* We divide the proof into two parts, based on whether or not there is a point in \(\zeta\) collinear to a point \(x \in \pi \setminus L\). Before heading off, note that \(\xi \cap \pi\) is empty. Indeed, suppose \(\xi \cap \pi\) is a point \(p\) (off \(L, \pi\), by assumption). By \((-1)\)-lacunarity, \(\xi \cap \pi\) contains a points \(p'\) (also off \(L\)). Then \(p'\) is not collinear to \(p\), as otherwise \(p \in \xi(p, p') = \xi\) for some point \(r \in L\) not collinear to \(p'\), a contradiction. However, if \(p\) and \(p'\) are not collinear, \(\zeta = \xi(p, p')\) contains a point on \(L\) after all, violating our assumption.
**Case I:** There is a point \( q \) of \( \zeta \) collinear to some point \( x \) of \( \pi \setminus L \).

**Claim.** Each point of \( \zeta \) is collinear to at least one point of \( \pi \).

Denote by \( Z \) be the subset of points of \( \zeta \) collinear to at least one point of \( \pi \). We (subsequently) show that \( Z \) is a subspace containing \( q \sqcap \zeta \) and at least one point of \( \zeta \) not belonging to \( q \sqcap \zeta \), as then Fact 3.3 implies that \( Z = \zeta \), proving the claim.

- **\( Z \) is a subspace of \( \zeta \):**
  Let \( q_1, q_2 \) be collinear points of \( Z \). Then either they are collinear to a common point of \( \pi \), in which case every point of \( q_1 q_2 \) is collinear to that point, or else they are collinear to distinct points \( x_1, x_2 \), respectively, with \( \delta(q_1, x_2) = \delta(q_2, x_1) = 2 \). But then, in the symp \( \xi(q_1, x_2) = \xi(q_2, x_1) \), every point of \( q_1 q_2 \) is collinear to a unique point of the line \( x_1 x_2 \subseteq \pi \).

- **\( Z \) contains \( q \sqcap \zeta \):**
  Let \( r \in \zeta \) be a point collinear to \( q \). We show that \( r \subseteq Z \). If \( r \perp x \), then there is nothing to prove, so suppose \( x \notin r \perp \). Then the symp \( \xi(r, x) \) intersects \( \zeta \) in at least some point \( p^* \). If \( p^* \notin L \), its distance to \( x \) is at least 2 (like above this follows from \( x \notin \xi \)) and hence by convexity, \( L \cap \xi(r, x) \) contains a point. Either way, \( \xi(r, x) \cap \pi \) contains a line, at least one of which points is collinear to \( r \).

- **At least one point \( r \) of \( \zeta \) not belonging to \( q \) belongs to \( Z \):**
  If some point \( p \) of \( \zeta \) is not collinear to \( q \), then we can take \( r = p \). Hence suppose \( p \perp q \) for all points \( p \in \zeta \cap q \). It suffices to find a point \( r \perp q, q \notin r \subseteq \zeta \), collinear to a point of \( \pi \setminus L \) (because interchanging the roles of \( q \) and \( r \) will then imply \( r \perp \subseteq Z \)). Assume for a contradiction that every point of \( \zeta \cap q \) is collinear to some point of \( L \). Then also \( q \) is; say \( p^* \subseteq q \). By assumption, \( p^* \notin \zeta \). If some point \( p \) of \( \zeta \cap q \) is not collinear to \( p^* \), then \( \xi = \xi(p, p^*) \) contains \( q \) (recall \( p \perp q \)) a contradiction. This arguments shows that \( \xi \cap \zeta \) is just a point, say \( p \), which is collinear to \( p^* \). It also shows \( q \perp \subseteq L \) is exactly \( p^* \). Consider \( r \subseteq \zeta \) with \( r \subseteq q \setminus q^\perp \). Then, since \( \zeta = \xi(p, r) \) does not contain \( p^* \), \( r \subseteq \pi \cap q^\perp \subseteq L \). Whence \( \xi(r, x) \) contains \( p^* \) and \( q \), and hence also \( p^* \perp q^\perp \subseteq q \perp \pi \subseteq \xi(r, x) \). But then, inside \( \xi(r, x) \), \( r \subseteq \pi \) is collinear to the points of a line \( M \subseteq L \) of \( \pi \) as \( p^* \subseteq r^\perp \). This shows that \( r \subseteq \pi \) is collinear to some point of \( \pi \setminus L \).

As mentioned above, this shows the claim. We now show that \( \zeta \) is a quad indeed.

Suppose for a contradiction that \( \zeta \) has rank at least 3. Let \( p \) be a point of \( \xi \cap q \) and let \( p' \) be the unique point on \( L \) collinear to \( p \). Then consider a plane \( \alpha \subseteq \zeta \), \( \xi \cap q \) and \( p^\perp \) in exactly the point \( p \). If a point \( z \in \alpha \subseteq \zeta \) were collinear with a point \( p^* \subseteq L \), then our choice of \( \alpha \) implies \( p' \neq p^* \), but then \( z \subseteq p^\perp \perp p^* \subseteq \xi(p, p^*) = \xi \), a contradiction. The above claim implies that each point of \( \alpha \setminus \{ p \} \) is collinear to a unique point of \( \pi \setminus L \). A standard argument now shows that the perp correspondence restricted from \( \alpha \) to \( \pi \) preserves collinearity and hence is an isomorphism of planes. Consequently some points of \( \alpha \) different from \( p \) are collinear to points of \( L \) after all, a contradiction. This proves the lemma in Case 1.

**Case II:** No point of \( \zeta \) is collinear to a point of \( \pi \setminus L \).

**Claim.** No line of \( \pi \) is contained in a symp of rank at least 3.

Suppose for a contradiction that some line of \( \pi \) were contained in a symp of rank at least 3. Lemma 3.7 then yields a symp \( \xi^* \) containing \( \pi \). Let \( q \subseteq \xi^* \cap \zeta \). By assumption, no point of \( \pi \setminus L \) is collinear to \( q \). Hence all points of \( L \) are collinear to \( q \). Let \( p \subseteq \xi \) be arbitrary and set \( p' = p^\perp \) \( \subseteq L \). Then \( p' \perp q \) and, consequently, \( q \perp p \) (as otherwise \( p' \) would belong to \( \zeta = \xi(p, q) \), a contradiction). Hence \( \xi \), which is defined by \( L \) and \( p \), also contains \( q \), a contradiction. The claim follows.

We now show that \( \zeta \) is a quad, distinguishing between the following two cases.
• Case IIA: $\zeta \cap \xi$ is a single point $p$.
Let $p'$ be the unique point on $L$ collinear with $p$. Pick an arbitrary point $y \in \pi \setminus L$ and an arbitrary point $z \in \xi' \setminus L$ such that $z$ is collinear to a point $z' \in L \setminus \{p\}$. Then $y$ and $z$ are not collinear as otherwise $\xi' = \xi(p', z)$ contains $y$. Set $\xi^* = \xi(y, z)$. Then $\xi^*$ contains a line of $\pi$, namely $M = yz'$. By the above claim, $\xi^*$ is a quad and hence $\xi^* \cap \pi = M$. Noting that $p \in \xi$ is collinear to $p' \in \pi \setminus L$, we can interchange the roles of $(\xi, L)$ and $(\xi^*, M)$ and then Assumption I applies again, showing that $\zeta$ is a quad.

• Case IIB: $\zeta \cap \xi$ is a line $K$.
Select $p \in K$ arbitrarily and set $p' = p^\perp \cap L$. Select a line $M \neq K$ of $\zeta$ through $p$ not contained in $p^\perp$ and consider the symp $\xi_1$ defined by $p'$ and $M$. If $\xi_1$ has a line in common with $\pi$, then the points of $M \setminus \{p\}$ are collinear to points of $\pi \setminus L$, contradicting our hypothesis. Hence there is a line $N \neq pp'$ of $\xi_1$ through $p'$ not contained in $\pi$.
Now either $N$ and $L$ are contained in a singular plane $\pi'$ or they determine a symp $\xi'$, which is in fact a quad by the above claim, since it shares the line $L$ with $\pi$. In the first case, we replace $\pi$ by $\pi'$ and observe that the points of $M \setminus \{p\}$ are collinear to points of $\pi' \setminus L$; in the second case we replace $\zeta$ by $\zeta'$ and observe that the points of $K \setminus \{p\}$ are collinear to the points of $\pi \setminus L$. In both cases, these replacements imply that Case I applies again, yielding that $\zeta$ has rank 2.

This completes the proof of the lemma.

Lemma 6.4. Let $(X, \mathcal{L})$ be a strong parapolar space of minimum symplectic rank 2 with lacunary index $-1$. Then every symp that intersects a quad in a line is a quad.

Proof. Suppose for a contradiction that a quad $\zeta$ and a symp $\xi$ of rank at least 3 intersect in a line $L$. Pick $x \in \zeta$ arbitrarily but not on $L$. As in the proof of Lemma 6.1, there is a line $M$ through $x$ not contained in $\zeta$. Let $M'$ be a line of $\xi$ through $x$ disjoint from $L$. Then Lemma 6.3 implies that $M$ and $M'$ are not contained in a plane. Hence there is a symp $\xi'$ containing $M$ and $M'$. Since $L$ is contained in some plane of $\zeta$, Lemma 6.3 again implies that $\xi'$ is a quad.

Claim 1: The intersection $\zeta \cap \xi'$ is a point $q$.
Note that our main assumption yields $\zeta \cap \xi' \neq \emptyset$. Assume for a contradiction that $\zeta \cap \xi'$ is a line $K$.
Since $\xi \cap \xi' = M'$, the lines $K$ and $L$ are disjoint. For every point $z \in K$, the unique point in $z^\perp \cap M'$ and every point in $z^\perp \cap L$ (recall $L \cup K \subseteq \xi$) are collinear as $z \notin \xi$ (implying that also $z^\perp \cap L$ is unique). It follows that each point $z \in K$ is contained in a unique plane $\alpha_z$ intersecting $M'$ and $L$ in collinear points. Since $\alpha_z$ contains a line of $\zeta$, and $\zeta$ has rank at least 3, Lemma 3.7 implies the existence of a symp of rank at least 3 containing $\alpha_z$ and hence intersecting $\xi'$ in the line $\alpha_z \cap \xi'$. Now, for $z \neq z' \in K$, the plane $\alpha_z$ intersects $\xi'$ in a line disjoint from $\alpha_z \cap \xi'$. This contradicts once again Lemma 6.3. The claim is proved.

Similarly as in the previous paragraph, $q^\perp \cap L = p$ and $q^\perp \cap M = q'$. Let $\pi$ be any plane of $\zeta$ containing $L$. Then there is a point $x \in \pi \setminus L$ collinear to $q$ and a point $r \in \xi' \cap q^\perp$ such that $rq$ does not intersect $M'$.

Claim 2: $r$ is collinear to some point of $\pi \setminus L$.
If $r \perp x$, then this is trivial. If not then there is a symp $\xi(r, x)$, which intersects $\xi$ and hence, by convexity (as in the previous proof), it has a line $R$ in common with $\pi$. Let $x' \in R \cap r^\perp$ and suppose for a contradiction that $x' \in L$. Then the unique point $x''$ of $M'$ collinear with $x'$ is collinear to $r$ too (since $x' \notin \xi'$) and hence $x'' \neq q'$. This also implies that $p \neq x'$ and hence $x' \notin q^\perp$. But then $\xi(r, x) = \xi(q, x') = \zeta$, a contradiction. Claim 2 is proved.

Now we replace $\pi$ by another plane $\pi'$ of $\zeta$ containing $L$ and such that $\pi$ and $\pi'$ are not contained in a common 4-space. Then $r$ is also collinear to a point $x''$ of $\pi' \setminus L$. This implies that $x'$ and $x''$ are collinear, contradicting our choice of $\pi'$. 

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Lemma 6.6. Let $\pi$ be a singular plane. Then $\pi$ contains a line. Hence as soon as it contains two lines, it is a projective plane.

Proof. Assume for a contradiction that there is a symp $\xi$ of rank at least 3. Since the minimum rank is 2, there is also a quad $\xi$ and by $(-1)$-lacunarity, $\xi \cap \xi \neq \emptyset$. Moreover, by Lemma 6.4, $\xi \cap \xi$ is a point $p$. Pick lines $L \subseteq \xi$ and $M \subseteq \xi$ both through $p$. If $L$ and $M$ are contained in a plane, then by Lemma 3.7, this plane is contained in a symp of rank at least 3 intersecting $\xi$ in the line $L$, contradicting Lemma 6.4. Hence, by strongness, $L$ and $M$ define a symp, which has a line in common with both $\xi$ and $\xi$ and hence, again by Lemma 6.4, it can neither have rank at least 3 nor rank 2. This impossibility shows the proof.

6.2 Strong parapolar spaces with uniform symplectic rank 2 and with lacunary index $-1$

We continue with analysing strong parapolar spaces $\Omega = (X, \mathcal{L})$ with lacunary index $-1$ of minimum symplectic rank 2. By Proposition 6.5, $\Omega$ has in fact uniform symplectic rank 2.

Due to the lack of symps of rank at least 3, the singular subspaces are not necessarily projective. However, Lemma 6.2 implies that they are. We can even say more.

Lemma 6.6. Let $\Omega = (X, \mathcal{L})$ be a strong parapolar space of symplectic rank 2 with lacunary index $-1$. Then every singular subspace properly containing a line is a projective plane.

Proof. By Lemma 6.2, every singular subspace does not contain disjoint lines (as there are no symps of rank at least 3). Hence as soon as it contains two lines, it is a projective plane.

The previous lemma allows us to speak about (singular) planes instead of “singular subspaces properly containing a line”. Note also that Lemma 6.1 implies the existence of many singular planes.

Lemma 6.7. Let $\Omega = (X, \mathcal{L})$ be a strong parapolar space of symplectic rank 2 with lacunary index $-1$. Then every symp and every singular plane that share a point, share a line.

Proof. Let $\xi$ be a symp and $\pi$ a singular plane an suppose for a contradiction that $\xi \cap \pi = p$, with $p \in X$. Let $L$ be a line in $\pi$ not containing $p$ (and hence disjoint from $\xi$) and let $\xi_L$ be a symp containing $L$. Since $\xi_L$ does not contain planes, $p \notin \xi \cap \xi_L$. Let $q$ be a point of $\xi \cap \xi_L$ and denote by $r$ the unique point of $L$ collinear to $q$. Then $p \parallel r \perp q$. If $p$ and $q$ are not collinear, then $r \in \xi$, contradicting $L \cap \xi = \emptyset$. So suppose $p$ and $q$ are collinear. Then $p, q, r$ are contained in a singular plane $\pi'$ which intersects $\pi$ in the line $pr$. Let $t$ be a point in $\pi \setminus pr$. By Lemma 6.6, $t$ is not collinear to $q$; but then $q$ and $t$ determine a symp, which contains the planes $\pi$ and $\pi'$, a contradiction to the symplectic rank being 2.

Lemma 6.8. Let $\Omega = (X, \mathcal{L})$ be a strong parapolar space of symplectic rank 2 with lacunary index $-1$. Then every point $p$ not contained in a singular plane $\pi$ is collinear to a unique point of $\pi$.

Proof. Let $\ell$ be the distance of $p$ to $\pi$ (connectivity implies that $\ell$ is finite). If $\ell = 1$, then it follows by an argument similar to the one used at the end of the proof of Lemma 6.7 that the point in $\pi$ collinear to $p$ is unique. Next, if $\ell = 2$, strongness implies that $p$ is contained in a symp, which, by Lemma 6.7, shares a line $L$ with $\pi$. But then $L$ contains a point collinear to $p$, contradicting $\ell = 2$. Since by (PPS1) parapolar spaces are connected, it follows that $\ell$ always equals 1, and we are done.
In case there is a singular plane intersecting every symp non-trivially, we can show that the parapolar space is a Segre geometry of type \((1, 2)\). We first show, under this assumption, that each symp is non-thick.

**Lemma 6.9.** Let \((X, \mathcal{L})\) be a strong parapolar space of symplectic rank 2 with lacunary index \(-1\). If there is a singular plane \(\pi\) intersecting every symp non-trivially, then each symps of \((X, \mathcal{L})\) is non-thick.

Proof. By Lemma 6.7, \(\pi\) intersects each symp in a line. Let \(\xi\) be an arbitrary symp. Set \(L = \pi \cap \xi\) and let \(q\) be a point in \(\xi \setminus L\). Let \(p\) be the unique point on \(L\) collinear to \(q\) and take a line \(K\) in \(\pi\) intersecting \(L\) in \(p\). Let \(L'\) be a line in \(\xi\) through \(q\) disjoint from \(L\). By Lemma 6.8, \(p\) is the unique point of \(K\) collinear to \(q\) and hence, as \((X, \mathcal{L}')\) is strong, there is a unique symp \(\xi_{K,q}\) through \(K\) and \(q\). Let \(K'\) be a line in \(\xi_{K,q}\) through \(q\) disjoint from \(K\) (hence \(K' \not\subseteq \xi\)). We claim that \(L'\) and \(K'\) are contained in a singular plane \(\pi'\). If not, then by strongness, \(L'\) and \(K'\) are contained in a unique symp \(\xi'\). Since \(\pi\) shares a line with \(\xi'\), the latter contains a point of \(L\). Hence \(\xi'\), containing \(L'\) and a point of \(L\), coincides with \(\xi\), violating \(K' \not\subseteq \xi\). This shows the claim. If there would be another line \(qr\) in \(\xi\) disjoint from \(L\), then repeating the above argument implies \(r \perp K'\), contradicting the fact that \(r\) is collinear to a unique point (namely \(q\)) of \(\pi'\). We conclude that \(\xi\) is non-thick. \(\square\)

**Proposition 6.10.** Let \((X, \mathcal{L})\) be a strong parapolar space of symplectic rank 2 with lacunary index \(-1\). If there is a singular plane \(\pi\) intersecting every symp non-trivially, then \((X, \mathcal{L})\) is isomorphic to a Segre geometry of type \((1, 2)\).

Proof. Again, Lemma 6.7 implies that \(\pi\) intersects each symp in a line, and by Lemma 6.9, each symp is non-thick. Let \(L\) be an arbitrary line intersecting \(\pi\) in a point \(t\) (which exists since there is a symp through \(t\)).

Claim. The line \(L\) is the unique line through \(t\) not contained in \(\pi\).

Indeed, suppose for a contradiction that there is a point \(x \notin \pi \cup L\) with \(x \perp t\). If \(x\) and \(L\) would belong to a singular plane \(\pi'\), we take a symp \(\xi'\) through a line \(L'\) of \(\pi'\) not containing \(t\). Then \(\xi' \cap \pi\) is a line \(L''\) by assumption, and since \(t\), if not already on \(L''\), is collinear to two non-collinear points of \(L'\) and \(L''\), respectively, we obtain \(t \in \xi'\). This however means that \(\pi' \subseteq \xi'\), a contradiction. So \(x\) is not collinear to \(L\), and then strongness implies a symp containing \(x\) and \(L\). By assumption this symp intersects \(\pi\) in a line, which contains \(t\), implying that the symp has three lines through \(t\), contradicting that it is non-thick. The claim is proved.

We now complete the lemma by showing that \((X, \mathcal{L})\) is isomorphic to the direct product space \(\pi \times L\). Let \(x \in X\) be arbitrary. If \(x \in \pi \cup L\), then \(x\) can be uniquely written in \(L \times \{t\} \cup \{t\} \times \pi\). Suppose \(x \notin L \cup \pi\). By Lemma 6.8, \(x\) is collinear to a unique point \(x_\pi\) of \(\pi\), which does not coincide with \(t\) by the above claim. Hence, by strongness, there is a unique symp \(\xi\) through \(x\) and \(t\) and, again by the above claim, \(\xi\) contains \(L\) as one of its two lines through \(t\). So there is a unique point \(x_L \in L\) collinear to \(x\), and \(x_L \neq t\). Just like \(L\) was the unique line through \(t\) not in \(\pi\), the line \(xx_\pi\) is the unique line through \(x_\pi\) not contained in \(\pi\). Therefore, since \(x_L\) is collinear with a unique point of \(xx_\pi\) (as \(x_L\) and \(x_\pi\) are not collinear), \(x_L\) and \(x_\pi\) determine \(x\) uniquely. Lastly, it follows from the argument in the previous proof that the lines distinct from \(L\) through any point \(x' \in L \setminus \{x\}\) belong to a singular plane.

The proposition is proved. \(\square\)

We now arrive at the crux of the proof.

**Lemma 6.11.** Let \((X, \mathcal{L})\) be a strong parapolar space of symplectic rank 2 with lacunary index \(-1\). If some plane \(\pi\) is disjoint from some symp \(\xi\), then \(\xi\) is non-thick and there exists a bijection from the point set of some line in \(\pi\) to one system of generators of \(\xi\) such that elements corresponding under this bijection are contained in a common singular plane.
Proof. Let $L$ be a line in $\xi$. Pick $p_1, p_2 \in L$ distinct. Let $q_i$ be the unique respective points in $\pi$ collinear to $p_i$, $i = 1, 2$. If $q_1 = q_2$, then $L$ is contained in a singular plane intersecting $\pi$ in a point; if $q_1 \neq q_2$, then $\xi(p_1, q_2)$ contains $L$ and $q_1 q_2$ and hence collinearity is a bijection between $L$ and $q_1 q_2$. In the first case we say that $L$ is $\pi$-triangular (with centre $q_1 = q_2$), in the second case $\pi$-quadrangular (with axis $q_1 q_2$). We show three properties.

(1) Each pencil of lines in $\xi$ contains at most one $\pi$-triangular line.
Let $L_1, L_2$ be two intersecting lines of $\xi$. If both are $\pi$-triangular, the planes meet in a line, contradiction Lemma 6.8 and showing the claim.

Now let $M_1$ and $M_2$ be two disjoint $\pi$-quadrangular lines of $\xi$.

(2) One or all lines meeting both $M_1$ and $M_2$ are $\pi$-triangular, according to whether the axes of $M_1, M_2$ are distinct or not.
Indeed, the axes of $M_1$ and $M_2$, being contained in a projective plane, have at least one point $r$ in common. Then $r$ is collinear to some points $s_1, s_2$ on $M_1, M_2$, respectively. If $s_1$ were not collinear to $s_2$, then $r \in \xi$, a contradiction. Hence $r, s_1, s_2$ are contained in a singular plane and the line $s_1 s_2$ of $\xi$ is $\pi$-triangular with centre $r$. If the axes intersect in a unique point, there is a unique $\pi$-triangular line meeting both $M_1$ and $M_2$; if they coincide, each line meeting both $M_1$ and $M_2$ is $\pi$-triangular. The claim is proved.

It is now easy to see that the previous claim yields at least two (necessarily disjoint, by the first claim) $\pi$-triangular lines (even if $\xi$ is non-thick), say $T_1, T_2$, with respective centres $t_1, t_2$. Let $U_1, U_2, U_3$ be three lines each intersecting both $T_1$ and $T_2$ non-trivially.

(3) The lines $T_1$ and $T_2$ define a (full) grid $G$ in $\xi$, one of which reguli consisting of $\pi$-triangular lines and the other of $\pi$-quadrangular lines.
For $j \in \{1, 2, 3\}$, the axis $B_j$ of $U_j$ is a line containing $t_1$ and $t_2$ and it follows that $t_1 \neq t_2$, so $B_j = t_1 t_2$. Let $t$ be an arbitrary point on $t_1 t_2$. Then the points on $U_1, U_2, U_3$ collinear to $t$ are pairwise collinear, as above. This implies that, varying $t \in t_1 t_2$, each line intersecting $U_1$ and $U_2$ non-trivially also intersects $U_3$ non-trivially, and, on top, is $\pi$-triangular. This shows the claim.

By (3), it suffices to show that $\xi$ is non-thick to finish the proof.

Suppose for a contradiction that $\xi$ is thick. Let $i \in \{1, 2\}$. Put $p_i = U_1 \cap T_i$ and take a line $L_i$ through $p_i$ distinct from $U_1$ and $T_i$. By (1), $L_i$ is $\pi$-quadrangular with axis $A_i \ni t_i$. By (2) and the fact that $U_1$ is $\pi$-quadrangular, exactly one line intersecting both $L_1$ and $L_2$ is $\pi$-triangular. Consequently, there is $\pi$-quadrangular line $U'_i$ distinct from $U_1$ intersecting both $L_1$ and $L_2$. Again, (2) implies a $\pi$-triangular line $T'_i$ intersecting both $U_1$ and $U'_i$. However, the grid $G$ determined by $T_1$ and $T_2$ already possessed a $\pi$-triangular line through the point $T'_i \cap U_1$, contradicting (1).

We can now show in general that every symp is non-thick.

Lemma 6.12. Let $(X, \mathcal{L})$ be a strong parapolar space of symplectic rank 2 of lacunary index $-1$. Then every symp is non-thick.

Proof. Suppose for a contradiction that there is thick symp $\xi$. By Lemmas 6.1 and 6.6, there exists some singular plane $\pi$. By Lemmas 6.11 and 6.7, $\pi \cap \xi$ is a line $L$. Let $M$ be a line in $\xi$ disjoint from $L$ and pick a point $p \in M$. Considering a symp through a point $x$ of $\pi \setminus \xi$ and $p$ (which exists since the
unique point of $\pi$ collinear to $p$ is contained in $L$ and $(X, \mathcal{L})$ is strong) we see that there exists some line $K \ni p$ not contained in $\xi$ (and some point of $K$ is collinear to $x$). Replacing $M$ by another line through $p$ disjoint from $L$ (which is possible by the thickness of $\xi$) if necessary, we may assume that $M$ and $K$ are contained in a unique symp $\xi'$. If $\xi' \cap \pi$ contained a point $q$, then $q$ being collinear to all points of $L$ and a unique point of $M$, would belong to $\xi$, and hence to $L$, a contradiction, as that point and $M$ define $\xi \neq \xi'$. So $\xi' \cap \pi$ is empty and Lemma 6.11 implies that $\xi'$ is non-thick.

We use the terminology of the proof of Lemma 6.11, applied to the pair $(\pi, \xi')$. Clearly, $M$ is $\pi$-quadragular with axis $L$, hence by Lemma 6.11, the line $K$, belonging to the other regulus, is contained in a singular plane with a unique point on $L$. But some point on $K$ was collinear to $x$, contradicting the uniqueness assertion in Lemma 6.8. This absurdity proves the lemma.

**Theorem 6.13.** Let $(X, \mathcal{L})$ be a strong parapolar space of symplectic rank $2$ with lacunary index $-1$. Then $(X, \mathcal{L})$ is isomorphic to a Segre geometry either of type $(1,2)$ or of type $(2,2)$.

**Proof.** By Proposition 6.10, we may assume that there is a singular plane disjoint from some symp. The existence of two singular planes $\pi_1$ and $\pi_2$ intersecting each other in a point $p$ then is an easy consequence of Lemma 6.11.

Let $x \in X \setminus (\pi_1 \cup \pi_2)$ be arbitrary. Then $x$ is not collinear to $p$ as otherwise a symp through $xp$ has a line through $x$ in common with both $\pi_1$ and $\pi_2$ by Lemma 6.7, contradicting non-thickness (cf. Lemma 6.12). Hence, using Lemma 6.8, $x$ is collinear to unique distinct points $x_1 \in \pi_1 \setminus \{p\}$ and $x_2 \in \pi_2 \setminus \{p\}$. Conversely, given points $x_1 \in \pi_1$ and $x_2 \in \pi_2$ distinct from $p$, there is a unique symp through $x_1, x_2$ (again using strongness and the fact that $x_1, x_2$ are not collinear), which is non-thick by Lemma 6.12 and therefore contains a unique point collinear to both $x_1$ and $x_2$ and not contained in $\pi_1 \cup \pi_2$. Consequently we already have that $X$ can be written as $\pi_1 \times \pi_2$ in a set-theoretic way. It remains to show that two points $x, x' \in X$ collinear to the same point $x_1 \in \pi_1$ are collinear themselves. But if $x$ and $x'$ were not collinear, then the symp through them (note that $x \perp x_1 \perp x'$) contains, by Lemma 6.7, a line in $\pi_1$, hence a third line through $x'$, a contradiction. Similarly for $x_2 \in \pi_2$. The theorem is proved.

Theorem 6.13 proves (ii) of the Main Result—Extended Version.

### 6.3 Parapolar spaces of symplectic rank at least 3 with lacunary index $-1$ having at least one sympthick line

In this subsection, $\Omega = (X, \mathcal{L})$ is a parapolar space of minimum symplectic rank $d$ with $d \geq 3$ with lacunary index $-1$, having at least one sympthick line (recall that this is a line contained in at least two symps). A symp not containing a sympthick line will be called isolated. Our main hypothesis still is that every pair of symps meets non-trivially.

We aim to prove the assumptions of the Haircut Theorem (cf. Theorem 3.20). Hence we need to show that $\Omega$ is locally connected, that its singular rank and symplectic rank are bounded (we will even show uniform symplectic rank), and finally we need to show that the Haircut Axiom (H) holds.

The first lemma is general and does not need $(-1)$-lacunarity. It does require a sympthick line. Note however that if $\Omega$ is a locally connected parapolar space of symplectic rank at least $3$, then each of its lines is sympthick (since $\Omega_p$ is a parapolar space in this case, cf. Facts 3.11 and 3.12).

**Lemma 6.14.** Let $\Omega = (X, \mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ having at least one sympthick line. Let $\xi$ be a non-isolated symp with rank $d_1$. Then, for every singular subspace $S$ of $\xi$ of dimension $d - 2$, there is a symp $\xi^* \neq \xi$ such that $S \subseteq \xi \cap \xi^*$. Furthermore, one of the following holds.
(i) The symp $\xi^*$ is non-thick, has rank $d$ and $\dim(\xi \cap \xi^*) = d - 1$.

(ii) For each singular subspace $M$ of $\xi$ of dimension $d - 1$ through $S$, there is a symp $\xi_M$ with $M \subseteq \xi \cap \xi_M$ (equality if $d_1 = d$).

Proof. By assumption, $\xi$ contains a line $L$ which is contained in a second symp. We first deal with singular subspaces through $L$; afterwards we show that this is not a restriction, by showing that each line of $\xi$ is symplactic. So consider a singular subspace $S$ of dimension $d - 2$ with $L \subseteq S \subseteq \xi$.

Claim 1: There is a symp $\xi^* \neq \xi$ such that $S \subseteq \xi \cap \xi^*$.

Let $U$ be a subspace of $S$ through $L$, maximal with respect to the property that there exists a symp $\xi^* \in \Omega$ with $U \subseteq \xi \cap \xi^*$ ($U$ is well defined since $L$ satisfies this requirement). Suppose for a contradiction that $U \subseteq S$, so there is a point $p \in S \setminus U$. The set $p^+ \cap \xi^*$ is a singular subspace of $\xi^*$, clearly containing $U$. Also $\xi \cap \xi^*$ is a singular subspace of $\xi^*$ containing $U$. Since $\xi^*$ is a symp of rank at least $d$ and $\dim(U) < d - 2$, there is a point $q \in \xi^* \setminus \xi$ collinear to $U$ with $q \notin p^+$. Then $q$ and $p$ are non-collinear and $U \subseteq p^+ \cap q^+$. Hence there is a symp $\xi^*$ through $p$ and $q$, which is distinct from $\xi$ since $q \notin \xi$. But now $\xi \cap \xi^*$ contains $\langle p, U \rangle$, contradicting the maximality of $U$. We conclude that there is a symp $\xi^* \neq \xi$ with $S \subseteq \xi \cap \xi^*$, showing the claim.

Now suppose that the above found symp $\xi^*$ is either thick, or has rank at least $d + 1$ or is such that $\xi \cap \xi^* = S$. Let $M$ be any singular subspace of $\xi$ through $S$ of dimension $d - 1$.

Claim 2: Under the above assumptions on $\xi^*$, there is a symp $\xi_M$ with $M \subseteq \xi \cap \xi_M$.

Take a point $p \in M \setminus S$. We may assume that $M \not\subseteq \xi \cap \xi^*$. Our assumptions on $\xi^*$ imply the existence of a subspace $M'$ of dimension $d - 1$ through $S$ in $\xi^*$ which is not contained in $p^+ \cap \xi^*$ (which is a singular subspace of $\xi^*$ through $S$) nor in $\xi \cap \xi^*$ (the latter coincides with $S$ if $\xi^*$ is non-thick and has rank $d$). Similarly as above, we take a point $q \in M' \setminus S$, which is then symplectic to $p$. The unique symp $\xi_M$ through $p$ and $q$ contains $M$. This shows the claim.

If $\xi^*$ does not satisfy those assumptions, then $\xi^*$ is non-thick, has rank $d$ and $S \subseteq \xi \cap \xi^*$. Since $\xi^*$ has rank $d$ and $\dim(S) = d - 2$, the latter implies that $\dim(\xi \cap \xi^*) = d - 1$. We now complete the lemma by showing that each line in $\xi$ is symplactic.

Claim 3: Each line in $\xi$ is symplactic.

Without loss of generality, we may consider a line $K$ in $\xi$ generating a plane $\pi$ together with $L$. If $d_1 > 3$, $\pi$ is contained in a $(d - 2)$-space of $\xi$, so by Claim 1 we may assume that $d_1 = 3$. Likewise, by Claim 2, we may assume that a symp $\xi^* \neq \xi$ through $L$ is non-thick, has rank $3$ (since $3 = d_1 \geq d \geq 3$) and is such that $\xi \cap \xi^*$ is a plane $\pi^*$ through $\pi$. Let $\pi'$ be the unique plane through $L$ in $\xi^*$ distinct from $\pi^*$. If $\pi \cap \pi'$ contains a pair of non-collinear points, these determine a symp containing $\pi \cup \pi'$, proving that $K$ is symplectic. So suppose $\pi$ and $\pi'$ are collinear. Let $q$ be a point of $\pi' \setminus L$ and note that $q^+ \cap \xi = \pi$ since $d_1 = 3$. Hence a point $p \in \xi \cap K^\perp \setminus \pi$ is not collinear to $q$. The pair $\{p, q\}$ is symplectic since $K \subseteq p^+ \cap q^+$, proving again that $K$ is symplectic, as required. □

[Note this provides a direct proof for Statement (xiv) of the Main Result—Extended Version.]

Under our condition that every pair of symps intersect non-trivially, we can show that no symp is isolated, and hence the previous lemma holds for all symps of $\Omega$.

Lemma 6.15. Let $\Omega = (X, \mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symplactic line. Then no symp is isolated.

Proof. Suppose for a contradiction that some symp $\xi$ is isolated, i.e., none of its lines is symplectic. Since $\Omega$ contains at least one symplactic line, there is a non-isolated symp $\xi'$. Then $(-1)$-lacunarity,
together with our assumption on $\xi$, implies that $\xi \cap \xi'$ is just a point $p$. Take a subspace $\Sigma$ in $\xi'$ of dimension $d - 2$ which is not contained in $\pi$. By one of the two cases occurring in Lemma 6.14, there is a sym $\xi'' \neq \xi'$ through $\Sigma$ such that $\dim(\xi' \cap \xi'') = d - 1$. Again, $(-1)$-lacunarity and our assumption on $\xi$ implies that $\xi \cap \xi''$ is just a point $p''$. Then $p'' \neq p$, as $\xi' \cap \xi''$ is a singular subspace of $\xi'$ and $p$ is not collinear with $\Sigma$. Since the rank of $\xi'$ is at least $3$, the intersection $\xi' \cap \xi''$ contains at least a point $q$ collinear to both $p$ and $p''$. The point $q$ does not belong to $\xi$ but is collinear to the distinct points $p, p''$, implying $p$ and $p''$ are collinear. Hence, since $p''$ is collinear to all points of the line $pq$ in $\xi'$, Lemma 3.7 says $p''$ and $pq$ are contained in a symp, in particular, there is a second symp containing $pp''$ after all, a contradiction.

Lemma 6.16. Let $\Omega = (X, \mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symthick line. Let $\xi$ be any symp of rank $d$. Then we have

(i) for each symp $\xi'$ with $\dim(\xi \cap \xi') \geq d - 2$, the rank of $\xi'$ is $d$ and $\dim(\xi \cap \xi') = d - 1$.

(ii) $\xi$ is hyperbolic (i.e., non-thick) of odd rank.

Proof. (i) Consider opposite subspaces $S_1$ and $S_2$ of $\xi$ of dimension $d - 2$ (note that $d - 2 \geq 1$). By Lemmas 6.14 and 6.15, there are symps $\xi_1^*$ and $\xi_2^*$ intersecting $\xi$ in maximal singular subspaces $M_1$ and $M_2$ of $\xi$ through $S_1$ and $S_2$, respectively. If $M_1 \cap M_2 = \emptyset$, then $\xi_1^* \cap \xi_2^*$, which contains at least a point $p$ by $(-1)$-lacunarity, is disjoint from $\xi$. But then $p$ is collinear to the non-collinear subspaces $S_1$ and $S_2$ of $\xi$, a contradiction. Hence $M_1 \cap M_2$ is a point (it cannot be more since $S_1$ and $S_2$ are opposite).

Observe that this implies that Possibility (ii) of Lemma 6.14 cannot occur, so any symp $\xi^*$ with $\dim(\xi \cap \xi^*) \geq d - 2$ is non-thick, has rank $d$ and $\dim(\xi \cap \xi^*) = d - 1$. This shows the first assertion, so we continue with the second one.

(ii) Firstly, suppose for a contradiction that $\xi$ is thick. Let $M^*_2$ be a $(d - 1)$-space in $\xi^*$ through $S_2$ distinct from $M_2$. Then $M^*_2$ is collinear to at most one of the maximal singular subspaces of $\xi$ through $S_2$ and, as there are at least three such subspaces, $M^*_2$ is contained in a sym with a maximal singular subspace $M_3^*$ of $\xi$ through $S_2$ which is disjoint from $M_1$, contradicting the first paragraph. We conclude that $\xi$ is hyperbolic. Secondly, suppose $\xi$ is hyperbolic of even rank $d$. Then $M_1$ and $M_2$, intersecting each other in a point, belong to different natural types of generators. By Lemma 3.2, there exists a subspace $S_3$ of $\xi$ of dimension $d - 2$ disjoint from $M_1$ and $M_2$. By Lemma 6.14, there is a symp $\xi_3^* \neq \xi$ with $S_3 \subseteq \xi \cap \xi_3^*$. By the above observation, $\xi \cap \xi_3^*$ is a maximal singular subspace $M_3$ of $\xi$ through $S_3$. The first paragraph implies that both $M_1 \cap M_3$ and $M_2 \cap M_3$ is a point, but then the types of $M_1$, $M_2$ and $M_3$ should all be distinct, which is clearly impossible.

For convenience we record a consequence of the proof of the previous lemma.

Corollary 6.17. Let $\Omega = (X, \mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symthick line. If $M_1$ and $M_2$ are opposite maximal singular subspaces in a symp $\xi$ of rank $d$, then at most one of them is contain in a second symp.

Proof. This follows directly from the first paragraph of the proof of Lemma 6.16.

Lemma 6.18. Let $\Omega = (X, \mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symthick line. Let $\xi$ be any symp of rank $d$. Then the set $\Phi$ of maximal singular subspaces of $\xi$ that are the intersection of $\xi$ with another symp is precisely the set of generators belonging to one natural type.
Proof. Suppose two generators $M_1$ and $M_2$ of $\xi$ belong to $\Phi$, and assume for a contradiction that they have distinct natural type. By Lemma 3.2, we can find a submaximal subspace $S$ in $\xi$ disjoint from $M_1$ and $M_2$. By Lemma 6.14 and 6.15, there is a symp $\xi^*$ through $S$. In view of Lemma 6.16, $\xi^* \cap \xi$ is a maximal singular subspace $M$. By Corollary 6.17 and our choice of $S$, $M$ intersects both $M_1$ and $M_2$ in exactly a point. Since $M_1$ and $M_2$ have distinct natural type, this is impossible.

We deduced that all members of $\Phi$ belong to the same natural type of generators. Conversely, to see that each generator of this type belongs to $\Phi$, we consider any submaximal singular subspace $S$ of $\xi$. As above, there is a symp $\xi^*$ such that $\xi \cap \xi^*$ is a maximal singular subspace $M$ of $\xi$ containing $S$. The lemma follows.

The following two lemmas are the basis to prove local connectivity and uniform rank.

**Lemma 6.19.** Let $(X,\mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symphick line. Then a generator of some symp of rank $d$ which is not contained in a second symp is contained in a singular $d$-space.

**Proof.** Let $\xi$ be an arbitrary symp of rank $d$ and $M$ an arbitrary generator of $\xi$ not contained in another symp (cf. Lemma 6.18). Let $M'$ be any generator of $\xi$ intersecting $M$ in a $(d-2)$-space $W$. Then $M' = \xi \cap \xi'$, for some $\xi' \in \Xi$. By Lemma 6.16(i), $\xi'$ is (just as $\xi$) hyperbolic of odd rank $d$. In $\xi'$, we consider the generator $M''$ containing $W$ and distinct from $M'$, and some point $p \in M'' \setminus M'$. If $p$ were not collinear to all points of $M$, then $\{p,q\}$ is a symplectic pair for every $q \in M \setminus M''$, and the corresponding symp contains $M$ and is different from $\xi$, contradicting to our assumption on $M$. Hence $p$ and $M$ generate a singular subspace of dimension $d$.

**Lemma 6.20.** Let $\Omega = (X,\mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symphick line. Let $\xi_1$ be a symp of rank $d$ and let $\xi_2$ be any symp intersecting $\xi_1$ in exactly a point $p$. Then there is a singular plane through $p$ intersecting both symps in a line.

**Proof.** Consider a generator $M_1$ in $\xi_1$ through $p$ not contained in a second symp of $\Omega$ (cf. Lemma 6.18). Then, by Lemma 6.19, there is a singular $d$-space $W$ containing $M_1$. If $W$ would intersect $\xi_2$ in more than $p$, the lemma follows immediately, so assume $W \cap \xi_2 = p$. We select a hyperplane $H$ of $W$ not containing $p$. Then, by Fact 3.8, $H$ is contained in a symp $\xi$. By $(-1)$-lacunarity, we obtain a point $x_2$ contained in $\xi_2 \cap \xi$. Then $x_2 \neq p$ since otherwise $\xi$ would contain the $d$-space $W$, whereas $\dim(\xi \cap \xi_1) \geq d - 2$ implies, by Lemma 6.16(i), that $\xi$ has rank $d$. Let $x_1 \in H \cap M_1$ be collinear to $x_2$ ($x_1$ exists since $\dim(H \cap M_1) \geq 1$). Since $x_2 \perp x_1 \perp p$ and both $x_2$ and $p$ belong to $\xi_2$ we deduce that $x_2 \perp p$, and by Lemma 3.7, $\langle p,x_1,x_2 \rangle$ is a singular plane intersecting both $\xi_1$ and $\xi_2$ in the lines $px_1$ and $px_2$, respectively.

Finally we can show that the symplectic rank is uniform.

**Lemma 6.21.** Let $\Omega = (X,\mathcal{L})$ be a parapolar space of minimum symplectic rank $d \geq 3$ with lacunary index $-1$, containing at least one symphick line. Then the symplectic rank is uniform and therefore each symp is hyperbolic of odd rank $d$.

**Proof.** Let $\xi$ be any symp of rank $d$. By Lemma 6.16, any symp $\xi'$ with $\dim(\xi \cap \xi') \geq d - 2$ has rank $d$ as well. Now let $\xi^*$ be an arbitrary symp. We claim that we can find (a finite) sequence of symps between $\xi^*$ and $\xi$ such that successive symps in the sequence intersect each other in a subspace of dimension at least $d - 2$, from which then follows that each symp in this sequence has rank $d$. By $(-1)$-lacunarity,
\(\xi \cap \xi^*\) is non-empty. If \(\xi \cap \xi^*\) is a point, Lemma 6.20 implies the existence of a plane \(\pi\) intersecting both \(\xi\) and \(\xi^*\) in a line, and since \(d \geq 3\), Fact 3.8 guarantees a symp through \(\pi\) which then shares at least a line with both \(\xi\) and \(\xi^*\). Hence, if \(d = 3\), we are done. If \(d > 3\), we may already assume that \(1 \leq \dim(\xi \cap \xi^*) \leq d - 3\). Under this assumption we can take points \(p\) and \(p^*\) in \(\xi\) and \(\xi^*\), respectively, collinear to \(\xi \cap \xi^*\) and not collinear to each other. The symp determined by \(p\) and \(p^*\) intersects both \(\xi\) and \(\xi^*\) in a subspace strictly bigger than \(\xi \cap \xi^*\). Recursively, the claim follows and hence each symp has rank \(d\). Lemma 6.16 now implies that each symp is hyperbolic of odd rank \(d\).

Henceforth we could therefore drop the word “minimum” from our assumptions on \(\Omega\), but we prefer to put it between brackets to remind the reader of the full context. Local connectivity now follows as a consequence of Lemma 6.20.

**Lemma 6.22.** Let \(\Omega = (X, L)\) be a parapolar space of (minimum) symplectic rank \(d \geq 3\) with lacunary index \(-1\), containing at least one symthick line. Then \(\Omega\) is locally connected.

**Proof.** Consider two lines \(L_1\) and \(L_2\) through \(p\). Let \(\xi_1\) and \(\xi_2\) be symps through \(L_1\) and \(L_2\), respectively. If \(\dim(\xi_1 \cap \xi_2) \geq 1\), it is clear that \(L_1\) and \(L_2\) are connected via a sequence of singular planes intersecting each other in lines. If \(\dim(\xi_1 \cap \xi_2) = 0\), then, as all symps have rank \(d\) now by Lemma 6.21, a link between \(\xi_1\) and \(\xi_2\) is provided by Lemma 6.20 (and inside the symps we are fine, as just mentioned before).

We proceed by showing boundedness of the singular rank.

**Lemma 6.23.** Let \(\Omega = (X, L)\) be a parapolar space of (minimum) symplectic rank \(d \geq 3\) with lacunary index \(-1\), containing at least one symthick line. Then the singular rank is at most \(2(d - 1)\).

**Proof.** Suppose there is a singular \((2d - 1)\)-space \(W\) in \(\Omega\). Let \(M_1\) and \(M_2\) be two disjoint \((d - 1)\)-subspaces in \(W\). By Fact 3.8, there are symps \(\xi_1\) and \(\xi_2\) containing \(M_1\) and \(M_2\), respectively. Then \((-1)\)-lacunarity yields a point \(p \in \xi_1 \cap \xi_2\). Since \(M_i\) is a maximal singular subspace in \(\xi_i\), \(i = 1, 2\), we know \(p \notin W\). In particular \(p \notin M_1 \cup M_2\) and so we can find points \(q_1 \in M_1\) and \(q_2 \in M_2\) with \(q_1 \notin p^\perp\) and \(q_2 \notin p^\perp\). Then \(q_2 \in p^\perp \cap q_1^\perp \subseteq \xi_1\), a contradiction.

Finally we prove the Haircut Axiom (H).

**Lemma 6.24.** Let \(\Omega = (X, L)\) be a parapolar space of (minimum) symplectic rank \(d \geq 3\) with lacunary index \(-1\), containing at least one symthick line. For any symp \(\xi\) and any point \(p\), the set \(p^\perp \cap \xi\) can never be a submaximal singular subspace of \(\xi\).

**Proof.** Assume for a contradiction that \(p^\perp \cap \xi = H\), with \(H\) a submaximal singular subspace of \(\xi\). Since \(\xi\) is hyperbolic, there are exactly two generators \(M_1, M_2\) containing \(H\). Pick \(p_i \in M_i \setminus H\), \(i = 1, 2\). By assumption, \(p_i \notin p^\perp\), \(i = 1, 2\). Then the symps \(\xi(p, p_1)\) and \(\xi(p, p_2)\) contain \(M_1\) and \(M_2\), respectively, contradicting Lemma 6.18 and the fact that \(M_1\) and \(M_2\) have distinct natural type.

We obtain that \(\Omega\) satisfies all assumptions of Shult’s Haircut Theorem. Hence \(\Omega\) is contained in the list given in this theorem, so we only have to determine which of those geometries have lacunary index \(-1\). To simplify this verification, we nail down the symplectic rank of \(\Omega\). It is not strictly necessary, but it makes the job easier. It also points at the possibility of finishing the job without invoking the Haircut Theorem.

We first show that two symps which intersect in a plane, intersect in a generator.
Lemma 6.25. Let \((X, \mathcal{L})\) be a parapolar space of (minimum) symplectic rank \(d \geq 3\) with lacunary index \(-1\), containing at least one sympthick line. Then two syms that have no generator in common intersect in either a point or a line.

Proof. Recall that we know from Lemma 6.21 that each sym has rank \(d\). The result is trivial if \(d = 3\), so let \(d \geq 4\). Suppose two generators \(\xi\) and \(\xi'\) intersect in a singular subspace \(U\) of dimension \(j\), \(0 \leq j \leq d - 2\). Select a generator \(M\) in \(\xi\) disjoint from \(U\) such that \(M = \xi \cap \xi'^*\), for some \(\xi^* \in \Xi\), which is possible by Lemma 6.18. By \((-1)\)-lacunarity, there is a point \(p \in \xi \cap \xi'^*\). Then \(p \notin \xi\) since \(M\) is disjoint from \(\xi'^*\). However, \(p\) is collinear to all points of a \((d-2)\)-space in \(M\) (since \(p \in \xi'^*\)) and \(\dim(p^+ \cap U) \geq j - 1\) (since \(p \in \xi'^*\)). Since \(p^+ \cap \xi\) is a singular subspace, its dimension \(\ell\) satisfies \((d-2)+(j-1)+1 \leq \ell \leq d - 1\), implying \(j \leq 1\). The lemma is proved. \(\square\)

Lemma 6.26. Let \((X, \mathcal{L})\) be a parapolar space of symplectic rank \(d \geq 3\) with lacunary index \(-1\), containing at least one sympthick line. Then \(d \in \{3, 5\}\), so the syms are either hyperbolic polar spaces of rank 3, or hyperbolic polar spaces of rank 5.

Proof. Suppose \(d \geq 5\), we show that \(d = 5\). Let \(\xi\) be a sym and choose two generators \(M, M'\) of \(\xi\) not contained in second syms and intersecting in a plane \(\pi\). Let \(W\) and \(W'\) be \(d\)-spaces through \(M, M'\), respectively (these exist by Lemma 6.19). If all points of \(W \setminus M\) were collinear to all points of \(W' \setminus M'\), then all points of \(M\) would be collinear to all points of \(M'\), a contradiction. So there are points \(p \in W \setminus M\) and \(p' \in W' \setminus M'\) which are not collinear. Since \(\pi\) belongs to \(p^+ \cap p'^+\), the pair \(\{p, p'\}\) is symplectic and the corresponding symp \(\xi^*\) intersects \(\xi\) in at least the plane \(\pi\), so by Lemma 6.25, \(\xi \cap \xi^*\) is a generator \(M^*\). Since \(p^+ \cap \xi = M\), we have \(p^+ \cap M^* \subseteq M\), likewise \(p'^+ \cap M^* \subseteq M'\). Both subspaces have dimension \(d - 2\) and are contained in \(M^*\), and hence intersect in a \(d - 3\)-space. On the other hand, they intersect in \(\pi\) only, so \(d - 3 \leq 2\), implying \(d \leq 5\). \(\square\)

Once we know \(d \in \{3, 5\}\), Theorem 3.20 reveals that the parapolar spaces with minimum symplectic rank \(d \geq 3\) and lacunary index \(-1\), containing at least one sympthick line, are precisely \(\text{A}_{4,2}(\mathbb{L}), \text{A}_{5,2}(\mathbb{L})\) (and in these cases \(d = 3\)) and \(E_{6,1}(\mathbb{K})\) (and then \(d = 5\)).

This proves (iii) of the Main Result—Extended Version.

7 Parapolar spaces with lacunary index 0

In this section \(\Omega = (X, \mathcal{L})\) is a parapolar space of symplectic rank at least \(d \geq 2\) and with lacunary index 0. Note that we do not assume local connectivity in this section, but we shall prove it.

7.1 Basic properties

Lemma 7.1. Let \(\Omega = (X, \mathcal{L})\) be a parapolar space of symplectic rank at least \(d \geq 3\) with lacunary index \(-1\). Then for every point \(p \in X\), the point-residual \(\Omega_p\) is a strong parapolar space of lacunary index \(-1\). In particular, \(\Omega\) is locally connected.

Proof. By Axiom (PPS1), there exists a point \(p \in X\) contained in at least two syms. By Fact 3.11 and the absence of syms of rank 2, every connected component of \(\Omega_p\) is a strong parapolar space, obviously with lacunary index \(-1\). Since any two syms through \(p\) have rank at least 3 and share a line, there is only one such component. Then by (ii) and (iii) of the Main Result—Extended Version (which we proved in the previous section), every line through \(p\) is contained in at least two syms. Hence every point of \(\Omega\) collinear to \(p\) is contained in at least two syms and we can interchange its role with that of \(p\). A connectivity argument now shows that \(\Omega\), is a strong parapolar space with lacunary index \(-1\), for every \(x \in X\). Fact 3.12 completes the proof of the lemma. \(\square\)
Lemma 7.2. Parapolar spaces with lacunary index 0 are strong.

Proof. Recall that in the case of symplectic rank 2 this is an assumption. When the symplectic rank is at least 3 this follows from Lemma 7.1 and the fact that strong parapolar spaces of lacunary index $-1$ all have diameter 2, see (ii) and (iii) of the Main result—Extended Version.

We now bound the diameter.

The following lemma is Exercise 13.26 in [15]. We provide a proof for completeness.

Lemma 7.3. Let $\Omega = (X, \mathcal{L})$ be a parapolar space with lacunary index 0. Then $\text{Diam}\, \Omega \leq 3$.

Proof. Suppose for a contradiction that $p_0 \perp p_1 \perp p_2 \perp p_3 \perp p_4$ are points with $\delta(p_0, p_4) = 4$. The symps $\xi(p_0, p_2)$ and $\xi(p_2, p_4)$ (which really are symps by Lemma 7.2) have $p_2$ and hence a line $L$ in common. In the symp $\xi(p_2, p_i)$ there is a point $q_i$ on $L$ collinear to $p_i$, $i = 0, 4$. Then we have $p_0 \perp q_0 \perp q_4 \perp p_4$ and $\delta(p_0, p_4) \leq 3$, a contradiction. This proves the lemma.

We now consider diameters 2 and 3 separately.

7.2 Diameter 2

7.2.1 Diameter 2 and minimum symplectic rank 2

Let $\Omega = (X, \mathcal{L})$ be a (strong) parapolar space with lacunary index 0, minimum symplectic rank 2, and diameter 2. We first prove that the symplectic rank is uniformly 2.

Lemma 7.4. Let $\Omega = (X, \mathcal{L})$ be a (strong) parapolar space with lacunary index 0, diameter 2 and minimum symplectic rank 2. Then $\Omega$ has uniform symplectic rank 2.

Proof. Suppose for a contradiction that there is a symp $\xi$ of rank at least 3. By connectivity, we may assume that some point $p \in \xi$ is also contained in a symp of rank 2. We first claim that every line $L$ through $p$ is contained in a symp of rank 3. Indeed, let $x \in L \setminus \{p\}$ and pick a point $q$ in $\xi$ opposite $p$. Then the symp $\xi(x, q)$ shares a line $K$ with $\xi$ and hence $K$ contains a point $y$ collinear to $x$. Since $K$ does not contain $p$, we obtain a line $yp$ all of whose points are collinear to $x$. The claim now follows from Lemma 3.7.

We next claim that $\Omega_p$ is a parapolar space. Indeed, our first claim implies that, if $L, M$ are lines through $p$, then they belong to respective symps of rank at least 3, and these share a line by 0-lacunarity. Consequently $\Omega_p$ is a connected point-line geometry. Hence Fact 3.11 implies that $\Omega_p$ is either a single line, or the set of lines of a single symp of rank at least 3, or a parapolar space. Clearly the first two possibilities are impossible since there are at least two symps through $p$ by assumption (one of rank 2 and one of rank at least 3). The claim is proved.

Clearly, $\Omega_p$ is strong and has lacunary index $-1$. By the previous section and as can be seen in Table 1, these all have diameter 2. Therefore every pair of lines of $\Omega$ through $p$ is contained in a symp of rank at least 3, contradicting the existence of a symp of rank 2 through $p$. This contradiction shows that all symps through $p$ have rank 2. This proves the lemma.

Hence, if $\Omega = (X, \mathcal{L})$ is a parapolar space with lacunary index 0, minimum symplectic rank 2, and diameter 2, then the symplectic rank is 2 and we clearly have an imbrex geometry. So we can apply Proposition 3.17.
If the symps are thick, then we obtain \((iv)\) of the Main Result—Extended version. An example that this really occurs is given by taking for \((X, \Sigma)\) (with \(\Sigma\) the family of maximal singular subspaces) the dual of any Hermitian quadrangle in a 4-dimensional projective space, and for \(\Xi\) the family of subquadrangles induced by the hyperplanes.

If the symps are non-thick, then \(\Omega\) is the direct product space of two linear spaces \(Y\) and \(Z\). Suppose both contain at least two lines, say \(L_Y, K_Y\) are lines of \(Y\) and \(L_Z, K_Z\) are lines of \(Z\). Obviously, we may choose \(L_Y \cap K_Y\) intersecting (say in the point \(p_Y\)) , and likewise \(L_Z \cap K_Z\) may assumed to have a point, say \(p_Z\), in common with each other. Then the symps \(L_Y \times L_Z\) and \(K_Y \times K_Z\) intersect only at the point \((p_Y, p_Z)\), which is impossible by 0-lacunarity. Hence one of \(Y, Z\) is trivial and isomorphic to a thick line.

This takes care of \((iv)\) and \((v)\) of the Main Result—Extended version.

\subsection*{7.2.2 Diameter 2 and symplectic rank at least 3}

In this subsection we show that parapolar spaces of diameter 2, symplectic rank at least 3 and lacunary index 0 satisfy the assumptions of Theorem 3.15.

\textbf{Lemma 7.5.} Let \(\Omega = (X, \mathcal{L})\) be a parapolar space of diameter 2, symplectic rank at least 3 and lacunary index 0. Then \(\Omega\) has uniform symplectic rank \(d \geq 3\) and satisfies Condition \((CC)_{d-2}\).

\textit{Proof.} By Lemma 7.1, the point-residual \(\Omega_p\), for any \(p \in X\), is a strong parapolar space with lacunary index \(-1\). By the Main Result—Extended Version, it is one of \(A_{1,1}(\ast) \times A_{2,1}(\ast), A_{2,1}(\ast) \times A_{2,1}(\ast), A_{4,2}(L), A_{6,2}(L), E_{6,1}(K)\). Clearly, if \(p \perp q \in \Omega\), then the parameters (singular ranks, symplectic rank; see Table 1) of \(\Omega_p\) and \(\Omega_q\) coincide, which implies, given the above list, that \(\Omega_p\) and \(\Omega_q\) are isomorphic. By connectivity we conclude that all point-residuals have the same uniform symplectic rank, say \(d - 1 \geq 2\). Hence \(\Omega\) has uniform symplectic rank \(d \geq 3\).

By Lemma 3.19, it suffices to check \((CC)_{d-3}\) in the point-residuals. This is a straightforward verification, given the list above (this also follows from Theorem 3.20).

Next, we verify the finite singular rank condition of Theorem 3.15.

\textbf{Lemma 7.6.} Let \(\Omega = (X, \mathcal{L})\) be a (strong) parapolar space with lacunary index 0 and symplectic rank at least 3. Then the maximal singular subspaces have finite dimension.

\textit{Proof.} The point-residuals of \(\Omega\) are (strong) parapolar spaces of lacunary index \(-1\), by Lemma 7.1. These all have maximal singular subspaces of finite projective dimension by our classification, see Table 1. Whence the lemma.

By Lemmas 7.5 and 7.6, we have proved the hypotheses of Theorem 3.15. Moreover, by Lemma 3.16, \(A_{2n-1,n}(L)/(\sigma)\) has diameter at least 5, \(D_{n,n}(K)\) has diameter at least 4 when \(n \geq 6\) (and hence cannot be 0-lacunary), \(E_{6,1}(K)\) has symps intersecting exactly in one point, and \(E_{7,7}(K)\) has diameter 3, where \(K\) is a commutative field, and \(L\) is a skew field.

This leads to conclusion (vii) of the Main Result—Extended version.
7.3 Diameter 3

In this case, we verify that the conditions of Theorem 3.14 are fulfilled. We begin by verifying Condition 3.14(i).

**Lemma 7.7.** Let \( \Omega = (X, \mathcal{L}) \) be a (strong) parapolar space with lacunary index 0 and diameter 3. Then for every point-symplecton pair \((p, \xi)\), we have \( p \perp \xi \neq \emptyset \). In particular, the distance between a point and a line in \( \Omega \) is at most 2.

**Proof.** Consider a point \( q \) not in \( \xi \), for which there is a path of length two, say \( q \perp r \perp s \), for \( r, s \in X \), with \( s \in \xi \). We may assume \( q \notin s \perp \). By assumption the symps \( \xi(q, s) \) and \( \xi(p, s) \) intersect in a line \( L \). But then \( q \) is collinear with a point on \( L \), which also lies in \( \xi \).

So we can keep shortening the path from \( p \) to \( \xi \), which exists by connectivity, and hence we have proved the first part of the lemma. For the second statement, let \( p \in X \) and \( L \in \mathcal{L} \), then we include \( L \) in symp \( \xi \), obtain a point \( q \in \xi \) with \( q \perp p \), and so any point on \( L \) collinear to \( q \) is at distance at most 2 from \( p \).

Next, we show Condition 3.14(ii).

**Lemma 7.8.** Let \( \Omega = (X, \mathcal{L}) \) be a (strong) parapolar space with lacunary index 0 and diameter 3. Then the points at distance at most 2 from a given point \( p \) form a subspace, which either is a geometric hyperplane or coincides with \( \Omega \).

**Proof.** Let \( L \) be a line containing at least two points at distance at most 2 from \( p \), say \( x, y \). If \( \delta(p, L) = 1 \) there is nothing to prove, so we may assume \( \delta(p, x) = \delta(p, y) = 2 \). Since \( \Omega \) is strong, we have the symps \( \xi(p, x) \) and \( \xi(p, y) \), which intersect in a line \( L_p \). If the points \( x_p, y_p \in L_p \) collinear to respectively \( x, y \), coincide, then \( x \perp y \perp x_p \perp x \) and hence, by Axiom (PPS1), \( x_p \) is collinear to each point of \( L \), completing the proof. If \( x_p \neq y_p \), then \( p \) and \( L \) are contained in the symp \( \xi(y, x_p) = \xi(x, y_p) \) and hence \( \delta(p, L) = 1 \), concluding the proof, noting that the last statement follows from the last statement of Lemma 7.7.

It now follows from Theorem 3.14 and Lemmas 7.6, 7.7, and 7.8 that a parapolar space with lacunary index 0 and diameter 3 is one of the following.

- \( D_{6,6}(\mathbb{K}), A_{5,3}(\mathbb{L}) \) or \( E_{7,7}(\mathbb{K}) \);
- a dual polar space of rank 3 (\( B_{3,3}(\ast) \));
- a product geometry \( L \times \Delta \), where \( L \) is a thick line, and \( \Delta \) is a non-degenerate polar space of rank at least 2.

This proves (vi) and (viii) of the Main Result—Extended Version.

8 Parapolar spaces with lacunary index 1

Let \( \Omega = (X, \mathcal{L}) \) be a locally connected parapolar space with lacunary index 1, strong if there are symps of rank 2. We start by showing that the symplectic rank is at least 3. Then, since the point-residuals are 0-lacunary parapolar spaces, we can distinguish between the diameters 2 and 3 in these point-residuals. Therefore, we first show that all such residuals are of the same type.

**Lemma 8.1.** Let \( \Omega = (X, \mathcal{L}) \) be a strong parapolar space of minimum symplectic rank 2. Then \( \Omega \) is not 1-lacunary.
Proof. Suppose for a contradiction that $\Omega$ is 1-lacunary. Let $\xi$ be any symp of $\Omega$ of rank 2. By connectivity, there is a symp $\xi'$ intersecting $\xi$ non-trivially. Then $\xi \cap \xi'$ is a point $p$, as otherwise 1-lacunarity forces $\xi \cap \xi'$ to contain a plane, which is impossible as $\xi$ has rank 2. Let $L$ and $L'$ be lines through $p$ in $\xi$ and $\xi'$, respectively. Since $\Omega$ is strong, there is a symp through $L$ and $L'$, which intersects $\xi$ in $L$, contradicting what we have just deduced. \hfill $\Box$

Lemma 8.1 implies (xiv) for minimum symplectic rank 2 of the Main Result—Extended Version.

One might wonder what happens if the strongness assumption is omitted. In this case, there is a classification, see Remark 10.9.

**Lemma 8.2.** Let $\Omega = (X, \mathcal{L})$ be a locally connected parapolar space with lacunary index 1 and symplectic rank at least 3. Then, for each two points $p, q \in X$, the point-residuals $\Omega_p$ and $\Omega_q$ are Lie incidence geometries of the same Coxeter type.

**Proof.** Take any point $p \in X$ and consider the point-residual $\Omega_p$. Since $\Omega$ is locally connected, Fact 3.12 implies that $\Omega_p$ is connected, and then we obtain from Fact 3.11 that $\Omega_p$ is a (strong) parapolar space. Clearly, $\Omega_p$ is 0-lacunary. Moreover, the singular subspaces of $\Omega_p$ are projective, since this is the case for $\Omega$ (cf. Lemma 3.7). Hence $\Omega_p$ is as in Tables 1 and 2 (columns corresponding to $k = 0$), except that the GQ-case does not occur, and that the linear space “LS” is a projective space. Consequently, $\Omega_p$ is a Lie incidence geometry. One also observes that we can distinguish the entries of these columns by their symplectic and singular ranks.

Now let $q \in X$ be collinear with $p$. We claim that $\Omega_q$ has the same symplectic and singular ranks as $\Omega_p$. Indeed, each symp through the line $pq$ corresponds with a unique symp of $\Omega_p$ through $q$ and with a unique symp $\Omega_q$ through $p$, and vice versa. Hence there is a bijective correspondence between the symps of $\Omega_p$ through $q$ and of $\Omega_q$ through $p$; likewise for the maximal singular subspaces. Each point in $\Omega_p$ (resp. $\Omega_q$) plays the same role, so the local parameters determine the global ones, proving the claim. So $\Omega_p$ and $\Omega_q$ have the same Coxeter type indeed, and by connectivity, the lemma follows. \hfill $\Box$

**Proposition 8.3.** Let $\Omega = (X, \mathcal{L})$ be a locally connected parapolar space with lacunary index 1 and symplectic rank at least 3. Assume that $\Omega_p$ has diameter 3, for some point $p \in X$. Then $\Omega$ is one of $B_{n,2}(*), D_{4,2}(\mathbb{K})$ or $F_{4,1}(\mathbb{K})$.

**Proof.** By Lemma 8.2, $\Omega_q$ has diameter 3, for any point $q \in X$. We now verify the conditions of Theorem 3.18.

(NP) It suffices to check the following property in any point-residual: Each point $x$ is collinear to at least one point of any given symp. Since every point-residual is 0-lacunary, and has diameter 3 by assumption, this follows from Lemma 7.7.

(F) This follows from Tables 1 and 2 observing that the parapolar space in the gray cells (these are the ones with diameter 3) corresponding to $k = 0$ all have finite singular rank.

Hence we can apply Theorem 3.18 and obtain the parapolar spaces in the statement, noting that $\Omega$ cannot be strong (since this would imply that every point-residual has diameter 2). \hfill $\Box$

**Proposition 8.4.** Let $\Omega = (X, \mathcal{L})$ be a locally connected parapolar space with lacunary index 1 and symplectic rank $d$ at least 3. Assume that $\Omega_p$ has diameter 2, for some point $p \in X$. Then $\Omega$ is one of $A_{n,2}(L), D_{5,5}(\mathbb{K})$ or $E_{6,1}(\mathbb{K})$.

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Proof. Take any point \( p \in X \) and consider the point-residual \( \Omega_p \). As in the proof of Lemma 8.2, we deduce that \( \Omega_p \) is a strong parapolar space with lacunary index 0, each of whose singular subspaces are projective. Hence Tables 1 and 2 and our assumption imply that \( \Omega_p \) is either \( A_{1,1}(\ast) \times A_{n,1}(\mathbb{L}) \), \( A_{4,2}(\mathbb{L}) \) or \( D_{5,5}(\mathbb{K}) \). Since all these have uniform symplectic rank, Lemma 8.2 implies that \( \Omega \) has uniform symplectic rank 3, 4 or 5, respectively. We want to apply Theorem 3.15. If \( d \geq 4 \), this requires that \( \Omega \) is strong and has finite singular rank, which is the case: the point-residuals \( A_{4,2}(\mathbb{L}) \) and \( D_{5,5}(\mathbb{K}) \) have diameter 2, implying that \( \Omega \) is strong; and as their singular rank is finite, so is \( \Omega \)’s. Finally, we verify the Condition \((CC)_{d-2}\). By Lemma 3.19, it suffices to verify Condition \((CC)_{d-3}\) (or \((H)\)) in the point-residuals. For \( d = 4, 5 \), this follows from the last statement of Theorem 3.20. For \( d = 3 \), this is an easy verification.

\[ \text{Propositions 8.3 and 8.4 imply (ix) of the Main Result—Extended Version.} \]

9 Parapolar spaces with lacunary index at least 2

Let \( \Omega = (X, \mathcal{L}) \) be a locally connected parapolar space with lacunary index \( k \geq 2 \). By definition, the symplectic rank is at least \( k + 1 \).

We verify the assumptions of Shult’s Haircut Theorem (cf. Theorem 3.20).

Lemma 9.1. Let \( \Omega = (X, \mathcal{L}) \) be a locally connected parapolar space with lacunary index \( k \geq 2 \), and symplectic rank at least \( d \) with \( d \geq k + 1 \). Then \( \Omega \) satisfies the Haircut Axiom \((H)\), has uniform symplectic rank and bounded singular rank.

Proof. By Lemma 3.19, it suffices to check the Haircut Axiom \((H)\) in every point-residual.

First note that the fact that \( \Omega \) is locally connected implies as before that, for each point \( p \in X \), the point-residual \( \Omega_p \) is a strong parapolar space, clearly with lacunary index \( k - 1 \) and of symplectic rank at least \( d - 1 \). We claim that \( \Omega_p \) is locally connected. Indeed, when \( d \geq 4 \), this follows from the fact that \( \Omega_p \) is a strong parapolar space of symplectic rank at least 3. If the minimum symplectic rank of \( \Omega \) is 3, which happens only if \( k = 2 \), then \( \Omega_p \) is a strong parapolar space of minimum symplectic rank 2 with lacunary index 1, which is impossible by Lemma 8.1. Hence we may assume \( d \geq 4 \) and the claim follows. This allows us to take subsequent point-residuals until we obtain a residual with lacunary index 1 (i.e., a \( K \)-residual of \( \Omega \) for some singular subspace \( K \) of \( \Omega \) of dimension \( k - 2 \)).

In the previous section we determined the locally connected parapolar spaces with lacunary index 1 (for the precise results, see Main Result—Extended Version (ix) or the columns in Tables 1 and 2 corresponding to \( k = 1 \)). According to these tables, the strong such parapolar spaces—which are written in black—are precisely \( D_{5,5}(\mathbb{K}) \), \( E_{6,2}(\mathbb{K}) \) or \( A_{n,2}(\mathbb{L}) \) with \( n \geq 4 \) (with \( \mathbb{K} \) a commutative field and \( \mathbb{L} \) a skew field, as usual). Hence these are exactly the possibilities for the \((k - 2)\)-residue of \( \Omega \). Since those geometries all appear in the conclusion of Theorem 3.20, they satisfy \((H)\) and hence so does \( \Omega \) by (successive applications of) Lemma 3.19. Moreover, they have uniform symplectic rank and bounded singular rank and hence it is clear that the same holds for \( \Omega \).

We can now conclude the proof of our Main Theorem, using Shult’s Haircut Theorem and an induction on the lacunary index \( k \), starting with \( k = 2 \).

Theorem 9.2. Let \( \Omega = (X, \mathcal{L}) \) be a locally connected parapolar space with lacunary index \( k \geq 2 \), and uniform symplectic rank \( d \geq k + 1 \). Then \( \Omega \) is one of the parapolar spaces mentioned in Tables 1 and 2 in the columns corresponding to \( k = 2, 3, 4, 5 \).
Proof. We already know by Lemma 9.1 that Ω satisfies all assumptions of Theorem 3.20. Hence we can apply the latter theorem and single out the k-lacunary parapolar spaces with point-residuals isomorphic to the black entries in Tables 1 and 2 in the columns corresponding to lacunary index k − 1 ≥ 1. This gives us the desired parapolar spaces. □

This takes care of (x), (xi), (xii), (xiii) and (xiv) of the Main Result—Extended Version.

10 The locally disconnected case

In this final section, we prove a general result that enables us to consider any parapolar space of symplectic rank at least 3, which is not locally connected as a certain (non-disjoint) union of locally connected (para)polar spaces. Our approach has an analogue for graphs, see Exercise 1.17 in [15]. Then, using our classification of locally connected lacunary parapolar spaces, this yields a universal construction of all lacunary parapolar spaces with symplectic rank at least 3 which are not locally connected. Recall that we classified the strong lacunary parapolar spaces of minimum symplectic rank 2, without any assumption on local connectedness (the latter’s definition becoming of no use in this case). Whence our restriction on the symplectic rank being at least 3.

Let Ω = (X, L) be an arbitrary parapolar space with symplectic rank at least 3. For each point p ∈ X, we denote by C_p the set of connected components of Ω_p (see Definition 3.10 and Fact 3.11). By Fact 3.12, Ω is locally connected if and only if Ω_p is connected. The following construction introduces a copy of a point p for each connected component of Ω_p.

Construction 10.1. Let Ω = (X, L) be an arbitrary parapolar space with symplectic rank at least 3. The unbuttoning of Ω is defined as the following point-line geometry Ω̃ = (X̃, L̃):

- X̃ = {(p, Y) : p ∈ X and Y ∈ C_p};
- for each line L ∈ L, we define L̃ = {(p, Y) ∈ X̃ : p ∈ L ∈ Y},
- L̃ = {L : L ∈ L}.

So two points (p_1, Y_1) and (p_2, Y_2), with Y_i ∈ Ω_p for i = 1, 2, are collinear in Ω̃ if and only if p_1 ⊥ p_2 and the line p_1p_2 is an element of both Y_1 and Y_2.

We now have the following result.

Proposition 10.2. Let Ω = (X, L) be a not necessarily locally connected parapolar space. Then its unbuttoning Ω̃ is the disjoint union of locally connected (para)polar spaces.

Proof. We verify the axioms of a parapolar space except that in Axiom (PPS1) we do not require that there is a point-line pair (p, L) such that no point of L is collinear to p, nor do we require that Ω̃ is connected, instead we will in the end consider its connected components.

(PPS1) Suppose (p, Y) ∈ X̃ and L ∈ L̃ are such that (p, Y) ̸∈ L̃ is collinear to at least two points of L̃. Let (x^*, Y^*) be any point of L̃. In Ω̃, at least two points of L̃ are collinear to p, so (p, L) is a plane π. This means that each line of π through p belongs to Y (in particular, px^* ∈ Y) and likewise each line of π through x^* is contained in Y^* (in particular, px^* ∈ Y^*). Consequently, (p, Y) and (x^*, Y^*) are contained in px^* and as such they are indeed collinear in Ω̃.
(PPS2) Let \((p_i, \Sigma_i) \in \tilde{\Omega}, i = 1, 2\) be two non-collinear points of \(\tilde{\Omega}\) collinear to at least one common point \((x_1, \Sigma_1)\) of \(\Omega\). We claim that \(p_1\) and \(p_2\) are not collinear in \(\tilde{\Omega}\). Indeed, suppose they are. Since \((p_1, \Sigma_1)\) is collinear to \((x_1, \Sigma_1)\), the line \(p_1x_1\) belongs to \(\Sigma_1\). As \(x_1\) is collinear to \(p_2\), the line \(p_1p_2\) lies in \(\Sigma_1\) too. Likewise we obtain \(p_1p_2 \in \Sigma_2\). But then the points \((p_i, \Sigma_i), i = 1, 2\) both belong to \(p_1p_2\), a contradiction. Our claim follows. Now suppose that both \((p_i, \Sigma_i), i = 1, 2\), are collinear to a second point \((x_2, \Sigma_2)\), with \((x_1, \Sigma_1) \neq (x_2, \Sigma_2)\). Since \(x_ip_i \in \Sigma_i\) for \(i = 1, 2\) and \(\Sigma_1 \cap \Sigma_2 = \{x_1\}\) if \(x_1 = x_2\), we deduce \(x_1 \neq x_2\).

We now show that the convex closure \(C\) of \((p_1, \Sigma_1)\) and \((p_2, \Sigma_2)\) is a polar space canonically isomorphic to the symplectic \(\xi := \xi(p_1, p_2)\). To that aim, we have to show two claims, the first one of which is the following.

- **Claim 1:** If \(x \in \xi(p_1, p_2)\) and if we denote by \(\Sigma_{x, \xi}\) the component of \(\Omega_x\) containing the lines of \(\xi\) through \(x\), then \((x, \Sigma_{x, \xi})\) belongs to \(C\).

Indeed, by Fact 3.4, it suffices to show that that \((x, \Sigma_{x, \xi}) \in C\) for all points \(x\) which are contained in a line joining \(p_1\) or \(p_2\) with a point of \(p_1^\perp \cap p_2^\perp\). Suppose first that \(x \in p_1^\perp \cap p_2^\perp\). Then, firstly, \(xp_1\) and \(xp_2\) belong to \(\Sigma_{x, \xi}\) by our assumption on \(\Sigma_{x, \xi}\). Secondly, \(p_x\) belongs to \(\Sigma_i, i = 1, 2\), because \(p_x\) lies in the same connected component of \(\Sigma_{\rho_i}\) as \(p_ix^1\) and \(p_ix^2\) (these lines all lie in \(\xi\)), \(i = 1, 2\). This shows that the point \((x, \Sigma_{x, \xi})\) is collinear to \((p_i, \Sigma_i)\), \(i = 1, 2\), and hence belongs to \(C\) indeed. Similarly one can now show that each point \(x'\) on the line \(p_x\) is such that \((x', \Sigma_{x', \xi})\) is on the line joining \((p_i, \Sigma_i)\) and \((x, \Sigma_{x, \xi})\), \(i = 1, 2\), and hence \((x', \Sigma_{x', \xi}) \in C\) too. This shows Claim 1.

The second claim is the following.

- **Claim 2:** If \((y, \Sigma) \in C\), then \(y \in \xi\), and \(\Sigma\) is the component of \(\Omega_y\) containing the lines of \(\xi\) through \(y\).

Indeed, let \(C'\) denote the set of points \((x, \Sigma_{x, \xi})\), with \(x \in \xi\). Let \(\rho\) be the projection map \(C' \rightarrow \xi : (x, \Sigma) \mapsto x\). Then \(\rho\) is an isomorphism of point-line geometries: the first paragraph implies that \(\rho\) preserves collinearity and is injective; surjectivity follows by definition of \(C'\). Hence \(C'\) is a polar space containing \((p_1, \Sigma_1)\) and \((p_2, \Sigma_2)\) and therefore \(C' = C\).

This concludes the verification of Axiom (PPS2).

(PPS3) Let \(\tilde{L}\) be a line of \(\tilde{\Omega}\). Then \(L \in \mathcal{L}'\) is contained in some symplectic \(\xi\). We consider two points \(p_1, p_2 \in \xi\) at distance 2 and with \(p_1 \in L\). We showed above that \((p_1, \Sigma_{p_1, \xi})\) and \((p_2, \Sigma_{p_2, \xi})\) determine a symplectic \(\tilde{\xi}\) in \(\tilde{\Omega}\), which contains precisely the points \((x, \Sigma_{x, \xi})\) with \(x \in \xi\), so in particular those with \(x \in L\). Since \(L \subseteq \xi\), we have that \(L \subseteq \Sigma_{x, \xi}\) for all \(x \in L\), showing that \(\tilde{L}\) belongs to \(\tilde{\xi}\).

This shows that each connected component \(\omega\) of \(\tilde{\Omega}\) is a (para)polar space. The fact that \(\omega\) is locally connected follows immediately from the definition of \(\tilde{\Omega}\). This proves the proposition.

We are now interested in a reverse procedure. Which parapolar spaces can we obtain by collecting connected locally (para)polar spaces and identifying certain points? As before we may restrict to the case of symplectic rank at least 3.

The following lemma is necessary to make the construction universal. Basically it says that, in \(\Omega\), you cannot walk from a point \(p\) to itself in less than five steps using two different components of \(\Omega_p\) to start and come back in.
Lemma 10.3. Let $\Omega = (X, L)$ be a not necessarily locally connected parapolar space with symplectic rank at least 3. Let $\Omega$ be its unbuttoning. Let $p \in X$ be such that $\Omega_p$ is disconnected and let $\Omega_1^{(p)}$ and $\Omega_2^{(p)}$ be two distinct connected components of $\Omega_p$. Let $q, r, s \in X \setminus \{p\}$ be arbitrary (not necessarily distinct) and let $\Omega_1^{(q)}, \Omega_2^{(q)}, \Omega_1^{(r)}, \Omega_2^{(r)}, \Omega_1^{(s)}, \Omega_2^{(s)}$ be not necessarily distinct respective connected components (with self-explaining notation) of $\Omega_q, \Omega_r, \Omega_s$. Then

$$\ell := \delta((p, \Omega_1^{(p)}), (q, \Omega_1^{(q)})) + \delta((q, \Omega_2^{(q)}), (r, \Omega_1^{(r)})) + \delta((r, \Omega_2^{(r)}), (s, \Omega_1^{(s)})) + \delta((s, \Omega_2^{(s)}), (p, \Omega_1^{(p)})) \geq 5 \quad \text{(points in different components of $\Omega$ have distance } \infty, \text{ which is by definition larger than any positive number).}$$

Proof. Suppose for a contradiction that $\ell \leq 4$. We examine the case $\ell = 4$, leaving the easier cases $\ell = 1, 2, 3$ to the interested reader. The assumption $\ell = 4$ allows us to also assume that

$$\delta((p, \Omega_2^{(p)}), (q, \Omega_1^{(q)})) = \delta((q, \Omega_2^{(q)}), (r, \Omega_1^{(r)})) = \delta((r, \Omega_2^{(r)}), (s, \Omega_1^{(s)})) = \delta((s, \Omega_2^{(s)}), (p, \Omega_1^{(p)})) \geq 1,$$

since, if some of these distances would be 0, then another distance must be at least 2 and we can insert a chain of points consecutively at distance 1, rename, and get the above assumption back.

By the definition of lines in $\Omega$ we then obtain $p \perp q \perp r \perp s \perp p$. First note that the lines $pq$ and $ps$ belong to $\Omega_2^{(p)}$ and $\Omega_1^{(p)}$, respectively. By assumption, $\Omega_1^{(p)} \neq \Omega_2^{(p)}$. This already implies $q \neq s$. It also implies that $q$ cannot be collinear to $s$, for then $\langle p, q, s \rangle$ would be a projective plane (cf. Fact 3.8), yielding $\Omega_1^{(p)} = \Omega_2^{(p)}$ after all. However, if $q$ and $s$ are not collinear, they determine a symp $\xi$ since $p \neq r$, clearly containing the lines $pq$ and $ps$, which again leads to $\Omega_1^{(p)} = \Omega_2^{(p)}$. This contradiction proves the lemma. \[\square\]

We now present a construction of the class of locally disconnected parapolar spaces with symplectic rank at least 3 and with lacunary index $k \geq -1, k \neq 0$ (recall that 0-lacunary spaces automatically locally connected by Lemma 7.1). For convenience we shall call a polar space $k$-lacunary whenever its rank is at least $k + 1$ and refer to its rank as its symplectic rank.

Construction 10.4. Let $\mathcal{F} = \{\Omega_i = (X_i, L_i) : i \in I\}$ be a family of (disjoint) $k$-lacunary locally connected (para)polar spaces of symplectic rank at least 3, over some nonempty index set $I$, $0 \neq k \geq -1$. If $k = -1$, then we additionally require that $\mathcal{F}$ only consists of polar spaces of rank at least 3. Let $\mathcal{R}$ be an equivalence relation on the union $\tilde{X} = \bigcup_{i \in I} X_i$ of the sets of points of all members of $\mathcal{F}$, satisfying the following two conditions (C1) and (C2).

(C1) Let $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ be four (not necessarily distinct, but $\tilde{p} \notin \{\tilde{q}, \tilde{r}, \tilde{s}\}$) equivalence classes with respect to $\mathcal{R}$, and let $p_1, p_2 \in \tilde{p}$, with $p_1 \neq p_2$. If $q_1, q_2 \in \tilde{q}, r_1, r_2 \in \tilde{r}$ and $s_1, s_2 \in \tilde{s}$, then

$$\delta(p_2, q_1) + \delta(q_2, r_1) + \delta(r_2, s_1) + \delta(s_2, p_1) \geq 5.$$

(C2) The graph with vertex set $\mathcal{F}$, where two vertices $\Omega_i$ and $\Omega_j$, $i, j \in I$, are adjacent if some point of $\Omega_i$ is contained in the same equivalence class as some point of $\Omega_j$, is connected. If $k = -1$, we additionally require that this graph is a complete graph.

Set $X = \tilde{X} / \mathcal{R}$. For each line $L$ contained in some member of $\mathcal{F}$, we put $\tilde{L} := \{\tilde{p} \mid p \in L\}$ and define $\mathcal{L}$ as $\{\tilde{L} \mid L \in L_i \text{ for some } i \in I\}$. Then we denote the geometry $\Omega = (X, \mathcal{L})$ by $\Omega(\mathcal{F}, \mathcal{R})$. If $\mathcal{R}$ is non-trivial, then we call $\Omega$ a $k$-buttoned geometry. \[\square\]
Remark 10.5. We claim that distinct lines $L$ and $L'$ define distinct sets $\tilde{L}$ and $\tilde{L}'$. Indeed, suppose $\tilde{L} = \tilde{L}'$ and take points $\tilde{p}, \tilde{q} \in \tilde{L} = \tilde{L}'$ with $\tilde{p} \neq \tilde{q}$. Let $p_1, q_2$ be (distinct) points on $L$ and $p_2, q_1$ (distinct) points on $L'$ with $p_1, p_2 \in \tilde{p}$ and $q_1, q_2 \in \tilde{q}$. Then Condition (C1), with $\tilde{r} = \tilde{s} = q_1, r_1 = s_2 = q_2$ and $r_2 = s_1 = q_1$ implies $p_1 = p_2$. Since $\tilde{p} \in L$ was arbitrary, we obtain $L = L'$. The claim follows.

Remark 10.6. Note that (C1) implies that an equivalence class of $R$ cannot contain two points of the same member of $F$ if that member’s diameter is smaller than 5. This situation in particular applies if all members of $F$ are ordinary polar spaces; which is always the case when $k = -1$ (by assumption) and when $k \geq 6$ (as follows from our classification of locally connected $k$-lacunary parapolar spaces).

We have the following crucial result.

Proposition 10.7. For every $k \geq -1$, $k \neq 0$, every $k$-buttoned parapolar space is a locally disconnected $k$-lacunary parapolar space. More exactly, let $\mathcal{F} = \{\Omega_i = (X_i, \mathcal{L}_i) : i \in I\}$ be a family of (disjoint) $k$-lacunary locally connected (para)polar spaces of symplectic rank at least 3, over some nonempty index set $I$ (if $k = -1$, we only allow polar spaces of arbitrary rank at least 3). Let $\mathcal{R}$ be a non-trivial equivalence relation on the union $X = \bigcup_{i \in I} X_i$ of the sets of points of all members of $\mathcal{F}$, satisfying Conditions (C1) and (C2). Then the geometry $\Omega(\mathcal{F}, \mathcal{R})$ is a locally disconnected $k$-lacunary parapolar space.

Proof. We verify the axioms of a parapolar space for $\Omega := \Omega(\mathcal{F}, \mathcal{R})$.

(PPS1) Condition (C2) implies immediately that $\Omega$ is connected. Let $\tilde{L} \in \mathcal{L}$ and $\tilde{p} \in X$ with $\tilde{p} \notin \tilde{L}$ be such that $\tilde{p}$ is collinear to at least two points $\tilde{q}, \tilde{r} \in \tilde{L}$. The definition of $\mathcal{L}$ yields (unique) points $p_1, p_2 \in \tilde{p}$, $q_1, q_2 \in \tilde{q}$, $r_1, r_2 \in \tilde{r}$ with $p_1 \perp q_2$, $q_1 \perp r_2$, $r_1 \perp p_2$. By Condition (C1), $p_1 = p_2$, $q_1 = q_2$ and $r_1 = r_2$. It follows that $q_1 r_1 = L$ and hence $p_1$ is collinear to each point of $L$. This implies that $\tilde{p}$ is collinear to each point of $L$. Condition (C1) implies that there exist $p \in \tilde{p}$, $q \in \tilde{q}$ and $r \in \tilde{r}$ with $p \perp q \perp r \perp p$ and so all of $p, q, r$ lie in a common member of $\mathcal{F}$, implying that $p$ is collinear to all points of $L$.

Now assume first that $\mathcal{F}$ contains a parapolar space. Let $\Omega_i, i \in I$ be such a member, let $p \in X_i$ be arbitrary and let $L \in \mathcal{L}_i$ be arbitrary but such that $\delta(p, L) = 2$. Suppose for a contradiction that $\tilde{p}$ is collinear to some point $\tilde{q}$ on $\tilde{L}$. Let $p \perp s \perp r \in L$. Then $\tilde{p} \perp \tilde{q} \perp \tilde{r} \perp \tilde{s} \perp \tilde{p}$, which implies by (C1) that all of $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ contain representatives in $\Omega_i$, and hence $p$ is collinear to some point of $L$ after all, a contradiction.

Next assume all members of $\mathcal{F}$ are polar spaces. Then there exist two members $\Omega_i, \Omega_j, i, j \in I$, and points $p_i \in X_i$ and $p_j \in X_j$ such that $p_i$ and $p_j$ are contained in the same equivalence class $\tilde{p}$. Choose a point $x \in X_i$ collinear to $p_i$ and a line $L$ in $\Omega_j$ not incident with $p_j$. One now easily checks that, with self-explaining notation, $\tilde{x}$ is not collinear to any point of $\tilde{L}$.

This completes the proof of (PPS1).

(PPS2) Let $\tilde{p} \perp \tilde{q} \perp \tilde{r} \perp \tilde{s} \perp \tilde{p}$ be a quadrangle in $\Omega$ with $\tilde{p}$ not collinear to $\tilde{r}$. Condition (C1) implies that there are unique representatives $p, q, r, s$ of $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$, respectively, contained in a common member $\Omega_i$ of $\mathcal{F}$, for a unique $i \in I$. Clearly, $p \perp q \perp r \perp s$ and $p$ and $r$ are not collinear. It follows that the image in $X$ of the unique symp $\xi(p, r)$ of $\Omega_i$ containing $p$ and $r$ is part of the convex closure $C$ of $\tilde{p}$ and $\tilde{r}$. But $C$ does not contain any further points since this would yield a circuit of length 4 and again a contradiction to Condition (C1). In particular this shows that the map $p \mapsto \tilde{p}$ is bijective when restricted to the point set of symps of $\mathcal{F}$ and hence symps of $\mathcal{F}$ correspond bijectively with symps in $\Omega$. 

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(PPS3) In view of the above, this follows immediately from the fact that every line of any member of \( \mathcal{F} \) is contained in a symp of that member.

The \( k \)-lacunarity for \( k \geq 1 \) now follows easily since symps of \( \Omega(\mathcal{F}, \mathcal{R}) \) that intersect in at least a line are contained in the same member of \( \mathcal{F} \). For \( k = -1 \), this follows directly from Condition (C2). Also, the relation \( \mathcal{R} \) is non-trivial and so there exists a class \( \tilde{p} \) with at least two elements, say \( p, p' \). By Condition (C1) \( p^\perp \cap p'^\perp = \emptyset \) and moreover, no pair of points in \( p^\perp \cup p'^\perp \) is in the same equivalence class. This implies that \( p^\perp \) and \( p'^\perp \) induce two connected components of \( \tilde{p} \), so \( \tilde{p} \) is locally disconnected at \( \tilde{p} \).

This implies the following classification of locally disconnected lacunary parapolar spaces of symplectic rank at least 3.

**Theorem 10.8.** Let \( \Omega = (X, \mathcal{L}) \) be a \( k \)-lacunary parapolar space with symplectic rank at least 3 and \( k \geq -1 \). Then either \( \tilde{\Omega} \) is locally connected (and hence is one of the parapolar spaces of rank at least 3 in Tables 1 and 2), or \( \Omega \) is a \( k \)-buttoned parapolar space.

**Proof.** If \( \Omega \) is locally connected, then this follows from the previous sections. If \( k = 0 \), then by Lemma 7.1, \( \tilde{\Omega} \) is automatically locally connected. If \( \Omega \) is not locally connected, then let \( \tilde{\Omega} \) be its unbuttoning. Let \( \mathcal{F} \) be the family of connected components of \( \tilde{\Omega} \) and let \( \mathcal{R} \) be the equivalence relation on the point set of \( \tilde{\Omega} \) defined by “sharing the first component” (remember points of \( \tilde{\Omega} \) are pairs \((p, \Upsilon)\) with \( p \in X \) and \( \Upsilon \) a connected component of \( \Omega_p \)). If \( \mathcal{R} \) satisfies Conditions (C1) and (C2), it is clear that \( \tilde{\Omega} \) is isomorphic to the \( k \)-buttoned geometry \( \Omega(\mathcal{F}, \mathcal{R}) \) arising from \( \mathcal{F} \) and \( \mathcal{R} \). Now, by Lemma 10.3, \( \mathcal{R} \) satisfies Condition (C1), and if \( k \neq -1 \), the connectivity of \( \Omega \) implies Condition (C2).

Hence suppose \( k = -1 \) now. Since we also assume that \( \tilde{\Omega} \) is not locally connected, Lemma 6.22 implies that every line of \( \tilde{\Omega} \) is contained in a unique symp. A moment’s thought reveals that all connected components of each point-residual are ordinary polar spaces. Consequently, the unbuttoning of \( \tilde{\Omega} \) only contains polar spaces, and then \((-1)\)-lacunarity implies Condition (C2). The theorem is completely proved.

This completes our classification of the (possibly locally disconnected) \( k \)-lacunary parapolar spaces \( \Omega \) of symplectic rank at least 3, and hence our classification of all \( k \)-lacunary parapolar spaces \( \Omega \), only assuming strongness if the minimum symplectic rank is 2. We have one final remark regarding this last assumption, at least when \( k = 1 \), as the existence of such parapolar spaces was excluded right away (cf. Lemma 8.1). It however appears that something can be said if it is not strong too.

**Remark 10.9.** Let \( \Omega = (X, \mathcal{L}) \) be a parapolar space with lacunary index 1 and with minimum symplectic rank 2. Then \( \Omega \) is not strong by Lemma 8.1. Set

\[
X' = \{x \in X : x \text{ is contained in a symp of rank at least } 3\}
\]

and

\[
\mathcal{L}' = \{L \in \mathcal{L} : L \text{ is contained in a symp of rank } 3\}.
\]

Then each connected component \( \omega \) of \( \Omega' = (X', \mathcal{L}') \) is a (para)polar space with symplectic rank at least 3 (use the observation that no member of \( \mathcal{L}' \) is contained in a symp of rank 2 belonging to \( \Omega \)). Moreover, if \( \omega \) is a parapolar space, it is 1-lacunary. Considering the disjoint union of all connected components of \( \Omega' \) and all symps of \( \Omega \) of rank 2 and then identifying all points which were the same in \( \Omega \), we see that this boils down to Construction 10.4. Hence, every 1-lacunary parapolar space with possibly symps of rank 2 and possibly non-strong, is a 1-buttoned geometry \( \Omega(\mathcal{F}, \mathcal{R}) \), where \( \mathcal{F} \) can also contain ordinary polar spaces of rank 2 besides locally connected 1-lacunary parapolar spaces and polar spaces of rank at least 3.
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