

# A uniform characterisation of the varieties of the second row of the Freudenthal-Tits magic square over arbitrary fields

Anneleen De Schepper<sup>\*</sup>    Jeroen Schillewaert<sup>†</sup>    Hendrik Van Maldeghem<sup>‡</sup>

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## Abstract

We characterize the projective varieties related to the second row of the Freudenthal-Tits magic square, for both the split and the non-split form, using a common, simple and short axiom system.

A special case of our result simultaneously captures the analogues over arbitrary fields of the Severi varieties, which comprise the quadric Veronese variety in 5 dimensions, the Segre variety in 8 dimensions, the line Grassmannian in 14 dimensions and the Cartan variety (27-dimensional  $E_6$  module) in 26 (projective) dimensions, as well as the Veronese representations of projective planes over composition division algebras, most notably the Cayley plane.

A particularly striking feature of our result is that we allow the quadrics which form the basic building bricks of our geometries not only to be non-isomorphic but even to have different Witt index. It is the culmination of 36 years of work since the original 1984 result by Mazzocca and Melone who characterised the quadric Veronese variety over a finite field of odd order. The latter result is a finite counterpart to the characterisation of the complex quadric Veronese surface by Severi from 1901.

## 1 Introduction

In 1993 Zak [24] proved the Hartshorne conjecture and in particular obtained a classification of the Severi varieties (over algebraically closed fields of characteristic 0), in particular generalizing Severi's characterization of the complex quadric surface [19]. In [18] the second and third author of this paper obtained an incidence geometric analogue of this classification over arbitrary fields. The method used starts from an axiomatic approach based on the immediate properties of Severi varieties: the existence of an abundance of split quadrics, the smoothness of the varieties and the dimension (of the tangent space at each point) is bounded in terms of the dimension of the aforementioned quadrics (see below for the precise axioms). The lack of any assumption on the dimension of the whole space implied a slightly longer list in the conclusion; basically also some specific subvarieties of the analogues of the Severi varieties over arbitrary fields satisfy the axioms. On the other side of the spectrum, namely when the Witt index is minimal, in [13] the same axiomatic setup for quadrics without lines was used to characterize the Veronese representations of projective planes over quadratic alternative division algebras. Now, these Veronese representations and the analogues of the Severi varieties over arbitrary fields are closely

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<sup>\*</sup>Department of Mathematics, Ghent University, Belgium, [Anneleen.DeSchepper@ugent.be](mailto:Anneleen.DeSchepper@ugent.be), supported by the Fund for Scientific Research Flanders—FWO Vlaanderen

<sup>†</sup>Department of Mathematics, University of Auckland, New Zealand, [j.schillewaert@auckland.ac.nz](mailto:j.schillewaert@auckland.ac.nz), supported by a UoA FRDF grant

<sup>‡</sup>Department of Mathematics, Ghent University, Belgium, [Hendrik.VanMaldeghem@ugent.be](mailto:Hendrik.VanMaldeghem@ugent.be)

related: generically they correspond to non-split and split, respectively, forms of the same algebraic groups, namely those of types  $A_2$  (only the split form),  $A_2 \times A_2$ ,  $A_5$  and  $E_6$ . Whence the need to check whether or not other forms of those groups give rise to varieties with similar behaviour. “Similar” means in a global setting encompassing the two separate ones.

An obvious way to achieve this global setting is to omit the assumption that the quadrics are split, or non-ruled, respectively. Intuitively, possible additional examples are expected to satisfy the property that all quadrics are isomorphic. However, we here consider the most general situation in which the quadrics not only can be non-isomorphic, they also need not have the same Witt index (but inherent to the axioms is the property that all quadrics span a subspace of equal dimension). In this most general setup, we show that only the aforementioned varieties occur. This yields a very neat and complete characterization of the varieties of the second row of the Freudenthal-Tits Magic Square.

There are two reasons why we are now able to prove the current Main Theorem although it was already stated as a conjecture in [18]. The first one is that an approach to include degenerate quadrics in the picture in [8] generated a new little technique, which seems to work particularly well in our setting. Roughly, it is demonstrated in Lemmas 4.1 and 4.2. The second reason is that we now have at our disposal a classification of parapolar spaces which are so-called 0-lacunary [6, 7], see Definition 3.6. We use that result in a crucial way.

Below we outline the axiomatic setup and we discuss the Main Theorem in some greater detail.

## 1.1 Axiomatic setup

**Projective quadrics and ovoids.** For a (commutative) field  $\mathbb{K}$  and a non-zero cardinal number  $n$ , we denote by  $\mathbb{P}^n(\mathbb{K})$  the  $n$ -dimensional projective space over  $\mathbb{K}$ . The subspace generated by a family  $\mathcal{F}$  of subsets of points is denoted by  $\langle S \mid S \in \mathcal{F} \rangle$ . A *non-degenerate quadric*  $Q$  in  $\mathbb{P}^n(\mathbb{K})$ ,  $n$  finite, is the null set of an irreducible quadratic homogeneous polynomial in the (homogeneous) coordinates of points of  $\mathbb{P}^n(\mathbb{K})$ . The *projective index* of  $Q$  is the (common) dimension of the maximal subspaces of  $\mathbb{P}^n(\mathbb{K})$  entirely contained in  $Q$  (in the literature, one finds more commonly the *Witt index*, which is the projective index plus one; we prefer to work in a projective setting and hence express all dimensions projectively instead of in the underlying vector space). A *tangent line to  $Q$  (at a point  $x \in Q$ )* is a line which has either only  $x$  or all its points in  $Q$ . The union of the set of tangent lines to  $Q$  at one of its points  $x$  is a hyperplane of  $\mathbb{P}^n(\mathbb{K})$ , denoted by  $T_x(Q)$ . An *ovoid*  $O$  of  $\mathbb{P}^n(\mathbb{K})$  is a set of points which behaves like a quadric of projective index 0: each line of  $\mathbb{P}^n(\mathbb{K})$  intersects  $O$  in at most two points, and the union of the set of tangent lines (defined as above) at each point is a hyperplane of  $\mathbb{P}^n(\mathbb{K})$ .

**Axiomatic Veronese varieties.** Consider a pair  $(X, \Xi)$ , where  $X$  is a spanning point set of a projective space  $\mathbb{P}^N(\mathbb{K})$  over some field  $\mathbb{K}$  and with  $N \in \mathbb{N} \cup \{\infty\}$ , and  $\Xi$  is a collection of at least two different  $(d+1)$ -dimensional subspaces of  $\mathbb{P}^N(\mathbb{K})$ , where  $1 \leq d < \infty$ , such that for each  $\xi \in \Xi$ , the set  $X(\xi) := X \cap \xi$  is a non-degenerate quadric or ovoid generating  $\xi$ . We denote  $T_x(X(\xi))$  also by  $T_x(\xi)$ . A subspace of  $\mathbb{P}^N(\mathbb{K})$  is called *singular* if it has all its points in  $X$ ; the set of singular lines is denoted by  $\mathcal{L}$ .

The *tangent space at  $x \in X$  to  $X$*  is the subspace  $T_x$  generated by the sets  $\{T_x(\xi) \mid x \in \xi \in \Xi\}$  and  $\{L \mid x \in L \in \mathcal{L}\}$ . Usually only the former set is used to define  $T_x$ , as in view of (MM1) below, the latter set is automatically contained in what is generated by the former set. The reason that we use both sets is that our inductive approach leads to structures in which (MM1) holds in a weaker form (see Section 3).

**Definition 1.1.** We say that the pair  $(X, \Xi)$  is an *axiomatic Veronese variety of type  $d$*  (or, briefly, an AVV of type  $d$ ) if it satisfies the following axioms:

- (MM1) Any pair of points  $x_1, x_2 \in X$  lies in at least one element of  $\Xi$ ;
- (MM2) if  $\xi_1, \xi_2 \in \Xi$  are distinct, then  $\xi_1 \cap \xi_2 \subseteq X$ ;
- (MM3) for each  $x \in X$ ,  $\dim T_x \leq 2d$ .

The letters MM refer to Mazzocca and Melone, as they introduced these axioms in 1984 ([11]) in their most simplified form, i.e., for quadrics which are finite conics, to characterise the quadric Veronese variety in  $\mathbb{P}^5(\mathbb{K})$  for finite fields  $\mathbb{K}$ . We refer to Section 2 for an overview of the evolution of a problem in finite geometry to the ultimate general setting introduced in the current paper. In that section we also provide explicit descriptions of some of the examples, and explain the context of the Freudenthal-Tits magic square.

## 1.2 Main Result

In [13] it has been shown that AVVs of type  $d$  such that all members of  $\Xi$  are ovoids, exist precisely if  $d$  is a power of 2; if  $\text{char}(\mathbb{K}) \neq 2$ , then  $d \leq 8$ . In these cases we call the AVV a *Veronese cap*, since the examples arise as the image of a projective plane over a quadratic alternative division algebra under the standard Veronese map. Moreover, it is shown in [18] that AVVs of type  $d$  such that all members of  $\Xi$  are *split* quadrics, that is, quadrics with projective index  $\lfloor \frac{d}{2} \rfloor$ , exist precisely for  $d = 1, 2, 4, 6, 8$ , and a complete classification is obtained. These AVVs are called *split*.

In the present paper, we show that there are no AVVs of type  $d$  other than these:

**Main Theorem.** *An AVV of type  $d$  is either split or a Veronese cap.*

Using the main results of [13] and [18], we can formulate the Main Theorem more explicitly. For the definitions and descriptions of the varieties we refer to Section 2.

**Theorem 1.2.** *An AVV of type  $d$  in  $\mathbb{P}^N(\mathbb{K})$  is projectively equivalent to one of the following:*

- $d = 1$ . *The quadric Veronese variety  $\mathcal{V}_2(\mathbb{K})$ , and then  $N = 5$ ;*
- $d = 2$ . *the Segre variety  $\mathcal{S}_{1,2}(\mathbb{K})$  ( $N = 5$ ),  $\mathcal{S}_{1,3}(\mathbb{K})$  ( $N = 7$ ) or  $\mathcal{S}_{2,2}(\mathbb{K})$  ( $N = 8$ );*
- $d = 4$ . *the line Grassmannian variety  $\mathcal{G}_{4,1}(\mathbb{K})$  ( $N = 9$ ) or  $\mathcal{G}_{5,1}(\mathbb{K})$  ( $N = 14$ );*
- $d = 6$ . *the half-spin variety  $\mathcal{D}_{5,5}(\mathbb{K})$ , and then  $N = 15$ ;*
- $d = 8$ . *the (Cartan) variety  $\mathcal{E}_{6,1}(\mathbb{K})$ , and then  $N = 26$ ;*
- $d = 2^\ell$ . *the Veronese variety  $\mathcal{V}_2(\mathbb{K}, \mathbb{A})$ , for some  $d$ -dimensional quadratic alternative division algebra  $\mathbb{A}$  over  $\mathbb{K}$ . Moreover, if the characteristic of the underlying field is not 2, then  $d \in \{1, 2, 4, 8\}$ . Here,  $N = 3d + 2$ .*

Note that the case  $d = 1$  is also included in the last case,  $d = 2^\ell$ . We repeat it though, as the quadric Veronese variety is both split and a Veronese cap.

## 1.3 Structure of the proof

Let  $(X, \Xi)$  be an AVV of type  $d$ . For each  $\xi \in \Xi$ , its *index*  $w_\xi$  is the projective index of  $X(\xi)$ , if  $X(\xi)$  is a quadric, and 0 if  $X(\xi)$  is an ovoid. For each point  $x \in X$ , the set  $W_x := \{w_\xi \mid x \in \xi \in \Xi\}$  is called a *local index set* of  $(X, \Xi)$ ; the global index set  $W$  is the union of all these  $W_x$ . We will distinguish cases depending on these index sets.

Our main technique uses an inductive argument to reduce both  $d$  and the index set, based on the local structure of the AVVs. Indeed, we derive conditions under which the point-residue of  $(X, \Xi)$  at a point  $x \in X$  is an AVV (the main problem being Axiom (MM3)), which then necessarily is of type  $d - 2$  and has index set  $\{w - 1 \mid w \in W_x, w \geq 1\}$ . In the cases that this technique fails, we require a totally different and more inventive approach. More precisely, we proceed as follows.

If  $W = \{0\}$ , then  $(X, \Xi)$  is a Veronese cap (as was proved in [13]), so we will assume that there is a point  $x \in X$  contained in at least one member of  $\Xi$  of index at least 1. Our first aim is to show that there are no points  $x \in X$  contained in exactly one member of  $\Xi$  of index at least 1 (cf. **Section 5**), which guarantees that all point-residues are sufficiently rich in order to deduce properties. Knowing this, we continue systematically:

- Suppose first that there is a point  $x \in X$  with  $\max W_x = 1$ , and so  $d \geq 2$  (**Case 1**). In this case, a rather general argument using normal rational cubic scrolls excludes values of  $d$  exceeding 3. The case  $d = 3$  can be ruled out by relying on a result of the second and third author ([17]). The case where  $d = 2$  leads to the existing cases where all quadrics have index 1 are split, and then the main result of [18] says that  $(X, \Xi)$  is one of the Segre varieties  $\mathcal{S}_{1,2}(\mathbb{K})$ ,  $\mathcal{S}_{2,2}(\mathbb{K})$  and  $\mathcal{S}_{1,3}(\mathbb{K})$  (cf. Section 2).
- Secondly, suppose that there is a point  $x \in X$  with  $\max W_x = 2$ , and so  $d \geq 4$  (**Case 2**). As in the previous case, a rather general argument rules out the cases  $d \geq 6$ . The case where  $d = 5$  does not require much additional effort. The case where  $d = 4$  leads to existing cases, this time its quadrics are all split and of projective index 2. Again, the main result of [18] says that  $(X, \Xi)$  is a line Grassmannians  $\mathcal{G}_{n,1}(\mathbb{K})$  for  $n \in \{4, 5\}$  (cf. Section 2).
- Finally, we may assume that for each point  $x \in X$  holds that either  $W_x = \{0\}$  or  $\max W_x \geq 3$  and the latter option occurs at least once, so  $d \geq 6$  (**Case 3**). We consider a point  $x \in X$  such that  $w^* := \max W \in W_x$  (note that  $w^* \geq 3$ ). The corresponding point-residue is a (possibly weak) AVV, and the induction hypothesis then reveals that all members of  $\Xi$  through  $x$  are split and of the same index  $w^*$ . From this, we will deduce that each member of  $\Xi$  is either of index  $w^*$  and split, or has index 0. Our final task is to get rid of the index 0 members. When this is accomplished, once again the main result of [18] then says that  $(X, \Xi)$  is either the half spin variety  $\mathcal{D}_{5,5}(\mathbb{K})$  (in which case  $d = 6$  and the quadrics have index 3) or the Cartan variety  $\mathcal{E}_{6,1}(\mathbb{K})$  (in which case  $d = 8$  and the quadric have index 4) (cf. Section 2).

Before embarking on the proof, we give an overview of the involved varieties and provide more motivation and background of the problem in **Section 2**. Afterwards, in **Section 3**, we fix notation and show some general properties of AVVs. In **Section 4** we gather technical properties of some specific varieties that we will encounter later on.

## 2 History, motivation and examples

In 1984, Mazzocca & Melone [11] introduced the axioms (MM1), (MM2) and (MM3) for  $d = 1$ ,  $N = 5$  and merely in the finite case, that is, for sets of points in a finite projective space of dimension 5. Using our present terminology, they show in [11] that finite AVVs of type 1 in Galois spaces of dimension 5 are quadric Veronese varieties. As noted by Hirschfeld & Thas [9], their proof for the case of even characteristic contains a flaw and this was corrected in [9]. Cooperstein, Thas & Van Maldeghem [5] introduced Hermitian Veronese caps over finite fields and, with the current terminology, classified finite AVVs of type 2 which are Veronese caps. Then the second and third author classified in [14] all AVVs of type 1, for the first time including in general the infinite case. The same authors also classified in [15] all Veronese caps of type 2. Thus far only Veronese caps had been classified. The first paper dealing with ruled quadrics is [16], where the authors classified all AVVs of type 2, even including a generalization using degenerate quadrics. Then Krauss [12] classified Veronese caps of type 4 over fields admitting exactly two quadratic residue classes, showcasing the hardness of the problem in general. Using some ideas of Krauss' thesis, and some additional ones, Krauss, Schillewaert & Van Maldeghem managed to classify all Veronese caps of arbitrary type (including the infinite-dimensional case rewording (MM3) slightly). Around the same time, the second and third author [18] classified all split AVVs, explicitly conjecturing the Main Result of the present paper.

One of the main reasons why the split AVVs were considered in the first place was because it became clear in [16] that this case has a link with the *Freudenthal-Tits Magic Square (FTMS)*. The split AVVs of type 1 and 2 are exactly the varieties appearing in the first two cells of the second row of Tits' geometric version of the FTMS, see page 142 in [22], hinting at the fact that the other varieties of the second row

also qualify as split AVVs. The eventual classification [18] revealed that certain subvarieties of those are also split AVVs. On top of that, the so-called *non-split* geometric version of the FTMS contains, in the second row, the Veronese representations of the projective planes over quadratic alternative division algebras. Since both the split AVVs and Veronese caps are strongly linked to the second row of the FTMS, it is highly desirable to find a unified form of the axiom systems. This is done in the present paper.

Now we introduce the varieties mentioned in Theorem 1.2. Let  $\mathbb{K}$  be an arbitrary field.

**Quadric Veronese varieties** — The *quadric Veronese variety*  $\mathcal{V}_n(\mathbb{K})$ ,  $n \geq 1$ , is the set of points in  $\mathbb{P}^{\binom{n+2}{2}-1}(\mathbb{K})$  obtained by taking the images of all points of  $\mathbb{P}^n(\mathbb{K})$  under the Veronese mapping, which maps the point  $(x_0, \dots, x_n)$  of  $\mathbb{P}^n(\mathbb{K})$  to the point  $(x_i x_j)_{0 \leq i < j \leq n}$  of  $\mathbb{P}^{\binom{n+2}{2}-1}(\mathbb{K})$ . If  $n = 2$ , then it is an AVV of type 1, and all AVVs of type 1 arise this way.

**Segre varieties** — The *Segre variety*  $\mathcal{S}_{k,\ell}(\mathbb{K})$  of  $\mathbb{P}^k(\mathbb{K})$  and  $\mathbb{P}^\ell(\mathbb{K})$  is the set of points of  $\mathbb{P}^{k\ell+k+\ell}(\mathbb{K})$  obtained by taking the images of all pairs of points, one in  $\mathbb{P}^k(\mathbb{K})$  and one in  $\mathbb{P}^\ell(\mathbb{K})$ , under the Segre map

$$\sigma(\langle (x_0, x_1, \dots, x_k), (y_0, y_1, \dots, y_\ell) \rangle) = (x_i y_j)_{0 \leq i \leq k; 0 \leq j \leq \ell}.$$

If  $(k, \ell) \in \{(1, 2), (1, 3), (2, 2)\}$ , then  $\mathcal{S}_{k,\ell}(\mathbb{K})$  is a split AVV of type 2, and all split AVVs of type 2 arise this way.

**Line Grassmannian varieties** — The *line Grassmannian variety*  $\mathcal{G}_{m,1}(\mathbb{K})$ ,  $m \geq 2$ , of  $\mathbb{P}^m(\mathbb{K})$  is the set of points of  $\mathbb{P}^{\frac{m^2+m-2}{2}}(\mathbb{K})$  obtained by taking the images of all lines of  $\mathbb{P}^m(\mathbb{K})$  under the Plücker map

$$\rho(\langle (x_0, x_1, \dots, x_m), (y_0, y_1, \dots, y_m) \rangle) = \left( \begin{array}{cc} x_i & x_j \\ y_i & y_j \end{array} \right)_{0 \leq i < j \leq m}.$$

If  $m \in \{4, 5\}$ , then  $\mathcal{G}_{m,1}(\mathbb{K})$  is a split AVV of type 4, and every split AVV of type 4 arises this way.

**Half-spin varieties** — An algebraic description of half-spin varieties in full generality is due to Chevalley [3], see also the recent reference [10]. A geometric approach was taken in [23]. Since we only need the case of type  $D_5$  it is more convenient to follow the latter approach.

Consider in  $\mathbb{P}^{15}(\mathbb{K})$  two disjoint 7-dimensional subspaces  $U_1, U_2$ , and in each of them a hyperbolic quadric  $Q_1, Q_2$ , respectively (i.e., a quadric of projective index 3). Let  $\tau$  be any triality of type  $I_{\text{id}}$  (with the terminology of [21]) of  $Q_1$  and let  $\iota$  be a linear isomorphism from  $Q_1$  to  $Q_2$ . Then set  $\varphi = \tau\iota$ . Note that, for each point  $p \in Q_1$ , the image  $p^\varphi$  is a 3-space belonging to one natural system of generators of  $Q_2$ .

The half-spin variety  $\mathcal{D}_{5,5}(\mathbb{K})$  consists of all points of  $\mathbb{P}^{15}(\mathbb{K})$  that lie on a line intersecting  $U_1$  in a point  $p \in Q_1$  and  $U_2$  in a point  $q \in p^\varphi$ . These varieties are the only split AVVs of type 6.

**The Cartan variety** — Since we will not need the precise definition of the variety  $\mathcal{E}_{6,1}(\mathbb{K})$ , which is the projective version of the well known 27-dimensional module of the (split) exceptional group of Lie type  $E_6$ , we simply refer to the literature here. Aschbacher [1] provides an algebraic description, Cohen [4] provides a construction using intersections of quadrics.

The varieties

$\mathcal{V}_2(\mathbb{K})$	$\mathcal{S}_{2,2}(\mathbb{K})$	$\mathcal{G}_{5,1}(\mathbb{K})$	$\mathcal{E}_{6,1}(\mathbb{K})$
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form the second row of the FTMS, split version.

**The Veronese varieties** — These varieties will be of little importance in the rest of the paper. Let us limit ourselves by mentioning that each finite-dimensional quadratic alternative division algebra  $\mathbb{A}$  over  $\mathbb{K}$ , say  $\dim_{\mathbb{K}} \mathbb{A} = j$ , defines a unique Veronese variety  $\mathcal{V}_2(\mathbb{K}, \mathbb{A})$  in  $\mathbb{P}^{3j+2}(\mathbb{K})$  using the standard Veronese map. Also,  $\mathcal{V}_2(\mathbb{K}, \mathbb{K}) = \mathcal{V}_2(\mathbb{K})$ , and if  $\mathbb{L}$  is a quadratic Galois extension of  $\mathbb{K}$ ,  $\mathbb{H}$  a quaternion division ring with center  $\mathbb{K}$  containing  $\mathbb{L}$ , and  $\mathbb{O}$  a Cayley algebra over  $\mathbb{K}$  containing  $\mathbb{H}$ , then the Veronese varieties

$\mathcal{V}_2(\mathbb{K}, \mathbb{K})$	$\mathcal{V}_2(\mathbb{K}, \mathbb{L})$	$\mathcal{V}_2(\mathbb{K}, \mathbb{H})$	$\mathcal{V}_2(\mathbb{K}, \mathbb{O})$
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form the second row of the FTMS, non-split version.

As can easily be observed, all examples of AVVs of type  $d$  in  $\mathbb{P}^N(\mathbb{K})$  satisfy  $N \leq 3d + 2$ . In fact the parameters of all Veronese caps satisfy the equality  $N = 3d + 2$ , as do most examples in the split case, except for the Segre varieties  $\mathcal{S}_{1,2}(\mathbb{K})$  and  $\mathcal{S}_{1,3}(\mathbb{K})$ , the line Grassmannian  $\mathcal{G}_{4,1}(\mathbb{K})$ , and the half-spin variety  $\mathcal{D}_{5,5}(\mathbb{K})$ . This is related to the theory of Severi varieties, from which we derive that, if  $\mathbb{K}$  is algebraically closed, then the inequality  $N < 3d + 2$  readily implies that every point of  $\mathbb{P}^N(\mathbb{K})$  is contained in a secant line of the variety (a *secant line*, in our case, is a line of  $\mathbb{P}^N(\mathbb{K})$  intersecting the variety in exactly two points). However, we will need this property for arbitrary fields and so we prove it in general:

**Proposition 2.1.** *Let  $(X, \mathfrak{E})$  be one of the following AVVs:  $\mathcal{S}_{1,2}(\mathbb{K})$ ,  $\mathcal{S}_{1,3}(\mathbb{K})$ ,  $\mathcal{G}_{4,1}(\mathbb{K})$ ,  $\mathcal{D}_{5,5}(\mathbb{K})$ . Then every point of the ambient projective space  $\mathbb{P}$  is contained in a secant, that is, a line of  $\mathbb{P}$  intersecting  $X$  in exactly two points.*

*Proof.* (i) If  $(X, \mathfrak{E}) \cong \mathcal{S}_{1,n}(\mathbb{K})$ , then  $X$  contains two disjoint singular  $n$ -spaces. It follows immediately that each point of  $\mathbb{P}^{2n+1}(\mathbb{K})$  not in  $X$  is contained in a unique line meeting both planes in a point. This holds for all  $n \geq 1$ .

(ii) If  $(X, \mathfrak{E}) \cong \mathcal{G}_{4,1}(\mathbb{K})$ , then we can select a member  $\xi \in \mathfrak{E}$  and a disjoint singular 3-space  $\Sigma$ . If we identify  $\mathcal{G}_{4,1}(\mathbb{K})$  with the line Grassmannian of the projective space  $\mathbb{P}^4(\mathbb{K})$ , then  $Q := X(\xi)$  corresponds to all lines in a 3-space  $U$  of  $\mathbb{P}^4(\mathbb{K})$ , whereas  $\Sigma$  corresponds to the set of lines through a point  $p$  of  $\mathbb{P}^4(\mathbb{K})$  not in  $U$ . Each line  $L$  of  $\mathbb{P}^4(\mathbb{K})$  not in  $U$  and not through  $p$  is contained in a unique planar line pencil with vertex  $x := L \cap U$  and containing the lines  $\langle x, p \rangle$  and  $\langle p, L \rangle \cap U$ . It follows that, if  $q \in Q$  corresponding to the line  $L_q$  in  $U$ , and if  $M$  is the singular line in  $\Sigma$  corresponding to the planar line pencil in  $\mathbb{P}^4(\mathbb{K})$  with vertex  $p$  in the plane  $\langle p, L_q \rangle$ , then the plane  $\langle q, M \rangle$  is entirely contained in  $X$ .

Now let  $z$  be any point of  $\mathbb{P}$  (and we may assume  $z \notin X$ ). If  $z \in \langle Q \rangle$ , then clearly  $z$  is on a secant of  $Q$ . If  $z \notin \langle Q \rangle$ , then it is contained in a unique line  $K$  intersecting  $\langle Q \rangle$  in a point  $z_Q$  and  $\Sigma$  in a point  $z_\Sigma$ . If  $z_Q \in Q$ , then we are done. If not, then  $z_Q$  is on some secant  $S$  of  $Q$ ; let  $u, v \in S \cap Q$ ,  $u \neq v$ . By the previous paragraph, there are planes  $\pi_u$  and  $\pi_v$  containing  $u, v$ , respectively, intersecting  $\Sigma$  in lines  $L_u, L_v$ , respectively. Note that  $L_u$  and  $L_v$  do not intersect as  $u$  and  $v$  are not collinear on  $Q$ . It follows that there exists a line  $K$  containing  $z_\Sigma$  and intersecting both  $L_u$  and  $L_v$  non-trivially, say in the points  $p_u$  and  $p_v$ , respectively. Hence there is a line through  $z$  intersecting the lines  $\langle u, p_u \rangle$  and  $\langle v, p_v \rangle$  non-trivially (as  $z$  and these lines are contained in the 3-space spanned by  $S$  and  $K$ ).

(iii) Let  $(X, \mathfrak{E}) \cong \mathcal{D}_{5,5}(\mathbb{K})$ . This case is treated similarly as the previous one, now using the construction above with the quadrics  $Q_1, Q_2$ . Each point  $x$  of  $Q_1$  defines a unique 4-space  $U_x = \langle x, x^\phi \rangle$  intersecting  $Q_2$  in the singular 3-space  $\langle x^\phi \rangle$ . A point  $z \notin (U_1 \cup U_2)$  is contained in a line  $\langle z_1, z_2 \rangle$ , with  $z_i \in \langle Q_i \rangle$ ,  $i = 1, 2$ . Then  $z_1$  is on a secant  $\langle u, v \rangle$ , with  $u, v \in Q_1$  (possibly  $z \in \{u, v\}$ ), and  $z_2$  is on a secant  $\langle p_u, p_v \rangle$ , with  $p_u \in U_u$  and  $p_v \in U_v$ . Then  $z$  is on a secant intersecting  $\langle u, p_u \rangle$  and  $\langle v, p_v \rangle$  non-trivially.  $\square$

### 3 Preliminaries

Let  $(X, \Xi)$  and  $d$  be as in the introduction. We start by introducing a more general version of AVVs by omitting axiom (MM3) and/or considering the following, weaker version of (MM1):

(MM1') *Any pair of non-collinear points  $x_1, x_2 \in X$  lies in at least one element of  $\Xi$ .*

**Definition 3.1.** We say that a pair  $(X, \Xi)$  is a *pre-AVV of type  $d$*  if it satisfies Axioms (MM1) and (MM2); we call it a *weak AVV of type  $d$*  if it satisfies Axioms (MM1'), (MM2) and (MM3). A *weak pre-AVV of type  $d$*  is then a pair  $(X, \Xi)$  which satisfies Axioms (MM1') and (MM2).

Henceforth, let  $(X, \Xi)$  be a weak pre-AVV of type  $d$  in  $\mathbb{P}^N(\mathbb{K})$ .

#### 3.1 Collinearity relations

Recall that a subspace of  $\mathbb{P}^N(\mathbb{K})$  is called *singular* if it has all its points in  $X$ . Two points  $x, y$  of  $X$  are called *collinear* if they are on a common singular line  $L$ , in which case we write  $x \perp y$  and, if  $x \neq y$ , we also write  $L = xy$ ; moreover,  $x^\perp$  denotes the set of points collinear to  $x$ .

**Lemma 3.2.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$ . Then each line of  $\mathbb{P}^N(\mathbb{K})$  containing at least three points of  $X$  is singular. Secondly, if  $x, y \in X$  are non-collinear points then there is a unique member of  $\Xi$  through them, denoted by  $[x, y]$ .*

*Proof.* Let  $L$  be a line of  $\mathbb{P}$  with  $|L \cap X| \geq 3$ . Let  $x_1, x_2$  be two points in  $L \cap X$ . If  $L$  is not singular, (MM1') yields a  $\xi \in \Xi$  containing  $x_1, x_2$ . Since  $X(\xi)$  is a quadric,  $L$  has to be singular after all. As for the second statement, (MM1') implies that there is at least one member of  $\Xi$  containing  $x$  and  $y$ ; uniqueness follows from (MM2).  $\square$

The next lemma should be compared to Lemmas 4.1 and 4.2 in [18].

**Lemma 3.3.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$ . Let  $L_1$  and  $L_2$  be singular lines, meeting each other in a point  $x$ . Then either  $L_1$  and  $L_2$  are contained in a singular plane, or there is a unique  $\xi \in \Xi$  (which we denote by  $[L_1, L_2]$ ) containing  $L_1 \cup L_2$ . Consequently, if  $x \in X$  and  $\xi \in \Xi$  with  $x \notin \xi$ , then  $x^\perp \cap X(\xi)$  is a singular subspace (possibly empty).*

*Proof.* Let  $x_1, x_2$  be points on  $L_1 \setminus \{x\}$  and  $L_2 \setminus \{x\}$ , respectively, and suppose that they are not collinear. Let  $x'_1, x'_2$  be points on  $L_1 \setminus \{x, x_1\}$  and  $L_2 \setminus \{x, x_2\}$ . Then the line  $\langle x'_1, x'_2 \rangle$  meets the line  $\langle x_1, x_2 \rangle$  in a point  $z$  not on  $L_1 \cup L_2$ . By Lemma 3.2,  $z \notin X$ , and by the same lemma  $x'_1$  and  $x'_2$  are not collinear. By (MM1')  $[x_1, x_2], [x'_1, x'_2] \in \Xi$  and since they both contain  $z$ , (MM2) implies that they are equal. So if  $L_1 \cup L_2$  contains a pair  $(x_1, x_2)$  of non-collinear points, then  $L_1 \cup L_2 \subseteq [x_1, x_2]$ . If not, then clearly, the plane  $\langle L_1, L_2 \rangle$  is singular.

Now consider  $x \in X$  and  $\xi \in \Xi$  with  $x \notin \xi$ . Suppose for a contradiction that  $x$  is collinear to two non-collinear points  $x_1, x_2$  in  $\xi$ . Set  $L_i = \langle x, x_i \rangle$ ,  $i = 1, 2$ . The previous paragraph implies that  $[x_1, x_2]$  contains  $L_1 \cup L_2$ , in particular  $x \in [x_1, x_2] = \xi$ , a contradiction.  $\square$

#### 3.2 The point-line geometry associated to $(X, \Xi)$

The set of singular lines of  $X$  is denoted by  $\mathcal{L}$ . In case  $\mathcal{L}$  is non-empty (which is not necessarily the case, for instance if  $\Xi$  has only quadrics of index 0), then the pair  $(X, \mathcal{L})$ , equipped with containment as incidence, is the natural *point-line geometry* associated to  $(X, \Xi)$ . Considering this point-line geometry carries a lot of information on  $(X, \Xi)$ , especially when we can invoke the theory of parapolar spaces.

In general a *point-line geometry*  $\Gamma$  is a pair  $\Gamma = (Y, \mathcal{M})$  where  $Y$  is a set of points and  $\mathcal{M}$  a non-empty set of lines, each of which is a subset of  $X$ . A *subspace*  $S$  is a subset with the property that each line not contained in  $S$  intersects  $S$  in at most one point. *Collinearity* between points again corresponds to being contained in a common line (not necessarily unique), and we also denote this by the symbol  $\perp$ . The *collinearity graph* of  $\Gamma$  is the graph on  $Y$  with collinearity as adjacency relation. The *distance*  $\delta(x, y)$  between two points  $x, y \in Y$  is the distance between  $x$  and  $y$  in the collinearity graph (possibly  $\delta(x, y) = \infty$  if there is no path between them). A path between  $x$  and  $y$  of length  $\delta(x, y)$  is called a *shortest path*. The diameter of  $\Gamma$  is the diameter of its collinearity graph. We say that  $\Gamma$  is *connected* if for every two points  $x, y$  of  $Y$ ,  $\delta(x, y) < \infty$ . A subspace  $S \subseteq Y$  is called *convex* if all shortest paths between points  $x, y \in S$  are contained in  $S$ . The *convex subspace closure* of a set  $S \subseteq Y$  is the intersection of all convex subspaces containing  $S$  (this is well defined since  $Y$  is a convex subspace itself).

Before moving on to the viewpoint of parapolar spaces, we need to consider each member of  $\Xi$  of index at least 1 as a convex subspace of  $(X, \mathcal{L})$  isomorphic to a so-called *polar space* (for a precise definition and background see Section 7.4 of [2]). Indeed, for each  $\xi \in \Xi$  with  $w_\xi \geq 1$ ,  $X(\xi)$  is an instance of a polar space, that is, a point-line geometry  $(X', \mathcal{L}')$  in which, apart from three non-degeneracy axioms, the *one-or-all axiom* holds: *Each point  $x \in X'$  is collinear to either exactly one or all points of any given line*. Still assuming  $w_\xi \geq 1$ , we also have that  $X(\xi)$  is a convex subspace: Obviously, for any pair of distinct collinear points  $x, x' \in X(\xi)$ , the line  $xx'$  belongs to  $X(\xi)$ , and for any pair of non-collinear points  $x, x' \in X(\xi)$ , Lemma 3.3 implies that  $x^\perp \cap x'^\perp$  belongs to  $X(\xi)$  and hence so do the shortest paths between  $x$  and  $x'$  in the collinearity graph of  $(X, \mathcal{L})$ . Observe that  $X(\xi)$  is the convex subspace closure of any pair of non-collinear points  $x, x' \in X(\xi)$ , since  $X(\xi)$  contains no convex subspaces other than singular subspaces and itself.

**Definition 3.4.** A connected point-line geometry  $\Gamma = (X, \mathcal{L})$  is a *parapolar space* if for every pair of non-collinear points  $p$  and  $q$  in  $\mathcal{P}$ , with  $|p^\perp \cap q^\perp| > 1$ , the convex subspace closure of  $\{p, q\}$  is a polar space, called a *symplecton* (a *symp* for short); moreover, each line of  $\mathcal{L}$  has to be contained in a symplecton and no symplecton contains all points of  $X$ .

The parapolar space is called *strong* if there are no pairs of points  $p, q$  with  $|p^\perp \cap q^\perp| = 1$ .

**Lemma 3.5.** *Suppose  $(X, \Xi)$  is a weak pre-AVV of type  $d$ . Then each connected component of the point-line geometry  $(X, \mathcal{L})$  associated to  $(X, \Xi)$  is one of the following:*

- (i) *A singular subspace of dimension at least 0 (no point of which is contained in member of  $\Xi$  of index  $\geq 1$ );*
- (ii) *A quadric  $\xi \in \Xi$  of index at least 1 (and all members of  $\Xi$  meeting  $\xi$  non-trivially have index 0);*
- (iii) *A strong parapolar space (which moreover has diameter 2 if  $\min(W) \geq 1$ ).*

*Proof.* Suppose  $x$  belongs to the connected component  $C$ . If all points of  $C$  are collinear with  $x$ , then all points of  $C$  are mutually collinear since otherwise Lemma 3.3(1) yields a member of  $\Xi$  through  $x$  of projective index  $\geq 1$ , which contains points (automatically in  $C$ ) not collinear to  $x$ . Hence we are in Case (i). If there is a point  $y$  in  $C$  not collinear with  $x$ , then by Lemma 3.3 there is a member  $\xi \in \Xi$  of index at least 1 containing  $x$ . If  $C = \xi$  then we are in Case (ii).

If  $C$  strictly contains  $\xi$ , then we wish to show that  $C$  is a strong parapolar space. Let  $p, q$  be points of  $C$  at distance 2, i.e., there are lines  $L_p$  and  $L_q$  through  $p, q$ , respectively, meeting each other in a point. From Lemma 3.3, it follows that  $L_p \cup L_q$  is contained in a unique member of  $\Xi$ , which, as noted before Definition 3.4, is the convex closure of  $p$  and  $q$ . In particular,  $|p^\perp \cap q^\perp| \neq 1$ , showing strongness. Finally, suppose  $L$  is a line in  $C$ . If  $L$  belongs to  $\xi$  there is nothing to prove; if  $L$  intersects  $\xi$  in a point, then by Lemma 3.3,  $L$  is contained in a member of  $\Xi$  together with a line of  $\xi$ . By connectivity we can repeat this argument to conclude that each line is contained in a member of  $\Xi$ . By assumption,  $C$  does not



coincide with a member of  $\Xi$ . We conclude that  $C$  is a strong parapolar space indeed. The claim about the diameter is obvious.  $\square$

**Definition 3.6.** Let  $k \in \mathbb{Z}_{\geq -1}$ . A parapolar space is called  $k$ -lacunary if  $k$ -dimensional singular subspaces never occur as the intersection of two symplecta, and all symplecta do possess  $k$ -dimensional singular subspaces.

In [6] and [7],  $k$ -lacunary parapolar spaces have been classified for  $k = -1$  and  $k \geq 0$ , respectively. At several points in the proof we will use the classification of 0-lacunary parapolar spaces, and also once that of  $(-1)$ -lacunary parapolar spaces. We extract from the Main Result of [7] the results that we will need, restricting our attention to strong parapolar spaces embedded in a projective space over a field  $\mathbb{K}$ .

**Fact 3.7.** Let  $\Gamma = (X, \mathcal{L})$  be a strong  $(-1)$ -lacunary parapolar space whose points are points of a projective space  $\mathbb{P}$  over a field  $\mathbb{K}$ , whose lines are lines of  $\mathbb{P}$  and whose symplecta are all isomorphic to each other. Then  $\Gamma = (X, \mathcal{L})$  is, as a point-line geometry, isomorphic to either a Segre variety  $\mathcal{S}_{n,2}(\mathbb{K})$  with  $n \in \{1, 2\}$ , a line Grassmannian variety  $\mathcal{G}_{n,1}(\mathbb{K})$  with  $n \in \{4, 5\}$ , or to the Cartan variety  $\mathcal{E}_{6,1}(\mathbb{K})$ . In particular, the symps of  $\Gamma$  are all hyperbolic quadrics.

**Fact 3.8.** Let  $\Gamma = (X, \mathcal{L})$  be a strong 0-lacunary parapolar space whose points are points of a projective space  $\mathbb{P}$  over a field  $\mathbb{K}$ , whose lines are lines of  $\mathbb{P}$  and whose symplecta are all isomorphic to each other. Then the symps of  $\Gamma$  are all hyperbolic quadrics. Moreover, if these quadrics all have projective index 1, then  $\Gamma = (X, \mathcal{L})$  is, as a point-line geometry, isomorphic to a Segre variety  $\mathcal{S}_{1,n}(\mathbb{K})$ , for some  $n \in \mathbb{N}$  with  $n \geq 2$ , or the direct product of a line and a hyperbolic quadric of projective index  $n$ , for some  $n \in \mathbb{N}$  with  $n \geq 2$ .

### 3.3 Point-residues of $(X, \Xi)$

Our main technique involves the use of local information coming from the point-residues, which are defined as follows.

**Definition 3.9.** Suppose  $(X, \Xi)$  is an AVV. Let  $x \in X$  be arbitrary and consider a subspace  $C_x$  of  $T_x$  of dimension  $\dim T_x - 1$  not containing  $x$ . Consider the set  $X_x$  of points of  $C_x$  which are contained in a singular line of  $X$  with  $x$ . Let  $\Xi_x$  be the collection of subspaces of  $C_x$  obtained by intersecting  $C_x$  with all  $T_x(\xi)$ , with  $\xi$  running through all members  $\xi$  of  $\Xi$  containing  $x$  together with at least two points of  $X_x$ . Note that the members of  $\Xi_x$  correspond precisely to the members of  $\Xi$  through  $x$  of index at least 1.

The next lemma is the counterpart of Lemma 4.6 in [18].

**Lemma 3.10.** Suppose  $(X, \Xi)$  is an AVV of type  $d$ ,  $d > 2$ , and with global index set  $W$ . Then for each  $x \in X$ , the pair  $(X_x, \Xi_x)$ , with  $X_x \subseteq C_x$  as above, is a weak pre-AVV of type  $d - 2$  and with global index set  $\{w - 1 \mid w \in W_x, w \geq 1\}$ , in the subspace  $C_x$  of dimension  $N_x \leq 2d - 1$  whose isomorphism type is independent of  $C_x$ .

*Proof.* By construction, a member  $\xi$  of  $\Xi_x$  has dimension  $d - 1$  and the quadric  $X(\xi)$  has index  $w_\xi - 1$ .

Let  $p_1$  and  $p_2$  be two non-collinear points of  $X_x$ . In  $X$ , they correspond to two non-collinear lines  $L_1$  and  $L_2$  through  $x$ , which are contained in a member of  $\Xi$  through  $x$  by Lemma 3.3, hence (MM1') holds.

For (MM2), let  $\xi$  and  $\xi'$  in  $\Xi_x$  and suppose that  $y \in \xi \cap \xi'$ . Then  $y$  is contained in  $T_x(\sigma) \cap T_x(\sigma')$ , where  $\sigma$  and  $\sigma'$  are two members of  $\Xi$  containing  $x$  together with at least two points of  $X_x$ . Hence in particular  $y \in \sigma \cap \sigma'$  and so by (MM2) for  $(X, \Xi)$  we obtain  $y \in X_x$ . Hence (MM2) holds in  $(X_x, \Xi_x)$ .

If  $C'_x$  is another hyperplane of  $T_x$ , and if we denote by  $X'_x$  the set of points of  $C'_x$  on a singular line with  $x$ , then the projection from  $x$  of  $C_x$  onto  $C'_x$  yields an isomorphism from  $(X_x, \Xi_x)$  to  $(X'_x, \Xi'_x)$ , where  $\Xi'_x$  is the collection of subspaces of  $C'_x$  obtained by intersecting  $C'_x$  with all  $T_x(\xi)$ , with  $\xi$  running through all quads  $\xi$  of  $\Xi$  containing  $x$  together with at least two points of  $X_x$ .  $\square$

Henceforth we denote by  $W'_x$  the index set  $\{w-1 \mid w \in W_x, w \geq 1\}$ . In case  $x$  satisfies  $\min(W_x \setminus \{0\}) \geq 2$ , we can prove that (MM1) holds. This relies on the following lemma.

**Lemma 3.11.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$  and let  $y \in X$  be arbitrary. Then the local index set  $W_y$  is non-empty (and hence  $\max W_y$  is well-defined). Moreover, if  $1 \notin W_y$  and  $\max W_y \geq 2$ , then each singular plane that contains  $y$  is contained in a member of  $\Xi$ .*

*Proof.* We have to show that there is at least one member of  $\Xi$  containing  $y$ . Suppose the contrary. By assumption, we can pick  $\xi \in \Xi$ . Lemma 3.3 yields a point  $y' \in \xi$  not collinear to  $y$ , and then (MM1') yields  $\xi' \in \Xi$  containing  $y$  and  $y'$ .

Next, suppose  $1 \notin W_y$  and  $\max W_y \geq 2$  and let  $\pi$  be a singular plane through  $y$ . Let  $\xi$  be any member of  $\Xi$  through  $y$ , with  $w_\xi \geq 2$ . If  $\pi \subseteq X(\xi)$  we are done, so suppose there is a point  $z \in \pi \setminus \xi$ . We applying Lemma 3.3 several times. Firstly, it implies that there is a point  $z' \in X(\xi)$  not collinear to  $z$ , but collinear to  $y$ . Then (MM1') yields  $[z, z']$ , which contains the line  $L = \langle y, z \rangle$ . Note that our assumptions imply that  $w_{[z, z']} \geq 2$ . Let  $u$  be a point in  $\pi \setminus L$ . Then  $u$  is collinear to a singular subspace of  $[z, z']$ , so there is a plane  $\pi'$  in  $X([z, z'])$  through  $L$  not all points of which are collinear to  $u$ . For a point  $u' \in \pi' \setminus L$ , we then have  $\pi \cup \pi' \subseteq [u, u']$ .  $\square$

**Corollary 3.12.** *Suppose  $(X, \Xi)$  is an AVV of type  $d$ . Then for each  $x \in X$  with  $\min(W_x \setminus \{0\}) \geq 2$ , the pair  $(X_x, \Xi_x)$  is a pre-AVV of type  $d-2$  with global index set  $W'_x$  in the projective space  $C_x$  of dimension  $N_x \leq 2d-1$ .*

*Proof.* Suppose  $x$  is a point with  $\dim W_x \geq 2$ . Note that this implies that  $d \geq 5$ . By Lemma 3.10, we only still need that each pair of collinear points of  $X_x$  is contained in a member of  $\Xi_x$ . By Lemma 3.11 and  $\min(W_x) \geq 2$ , this is the case.  $\square$

### 3.4 Basic general properties of weak pre-AVVs

Many of the following properties are similar to the split case in [18]. However, since we want to include weak pre-AVVs (which were not defined in [18]), some proofs must be modified. Hence we provide detailed proofs of all statements for completeness.

The next lemma generalizes Lemmas 4.9 and 4.10 of [18] from split quadrics to arbitrary ones.

**Lemma 3.13.** *Let  $Q$  be a non-degenerate quadric in  $\mathbb{P}^{d+1}(\mathbb{K})$  of projective index  $w$ . Consider a subspace  $D$  of  $\mathbb{P}^{d+1}(\mathbb{K})$ , with  $\dim D = d+1-w$ . Then the following hold.*

- (i) *The subspace  $D$  contains at least two non-collinear points of  $Q$ .*
- (ii) *The intersection  $D \cap Q$  spans  $D$ . Equivalently, for each hyperplane  $H$  of  $D$ , the complement  $D \setminus H$  contains a point of  $Q$ .*

*Proof.* (i) We prove this by induction on  $w$ , the result for  $w=0$  being trivial, since  $D$  coincides with  $\mathbb{P}^{d+1}(\mathbb{K})$  in this case. Suppose now that  $w > 0$ . Notice that  $Q \cap D \neq \emptyset$  since a dimension argument implies that  $D$  intersects every singular  $w$ -space of  $Q$  non-trivially. Select  $x \in D \cap Q$ . If some line in  $D$  through  $x$  has exactly two points in common with  $Q$ , then we found a pair of non-collinear points of  $Q$  in  $D$ . So assume that any line in  $D$  through  $x$  either intersects  $Q$  in a unique point or is entirely contained in

$Q$ . Then  $D$  belongs to the tangent space  $T_x(Q)$  at  $x$  to  $Q$ . Considering the residue at  $x$  we obtain a quadric  $Q'$  in  $\mathbb{P}^{d-1}(\mathbb{K})$  of projective index  $w-1$  and a subspace  $D'$  of  $D$  with  $\dim D' = d-w = (d-1) - (w-1)$  which, by induction, contains two non-collinear points  $y', z'$  of  $Q'$ . These points correspond to two singular lines of  $Q$  through  $x$  and in  $D$ , not contained in a singular plane of  $Q$ . This shows the assertion.

(ii) This follows from the fact that quadrics containing two non-collinear points span the ambient projective space of their corresponding quadratic form. An explicit geometric proof goes as follows. Let  $H$  be a hyperplane of  $D$  and suppose that  $D \cap Q \subseteq H$ . By (i),  $H$  contains two non-collinear points  $y$  and  $z$  of  $Q$ . Let  $\alpha$  be any plane in  $D$  through  $y$  and  $z$  with  $\alpha \not\subseteq H$ . Then  $T_z(Q) \cap \alpha$  is precisely one line  $L$ , as  $y$  is not collinear to  $z$ . Then each line  $L'$  in  $\alpha$  through  $z$  distinct from  $L$  contains a second point of  $Q$ . Taking  $L' \neq \langle y, z \rangle$ , this yields a point in  $(D \cap Q) \setminus H$ .  $\square$

The following lemma generalizes Lemma 4.12 of [18].

**Lemma 3.14.** *Suppose  $(X, \Xi)$  is an AVV of type  $d$ . If (distinct)  $\xi_1, \xi_2 \in \Xi$  share a point  $x \in X$ , then  $\langle T_x(\xi_1), T_x(\xi_2) \rangle \cap X \subseteq x^\perp$ .*

*Proof.* Suppose for a contradiction that there are (distinct)  $\xi_1, \xi_2$  through  $x$  such that  $\langle T_x(\xi_1), T_x(\xi_2) \rangle$  contains a point  $y \in X \setminus x^\perp$ . A dimension argument yields points  $a_i \in T_x(\xi_i)$ ,  $i \in \{1, 2\}$ , such that  $y \in \langle a_1, a_2 \rangle$ . If  $a_1 \in X$ , then considering a member of  $\Xi$  through  $y$  and  $a_1$  and using (MM2) implies that  $a_2 \in X$  too, and so the plane  $\langle x, a_1, a_2 \rangle$ —containing two singular lines and an extra point  $y \in X$ —must be singular, contradicting the fact that  $y$  is not collinear to  $x$ . Hence we may assume  $a_1, a_2 \notin X$ . We claim that we can (re)choose the points  $y$  and  $a_1$  in such a way that  $a_1 \in X$ .

Put  $w_i := w_{\xi_i}$  for  $i = 1, 2$ . Without loss of generality,  $w_1 \geq w_2$ . If  $w_2 = 0$ , a dimension argument implies that  $\xi_2 \cap [x, y]$  contains a line through  $x$ , which has to be singular by (MM2), a contradiction. So we may assume  $w_2 \geq 1$ . Put  $U := \xi_1 \cap \xi_2$  and  $\ell := \dim U$ . Then  $0 \leq \ell \leq w_2$ . Since  $\langle T_x(\xi_1), T_x(\xi_2) \rangle$  and  $T_x([x, y])$  are subspaces of respective dimensions  $2d - \ell$  and  $d$  in the  $2d$ -space  $T_x$ , we get that  $\dim(T_x([x, y]) \cap \langle T_x(\xi_1), T_x(\xi_2) \rangle) \geq d - \ell$ . Note that, for  $i = 1, 2$ ,  $T_x([x, y]) \cap T_x(\xi_i)$  has dimension at most  $w_1$  (recall  $w_1 \geq w_2$ ), so there is a (not necessarily singular) subspace  $Z$  in  $\langle T_x(\xi_1), T_x(\xi_2) \rangle$  of dimension  $d - \ell - w_1$  through  $x$  in  $T_x([x, y])$  that intersects  $T_x(\xi_1) \cup T_x(\xi_2)$  exactly in  $\{x\}$ . We consider the subspace  $Z^* = \langle Z, y \rangle$ , and since  $y \notin Z$  we have  $\dim Z^* = d - \ell - w_1 + 1$ . Every line in  $Z^* \subseteq [x, y]$  through  $x$  outside  $Z$  contains a unique point of  $(T_x \cap X) \setminus x^\perp$ . Together with  $Z \cap \xi_i = \{x\}$ , it then follows by (MM2) that  $Z^* \cap \xi_i = \{x\}$ ,  $i \in \{1, 2\}$ . A dimension argument yields unique  $(d - w_1 + 1)$ -spaces  $U_i \subseteq T_x(\xi_i)$  containing  $U$ ,  $i = 1, 2$  such that  $Z^* \subseteq \langle U_1, U_2 \rangle$ . Let  $U'_1$  be the  $(d - w_1)$ -space obtained by intersecting  $\langle U_2, Z \rangle$  with  $U_1$ . By Lemma 3.13(2), there exists a point  $a_1 \in (X \cap U_1) \setminus U'_1 \subseteq (X \cap U_1) \setminus U'_1$ . Since  $U_2$  and  $Z^*$  meet in only  $x$ , and  $a_1 \in \langle U_2, Z^* \rangle$ , there is a unique plane  $\pi$  containing  $x, a_1$  and intersecting both  $Z^*$  and  $U_2$  in (distinct) lines. By our choice of  $a_1$  outside  $U'_1$ , the line  $\pi \cap Z^*$  is not contained in  $Z$  and intersects  $[x, y]$  in a point  $y'$  not collinear to  $x$ . Hence  $y' \in \langle T_x(\xi_1), T_x(\xi_2) \rangle \cap (X \setminus x^\perp)$  and the claim follows. The lemma is proved.  $\square$

The following lemma should be compared with Lemma 4.13 of [18].

**Lemma 3.15.** *Suppose  $(X, \Xi)$  is an AVV of type  $d$  and with global index set  $W$ . Suppose  $x \in X$  is such that  $\min(W_x) \leq 1$ . Then  $T_x \cap X \subseteq x^\perp$ .*

*Proof.* Suppose for a contradiction that there exists  $z \in (T_x \cap X) \setminus x^\perp$ . We claim that we can find  $\xi_1, \xi_2$  in  $\Xi$  through  $x$  such that  $\langle T_x(\xi_1), T_x(\xi_2) \rangle \cap X$  contains a point non-collinear to  $X$ , which contradicts Lemma 3.14 and proves the assertion.

Suppose first that  $\min(W_x) = 0$ . Consider any  $\xi_1 \in \Xi$  through  $x$  of index 0, and any  $\xi_2 \in \Xi$  containing  $x$ . Then by (MM2) and (MM3) we obtain  $z \in T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$ , showing the claim in this case. So suppose that  $\min(W_x) = 1$ . Note that in this case  $T_x$  is generated by all singular lines through  $x$ .

Let  $x \in \xi_1 \in \Xi$  with  $w_{\xi_1} = 1$ . Put  $\xi^* := [x, z]$  and note that  $\xi^* = \langle T_x(\xi^*), z \rangle \subseteq T_x$ . Since  $\dim T_x \leq 2d$ , the intersection  $T_x(\xi_1) \cap \xi^*$  is at least a line. By (MM2),  $T_x(\xi_1) \cap \xi^*$  is singular and as  $\xi_1$  has index 1, it is a line  $L$  (through  $x$ ). In particular,  $\dim T_x = 2d$  and  $\langle T_x(\xi_1), T_x(\xi^*) \rangle = 2d - 1$ . This means that there is a point  $u \in (T_x \cap X) \setminus \langle T_x(\xi_1), T_x(\xi^*) \rangle$  with  $\langle x, u \rangle$  singular. As  $\dim \langle \xi^*, u \rangle = d + 2$ , we obtain that  $\langle \xi^*, u \rangle \cap T_x(\xi_1)$  is a plane  $\pi$  through  $L$ . Let  $v \in \pi \setminus L$  be a point. Inside the  $(d + 2)$ -space  $\langle \xi^*, u \rangle$ , the line  $\langle u, v \rangle$  meets  $\xi^*$  in a point  $y'$ . Since  $u \notin \langle T_x(\xi_1), T_x(\xi^*) \rangle$  we have  $y' \in \xi^* \setminus T_x(\xi^*)$  and hence the line  $\langle x, y' \rangle$  contains a unique point  $y$  on  $X(\xi^*) \setminus \{x\}$ . Clearly,  $y \in X \cap \langle T_x(\xi_1), \langle x, u \rangle \rangle$ . Hence, for an arbitrary member  $\xi_2$  through  $\langle x, u \rangle$  holds  $y \in \langle T_x \xi_1, T_x(\xi_2) \rangle$  and  $y \notin x^\perp$ . The lemma is proved.  $\square$

### 3.5 Projections of $(X, \Xi)$ from a member $\xi \in \Xi$

Projection from a member of  $\Xi$  is a successful tool in the proof of the classification of the case  $W = \{0\}$ , see [13]. Here, we extend its use to members with index  $\geq 1$ .

**Definition 3.16.** If  $(X, \Xi)$  is a (possibly weak) pre-AVV, we can consider the projection  $\rho_\xi$  from some  $\xi \in \Xi$  onto a subspace  $\Pi$  of  $\mathbb{P}^N(\mathbb{K})$  complementary to  $\xi$ , i.e.,

$$\rho_\xi : \mathbb{P}^N(\mathbb{K}) \setminus \xi \rightarrow \Pi : z \mapsto \langle \xi, z \rangle \cap \Pi.$$

For any set  $Z \subseteq \mathbb{P}^N(\mathbb{K})$ , we write  $Z^{\rho_\xi}$  instead of  $(Z \setminus S)^{\rho_\xi}$  for ease of notation.

**Lemma 3.17.** Suppose  $(X, \Xi)$  is a (possibly weak) pre-AVV and  $\xi \in \Xi$  arbitrary. If  $p, q \in X \setminus \xi$  have the same image under  $\rho_\xi$ , then  $\langle p, q \rangle$  is a singular line meeting  $X(\xi)$  in a point.

*Proof.* Put  $\rho = \rho_\xi$ . Then  $p^\rho = q^\rho$  implies that  $\xi$  is a hyperplane of the subspace  $\langle \xi, p, q \rangle$ . If  $\langle p, q \rangle$  is singular, then clearly it intersects  $\xi$  in a point of  $X$ . Suppose  $p$  and  $q$  are not collinear. Then, by (MM1') and (MM2),  $\langle p, q \rangle \cap \xi$  is a point of  $X$ , which by Lemma 3.2 implies that  $\langle p, q \rangle$  is singular after all.  $\square$

The following properties of  $\rho_\xi$  will be used several times, mainly for  $s \in \{0, 1\}$ .

**Lemma 3.18.** Suppose  $(X, \Xi)$  is a (possibly weak) pre-AVV and  $\xi \in \Xi$  arbitrary. Suppose  $\xi' \in \Xi$  meets  $\xi$  in a singular subspace  $S$  of dimension  $s \geq 0$ . Then:

- (i) The image of  $\xi'$  under  $\rho_\xi$  is a  $(d - s)$ -space  $\Pi_{\xi'}$ , in which  $T_S(\xi')^{\rho_\xi}$  is a subspace  $H_{\xi'}$  of dimension  $d - 2s - 1$ ;
- (ii) For any point  $q$  in  $X(\xi')^{\rho_\xi}$ , there is a point  $p$  on  $X(\xi') \setminus S$  such that  $\rho_\xi^{-1}(q) \cap X(\xi') = \langle p, p^\perp \cap S \rangle \setminus S$ , and  $p \in S^\perp$  if and only if  $q \in H_{\xi'}$ ;
- (iii) For any point  $q \in \Pi_{\xi'} \setminus X(\xi')^{\rho_\xi}$ , the set  $\rho_\xi^{-1}(q) \cap \xi'$  is an  $(s + 1)$ -space through  $S$  inside  $T_S(\xi')$  which only has  $S$  in  $X$  (in particular,  $\Pi_{\xi'} \setminus H_{\xi'} \subseteq X(\xi')^{\rho_\xi}$ );
- (iv) If  $s = 0$  and  $L$  is any line in  $\Pi_{\xi'}$  containing a unique point  $z$  in  $H_{\xi'}$ , then the union of  $\rho_\xi^{-1}(L) \cap X(\xi')$  with  $S$  is one of the following:
  - (a) A conic  $C$  through  $S$  if  $z \notin X(\xi')^{\rho_\xi}$ . The image under  $\rho_\xi$  of the tangent line  $T_S(C)$  is  $z$ ;
  - (b) The union of two intersecting non-collinear singular lines if  $z \in X(\xi')^{\rho_\xi}$ . Exactly one of these lines contains  $S$  and is projected by  $\rho_\xi$  onto  $z$ .

*Proof.* (i) Since  $\dim(\xi') = d + 1$  and  $\dim(\xi \cap \xi') = s$ , we get that  $\Pi_{\xi'}$  has dimension  $(d + 1) - s - 1 = d - s$  indeed. The dimension of the tangent space  $T_S(\xi')$  is  $d - s$ , and  $T_S(\xi') \cap \xi = S$ , so likewise  $H_{\xi'}$  has dimension  $(d - s) - s - 1 = d - 2s - 1$ .

(ii) Take any  $q \in X(\xi')^{\rho_\xi}$  and let  $p$  be a point in  $X(\xi')$  with  $\rho_\xi(p) = q$ . By definition of  $\rho_\xi$  and the choice of  $p$ , we have  $\rho_\xi^{-1}(q) \cap X(\xi') = \langle p, \xi \rangle \cap X(\xi')$ . Looking inside  $X(\xi')$ , it follows that the latter set coincides with  $\langle p, p^\perp \cap S \rangle$ . Moreover,  $p \in S^\perp$  if and only if  $p \in T_S(\xi')$  if and only if  $q \in H_{\xi'}$ .

(iii) This follows from the fact that an  $(s+1)$ -space of  $\xi'$  through  $S$  contains a point of  $X(\xi') \setminus S$  if and only if it does not belong to  $T_S(\xi')$ . In particular we obtain that each point of  $\Pi_{\xi'} \setminus H_{\xi'}$  is the image of some point of  $X(\xi') \setminus S^\perp$ .

(iv) Now let  $s = 0$  and take a line  $L$  in  $\Pi_{\xi'}$  containing a unique point  $z$  in  $H_{\xi'}$ . Then  $\rho_\xi^{-1}(L) \cap \xi'$  is a plane  $\pi$  through  $S$ , and by the above, each point  $q \in L \setminus \{z\}$  corresponds to a point  $p$  in  $(\pi \cap X) \setminus S$  not collinear to the point  $S$ . Hence the intersection of  $\pi$  with the quadric  $X(\xi')$  contains at least three points not on a line, and therefore it is either a conic or the union of two intersecting singular lines. Note that in the latter case, each point of  $L$  belongs to  $X(\xi')^{\rho_\xi}$ , i.e.  $z \in X(\xi')$  too. Conversely,  $z \in X(\xi')$  implies by (ii) that  $\rho_\xi^{-1}(z) \cap X(\xi')$  is a singular line through  $S$ . So  $z \in X(\xi')$  corresponds to case (b) indeed. Now, if  $z \notin X(\xi')$ , then  $(\rho_\xi^{-1}(L) \cap X(\xi')) \cup S$  is a conic  $C$  through  $S$ , and the tangent line  $T_S(C)$  is mapped onto  $z$  by  $\rho_\xi$  (cf. assertion (iii)).  $\square$

## 4 Technical lemmas concerning specific situations

Some rather technical work needed for the later sections is done here. The main common goal is often to construct additional singular lines joining members of  $\Xi$  (Lemmas 4.1, 4.2 and 4.4) or, in one case, even prove that there are members with large enough index (Lemma 4.5). We put this in a separate section since all of these results will be used in quite different situations. However, the reader may wish to skip this section during a first reading as it is very technical and will seemingly be out of context. It is probably easier to refer back to the results here when they are used in subsequent sections.

In order to state the first lemma, we need the following concept. In  $\mathbb{P}^4(\mathbb{K})$ , consider a line  $L$  and a conic  $C$  in a plane complementary to  $L$ , and suppose  $\varphi : L \rightarrow C$  is a bijection preserving the cross-ratio. Then the union of the *transversal lines*  $\langle x, \varphi(x) \rangle$ , with  $x \in L$  is called a *normal rational cubic scroll*, denoted  $\mathcal{N}_{1,2}(\mathbb{K})$ , and  $L$  is called the *axis*. Then for each two points not on  $L$  which are on distinct transversal lines of  $\mathcal{N}_{1,2}(\mathbb{K})$ , there is a unique conic through them intersecting all transversal lines (also in points not on  $L$ ). Every pair  $C_1, C_2$  of such conics intersect in precisely one point  $p$  and  $\langle C_1 \rangle \cap \langle C_2 \rangle = \{p\}$ . Conversely, given two arbitrary conics  $C_1$  and  $C_2$  in  $\mathbb{P}^4(\mathbb{K})$  intersecting in a unique point  $p$ , with  $\langle C_1 \rangle \cap \langle C_2 \rangle = \{p\}$ , and given a bijection  $\psi : C_1 \rightarrow C_2$  fixing  $p$  and preserving the cross-ratio, there is a unique normal rational cubic scroll  $\mathcal{N}$  containing all transversal lines  $\langle x, \psi(x) \rangle$ , with  $x \in C_1 \setminus \{p\}$ . In particular, there exists a unique line  $L$  intersecting all said transversal lines. Naturally, the line  $L$  and  $C_1$  determine  $\mathcal{N}$ , and the latter is defined by the map  $\varphi : C_1 \rightarrow L$  taking a point  $x \in C_1 \setminus \{p\}$  to  $\langle x, \psi(x) \rangle \cap L$ , and  $p$  to the unique ‘remaining’ point of  $L$ .

**Lemma 4.1.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$ . Suppose  $\xi_1$  and  $\xi_2$  are two members of  $\Xi$  of index 0, meeting each other in a point  $p$  and meeting some  $\xi \in \Xi$  not through  $p$  in distinct points  $p_1$  and  $p_2$ . If there is a singular line  $K$  meeting  $\xi_1$ ,  $\xi_2$  and  $\xi$  in three distinct points, then for  $i \in \{1, 2\}$ , there is a conic  $C_i$  on  $X(\xi_i)$  through  $p$  and  $p_i$  such that  $C_1$  and  $C_2$  are on a normal rational cubic scroll, and if  $|\mathbb{K}| > 4$ , all transversal lines except possibly the one through  $p$  are singular, as is the axis of the scroll.*

*Proof.* We consider the projection  $\rho = \rho_\xi$  of  $(X, \Xi)$  from  $\xi$  onto a complementary subspace  $\Pi$ . By assumption, the respective images of  $\xi_1$  and  $\xi_2$  under  $\rho$  share at least two points:  $p^\rho$  and  $K^\rho$  (distinct by Lemma 3.17). Let  $L$  be the projective line  $\langle p^\rho, K^\rho \rangle$ . Then  $L$  contains exactly one point  $t_i$  contained in  $T_{p_i}(\xi_i)^\rho$ ,  $i = 1, 2$ , which does not belong to  $X(\xi_i)^\rho$  since  $\xi_i$  has index 0. According to Lemma 3.18(iv),  $L$  corresponds to a conic  $C_i$  on  $X(\xi_i)$  through the points  $p_i$  and  $p$ , for  $i = 1, 2$ .

For  $i = 1, 2$ , let  $\mathcal{S}(p_i)$  denote the planar line pencil through  $p_i$  in  $\langle C_i \rangle$  and let  $\sigma_i$  be the projectivity taking a line  $M \in \mathcal{S}(p_i)$  to  $p_i$  if  $M$  is tangent to  $C_i$ , and to the unique point of  $M$  on  $C_i \setminus \{p_i\}$  if  $M$  is a secant of  $C_i$ . Each line of  $\mathcal{S}(p_i)$  corresponds to a unique point of  $L$  via  $\rho$  and hence we can consider the bijection  $\tau$  taking a line of  $\mathcal{S}(p_1)$  to the unique line of  $\mathcal{S}(p_2)$  with the same image under  $\rho$ . Since  $\langle \mathcal{S}(p_i) \rangle \cap \xi = \{p_i\}$ ,  $\tau$  is a projectivity. As such,  $\sigma_2 \circ \tau \circ \sigma_1^{-1}$  is a projectivity too, i.e., it preserves the cross-ratio.

We conclude that  $C_1$  and  $C_2$  are on a normal rational cubic scroll indeed. Let  $R$  denote the unique line intersecting all its transversal lines. Since the transversal lines  $z_1 z_2$ , with  $z_i \in C_i \setminus \{p_i, p\}$  for  $i = 1, 2$ , are such that  $\rho(z_1) = \rho(z_2)$ , Lemma 3.17 implies that  $\langle z_1, z_2 \rangle$  is singular. Note that this excludes at most three of the transversal lines, namely the ones through  $p, p_1, p_2$ , say  $T, T_1, T_2$  (possibly  $T_1 = T_2$ ). Hence, if  $|\mathbb{K}| > 4$ , we obtain at least three singular transversals that meet  $R$  in three distinct points. Hence  $R$  is singular. But then both  $T_1$  and  $T_2$  contain at least three points of  $X$  and are also singular.  $\square$

The previous lemma assumes the existence of a singular line meeting three members of  $\Xi$ . The next lemma creates a possibility of finding such a line.

**Lemma 4.2.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$  with  $d \geq 2$  and, if  $d = 2$ , we also require  $|\mathbb{K}| > 2$ . Let  $\xi, \xi_1, \xi_2$  be three distinct members of  $\Xi$  with  $\dim \langle \xi, \xi_1, \xi_2 \rangle \leq 2d + 3$ ,  $\xi_1 \cap \xi_2 = \{p\}$  and  $\xi \cap \xi_i = \{p_i\}$ ,  $i = 1, 2$ , where  $p_1, p_2, p$  are three distinct points of  $X$  with  $p \notin p_1^\perp \cup p_2^\perp$ . Then there exists a singular line meeting  $\xi, \xi_1, \xi_2$  in three distinct points  $z, z_1, z_2$ , respectively, with  $z_i$  and  $p_i$  non-collinear, for  $i = 1, 2$ .*

*Proof.* We again consider the projection  $\rho = \rho_\xi$  of  $(X, \Xi)$  from  $\xi$  onto a subspace  $\Pi$  in  $\langle \xi, \xi_1, \xi_2 \rangle$  complementary to  $\xi$ . By Lemma 3.18, the respective images  $\Pi_{\xi_1}$  and  $\Pi_{\xi_2}$  of  $\xi_1$  and  $\xi_2$  under  $\rho$  are  $d$ -spaces of  $\Pi$ , which share the point  $p^\rho$ . Since  $\dim \langle \xi, \xi_1, \xi_2 \rangle \leq 2d + 3$ , we have  $\dim \Pi \leq d + 1$ , and hence  $\Pi_{\xi_1} \cap \Pi_{\xi_2}$  has dimension at least  $d - 1 \geq 1$ . Recall that  $T_{p_i}(\xi_i)^\rho$  is a hyperplane of  $\Pi_{\xi_i}$  and that  $p^\rho$  is not contained in it since  $p \notin p_i^\perp$ ,  $i = 1, 2$ . This means that any line  $L$  in  $\Pi_{\xi_1} \cap \Pi_{\xi_2}$  through  $p^\rho$  contains at most one point  $t_i$  of  $T_{p_i}(\xi_i)^\rho$  for  $i = 1, 2$ . Since  $T_{p_1}(\xi_1)^\rho \cap T_{p_2}(\xi_2)^\rho$  has dimension at least  $d - 3$ , we can choose  $L$  in such a way that  $t_1 = t_2$  if  $d \geq 3$ . Note that, if  $d = 2$  and  $|\mathbb{K}| = 2$ , it might be that  $t_1, t_2, p^\rho$  are the only points of  $L$ .

Let  $q$  be a point in  $L \setminus \{p^\rho, t_1, t_2\}$  (which is non-empty by our assumptions on  $d$  and  $|\mathbb{K}|$ ). Lemma 3.18(ii) yields points  $z_1, z_2$  on  $X(\xi_1), X(\xi_2)$ , respectively, which are not collinear to  $p_1$  and  $p_2$ , respectively (recall  $q \notin \{t_1, t_2\}$ ) and with  $z_1^\rho = z_2^\rho = q$ . By Lemma 3.17, the latter implies that  $\langle z_1, z_2 \rangle$  is a singular line meeting  $X(\xi)$  in a point  $z$ .  $\square$

Here is an example of how Lemma 4.2 can be used to make an application of Lemma 4.1 possible.

**Lemma 4.3.** *Let  $(X, \Xi)$  be an AVV of type 2 containing a connected component  $\mathcal{C}$  of  $(X, \mathcal{L})$  isomorphic as a point-line geometry to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ . Then, for any two points  $x, y \in \mathcal{C}$  at distance 3 in  $(X, \mathcal{L})$ , the member  $[x, y] \in \Xi$  is not contained in the subspace  $\langle \mathcal{C} \rangle$ .*

*Proof.* First note that, if  $a, b \in \mathcal{C}$  are at distance 2 in  $\mathcal{C}$ , then  $[a, b] \in \Xi$  has index 1 and  $X([a, b]) \subseteq \mathcal{C}$  is a hyperbolic quadric of rank 2 which we will refer to as a *grid*. Also,  $\mathcal{C}$ , being isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ , contains disjoint grids, and so, by (MM2),  $\dim \langle \mathcal{C} \rangle = 7$ .

Now let  $x, y \in \mathcal{C}$  be points at distance 3. Then  $\xi := [x, y] \in \Xi$  has index 0. Suppose for a contradiction that  $[x, y]$  belongs to  $\langle \mathcal{C} \rangle$ . Consider any grid  $G$  of  $\mathcal{C}$  through  $x$ . Then by (MM3),  $T_x = \langle T_x(\xi), T_x(G) \rangle \subseteq \langle \mathcal{C} \rangle$ . Suppose first that  $|\mathbb{K}| = q < \infty$ . Let  $G' \subseteq \mathcal{C}$  be a grid not through  $x$ . Then  $G'$  contains  $q^2$  points at distance 3 from  $x$ , and for each such point  $z$ , we have that  $[x, z] \in \Xi$  has index 0. Noting that  $[x, z] \cap G' = \{z\}$  by (MM2), each point of  $G'$  at distance 3 from  $x$  determines a different member of  $\Xi$ , which results in  $q^2$

tangent planes that pairwise intersect each other in  $x$ . In addition, there are the three tangent planes of the grids of  $\mathcal{C}$  through  $x$  (which intersect each other pairwise in a line and the other tangent planes in only  $x$ ). This yields  $q^3 + q^2 + 3q > q^3 + q^2 + q + 1$  distinct lines through  $x$  in the 4-space  $T_x$ , a contradiction.

So suppose that  $|\mathbb{K}| = \infty$ . Let  $z$  be a point of  $\mathcal{C} \setminus \xi$  at distance 3 from both  $x$  and  $y$ . Then  $\xi_1 := [x, z]$  and  $\xi_2 := [y, z]$  are members of  $\Xi$  of index 0. Recalling that  $T_x \subseteq \langle \mathcal{C} \rangle$ , we get  $\xi_1 = \langle T_x(\xi_1), z \rangle \subseteq \langle \mathcal{C} \rangle$ ; likewise for  $\xi_2$ , from which it follows that  $\dim \langle \xi, \xi_1, \xi_2 \rangle \leq 7$ . Therefore  $(X, \Xi)$  and the triple  $\xi, \xi_1, \xi_2$  meet the conditions of Lemma 4.2, and hence also those of Lemma 4.1. The latter lemma implies that there are conics  $C_1$  and  $C_2$  on  $\xi_1$  through  $x, z$  and on  $\xi_2$  through  $y, z$ , respectively, such that  $C_1$  and  $C_2$  are on a normal rational cubic scroll, and each transversal line joining a point  $C_1 \setminus \{z\}$  with its image on  $C_2$  is singular. The line  $R$  meeting all these transversal lines, containing at least three points in  $X$ , is also singular (cf. Lemma 3.2). As a consequence all points of  $C_1$  belong to the same connected component as  $x$ , hence to  $\mathcal{C}$ . We now show that this is not possible.

Let  $p_1, p_2, p_3, p_4$  be four distinct points of  $C_1$ , which are pairwise at distance 3. Take grids  $G_i$  through  $p_i$ ,  $i = 1, 2, 3$ , with  $G_1, G_2, G_3$  pairwise intersecting in a line. Then  $G_1 \cap G_2 \cap G_3$  is a unique point  $p$ . We claim that  $\langle G_1, G_2, G_3, p_4 \rangle = \langle \mathcal{C} \rangle$ . Indeed, clearly any line of  $\mathcal{C}$  through  $p_4$  intersects one of  $G_1, G_2, G_3$  and so is contained in  $\langle G_1, G_2, G_3, p_4 \rangle$ . By connectivity of  $\mathcal{C} \setminus (G_1 \cup G_2 \cup G_3)$ , the claim follows. However,  $\dim \langle G_1, G_2, G_3, p_4 \rangle = 6$  as  $p_4 \in \langle p_1, p_2, p_3 \rangle$ , contradicting  $\dim \langle \mathcal{C} \rangle = 7$ .  $\square$

We collect a further application of Lemma 4.2.

**Lemma 4.4.** *Let  $(X, \Xi)$  be a pre-AVV of type  $d$  with  $d \geq 2$ . If  $d = 2$ , we also require  $|\mathbb{K}| > 2$ . Suppose  $\langle X \rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$ . If  $\xi, \xi_1$  are two members of  $\Xi$  intersecting each other in precisely a point  $p_1$ , then there is a point  $z_1$  in  $X(\xi_1) \setminus p_1^\perp$  collinear to a point  $z$  of  $X(\xi) \setminus p_1^\perp$ .*

*Proof.* Suppose that there are no such points  $z, z_1$ . Let  $p$  be any point in  $X(\xi_1) \setminus p_1^\perp$ , and  $p_2$  any point in  $X(\xi) \setminus p_1^\perp$ . By our assumption,  $p$  and  $p_2$  are non-collinear points. Moreover,  $\xi_2 := [p, p_2]$  meets  $X(\xi)$  in  $p_2$  only: if  $\xi_2 \cap \xi$  is at least a line, then  $p$  is collinear with at least a point  $p'$  of it, and  $p' \in p_1^\perp$  by our assumption, but then  $p' \in \xi_1$ , a contradiction. Likewise,  $\xi_2$  meets  $X(\xi_1)$  in  $p$  only. Lemma 4.2 yields a singular line meeting  $\xi, \xi_1, \xi_2$  in three distinct points  $z, z_1, z_2$ , respectively, with  $z_i$  and  $p_i$  non-collinear for  $i = 1, 2$ . If  $z$  would be collinear to  $p_1$ , then  $z \in \xi_1$ , which is not the case as  $\xi \cap \xi_1 = \{p_1\} \neq \{z\}$ . Hence we found a pair of points as described in the statement of the lemma after all, a contradiction.  $\square$

**Lemma 4.5.** *Let  $(X, \Xi)$  be a pre-AVV with  $d \geq 3$  and  $\langle X \rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$ . Suppose that  $\xi, \xi_1, \xi_2 \in \Xi$  are such that  $w_\xi = 1$ ,  $w_{\xi_1} \leq 1$ ;  $\xi \cap \xi_1$  is a point  $p_1$ ,  $\xi \cap \xi_2$  is a line  $L_2$  and  $\xi_1 \cap \xi_2$  contains a point  $p$  with  $p \notin p_1^\perp \cap L_2^\perp$ . If either  $d \geq 4$ , or  $d = 3$ ,  $|\mathbb{K}| > 2$  and  $\xi_1 \cap \xi_2 = \{p\}$ , then  $w_{\xi_2} \geq 2$  and not all members of  $\Xi$  of index at least 2 contain a common point.*

*Proof.* Again let  $\rho := \rho_\xi$  be the projection operator from  $\xi$  onto a complementary  $(d+1)$ -space  $\Pi$ . We claim that there exists a point  $q$  in  $U := \xi_1^\rho \cap \xi_2^\rho$  neither contained in  $(\xi_1 \cap \xi_2)^\rho$  nor in  $T_{p_1}(\xi_1)^\rho \cup T_{L_2}(\xi_2)^\rho$ .

Indeed, by Lemma 3.18(i),  $\xi_1^\rho$  is a  $d$ -space and  $\xi_2^\rho$  is a  $(d-1)$ -space. Hence  $\dim U \geq d-2 \geq 1$ . Obviously  $U$  contains  $(\xi_1 \cap \xi_2)^\rho$  and  $0 \leq \dim(\xi_1 \cap \xi_2) \leq w_{\xi_1} \leq 1$ . Since  $p \notin p_1^\perp \cup L_2^\perp$ , we have  $p^\rho \notin T_{p_1}(\xi_1)^\rho \cup T_{L_2}(\xi_2)^\rho$  and hence  $H := U \cap (T_{p_1}(\xi_1)^\rho \cup T_{L_2}(\xi_2)^\rho)$  is contained in the union of two hyperplanes of  $U$ . Now, we can find  $q$  in the complement of  $H' := H \cup (\xi_1 \cap \xi_2)^\rho$ , since if  $|\mathbb{K}| > 2$ , the set  $H'$  is contained in the union of three hyperplanes of  $\xi_1^\rho \cap \xi_2^\rho$ , and otherwise our conditions imply either that  $H'$  is the union of  $H$  and a subspace of codimension at least 2—and the claim follows—or  $d = 4$ . In the latter case the only situation in which no such point  $q$  can be found is when  $U$  is a plane and  $(\xi_1 \cap \xi_2)^\rho$ ,  $U \cap T_{p_1}(\xi_1)^\rho$  and  $U \cap T_{L_2}(\xi_2)^\rho$  are three distinct lines in  $U$  through a common point  $u$ . But then, if  $u' \in X(\xi_1) \cap X(\xi_2)$  is such that  $u'^\rho = u$ , then  $u' \in p_1^\perp \cap L_2^\perp$ , contradicting  $u' \notin \xi$ . The claim is proved.

Now, Lemma 3.18(ii) yields a point  $q_1 \in X(\xi_1) \setminus (\xi_1 \cap \xi_2)$  with  $q_1^\perp = q$  and  $q_1$  not collinear to  $q$ , and a line  $L$  in  $X(\xi_2)$  intersecting  $L_2$  in a unique point  $p_2$ , not collinear to  $L_2$  and disjoint from  $\xi_1 \cap \xi_2$ , with  $L^\perp = q$ . By Lemma 3.17,  $\langle q_1, L \rangle$  is a singular plane. Let  $q_2$  be a point on  $L \cap p^\perp$ . Then  $q_2 \in q_1^\perp \cap p^\perp$ , from which we deduce that  $q_1 \perp p$  (otherwise  $q_2 \in \xi_1$ , a contradiction). Now let  $q'_2$  be any point of  $L \setminus \{q_2\}$ . Then, likewise,  $p \perp q'_2$ , for otherwise  $q_1 \in p^\perp \cap q'^{\perp}_2 \in \xi_2$ , contradicting our choice of  $q$ . We conclude that  $\langle p, L \rangle$  is a singular plane  $\pi$  in  $\xi_2$ , collinear to  $q_1$ . This already implies that  $\xi_2$  has index at least 2.

Finally, suppose for a contradiction that all members of  $\Xi$  of index  $\geq 2$  contain a certain point  $x$  (which hence belongs to  $\xi_2$ ). Consider a point  $q'_1$  in  $X(\xi_2)$  which is not contained in the singular subspace  $q_1^\perp \cap X(\xi_2)$  and not collinear to  $x$ . Then  $[q_1, q'_1]$  does not contain  $x$  and has index at least 2 since  $q_1^\perp \cap X(\xi_2)$  contains  $\pi$ , the sought contradiction.  $\square$

The previous lemmas assume the existence of certain members of  $\Xi$  intersecting precisely in one point. In order to meet this condition, the next lemma, applied in a residue, will be helpful. It will also be crucial in proving Proposition 5.2 below.

**Lemma 4.6.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$  and suppose  $\Xi$  contains a unique member  $\xi^*$  of index at least 1. Then there exist two disjoint members of  $\Xi$  intersecting  $\xi^*$  non-trivially.*

*Proof.* Suppose for a contradiction that every pair of members of  $\Xi$  intersecting  $\xi^*$  non-trivially mutually intersect non-trivially. We will use the observation that no singular line in  $X$  intersects  $\xi^*$  in a point, for this would yield a second member of  $\Xi$  with index  $> 0$ .

If  $\mathbb{K} = \mathbb{F}_q$  is finite, then  $d = 2$  since quadrics of projective index 0 only exist in dimensions  $d + 1 = 2$  and  $d + 1 = 3$ , and quadrics of projective index at least 1 require  $d \geq 2$ . Hence  $\xi^*$  has exactly  $(q + 1)^2$  points, and every other member of  $\Xi$  has exactly  $q^2 + 1$  points. Now pick  $\xi \in \Xi$  intersecting  $\xi^*$  in a point  $x$  and let  $p \in X \setminus (\xi \cup \xi^*)$ . Note that  $p^\perp \cap \xi^* = \emptyset$  by the above observation. Hence the mapping  $X(\xi^*) \setminus \{x\} \rightarrow X(\xi) \setminus \{x\}$  taking  $z$  to  $[p, z] \cap \xi$  is well defined and clearly injective. But  $|X(\xi^*) \setminus \{x\}| = q^2 + 2q > q^2 = |X(\xi) \setminus \{x\}|$ , a contradiction.

So suppose  $|\mathbb{K}| = \infty$ . Let  $L$  be a singular line of  $\xi^*$ . Using the above observation and our assumption, it is easy to see that there are three members  $\xi, \xi_1, \xi_2$  of  $\Xi \setminus \{\xi^*\}$  intersecting  $L$  in three distinct points and pairwise intersecting in distinct points. Then Lemma 4.1 and  $|\mathbb{K}| = \infty$  yields the existence of a singular line intersecting  $\xi^*$  in a point—either the axis of the normal rational cubic scroll guaranteed by Lemma 4.1, or if the axis were contained in  $\xi^*$ , any singular transversal of the scroll distinct from  $L$ —contradicting our observation above.  $\square$

## 5 Connectivity

In this section we generate some arguments that will be crucial to show that certain geometries are connected, or certain connected components are large enough. In particular, they will enable us to conclude that appropriate point-residues are connected when a member of index at least 2 exists in  $\Xi$ . When the maximal index is 1, they will imply that there are connected components of  $(X, \mathcal{L})$  that are induced by at least two members of index 1 of  $\Xi$ . Recall that we assume that the global index set  $W$  of  $(X, \Xi)$  is not  $\{0\}$ .

The first result will be used in point-residues and creates members of index at least 1 therein.

**Lemma 5.1.** *Let  $(X, \Xi)$  be a weak pre-AVV of type  $d$ , with  $d \geq 2$ . Suppose  $\langle X \rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$ . Let  $\xi$  be a member of  $\Xi$  of index at least 1 and suppose that  $z \in X \setminus \xi$  is such that  $T_z \cap X(\xi) = \emptyset$  and  $\dim(T_z) \leq 2d + 1$ . Then there is a member of  $\Xi$  of index at least 1 containing  $z$  and intersecting  $\xi$  non-trivially.*



*Proof.* Suppose for a contradiction that  $\xi_p := [z, p] \in \Xi$  has index 0 for each  $p \in X(\xi)$  (note that  $p \notin z^\perp$  indeed since  $T_z \cap X(\xi) = \emptyset$ ). Take any  $p \in X(\xi)$ . We consider the projection  $\rho := \rho_{\xi_p}$  from  $\xi_p$  onto a subspace  $\Pi$  complementary to  $\xi_p$  in  $\mathbb{P}^{2d+3}(\mathbb{K})$ ; so  $\dim \Pi = d + 1$ . Moreover,  $\dim T_z^\rho \in \{d - 1, d\}$  (because  $0 \in W_z$  implies  $\dim T_z \geq 2d$ ) and  $\dim \xi^\rho = \dim \xi_u^\rho = d$  for all  $u \in X(\xi) \setminus \{p\}$  (cf. Lemma 3.18). For each  $u \in X(\xi) \setminus \{p\}$ , we have that  $\xi_u^\rho$  is determined by the  $(d - 1)$ -space  $T_z(\xi_u)^\rho \subseteq T_z^\rho$  and the point  $u^\rho \in X(\xi)^\rho$ .

*Claim 1:*  $T_z^\rho$  is disjoint from  $(p^\perp \cap \xi)^\rho$ .

By way of contradiction, let  $r$  be a point in  $(p^\perp \cap \xi)^\rho \cap T_z^\rho$ . Then  $r$  corresponds to a singular line  $L$  of  $\xi$  through  $p$  (cf. Lemma 3.18(ii)) and  $L \subseteq \langle T_z, \xi \rangle$ . However,  $T_z$  is at least a hyperplane of  $\langle T_z, \xi \rangle$  and hence contains at least a point of  $L \subseteq X$ , contradicting  $T_z \cap X(\xi) = \emptyset$ . This shows the claim.

*Claim 2:* For each point  $u \in X(\xi) \setminus \{p\}$ , the subspace  $\xi_u^\rho$  contains no points of  $(p^\perp \cap \xi)^\rho \setminus \{u^\rho\}$ .

Let  $u \in X(\xi) \setminus \{p\}$  be arbitrary and suppose that  $\xi_u^\rho$  contains a point  $r \in (p^\perp \cap \xi)^\rho$  with  $r \neq u^\rho$ . By Claim 1,  $r \notin T_z^\rho$ , so Lemma 3.18(ii) implies that there is a point  $r_u$  on  $X(\xi_u) \setminus \{z, u\}$  with  $r_u^\rho = r$ . By the same token, there is a point  $r'$  on  $X(\xi)$  collinear to  $p$  with  $r'^\rho = r$ . By Lemma 3.16, the line  $\langle r_u, r' \rangle$  is singular and meets  $X(\xi_p)$  in a point, say  $p'$ . Observe that  $r_u \neq u$  because  $r \neq u^\rho$ ; in particular,  $r_u \notin \xi$  and hence  $p \neq p'$ . This however means that  $r'$  is collinear to both  $p$  and  $p'$ , contradicting  $w_{\xi_p} = 0$ . The claim follows.

*Claim 3:* For each point  $u \in X(\xi) \setminus \{p\}$ , the subspace  $T_z(\xi_u)^\rho$  is disjoint from  $X(\xi)^\rho \setminus \{u^\rho\}$ .

Let  $u \in X(\xi) \setminus \{p\}$  be arbitrary and suppose that  $T_z(\xi_u)^\rho$  contains a point  $r \in X(\xi)^\rho$  with  $r \neq u^\rho$ . Then, on the one hand,  $r$  corresponds to a line  $L$  through  $z$  in  $T_z(\xi_u)$  (cf. Lemma 3.18(iii); note  $L \cap X = \{z\}$  since  $w_{\xi_u} = 0$ ), and on the other hand,  $r$  corresponds to a point  $r'$  on  $X(\xi) \setminus \{p\}$  (cf. Lemma 3.18(ii)). Since  $L^\rho = r'^\rho = r$ , the plane  $\langle L, r' \rangle$  meets  $\xi_p$  in a line  $M$  through  $z$ . If  $M \subseteq T_z(\xi_p)$ , then we obtain  $r' \in \langle M, L \rangle \subseteq T_z$ , contradicting the assumption  $T_z \cap X(\xi) = \emptyset$ ; if  $M \not\subseteq T_z(\xi_p)$ , then  $M$  contains a point  $z'$  in  $X$  other than  $z$ , and using (MM2) we deduce that the point  $\langle z', r' \rangle \cap L$  belongs to  $X$ , contradicting  $L \subseteq T_z(\xi_p)$ . This shows the claim.

Now let  $u$  be a point of  $X(\xi)$  collinear to  $p$ . Then, since  $\dim \Pi = d + 1$ , the  $(d - 1)$ -space  $T_z(\xi_u)^\rho$  has as space  $T$  in common with  $\xi^\rho$  of dimension at least  $d - 2$ . By Claim 3,  $T$  contains no points of  $X(\xi)^\rho$ , which by Lemma 3.18(iii) means that  $T \subseteq T_p(\xi)^\rho$ . Moreover, since  $u \perp p$ , we have  $u^\rho \in T_p(\xi)^\rho$  as well, so  $U := \langle T, u^\rho \rangle = T_p(\xi)^\rho$ , where the equality follows from  $\dim U = \dim T_p(\xi)^\rho = d - 1$  (cf. Lemma 3.18(i)). Consequently,  $U \subseteq \xi_u^\rho$  contains points of  $(p^\perp \cap \xi)^\rho \setminus \{u^\rho\}$ , violating Claim 2. This contradiction shows the lemma.  $\square$

The next proposition will be used globally, but also locally, in residues. It shows that we may often assume that there are at least two members of index at least 1, be it in  $(X, \Xi)$  or in a residue.

**Proposition 5.2.** *Let  $(X, \Xi)$  be a weak AVV of type  $d$ . Then each point of  $X$  is contained in either zero or at least two members of  $\Xi$  having index greater than 0.*

We will show this proposition in a series of lemmas. Suppose for a contradiction that  $(X, \Xi)$  is a weak AVV of type  $d$  containing a point  $x \in X$  through which there is a unique member  $\xi^* \in \Xi$  of index  $w > 0$ .

**Lemma 5.3.** *No singular line meets  $\xi^*$  in a unique point, and consequently for any point  $p$  in  $X(\xi^*)$ ,  $\xi^*$  is the unique member of  $\Xi$  of index  $w > 0$  containing  $p$ .*

*Proof.* If there were a singular line  $L_x$  through  $x$  not in  $\xi^*$ , then by Lemma 3.3, there exists a singular line  $L'_x$  of  $\xi^*$  through  $x$  such that the plane spanned by  $L_x$  and  $L'_x$  is not singular. The same lemma then implies that  $L_x$  and  $L'_x$  are contained in a unique member of  $\Xi \setminus \{\xi^*\}$  through  $x$ , which is of index at least 1, contradicting our assumption on  $x$ . Now let  $z$  be a point in  $X(\xi^*)$  collinear to  $x$  and suppose that  $L_z$

is a singular line meeting  $\xi^*$  in precisely  $z$ . Again by Lemma 3.3, we either have that the lines  $L_z$  and  $\langle x, z \rangle$  determine a member of  $\Xi$  (which is of index at least 1) or a singular plane. Both options yield a singular line through  $x$  not in  $\xi^*$ , a possibility we already ruled out. Hence no lines through  $z$  outside  $\xi^*$  exists; consequently  $\xi^*$  is the unique member of  $\Xi$  through  $z$ . Now a connectivity argument completes the proof of the lemma.  $\square$

We first get rid of the finite case.

**Lemma 5.4.** *The field  $\mathbb{K}$  is infinite.*

*Proof.* Suppose for a contradiction that  $\mathbb{K}$  is finite and has order  $q$ . As in the proof of Lemma 4.6, this implies  $d = 2$  and  $|X(\xi^*)| = (q + 1)^2$ . Let  $p$  be a point of  $X \setminus \xi^*$ . By Lemma 5.3, for each point  $z$  of  $X(\xi^*)$ , the points  $p$  and  $z$  determine a unique member of  $\Xi$ , which has index 0. This yields  $(q + 1)^2$  tangent planes at  $p$  which intersect each other pairwise in  $p$  (by (MM2)). Hence they account for  $(q + 1)^2(q^2 + q) + 1 > q^4 + q^3 + q^2 + q + 1$  points of the 4-dimensional subspace  $T_p$ , a contradiction.  $\square$

The next three lemmas and corollary are generalisations of three lemmas and a proposition in [13] (there, all members of  $\Xi$  had index 0).

**Lemma 5.5.** *For each point  $p \in X$  with  $p \notin \xi^*$ , the subspaces  $T_p$  and  $\xi^*$  are disjoint.*

*Proof.* Suppose for a contradiction that some point  $z$  belongs to both  $T_p$  and  $\xi^*$ . Pick two distinct points  $r, q \in X(\xi^*)$  (and we can assume that  $\dim\langle r, q, z \rangle = 2$ ). Then  $[r, p]$  and  $[q, p]$  have index 0 by Lemma 5.3 and so they intersect only in  $p$ , implying by (MM3) that  $T_p = \langle T_p([r, p]), T_p([q, p]) \rangle$ . Hence there is a line  $L$  through  $z$  intersecting  $T_p([r, p])$  in a point  $u$  and intersecting  $T_p([q, p])$  in a point  $v$ . The line  $\langle u, p \rangle$  intersects  $T_r([r, p])$  in a point  $a$  and the line  $\langle v, p \rangle$  intersects  $T_q([q, p])$  in a point  $b$ . By Lemma 5.4,  $|\mathbb{K}| > 2$ , so we find two points  $a' \in \langle u, p \rangle$  and  $b' \in \langle v, p \rangle$  such that  $z \in \langle a', b' \rangle$  and  $a' \neq a, b' \neq b$ . Since  $a' \notin T_r([r, p])$ , there is a point  $a'' \in X \setminus \{r\}$  on  $\langle r, a' \rangle$  and a point  $b'' \in X \setminus \{q\}$  on  $\langle q, b' \rangle$ . The line  $\langle a'', b'' \rangle$  belongs to the 3-space  $\langle r, q, z, a' \rangle$ , hence it intersects the plane  $\langle r, q, z \rangle$  in some point  $z'$ , which consequently belongs to  $\xi^*$ . The line  $\langle a'', b'' \rangle$  is not singular by Lemma 5.3. Then (MM1') and (MM2) imply that  $z' \in \xi^* \cap [a'', b''] \subseteq X$ , implying that  $\langle a'', b'' \rangle$  is singular after all, a contradiction.  $\square$

For the rest of this section, set  $\rho := \rho_{\xi^*}$  (see Definition 3.16).

**Corollary 5.6.** *The projection  $\rho$  is injective on  $X \setminus \xi^*$ .*

*Proof.* Suppose for a contradiction that  $x, y \in X \setminus \xi^*$  have the same image under  $\rho$ . Then, by Lemma 3.17, the line  $\langle x, y \rangle$  is singular and intersects  $X(\xi^*)$  non-trivially, contradicting Lemma 5.3.  $\square$

**Lemma 5.7.** *Let  $z \in X(\xi^*)$  be arbitrary. Then the subspace  $\langle \xi^*, T_z \rangle$  does not contain any point of  $X \setminus \xi^*$ .*

*Proof.* Put  $S = \langle \xi^*, T_z \rangle$ . Suppose for a contradiction that there is a point  $u \in (S \cap X) \setminus \xi^*$ . Since  $u$  is not collinear with  $z$  by Lemma 5.3, we see that  $[z, u] = \langle T_z[z, u], u \rangle \subseteq S$ . By (MM3),  $\dim S \leq 2d + 1$ , so  $[z, u] \cap \xi^*$  contains a line. This contradicts Lemma 5.3.  $\square$

**Lemma 5.8.** *The geometry  $(X, \Xi)$  is a projective plane and all members of  $\Xi \setminus \{\xi^*\}$  have index 0.*

*Proof.* Take any  $\xi \in \Xi \setminus \{\xi^*\}$  meeting  $\xi^*$  non-trivially. By Lemma 5.3,  $\xi$  has index 0 and intersects  $\xi^*$  in a unique point  $z \in X$ . We will use the following notation. By Lemma 3.18,  $X(\xi)^\rho$  is an affine  $d$ -space  $\alpha_\xi$ . By (MM3), its  $(d-1)$ -space at infinity only depends on  $z$ , and we denote it by  $\Pi_z$ . Finally, we set  $\Pi_\xi := \alpha_\xi \cup \Pi_z$ .

Now let  $z_1$  and  $z_2$  be two non-collinear points of  $X(\xi^*)$  and fix an arbitrary point  $p \in X \setminus \xi^*$ . For  $i = 1, 2$ , set  $\xi_i := [p, z_i]$ . By Lemma 5.5, we may assume that  $T_p \subseteq \Pi$ , the target space of  $\rho$ . Then the subspace  $\Pi_{\xi_i}$  coincides with  $T_p(\xi_i)$ . By Corollary 5.6,  $\alpha_{\xi_1} \cap \alpha_{\xi_2} = \{p\}$ , in particular  $\Pi_{z_1} \cap \Pi_{z_2} = \emptyset$ . We denote  $\Sigma = \langle \Pi_{z_1}, \Pi_{z_2} \rangle$  and note that this is a hyperplane in the subspace  $T_p$ . Also,  $\Pi_{z_1}$  and  $\Pi_{z_2}$  are complementary subspaces in  $T_p$ .

Let  $q$  be an arbitrary point of  $T_p \setminus (\Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Sigma)$ , which is indeed always non-empty. Then the subspace  $\langle q, \Pi_{z_1} \rangle$  intersects  $\Pi_{\xi_2}$  in a point  $q_2 \in \alpha_{\xi_2} \setminus \{p\}$ . Let  $u_2 \in X$  be the inverse image under  $\rho$  of  $q_2$  (cf. Lemma 5.6). Then the projection of  $[z_1, u_2]$  clearly coincides with  $\langle \Pi_{z_1}, q_2 \rangle$ , and so  $q$  can be written as  $u^\rho$  with  $u \in X([z_1, u_2])$ . We claim that  $[p, u]$  intersects  $\xi^*$  non-trivially. Indeed, suppose for a contradiction that  $[p, u] \cap \xi^* = \emptyset$ . Then  $\rho$  induces an isomorphism between  $[p, u]$  and  $[p, u]^\rho$ , and hence  $[p, u]^\rho = \langle T_p([p, u]), u \rangle^\rho = \langle T_p([p, u])^\rho, u^\rho \rangle = \langle T_p([p, u]), q \rangle \subseteq T_p$ . This implies that  $[p, u]^\rho$  and  $\Pi_{\xi_1}$  intersect in a line  $L$  containing  $p$ . This line is not contained in  $T_p([p, u])$ , as  $T_p([p, u])$  and  $\Pi_{\xi_1} = T_p(\xi_1)$  intersect in  $p$  only by (MM2). Hence  $L$  contains a second point  $y$  of  $X([p, u])^\rho$ ,  $y \neq p$ . By Corollary 5.6,  $\{y\} = L \cap \Pi_{z_1}$ . This, however, contradicts Lemma 5.7. The claim is proved.

It follows that  $q$  is contained in  $T_p([p, u]) = [p, u]^\rho$ , and so every point of  $T_p \setminus (\Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Sigma)$ , and hence every point of  $T_p \setminus \Sigma$ , is contained in a tangent subspace at  $p$  to some member of  $\Xi$  containing  $p$  and intersecting  $\xi^*$  in a point. Axioms (MM2) and (MM3) imply that there is no room for additional tangent spaces. We conclude that every member of  $\Xi$  through  $p$  meets  $\xi^*$  non-trivially. Since  $p \in X \setminus \xi^*$  was arbitrary, this shows that every member of  $\Xi \setminus \{\xi^*\}$  intersects  $\xi^*$  in a point. This also implies that every point of  $X \setminus \xi^*$  is projected into  $T_p$  and so  $T_p$  coincides with  $\Pi$ . From that we then deduce, by a dimension argument, that also each pair of members of  $\Xi \setminus \{\xi^*\}$  has a non-trivial intersection. The proposition follows.  $\square$

*Proof of Proposition 5.2.* By Lemma 5.8,  $(X, \Xi)$  is a projective plane in which  $\xi^*$  is the unique member of  $\Xi$  having index greater than 0. This contradicts Lemma 4.6.  $\square$

## 6 Case 1: there is an $x \in X$ with $\max(W_x) = 1$

Suppose  $(X, \Xi)$  is an AVV of type  $d$  and with global index set  $W$  in  $\mathbb{P}^N(\mathbb{K})$ , with  $\max(W_x) = 1$  for some  $x \in X$ . Our aim is to show the following proposition.

**Proposition 6.1.** *Let  $(X, \Xi)$  be an AVV of type  $d$  with global index set  $W$  in  $\mathbb{P}^N(\mathbb{K})$ , containing a point  $x \in X$  with  $\max(W_x) = 1$ . Then  $d = 2$  and  $W = \{1\}$ , and hence  $(X, \Xi)$  is isomorphic to  $\mathcal{S}_{1,2}(\mathbb{K})$ ,  $\mathcal{S}_{2,2}(\mathbb{K})$  or  $\mathcal{S}_{1,3}(\mathbb{K})$ .*

Note that the existence of  $\xi \in \Xi$  through  $x$  with  $w_\xi = 1$  implies  $d \geq 2$ . If  $d > 2$ , we can consider the residue  $(X_x, \Xi_x)$  (cf. Lemma 3.10); if  $d = 2$ , this makes no sense (a member  $\xi \in \Xi$  with  $w_\xi = 1$  would correspond to two points in the residue). Our technique for general  $d$  will work for  $d \geq 4$ , so we treat the cases  $d = 2$  and  $d = 3$  separately. The case  $d = 2$  takes quite some effort, compared to the other cases; however, it is precisely this case that leads to the actual examples, so we begin with it.

### 6.1 The case $d = 2$

Note that  $d = 2$  implies that  $W \subseteq \{0, 1\}$ . If  $W = \{1\}$  we are in the split case and we reach our desired conclusion, so assume that  $W = \{0, 1\}$ . Hence we may assume that  $W_x = \{0, 1\}$ . By Proposition 5.2,

there are at least two members of  $\Xi$  through  $x$  of index 1. Henceforth,  $\mathcal{C}_x$  is the connected component in  $(X, \mathcal{L})$  containing  $x$ .

The approach we take is inspired by [18], where the case in which all members of  $\Xi$  are split quadrics was treated.

Our first goal is to show that there are no singular planes in  $\mathcal{C}_x$ .

**Lemma 6.2.** *Suppose  $\pi$  is a singular plane in  $\mathcal{C}_x$  and let  $z \in \pi$ . If there are three singular lines  $L_1, L_2, L_3$  through  $z$  not in  $\pi$ , then a pair of them is contained in a singular plane  $\pi'$ . Moreover, either  $L_1 \cup L_2 \cup L_3 \subseteq \pi'$ , or  $\langle \pi, \pi' \rangle$  is a singular 3-space.*

*Proof.* Set  $\Sigma := \langle L_1, L_2, L_3 \rangle$ . If  $\Sigma$  is a plane, then, by Lemma 3.2, it is a singular plane and the assertion is proved. So we assume henceforth that  $\dim \Sigma = 3$ . By (MM3),  $\dim T_z \leq 4$  and hence the 3-space  $\Sigma$  has a line  $L_4$  in common with  $\pi$ . The planes  $\langle L_1, L_2 \rangle$  and  $\langle L_3, L_4 \rangle$  are distinct and hence meet in a line  $L_5$ . Using Lemma 3.3 and (MM2), we deduce that  $L_5$  is singular. If  $L_5 \notin \{L_1, L_2\}$ , then the plane  $\langle L_1, L_2 \rangle$  is singular. Else,  $\langle L_3, L_5 \rangle$  is singular and we may renumber subscripts so that  $\langle L_1, L_2 \rangle$  is singular again. Set  $\pi' := \langle L_1, L_2 \rangle$ .

Now suppose that  $\pi \cap \pi' = \{z\}$ . We claim that  $L_3 \subseteq \pi'$ . Indeed, if not, then  $L_3$  is contained in a unique plane  $\pi_3$  intersecting  $\pi$  and  $\pi'$  in respective lines  $L$  and  $L'$  through  $z$ . The plane  $\pi_3$ , containing three singular lines, is singular too. But then  $\langle \pi, \pi_3 \rangle$  is a singular 3-space  $\Pi$ : If not, then (MM1) and Lemma 3.3 imply that  $\pi \cup \pi_3$  is contained in a member of  $\Xi$ , which violates the assumption  $\max W \leq 1$ . Likewise,  $\langle \pi', \pi_3 \rangle$  is a singular 3-space  $\Pi'$  and, again likewise,  $\langle \Pi, \Pi' \rangle$  is a singular 4-space. Since  $\langle \Pi, \Pi' \rangle = T_z$ , this is not possible (no tangent plane to a member of  $\Xi$  is singular). We conclude that  $\pi \cap \pi'$  is a line, and as before, this means that  $\langle \pi, \pi' \rangle$  is a singular 3-space.  $\square$

**Lemma 6.3.** *The connected component  $\mathcal{C}_x$  contains no singular planes.*

*Proof.* Suppose first for a contradiction that  $\pi$  is a singular plane through  $x$ . For any line  $L$  of  $\pi$  through  $x$ , let  $\xi'$  be a member of  $\Xi$  through it (which exists by (MM1)). Denote by  $M$  the unique singular line of  $\xi'$  through  $x$  distinct from  $L$ . Inside  $T_x$  (which has dimension 4 by (MM3) and  $0 \in W_x$ ), a dimension argument implies that there is a plane  $\pi'$  through  $M$  meeting both planes  $\pi$  and  $T_x(\xi)$  in respective lines. Lemma 3.3(1) implies that  $\pi' \cap T_x(\xi)$  is a singular line, contradicting  $w_\xi = 0$ . We conclude that  $x$  is not contained in a singular plane.

Now let  $y$  be an arbitrary point collinear to  $x$  and suppose for a contradiction that there is a singular plane  $\pi$  through  $y$ . Denote by  $L$  the line  $\langle x, y \rangle$ . We establish three singular lines through  $y$  not in  $\pi$ . To that end, take three points  $y_1, y_2, y_3$  on a line  $M$  in  $\pi$  with  $y \notin M$ . By the above, there are no singular planes through  $x$ , so we can consider  $\xi_i := [x, y_i] \in \Xi$ ,  $i = 1, 2, 3$ . Note that  $\xi_i$  contains the lines  $L$  and  $\langle y, y_i \rangle$ . Let  $L_i$  be the unique singular line of  $\xi_i$  through  $y_i$  distinct from  $\langle y, y_i \rangle$ ; let  $L'_i$  be the unique singular line of  $\xi_i$  through  $x$  distinct from  $L$ ,  $i = 1, 2, 3$ . For each  $i \in \{1, 2, 3\}$ , Lemma 3.3(2) implies that  $\langle M, L_i \rangle$  is not singular. So we can put  $\xi'_i := [L_i, M]$ . Since there are no singular planes through  $x$ , no pair of lines in  $\{L'_1, L'_2, L'_3\}$  spans a singular plane. In particular, the points  $p_1 := L_1 \cap L'_1$  and  $p_2 := L_2 \cap L'_2$  are not collinear. So, if  $\xi'_1$  coincided with  $\xi'_2$ , then  $M \subseteq \xi'_1 = [p_1, p_2] \ni x$ , a contradiction, since  $x$  is then collinear to some point of  $M$ , yielding a singular plane through  $x$ . Similarly,  $\xi'_2 \neq \xi'_3 \neq \xi'_1$ . Let  $M'_i$  be the unique singular line in  $\xi'_i$  through  $y_1$  distinct from  $M$ ,  $i = 1, 2, 3$ . Then  $M'_1 = L_1, M'_2, M'_3$  are three distinct singular lines through  $y_1$ , not any belonging to  $\pi$ . If  $y$  were collinear to  $M'_i$ , then  $y \in \xi'_i$  and hence  $\xi'_i$  contains the singular plane  $\pi$ , a contradiction to  $w_{\xi'_i} = 1$ . So we can consider  $\xi''_i := [y_1, M'_i]$ , which yields three distinct members of  $\Xi$ . For each  $i \in \{1, 2, 3\}$ , we now take the unique singular line  $M_i$  in  $\xi''_i$  through  $y$  distinct from  $\langle y, y_2 \rangle$ . We obtain three distinct singular lines  $M_1, M_2, M_3$  through  $y$  not in  $\pi$ .

Renumbering if necessary, we may assume that  $L \notin \{M_1, M_2\}$ . Applying Lemma 6.2 to the triple  $\{L, M_1, M_2\}$  and using that  $L$  is not contained in a singular plane, we obtain that  $\langle M_1, M_2, \pi \rangle$  is a singular 3-space. This however contradicts the fact that  $M_1$  and  $M_2$  are not collinear with  $y_1$ . We conclude that there is no singular plane through  $y$ .

Now by connectivity, the lemma follows.  $\square$

The following proposition, which we will prove in a series of lemmas, is slightly more general, for it allows  $W_x = \{1\}$  (this will be useful in the next section). It assumes that there are no singular planes, which is something we have already proved in case  $W_x = \{0, 1\}$ .

**Proposition 6.4.** *Let  $(X, \Xi)$  be an AVV of type 2 with global index set  $W$  and such that  $\max W = 1$ . Then each connected component of  $(X, \mathcal{L})$  not containing singular planes, is either a point (if there are only members of index 0 through it) or isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ .*

We will prove this proposition in a series of lemmas. From now on, we let  $(X, \Xi)$  be an AVV of type 2 with  $\max W = 1$ . We select an arbitrary point  $x \in X$ , consider its connected component  $\mathcal{C}_x$ , and assume that it does not contain singular planes. If all members of  $\Xi$  through  $x$  have index 0, then  $\mathcal{C}_x = \{x\}$ . By Proposition 5.2, we may henceforth assume that there are at least two members of  $\Xi$  through  $x$  of index 1 (and every such member intersects  $X$  is a non-thick quadrangle).

**Lemma 6.5.** *If every pair of members of  $\Xi$  of index 1, inside  $\mathcal{C}_x$ , sharing a point, have a line in common, then  $\mathcal{C}_x \cong \mathcal{S}_{1,1,1}(\mathbb{K})$ .*

*Proof.* Our assumption implies that  $\mathcal{C}_x$  is a 0-lacunary parapolar space whose symps are quadrics of projective index 1 (see Definition 3.6). Since the symps of  $\mathcal{C}_x$  are all hyperbolic quadrics in dimension 3, Fact 3.8 implies that  $\mathcal{C}_x \cong \mathcal{S}_{1,1,1}(\mathbb{K})$ .  $\square$

Henceforth we may assume that  $\mathcal{C}_x$  contains a pair of members of  $\Xi$  of index 1 sharing exactly a point.

**Lemma 6.6.** *Let  $y$  be a point of  $\mathcal{C}_x$ . Then there are four singular lines through  $y$ . Moreover, any four singular lines through  $y$  span a 4-space, which coincides with  $T_y$ .*

*Proof.* Let  $z \in \mathcal{C}_x$  be a point contained in two members  $\xi_1, \xi_2$  of  $\Xi$  intersecting in precisely  $\{z\}$ . This yields four singular lines  $L_1, L_2, L_3, L_4$  through  $z$ . If  $y = z$ , the first assertion is proved. Now suppose that  $y$  is collinear to  $z$ , and put  $L = \langle y, z \rangle$ . Renumbering if necessary, we may assume that  $L \notin \{L_1, L_2, L_3\}$ . As there are no singular planes,  $[L, L_i] \in \Xi$  for  $i \in \{1, 2, 3\}$ . Considering the singular lines  $M_i$  in  $[L, L_i]$  through  $y$  distinct from  $L$ ,  $i = 1, 2, 3$ , we obtain four singular lines through  $y$ , too. By connectivity, the first assertion follows. .

Next let  $K_1, K_2, K_3, K_4$  be four singular lines through  $y$ . The absence of singular planes implies that  $[K_1, K_2]$  and  $[K_3, K_4]$  belong to  $\Xi$ , and hence (MM2) implies that  $\langle K_1, K_2 \rangle \cap \langle K_3, K_4 \rangle \subseteq [K_1, K_2] \cap [K_3, K_4] = \{y\}$ . So  $\dim \langle K_1, K_2, K_3, K_4 \rangle = 4$ . According to (MM3),  $T_y = \langle K_1, K_2, K_3, K_4 \rangle$ .  $\square$

Henceforth we fix an index 1 member  $\xi$  of  $\Xi$  contained in  $\mathcal{C}_x$ .

The following lemmas will be helpful to study the projection  $\rho_\xi$  of  $X \setminus \xi$  from  $\xi$  onto a complementary subspace.

**Lemma 6.7.** *Let  $L_1$  and  $L_2$  be two distinct singular lines of  $X$  meeting  $\xi$  exactly in (not necessarily distinct) points  $x_1, x_2$ , respectively. Then  $\dim \langle \xi, L_1, L_2 \rangle = 5$ .*

*Proof.* If  $x_1 = x_2$ , then this follows from Lemma 6.6. So assume  $x_1 \neq x_2$ . Suppose for a contradiction that  $\dim\langle\xi, L_1, L_2\rangle = 4$ . Then we claim that  $L_1$  and  $L_2$  do not intersect. Indeed, if they do, say in a point  $p$ , then, if  $x_1$  and  $x_2$  are collinear then we get a singular plane  $\langle p, x_1, x_2\rangle$ , contradicting our assumption; if not, then  $\xi = [x_1, x_2]$  contains  $L_1$  and  $L_2$  by Lemma 3.3, a contradiction. This shows the claim. Hence  $\langle L_1, L_2\rangle$  is a 3-space, which hence intersects  $\xi$  in a plane  $\pi$ . Take a point  $y$  on  $\pi \setminus \langle x_1, x_2\rangle$  not in  $X$ . Then  $y$  lies on a line  $M$  meeting both  $L_1$  and  $L_2$  in points, say  $z_1, z_2$ , respectively. Since  $y \notin \langle x_1, x_2\rangle$ , we have  $\langle z_1, z_2\rangle \not\subseteq \xi$ . This means that  $y \in [z_1, z_2] \cap \xi$ , with  $[z_1, z_2] \neq \xi$ , contradicting (MM2).  $\square$

**Lemma 6.8.** *Suppose  $\xi_1, \xi_2$  are distinct members of  $\Xi \setminus \{\xi\}$  with  $\xi_1 \cap \xi_2 \cap \xi$  a singular line  $L$ . Then  $W := \langle\xi, \xi_1, \xi_2\rangle$  has dimension 7.*

*Proof.* Since  $\xi, \xi_1, \xi_2$  share  $L$ , we already have  $\dim W \leq 7$ . Suppose for a contradiction that  $\dim W \leq 6$ . For  $i = 1, 2$ , put  $W_i := \langle\xi, \xi_i\rangle$  and note that  $\dim W_i = 5$ . So either  $W = W_1 = W_2$  or  $W_1 \cap W_2$  has dimension 4. In the first case, we take any 4-space  $U$  in  $W$  through  $\xi$ ; in the second case we put  $U = W_1 \cap W_2$ . In both cases,  $U$  is a hyperplane in  $W_i$ . We now take any singular line  $M_i$  on  $\xi_i$  disjoint from  $L$ . By choice of  $U$ , the line  $M_i$  has exactly one point  $m_i$  in common with  $U$ . For  $i \in \{1, 2\}$ , we denote by  $R_i$  the unique singular line of  $\xi_i$  through  $m_i$  distinct from  $M_i$ , and note that  $R_i$  intersects  $L$ , and hence  $\xi$ , in a point. Lemma 6.7 implies that  $R_1 = R_2$ . However, then  $\xi_1 = [L, R_1] = [L, R_2] = \xi_2$ , a contradiction.  $\square$

**Lemma 6.9.** *Let  $x_1$  and  $x_2$  be two distinct collinear points on  $X(\xi)$ . Then the 5-spaces  $U_1 = \langle\xi, T_{x_1}\rangle$  and  $U_2 = \langle\xi, T_{x_2}\rangle$  meet exactly in  $\xi$  and hence  $\dim\langle\xi, T_{x_1}, T_{x_2}\rangle = 7$ .*

*Proof.* Let  $L_1$  and  $L'_1$  be two singular lines through  $x_1$  not in  $\xi$ . By Lemma 6.6,  $T_{x_1}$  is generated by  $L_1, L'_1$  and the two singular lines of  $\xi$  passing through  $x_1$ . Set  $L := \langle x_1, x_2\rangle$ . As there are no singular planes,  $\xi_1 := [L, L_1]$  and  $\xi'_1 := [L, L'_1]$  belong to  $\Xi$ . Let  $L_2$  and  $L'_2$  be the respective singular lines of  $\xi_1$  and  $\xi'_1$  through  $x_2$  distinct from  $L$ . Then  $T_{x_2}$  is generated by  $L_2, L'_2$  and the two singular lines of  $\xi$  through  $x_2$ . As such,  $\langle\xi, \xi_1, \xi'_1\rangle = \langle U_1, U_2\rangle$ . By Lemma 6.8, the latter is 7-dimensional, from which it follows that  $U_1 \cap U_2$  is 3-dimensional, and hence coincides with  $\xi$ .  $\square$

**Lemma 6.10.** *Let  $p \in X(\xi)$  be arbitrary. Then each point of  $\langle\xi, T_p\rangle \cap X$  either belongs to  $\xi$ , or is on a singular line together with  $p$ .*

*Proof.* Suppose by way of contradiction that some point  $y$  not collinear to  $p$  is contained in  $\langle\xi, T_p\rangle \setminus \xi$ . Put  $\xi' := [p, y]$ . Then  $\xi' = \langle T_p(\xi'), y\rangle \subseteq \langle\xi, T_p\rangle$ . Since the latter has dimension 5, (MM2) implies that  $\xi \cap \xi'$  is a singular line  $L$  through  $p$ . Let  $M$  be the singular line of  $\xi'$  through  $y$  meeting  $L$  in a point, say  $z$ . By assumption,  $z \neq p$ . So, recalling  $y \in \langle\xi, T_p\rangle$  we get  $M \subseteq \langle\xi, T_x\rangle \cap T_z \subseteq \langle\xi, T_x\rangle \cap \langle\xi, T_z\rangle$ , and as  $M \not\subseteq \xi$ , this contradicts Lemma 6.9.  $\square$

We are now ready to use the projection  $\rho_\xi$  from  $\xi$  onto some subspace  $\Pi$  complementary to  $\xi$  in the subspace  $S$  generated by the points of  $\mathcal{C}_x$ .

**Lemma 6.11.** *The dimension  $n$  of the subspace  $S$  generated by the points of  $\mathcal{C}_x$  is 8.*

*Proof.* Let  $x_1, x_2$  be distinct points on a singular line  $L$  of  $\xi$ . Let  $i = 1, 2$ . By Lemma 6.6, there are two singular lines  $L_i, L'_i$  through  $x_i$  outside  $\xi$ . Recalling that there are no singular planes,  $\xi_i := [L_i, L'_i]$  is well defined. By Lemma 3.18, the image of  $X(\xi_i) \setminus (L_i \cup L'_i)$  under  $\rho_\xi$  is an affine plane  $\pi_i^*$  in  $\Pi$  with projective extension  $\pi_i$ . By the same lemma,  $T_i := \pi_i \setminus \pi_i^*$  is the image of  $T_{x_i}(\xi_i)$ , which coincides with  $T_{x_i}^{\rho_\xi}$  by (MM3). According to Lemma 6.9,  $W := \langle\xi, T_{x_1}, T_{x_2}\rangle$  is 7-dimensional, so  $T_1$  and  $T_2$  are skew lines and  $\dim \Pi \geq 3$ .

First suppose for a contradiction that  $\dim \Pi = 3$ . In this case, the plane  $\pi_1$  has a point  $z$  in common with the line  $T_2$ , and  $z \notin T_1$  by the above (so  $z \in \pi_1^*$ ). Let  $y$  be the unique inverse image in  $\xi_1$  of  $z$  in  $X$ ; then  $y \in X(\xi_1) \setminus x_1^\perp$  and  $y \in \langle \xi, T_{x_2} \rangle$ . By Lemma 6.10,  $y \perp x_2$ , implying that  $x_2 \in [x_1, y] = \xi_1$ , a contradiction. Hence  $\dim \Pi \geq 4$ . We need to show that  $\dim \Pi = 4$ , so suppose now by way of contradiction that  $\dim \Pi > 4$  (this means that there are projective lines in  $\langle \mathcal{C}_x \rangle$  skew to  $W$ ).

We distinguish two cases. First we assume that there is some singular line  $R$  disjoint from  $W$ . In particular,  $R$  is disjoint from  $T_{x_i}$ , and so no point of  $R$  is collinear to  $x_i$ ,  $i = 1, 2$ . We claim that  $R$  contains a point  $v$  with  $v^\perp \cap L = \emptyset$ . Indeed, if not then let  $y_1, y_2$  be two distinct points of  $R$  and let  $z_i \perp y_i$ , with  $z_i \in L$ ,  $i = 1, 2$ . Since there are no singular planes  $z_i$  is unique,  $i = 1, 2$ , and  $z_1 \neq z_2$ . Hence  $[y_1, z_2]$  contains  $L$  and  $R$  and so  $x_1$  is collinear to some point of  $R$  after all. This shows the claim.

So let  $v \in R$  be such that  $v^\perp \cap L = \emptyset$ . The members  $[v, x_1]$  and  $[v, x_2]$  of  $\Xi$  do not contain  $L$ . Let  $M_i$  denote the line  $T_v([v, x_i]) \cap T_{x_i}([v, x_i])$ ,  $i = 1, 2$ . Then  $M_1 \cap M_2 \subseteq T_{x_1} \cap T_{x_2}$  and Lemma 6.9 implies  $M_1 \cap M_2 \subseteq L$ . Since  $v$  is not collinear to any point of  $L$ , we deduce that  $M_1$  and  $M_2$  are disjoint. This implies that  $T_v$  contains a 3-dimensional subspace of  $W$ , namely  $\langle M_1, M_2 \rangle$ . Since, by (MM3),  $\dim T_v \leq 4$ ,  $R \subseteq T_v$  intersects  $W$  non-trivially, a contradiction.

Secondly, assume that no singular line is disjoint from  $W$ . Let  $x_1^*, x_2^* \in X$  be such that  $\langle x_1^*, x_2^* \rangle$  is disjoint from  $W$ . Put  $\xi^* := [x_1^*, x_2^*]$ . If  $\xi^*$  has index 1, then, as  $R$  is skew to  $W$  we see that  $\xi^*$  meets  $W$  in at most a line, and hence we find a singular line  $R^*$  on  $\xi^*$  skew to  $W$ , a contradiction. So  $\xi^*$  has index 0. Since  $x_i^* \notin W$ , in particular  $x_i^*$  is not collinear to  $x_i$ , we can consider  $\xi_i' := [x_i, x_i^*] \in \Xi$ . Then  $T_{x_1^*}(\xi_1') \cap T_{x_2^*}(\xi_2')$  is empty, for if it would contain a point  $z$ , then by (MM2),  $z \in X$  and hence  $\langle x_1^*, z \rangle \cup \langle x_2^*, z \rangle$  would be two singular lines in  $\xi^*$  by Lemma 3.3, a contradiction to  $w_{\xi^*} = 0$ . By Lemma 6.6, there are four singular lines through  $x_i^*$  spanning  $T_{x_i^*}$ , and by assumption, each of these lines has a point in common with  $W$ . This implies that  $T_{x_i^*}$  is a singular 4-space sharing a 3-space  $W_{x_i^*}$  with  $W$ ,  $i = 1, 2$ . Clearly,  $W_{x_1^*}$  and  $W_{x_2^*}$  share the line  $T_{x_1^*}(\xi^*) \cap T_{x_2^*}(\xi^*)$ , so using  $T_{x_1^*}(\xi_1') \cap T_{x_2^*}(\xi_2') = \emptyset$ , we obtain  $\dim \langle W_{x_1^*}, W_{x_2^*} \rangle = 5$ . This means that  $W^* := \langle T_{x_1^*}, T_{x_2^*}, \xi^* \rangle = \langle W_{x_1^*}, W_{x_2^*}, x_1^*, x_2^* \rangle$  has dimension 7; and hence  $\langle T_{x_1^*}, \xi^* \rangle \cap \langle T_{x_2^*}, \xi^* \rangle = \xi^*$ .

In what follows we interchange the roles of  $\xi, x_i, W$  and  $\xi^*, x_i^*, W^*$ , respectively. Since  $\xi$  and  $\xi^*$  have different index, the arguments are not entirely identical, hence we present them in detail.

Let  $R^*$  be a singular line of  $\xi$  disjoint from  $L$ . Let  $v^*$  be an arbitrary point on  $R^*$  and note that  $v^* \notin y_i^\perp$  since  $v^* \notin T_{y_i}$ ,  $i = 1, 2$ . Let  $M_i^*$  denote the line  $T_{v^*}([v^*, x_i^*]) \cap T_{x_i^*}([v^*, x_i^*])$ ,  $i = 1, 2$ . There are two cases: Suppose first that  $M_1^* \cap M_2^* \neq \emptyset$ . Then  $M_1^* \cap M_2^* \subseteq T_{x_1^*} \cap T_{x_2^*}$  and since by the previous paragraph  $\langle T_{x_1^*}, \xi^* \rangle \cap \langle T_{x_2^*}, \xi^* \rangle = \xi^*$ , we obtain  $M_1^* \cap M_2^* \subseteq \xi^*$ . By (MM2), the intersection  $M_1^* \cap M_2^*$  belongs to  $X$  and hence to  $x_1^{*\perp} \cap x_2^{*\perp}$ , contradicting  $w_{\xi^*} = 0$ . Suppose now that  $M_1^* \cap M_2^* = \emptyset$ . This implies that  $T_{v^*}$  contains a 3-dimensional subspace of  $W^*$ , namely  $\langle M_1^*, M_2^* \rangle$ . Since, by (MM3),  $\dim T_{v^*} \leq 4$ ,  $R^* \subseteq T_{v^*}$  intersects  $W^*$  non-trivially, a contradiction.  $\square$

We are now ready to prove Proposition 6.4.

*Proof of Proposition 6.4.* Again, let  $x_1, x_2$  be two points on  $\xi$  on a common singular line  $L$ . Let  $L_1, L_1'$  be two distinct singular lines through  $x_1$ , not inside  $\xi$ . Put  $\bar{\xi} := [L, L_1]$  (recall that there are no singular planes) and let  $L_2$  be the singular line of  $\bar{\xi}$  through  $x_2$  distinct from  $L$ . Finally, let  $L_2'$  be an arbitrary singular line through  $x_2$ , distinct from  $L_2$  and not in  $\xi$ . Put  $\xi_i' := [L_i, L_i']$  for  $i = 1, 2$ .

By Lemma 6.11, there is a 4-dimensional subspace  $\Pi$  in  $\langle \mathcal{C}_x \rangle$  complementary to  $\xi$ . Then the image of  $X(\xi_i') \setminus (L_i \cup L_i')$  under  $\rho_\xi$  is the set of points of an affine plane  $\pi_i^*$  in  $\Pi$ , with projective completion  $\pi_i$ , and the line  $T_i := \pi_i \setminus \pi_i^*$  is the projection of  $T_{x_i}$ ,  $i = 1, 2$ . The projective planes  $\pi_1$  and  $\pi_2$  meet non-trivially by a dimension argument. According to Lemma 6.9, the lines  $T_1$  and  $T_2$  are skew; also, the arguments of the second paragraph of the proof of Lemma 6.11 imply that  $T_1$  does not meet  $\pi_2$ , and  $T_2$  does not meet  $\pi_1$ . Hence the affine planes  $\pi_1^*$  and  $\pi_2^*$  meet in a unique point  $z$  and so we have points

$z_1$  in  $X(\xi'_1) \setminus (L_1 \cup L'_1)$  and  $z_2$  in  $X(\xi'_2) \setminus (L_2 \cup L'_2)$  lying in a common 4-space  $U$  with  $\xi$ . We claim that  $z_1 = z_2$ , so suppose for a contradiction that  $z_1 \neq z_2$ . Let  $\xi^*$  be a member of  $\Xi$  containing  $z_1, z_2$ .

Considering  $\xi^* \cap \xi$  and (MM2), we see that  $\langle z_1, z_2 \rangle$  is a singular line meeting  $\xi$  in some point  $u \in X$ . Note that  $u \notin L$  because otherwise  $L \subseteq \xi'_1$  by Lemma 3.3, a contradiction. So, possibly interchanging the roles of  $x_1$  and  $x_2$ , we may assume that  $\langle u, x_2 \rangle$  is not a singular line; let  $\{v\} = u^\perp \cap M_2$ , with  $M_2$  the singular line of  $\xi$  through  $x_2$  distinct from  $L$ . Recall that  $z_1 \notin T_{x_2}$ , as  $z \notin T_2$ , so we can consider  $\xi_{12} := [z_1, x_2]$ . Then we show  $\xi_{12} \cap \xi = \{x_2\}$ . Indeed, if  $L \subseteq \xi_{12}$ , then  $x_1 \in \xi_{12}$  and hence  $\xi_{12} = \xi'_1$ , a contradiction; if  $M_2$  belongs to  $\xi_{12}$ , then  $u \in v^\perp \cap z_1^\perp$  (note  $v \notin z_1^\perp$  by absence of singular planes), and then  $\xi_{12} = \xi$ , a contradiction. Thus we obtain that the image under  $\rho_\xi$  of  $X(\xi_{12}) \setminus x_2^\perp$  coincides with the plane  $\pi_2^*$  (for the projection is determined by  $z_1^{\rho_\xi} = z$  and  $T_{x_2}(\xi_{12})^{\rho_\xi} = T_{x_2}^{\rho_\xi} = T_2$ ). Noting that  $T_{z_1}(\xi'_1)^{\rho_\xi} = \pi_1$  and  $T_{z_1}(\xi_{12})^{\rho_\xi} = \pi_2$ , and recalling that  $\langle \pi_1, \pi_2 \rangle = \Pi$ , we obtain that  $T_{z_1}$  is a 4-space disjoint from  $\xi$ . However,  $u \in T_{z_1} \cap \xi$ , a contradiction. The claim follows.

Now let  $M_i$  be the singular line in  $\xi'_i$  through  $z_i$  meeting  $L_i$ ,  $i = 1, 2$ . Let  $m_i$  denote the point  $M_i \cap L_i$ . Remember that  $L_1, L, L_2$  are contained in  $\xi$ ; let  $L'$  be the singular line of  $\xi$  through  $m_1$ . Note that  $M_1 \neq M_2$ , for otherwise  $M_1 = L' = M_2$  and hence  $\xi'_1 = \xi = \xi'_2$ , a contradiction. Moreover,  $m_1$  and  $m_2$  are not collinear, for otherwise  $\langle z_1, m_1, m_2 \rangle$  would be a singular plane, a possibility we excluded by assumption. However, this means that  $z_1 \in m_1^\perp \cap m_2^\perp \subseteq \xi$ , a contradiction, recalling  $\xi'_1 \cap \xi = L_1$  and  $z_1 \notin L_1$ .

We conclude that our initial assumption that  $\mathcal{C}_x$  contains a pair of members of  $\Xi$  of index 1 sharing exactly one point is false. Hence by Lemma 6.5,  $\mathcal{C}_x$  is isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ .  $\square$

Finally, we rule out the existence of connected components isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ .

*Proof of Proposition 6.1 in case  $d = 2$ .* Suppose for a contradiction that  $W = \{0, 1\}$ . By assumption, there is a  $\xi \in \Xi$  through  $x$  of index 1, so the connected component  $\mathcal{C}_x$  of  $x$  in  $(X, \mathcal{L})$  is more than a point. By Lemma 6.3,  $\mathcal{C}_x$  does not contain singular planes and hence Proposition 6.4 then implies that  $\mathcal{C}_x$  is isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ . Pick two points  $x$  and  $y$  at distance 3 in  $\mathcal{C}_x$  and let  $\Sigma = \langle \mathcal{C}_x \rangle$ . Then  $[x, y]$  is a member of  $\Xi$  of index 0. Note that the presence of  $[x, y]$  implies (by (MM3)) that  $\dim T_x = \dim T_y = 4$ . Inside  $[x, y]$ , we see that the tangent planes  $T_x([x, y])$  and  $T_y([x, y])$  intersect each other in a line  $L$ . On the other hand, inside  $\mathcal{C}_x \cong \mathcal{S}_{1,1,1}(\mathbb{K})$ , the tangent spaces of  $x$  and  $y$  are disjoint 3-spaces (as indeed they span the 7-space, which follows from (P1) in Section 17.2 in [20]), which consequently both contain a unique point of  $L$ . Thus  $L$  belongs to  $\Sigma$ , and hence so does  $[x, y] = \langle x, y, L \rangle$ . However, this contradicts Lemma 4.3. We conclude that  $W = \{1\}$ . The main results from [18] then reveal that  $(X, \mathcal{L})$  is isomorphic to one of  $\mathcal{S}_{1,2}(\mathbb{K}), \mathcal{S}_{2,2}(\mathbb{K}), \mathcal{S}_{1,3}(\mathbb{K})$  indeed.  $\square$

## 6.2 The case $d = 3$

*Proof of Proposition 6.1 in case  $d = 3$ .* Let  $x$  be a point with  $\max W_x = 1$ . Then let  $\Xi(\mathcal{C}_x)$  be the set of members  $\xi \in \Xi$  of index 1 contained in  $\mathcal{C}_x$ , and let  $\mathcal{L}(\mathcal{C}_x)$  be the set of singular lines of  $\mathcal{C}_x$ . Due to Proposition 5.2,  $|\Xi(\mathcal{C}_x)| \geq 2$ . Now we observe that  $(\mathcal{C}_x, \Xi(\mathcal{C}_x))$  is a so-called *Lagrangian Grassmannian set*, as introduced in [17]. Indeed, the members of  $\Xi(\mathcal{C}_x)$  are projective 4-spaces intersecting  $\mathcal{C}_x$  in a non-singular parabolic quadric (and all such quadrics are isomorphic). Moreover, two points  $x, y$  at distance at most 2 in the point-line geometry  $(\mathcal{C}_x, \mathcal{L}(\mathcal{C}_x))$  are contained in at least one member of  $\Xi(\mathcal{C}_x)$ ; indeed, by (MM1) there is a member  $\xi$  of  $\Xi$  containing  $x$  and  $y$ , which obviously belongs to  $\Xi(\mathcal{C}_x)$  if  $x \perp y$ . If  $x$  is at distance 2 from  $y$ , then  $\emptyset \neq x^\perp \cap y^\perp \subseteq X(\xi)$ , so  $\xi$  has index 1. Furthermore, Axioms (MM2) and (MM3) hold in  $(\mathcal{C}_x, \Xi(\mathcal{C}_x))$ . This now implies by Main Result 2 of [17] that  $(\mathcal{C}_x, \Xi(\mathcal{C}_x))$  is isomorphic to the Lagrangian Grassmannian  $\text{LG}(3, 6)(\mathbb{K})$ , which, as a point-line geometry, is isomorphic to the dual polar space  $\text{C}_{3,3}(\mathbb{K})$ , and which lives in projective 13-space (hence  $\dim \langle \mathcal{C}_x \rangle = 13$ ).



This also implies that  $(\mathcal{C}_x, \mathcal{L}(\mathcal{C}_x))$  has diameter 3. Hence there exist two points  $y, z \in \mathcal{C}_x$  at distance 3. Then  $[y, z]$  is a member of  $\Xi$  of index 0. Clearly,  $T_y([y, z]) \cap T_z([y, z])$  is a plane. However, in  $\text{LG}(3, 6)(\mathbb{K})$ , the (6-dimensional) tangent spaces of points at distance 3 are disjoint (due to the fact that they span the 13-space  $\langle \mathcal{C}_x \rangle$ , which follows from (P1) in Section 17.2 of [20]), so  $T_y \cap T_z = \emptyset$ , a contradiction.  $\square$

### 6.3 The case $d \geq 4$

We now aim to show Proposition 6.1 for  $d \geq 4$ . Let  $x \in X$  be a point with  $\max W_x = 1$ . By Corollary 3.10,  $(X_x, \Xi_x)$  is a weak pre-AVV (use also Proposition 5.2 to see that  $|\Xi_x| \geq 2$ ).

**Lemma 6.12.** *The maximal singular subspaces of  $(X_x, \Xi_x)$  are pairwise disjoint.*

*Proof.* Suppose now for a contradiction that two distinct maximal singular subspaces  $\mathcal{M}_1$  and  $\mathcal{M}_2$  of  $(X_x, \Xi_x)$  have a point  $p$  in common. Note that this implies  $\dim \mathcal{M}_i \geq 1$ ,  $i = 1, 2$ . Let  $L_1$  and  $L_2$  be lines in  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively, through  $p$ . In  $(X, \Xi)$ , this corresponds to planes  $\pi_1$  and  $\pi_2$  intersecting each other in a line  $M$  through  $x$ . Consider points  $p_i \in \pi_i \setminus M$ , for  $i = 1, 2$ . If  $p_1$  were not collinear with  $p_2$ , then Axiom (MM1') together with Lemma 3.3 implies that there is a member of  $\Xi$  containing  $\pi_1 \cup \pi_2$ , contradicting  $\max(W_x) = 1$ . Hence  $p_1 \perp p_2$ , which means that  $L_1$  and  $L_2$  span a singular plane. Varying  $L_1$  and  $L_2$  yields the singular subspace  $\langle \mathcal{M}_1, \mathcal{M}_2 \rangle$ , contradicting the maximality of  $\mathcal{M}_i$ ,  $i = 1, 2$ .  $\square$

It is now convenient to distinguish between the finite and infinite case, noting that in the infinite case, we really only need the field  $\mathbb{K}$  to have at least 5 elements, but the counting arguments for  $|\mathbb{K}| \leq 4$  are uniform and hold for all finite fields.

#### 6.3.1 The infinite case

*Proof of Proposition 6.1 for  $d \geq 4$  and  $|\mathbb{K}|$  infinite.* Take  $\xi \in \Xi_x$  arbitrary. Since  $|\Xi_x| \geq 2$ , there exists  $p \in X_x \setminus \xi$ . By Lemma 3.3,  $p^\perp \cap \xi$  is a singular subspace and hence there are distinct points  $p_1, p_2 \in X(\xi)$  not collinear to  $p$ . Set  $\xi_i := [p, p_i]$ ,  $i = 1, 2$ . By Lemma 4.2 (applied to  $(X_x, \Xi_x)$ , recalling  $\dim T_x \leq 2d$  and  $|\mathbb{K}| > 2$ ) there is a singular line meeting  $\xi_1, \xi_2$  and  $\xi$  in three distinct points. Lemma 4.1 then yields conics  $C_1 \subseteq X(\xi_1)$  and  $C_2 \subseteq X(\xi_2)$  through  $p$  on a common normal rational cubic scroll. Moreover, since  $|\mathbb{K}| > 4$ , all its transversal lines, except possibly the one through  $p$ , are singular; and so is the unique line  $M$  meeting all these transversal lines. Let  $L_1$  and  $L_2$  be two such singular transversal lines. Since both of them intersect  $M$  in a point, Lemma 6.12 implies that they are collinear with  $M$  and, repeating this argument,  $\langle L_1, L_2 \rangle$  is a singular 3-space. This however contradicts the fact that the points  $L_1 \cap C_1$  and  $L_2 \cap C_1$  are not collinear.  $\square$

#### 6.3.2 Finite case

In this subsection, we assume that  $\mathbb{K}$  is the finite field  $\mathbb{F}_q$ . Since over a finite field quadrics of index 0 only exist in dimensions 2 and 3, we deduce  $d = 4$ . Hence, by (MM3),  $N_x := \dim \langle X_x \rangle \leq 7$ .

**Lemma 6.13.** *The maximal singular subspaces of  $(X_x, \Xi_x)$  have dimension at most 1, and at least one singular line in  $X_x$  exists.*

*Proof.* Let  $M$  be a maximal singular subspace of  $(X_x, \Xi_x)$ . Let  $p \in X_x$  be a point outside  $M$  (this exists, as there are non-collinear points in  $(X_x, \mathcal{L}_x)$ ). For each point  $z$  of  $M$ , it follows from Lemma 6.12 that the points  $p$  and  $z$  are non-collinear, and hence they define a unique member  $\xi_z$  of  $\Xi_x$ . Consider the projection  $\rho$  of  $X_x$  from  $M$  onto a complementary subspace  $\Pi$  in  $\langle X_x \rangle$ . This projection is injective, since points with the same image are necessarily collinear to a point of  $M$ , contradicting Lemma 6.12. For each member  $\xi_z$  of  $\Xi_x$ ,  $z \in M$ , the projection of  $X(\xi_z) \setminus \{z\}$  is an affine plane  $\pi_z^* \ni p^\rho$  with projective completion  $\pi_z$  and

we have  $L_z := \pi_z \setminus \pi_z^* = T_z(\xi_z)^\rho$ . We claim that, for  $z_1, z_2 \in M$ ,  $z_1 \neq z_2$ ,  $\pi_{z_1} \cap \pi_{z_2} = \{p^\rho\}$ . Indeed, suppose for a contradiction that  $p^\rho \neq u \in \pi_{z_1} \cap \pi_{z_2}$ . By possibly considering  $\langle p^\rho, u \rangle \cap L_{z_1}$ , we may assume  $u \in L_{z_1}$ . If  $u \in L_{z_2}$ , then there exists a point  $v \in \langle p^\rho, u \rangle$ , with  $p^\rho \neq v \in \pi_{z_1}^* \cap \pi_{z_2}^*$ , contradicting injectivity of  $\rho$ . So  $u \notin L_{z_2}$  and hence there exists  $u_2 \in X(\xi_{z_2})$  with  $u_2^\rho = u$ . Since  $u \in L_{z_1}$ , there exists a line  $U_1$  in  $T_{z_1}(\xi_{z_1})$  through  $z_1$  with  $U_1^\rho = u$ . This implies that the plane  $\langle U_1, u_2 \rangle$  contains a point  $u' \in M \setminus \{z_1\}$ . Then  $[u', u_2]$  exists and intersects  $\xi_{z_1}$  in a point of  $U_1 \setminus \{z_1\}$  not belonging to  $X_x$ , contradicting (MM2). The claim is proved.

Suppose for a contradiction that  $\dim M \geq 2$ . Then  $\dim \Pi \leq 4$  (recall  $N_x \leq 7$ ). Then the number of points in the union of the  $n := |M|$  planes  $\pi_z$ ,  $z \in M$ , is at least  $n(q^2 + q) + 1$ . Since  $n \geq q^2 + q + 1$  by assumption, this exceeds the number of points of  $\Pi$ . Hence  $\dim M \leq 1$ .

The first assertion follows.

For the second assertion, suppose for a contradiction that there are no singular lines in  $X_x$ . Take any  $\xi \in \Xi_x$  and consider the projection  $\rho := \rho_\xi$  onto  $\Pi$  (recall that  $N_x \leq 7$  so  $\dim(\Pi) \leq 3$ ). For any  $\xi' \in \Xi_x$  meeting  $\xi$  in a point  $p \in X_x$ , the  $q^2$  points of  $X_x(\xi') \setminus \{p\}$  determine distinct members of  $\Xi_x$  with any point  $p' \in X_x(\xi) \setminus \{p\}$ . The number of points of  $X_x$  on these  $q^2$  members of  $\Xi_x$  distinct from  $p'$  is  $q^4$ , whereas  $\Pi$  contains at most  $q^3 + q^2 + q + 1$  points. By Lemma 3.17, this gives rise to a singular line in  $(X_x, \Xi_x)$  after all, a contradiction.  $\square$

**Lemma 6.14.** *Each point of  $X_x$  is contained in precisely one singular line and in at least  $q^2 + q$  members of  $\Xi_x$ . Also,  $|X_x| \geq q^4 + q^3 + q + 1$ .*

*Proof.* Suppose for a contradiction that there is a point  $p$  of  $X_x$  through which there are no singular lines. By Lemmas 6.12 and 6.13, there is a point  $r \in X_x$  contained in a unique singular line. Let  $\alpha_p$  and  $\alpha_r$  be the respective numbers of members of  $\Xi_x$  through  $p$  and  $r$ . Note that members of  $\Xi_x$  have  $q^2 + 1$  points. Then  $|X_x| = \alpha_p q^2 + 1 = \alpha_r q^2 + (q + 1)$ . It follows that  $(\alpha_p - \alpha_r)q = 1$ , a contradiction.

Hence each point  $p \in X_x$  is contained in a unique singular line. This means that  $|X_x| = |\mathcal{L}_x| \cdot (q + 1)$ . Since  $|X_x| = \alpha_p q^2 + q + 1$ , it then follows that  $q + 1$  divides  $\alpha_p$ . Taking a member of  $\Xi_x$  not through  $p$ , we also see that  $\alpha_p \geq q^2$ . Combined, this implies  $\alpha_p \geq q^2 + q$ . It now also follows that  $|X_x| = \alpha_p q^2 + q + 1 \geq q^4 + q^3 + q + 1$ .  $\square$

*Proof of Proposition 6.1 for  $d \geq 4$  and  $|\mathbb{K}|$  finite.* We consider the projection  $\rho := \rho_\xi$  from any  $\xi \in \Xi_x$  onto a complementary subspace  $\Pi$  (which has dimension at most 3). By Lemma 3.17, two points of  $X_x \setminus \xi$  have the same image under  $\rho$  precisely if they are on a singular line meeting  $X(\xi)$ . Hence the number of points in  $\rho(X_x)$  is, by Lemma 6.14, at least  $(q^4 + q^3 + q + 1) - (q^2 + 1)q = q^4 + 1$ . This is strictly more than the number of points in  $\Pi$  however, a contradiction.  $\square$

## 7 Case 2: there is a point $x \in X$ with $\max W_x = 2$

Suppose  $(X, \Xi)$  is an AVV of type  $d$  with global index set  $W$  in  $\mathbb{P}^N(\mathbb{K})$  and with  $\max W_x = 2$  for some  $x \in X$ . The existence of  $\xi \in \Xi$  through  $x$  with  $w_\xi = 2$  implies  $d \geq 4$ . We show the following proposition.

**Proposition 7.1.** *Let  $(X, \Xi)$  be an AVV of type  $d$  with global index set  $W$  in  $\mathbb{P}^N(\mathbb{K})$  containing a point  $x \in X$  with  $\max W_x = 2$ . Then  $W = \{2\}$ ,  $d = 4$  and hence  $(X, \Xi)$  is isomorphic to  $\mathcal{G}_{n,1}(\mathbb{K})$  for  $n \in \{4, 5\}$ .*

As in the previous case, we need a different approach for the case  $d = 4$ . We start with the generic case  $d \geq 5$ . Henceforth let  $x \in X$  be such that  $\max W_x = 2$ .

## 7.1 The case $d \geq 5$

**Lemma 7.2.** *Suppose  $\xi$  and  $\xi_1$  are members of  $\Xi$  of index 2 and index at most 2, respectively, intersecting each other in precisely a line  $L \ni x$ . Then  $d = 5$  and  $\xi_1$  has index 2.*

*Proof.* In the residue  $(X_x, \Xi_x)$ , the members  $\xi, \xi_1 \in \Xi$  correspond to members  $\xi', \xi'_1 \in \Xi_x$  of index 1 and index at most 1, respectively, intersecting each other in precisely a point  $p_1$ . By Lemma 4.4 (note that  $d - 2 \geq 3$ ), we may assume that there is a singular line  $\langle z_1, z \rangle$  with  $z_1 \in X(\xi'_1) \setminus p_1^\perp$  and  $z \in X(\xi') \setminus p_1^\perp$ . As  $\xi'$  has index 1, we can take a singular line  $M$  through  $z$  in  $\xi'$  that is not collinear to  $z_1$ . Then  $M$  and  $z_1$  determine a unique member  $\xi'_2$  of  $\Xi_x$ .

Suppose first  $d \geq 6$ . Then Lemma 4.5 implies that  $\xi'_2$  has index at least 2, and hence the corresponding member  $\xi_2$  of  $\Xi$  has index at least 3, contradicting  $\max(W_x) \leq 2$ . Next suppose  $d = 5$ ,  $|\mathbb{K}| > 2$  and the index of  $\xi'_1$  is equal to 0 (the latter implies  $\xi'_1 \cap \xi'_2$  is exactly  $z_1$ ). Then Lemma 4.5 yields the same contradiction as just before.

At last suppose  $d = 5$  and  $|\mathbb{K}| = 2$ . Then  $W = \{2\}$ , as the only non-degenerate quadrics in finite 6-dimensional projective space are split. This possibility is excluded by the Main Result of [18].  $\square$

This has the following corollary.

**Corollary 7.3.** *We have  $2 \in W_x \subseteq \{0, 2\}$ .*

*Proof.* Suppose for a contradiction that  $\Xi_x$  has a member  $\xi^0$  of index 0. Recall that by assumption  $\max(W_x) = 2$ , and hence  $\Xi_x$  has at least one member  $\xi^1$  of index 1 too. By Lemma 7.2,  $\xi^0 \cap \xi^1$  is empty. We take a pair of non-collinear points  $p^0 \in X(\xi^0)$  and  $p^1 \in X(\xi^1)$  and obtain  $[p^0, p^1] \in \Xi_x$  with  $\xi^0 \cap [p^0, p^1] = \{p^0\}$ . By Lemma 7.2, the index of  $[p^0, p^1]$  is 0. But then  $\xi^1 \cap [p^0, p^1] = \{p^1\}$ , contradicting Lemma 7.2.  $\square$

*Proof of Proposition 7.1 in the case  $d = 5$ .* We proceed by showing some claims.

*Claim 1.* *The residue  $(X_x, \Xi_x)$  is a pre-AVV of type 3 and global index set  $\{1\}$ .*

Indeed, by Corollary 7.3,  $\Xi_x$  only contains members of index 1. By Proposition 5.2,  $|\Xi_x| \geq 2$ . The claim now follows from Corollary 3.12.

*Claim 2.* *For each point  $p \perp x$ , we have  $T_p \cap X \subseteq p^\perp$ .*

Indeed, this follows immediately from Lemma 3.14 if there exist two members of  $\Xi$  which intersect each other precisely in  $p$ , in particular if  $0 \in W_p$ . So we may assume that  $(X_p, \mathcal{L}_p)$  is a  $(-1)$ -lacunary parapolar space. By Fact 3.7, all members of  $\Xi_p$  are split, a contradiction. This shows the claim.

*Claim 3.* *For each point  $p \perp x$ , there exists  $\xi \in \Xi$  with  $x \in \xi$ ,  $p \notin \xi$  and  $w_\xi = 2$ .*

Indeed, by Claim 1, there exist two members  $\xi_1, \xi_2 \in X$  of index 2 containing  $x$ . Suppose they both contain  $p$ . In  $\xi_1$  we select a singular line  $L_1$  through  $x$  not collinear to  $p$ . In  $\xi_2 \setminus \xi_1$ , we select a singular line  $L_2$  through  $x$  not in a plane with  $L_1$ . Then the unique member of  $\Xi$  determined by  $L_1$  and  $L_2$  has index 2 (by Claim 1) and does not contain  $p$ , showing the claim.

*Claim 4.* *For each point  $p \perp x$ , we have  $\dim(T_x \cap T_p) \leq 8$ .*

The following argument is inspired by the proof of Corollary 4.15 of [18]. By Claim 3, there exists  $\xi \in \Xi$  of index 2 with  $p \notin \xi \ni x$ . Set  $W_p := T_p \cap \xi$ . By Claim 2,  $W_p \cap X(\xi) = p^\perp \cap \xi$ , so  $W_p \subseteq T_x(\xi)$  and also  $\dim W_p \geq 3$ . As such there exists a line  $L$  in  $T_x(\xi)$  disjoint from  $W_p$ . Noting that  $T_p$  and  $T_x$  are at most 10-dimensional, and that  $L \subseteq T_x \setminus T_p$ , the claim follows.

Now, the content of Section 6.1 of [18] is to prove non-existence of pre-AVV of type 3 with global index set  $\{1\}$  for which each tangent spaces has dimension at most 7. By Claims 1 and 4, this completes the proof of Proposition 7.1 in the case  $d = 5$ .

*Proof of Proposition 7.1 in the case  $d \geq 6$ .* By Corollary 7.3, all members of  $\Xi_x$  have index 1 and, by Lemma 7.2 and  $d \geq 6$ , no two of them share precisely a point. Note that  $|\Xi_x| \geq 2$ , according to Proposition 5.2, and the absence of members of index 0 in  $\Xi_x$  implies that  $(X_x, \Xi_x)$  is connected. We conclude that  $(X_x, \mathcal{L}_x)$  is a strong 0-lacunary parapolar space of diameter 2 whose symps are quadrics of projective index 1. By Fact 3.8,  $(X_x, \mathcal{L}_x)$  is the direct product of a line and a projective  $n$ -space. This however implies that the members of  $\Xi_x$  are hyperbolic quadrics in 3 dimensions, i.e. that  $d - 1 = 3$ , a contradiction.  $\square$

## 7.2 The case $d = 4$

Recall that  $d = 4$  implies that  $\max W \leq 2$ . If  $W = \{2\}$  then the Main Result of [18] proves Proposition 7.1, hence we assume for a contradiction that  $W \neq \{2\}$ . Let  $z \in X$  be a point with  $\max W_z = 2$ . If  $W_z = \{2\}$ , then our assumption  $W \neq \{2\}$  implies that there exists  $y \in X \setminus \{z\}$  with  $\min W_y < 2$ . Using (MM1) on the pair  $y, z$ , we have  $2 \in W_y$ , and so by  $d = 4$  we obtain  $\max W_y = 2$ . We may hence assume that there are points  $x \in X$  with  $m_x := \min W_x < \max W_x = 2$ .

**Lemma 7.4.** *For each  $x \in X$  with  $m_x < \max W_x = 2$ , the residue  $(X_x, \Xi_x)$  contains no singular subspaces of dimension  $2 + m_x$ .*

*Proof.* Let  $\xi_1$  and  $\xi_2$  be two members of  $\Xi$  through  $x$  of index  $m_x$  and 2, respectively. Suppose for a contradiction that there is a singular subspace  $S$  of dimension  $3 + m_x$  through  $x$ . Since  $S \cap \xi_2$  is contained in a plane of  $\xi_2$  and  $\xi_1 \cap \xi_2$  is contained in a line of  $\xi_2$ , we can select a singular plane  $\pi \subseteq \xi_2$  through  $x$  which intersects  $S$  and  $\xi_1$  in precisely  $x$ . Inside  $T_x$  (which has dimension at most 8 by (MM3)), the subspace  $\langle \pi, S \rangle$  has dimension  $5 + m_x$  and therefore it intersects the 4-space  $T_x(\xi_1)$  in a subspace  $S'$  of dimension at least  $1 + m_x$ . Since each point of  $S' \setminus \{x\}$  is on a line meeting both  $S$  and  $\pi$ , (MM2) implies that  $S'$  is singular. However,  $\dim S' = 1 + m_x > w_{\xi_1}$ , a contradiction.  $\square$

**Lemma 7.5.** *Let  $x \in X$  be a point with  $m_x < \max W_x = 2$ . Then the point-line geometry  $(X_x, \mathcal{L}_x)$  is a strong parapolar space whose symps are quadrics of projective index 1.*

*Proof.* By assumption on  $x$ , there is at least one member  $\xi^* \in \Xi$  of index 2 through  $x$ . In  $(X_x, \mathcal{L}_x)$ ,  $\xi^*$  corresponds to a member  $\xi$  of  $\Xi_x$  of index 1. Observe that  $X_x$  contains a point  $z \notin X_x(\xi)$ , because either there is a second member of index 2 containing  $x$ , or, if not, then Proposition 5.2 yields at least one member of  $\Xi$  of index 1 containing  $x$ .

We show that  $z$  belongs to the connected component  $\mathcal{C}_\xi$  of  $\xi$  in  $(X_x, \mathcal{L}_x)$ . Suppose not. Then  $z^\perp \cap \xi = \emptyset$  and  $[z, p] \in \Xi_x$  has index 0 for each  $p \in X_x(\xi)$ . Moreover, we claim that  $z$  satisfies the following two conditions:  $T_z \cap X_x(\xi) = \emptyset$  and  $\dim T_z \leq 5$ . To that end, we consider the situation in  $(X, \Xi)$ , where  $\xi$  corresponds to  $\xi^*$  and  $z$  to a singular line  $L$  containing  $x$ , on which we select a point  $z' \neq x$ . Then, since  $1 \in W_{z'}$  by the above, Lemma 3.15 implies that  $T_{z'} \cap X \subseteq z'^\perp$  and hence  $T_{z'} \cap X(\xi^*) \subseteq z'^\perp \cap X(\xi^*) = \{x\}$ . This shows the first part of the claim. It also implies that  $T_{z'} \cap \xi$  is contained in  $T_x(\xi)$  and is at most 2-dimensional for it contains no points of  $X$  other than  $x$ . Hence there is a singular line in  $T_x(\xi)$  disjoint from  $T_{z'}$ , which means that  $\dim(T_{z'} \cap T_x) \leq 2d - 2 = 6$ . The claim follows.

Lemma 5.1 now implies that there is a member of  $\Xi_x$  of index at least 1 through  $z$  meeting  $\xi$  non-trivially, and hence  $z \in \mathcal{C}_\xi$  after all, a contradiction. We conclude that  $(X_x, \mathcal{L}_x)$  is connected and non-trivial (i.e., not a single point or a single member of  $\Xi_x$ ). The lemma now follows from Lemma 3.5.  $\square$

**Lemma 7.6.** *Let  $x \in X$  be a point with  $m_x < \max W_x = 2$ . Then either  $(X_x, \mathcal{L}_x)$  is isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$  or each point  $p \in X_x$  is contained in four singular lines  $L_1, L_2, L_3, L_4$  of  $\mathcal{L}_x$  not in a common singular plane. In the latter case,*

- (i) *either  $\dim \langle L_1, L_2, L_3, L_4 \rangle = 4$  and  $\{L_1, L_2, L_3, L_4\}$  contains at most two (necessarily disjoint) pairs of collinear lines;*
- (ii) *or  $\dim \langle L_1, L_2, L_3, L_4 \rangle = 3$  and three lines of  $\{L_1, L_2, L_3, L_4\}$  lie in a singular plane.*

*Proof.* By Lemma 7.5,  $(X_x, \mathcal{L}_x)$  is a strong parapolar space whose symps are quadrics of projective index 1. Suppose first that  $(X_x, \mathcal{L}_x)$  is 0-lacunary. By Fact 3.8, and the fact that  $X_x$  does not contain singular 3-spaces by Lemma 7.4,  $(X_x, \mathcal{L}_x)$  is then isomorphic to either the direct product  $\mathcal{S}_{1,1,1}(\mathbb{K})$ , or to  $\mathcal{S}_{1,2}(\mathbb{K})$ . In the latter case, each point  $p \in X_x$  is contained in four singular lines  $L_1, L_2, L_3, L_4$  satisfying (ii).

So next, we suppose that there is a point  $p \in X_x$  contained in two index 1 members of  $\Xi_x$  that intersect each other in  $p$  only. Hence there are four singular lines through  $p$  not all in one singular plane. Let  $q$  be a point of  $X_x$  collinear to  $p$ . Let  $L_1, L_2, L_3$  be three singular lines through  $p$  distinct from  $pq$ . For each  $i \in \{1, 2, 3\}$ , the lines  $pq$  and  $L_i$  determine either a member of  $\Xi_x$  in  $X_x$  or a singular plane. Hence it is clear that there are at least four singular lines, not in a common plane, through  $q$  as well. By connectivity, there are four singular lines not in a common plane through each point of  $X_x$ .

Now assume that  $p \in X_x$  is contained in four singular lines  $L_1, L_2, L_3, L_4$ , not in one singular plane. First suppose that  $\dim \langle L_1, L_2, L_3, L_4 \rangle = 3$ . Then the planes  $\langle L_1, L_2 \rangle$  and  $\langle L_3, L_4 \rangle$  have a line in common. By (MM2), at least one of these planes, say  $\langle L_1, L_2 \rangle$  is singular, and one of  $L_3, L_4$  is contained in  $\langle L_1, L_2 \rangle$ . Hence (ii) holds. Next, suppose  $\dim \langle L_1, L_2, L_3, L_4 \rangle = 4$ . Then  $L_1$  is collinear with at most one member of  $\{L_2, L_3, L_4\}$ , as otherwise we obtain two singular planes sharing a line, and by absence of quadrics of index higher than 1, this yields a singular 3-space, violating Lemma 7.4. Hence (i) holds.  $\square$

**Lemma 7.7.** *Let  $x \in X$  be a point with  $m_x < \max W_x = 2$ . Then  $(X_x, \mathcal{L}_x)$  is isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ .*

*Proof.* Recall that, by Lemma 3.10,  $(X_x, \Xi_x)$  is a weak pre-AVV of type 2 with global index set  $W'_x$  in  $\mathbb{P}^N(\mathbb{K})$  with  $N \leq 7$ . We claim that Axioms (MM1) and (MM3) hold in  $(X_x, \Xi_x)$ . Indeed, (MM1') holds, and the same argument that we used to show Axiom (PPS3) in the proof of Lemma 3.5(iii), completes the proof of (MM1). Suppose now for a contradiction that there is a point  $p \in X_x$  with  $\dim \langle T_p(\xi) \mid p \in \xi \in \Xi_x \rangle \geq 5$ .

*Claim: there exist  $\xi \in \Xi_x$  containing  $p$ , and three singular lines of  $\mathcal{L}_x$ , also containing  $p$  and generating a 3-space  $S$ , such that  $S \cap \xi = \{p\}$ .*

We may assume that  $(X_x, \mathcal{L}_x)$  is not isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ , and hence Lemma 7.6 implies there are four singular lines  $L_1, L_2, L_3, L_4 \in \mathcal{L}_x$  through  $p$ . Set  $\Pi := \langle L_1, L_2, L_3, L_4 \rangle$ . Also by Lemma 7.6 either  $\dim \Pi = 4$  and, up to renumbering,  $[L_1, L_2], [L_3, L_4] \in \Xi_x$  (case (i)); or  $\dim \Pi = 3$  and, up to renumbering,  $L_2, L_3, L_4$  are in a singular plane  $\pi$  (case (ii)). By assumption on  $p$ , there exists  $\xi^* \in \Xi_x$  through  $p$  such that  $T_p(\xi^*)$  has at most a line in common with  $\Pi$ . We distinguish two cases.

*Case 1: Suppose that  $\xi^*$  has index 1.* Then we obtain that  $T_p(\xi^*)$  contains a singular line  $L_5$  with  $L_5 \not\subseteq \Pi$ . In case (ii), the lines  $L_1, L_2, L_3, L_5$  span a 4-space and, as  $\max W_x \leq 2$  and as there are no singular 3-spaces by Lemma 7.4, we deduce that  $[L_1, L_2], [L_3, L_5] \in \Xi_x$ , which brings us to case (i). So let us consider case (i) now, where we assume that  $[L_1, L_2], [L_3, L_4] \in \Xi_x$ . Since  $L_5 \not\subseteq \Pi$ , the lines  $L_1, L_2, L_3, L_4, L_5$  generate a 5-space. We consider the 3-space  $S := \langle L_3, L_4, L_5 \rangle$  and  $\xi := [L_1, L_2] \in \Xi_x$ . Clearly,  $\dim \langle \xi, S \rangle \geq \dim \langle L_1, \dots, L_5 \rangle = 5$ . If  $\dim \langle \xi, S \rangle = 6$ , then  $(L_3, L_4, L_5)$  and  $\xi$  are as required by the claim. If  $\dim \langle \xi, S \rangle = 5$ , then  $\xi \cap S$  is a line  $L$ . Since  $L$  does not belong to  $\langle L_1, L_2 \rangle = T_p(\xi)$ , it contains a unique point  $p' \in X_x(\xi) \setminus \{p\}$  (note that in particular,  $L \notin \{L_3, L_4, L_5\}$ ). Since  $p \notin p'^\perp$ , there is a point

$p_5 \in L_5 \setminus \{p\}$  with  $p_5 \notin p'^\perp$ . Using (MM1) and (MM2), we deduce that  $\langle p', p_5 \rangle$  meets  $\langle L_3, L_4 \rangle$  in a point of  $X_x$ . This however implies that the line  $\langle p', p_5 \rangle$  is singular, contradicting our choice of  $p_5 \notin p'^\perp$ . This concludes Case 1.

*Case 2: Suppose now that  $\xi^*$  has index 0.* In Case (ii), we immediately obtain that  $\xi^*$  cannot share a (necessarily non-singular) line  $L$  with  $\Pi$ , for  $\langle L_1, L \rangle \cap \pi$  would be a singular line  $L'$  through  $p$  and hence  $L \subseteq \xi \cap [L_1, L']$ , contradicting (MM2). As such,  $L_1, L_2, L_3$  and  $\xi^*$  are as required by the claim. So we may assume Case (i). If  $\xi^* \cap \Pi = \{p\}$ , we are done, so suppose  $\dim(\xi^* \cap \Pi) \geq 1$ . If  $\xi^* \cap \Pi$  is exactly a line  $L$ , then renumbering if necessary, we have  $L \not\subseteq \langle L_1, L_2, L_3 \rangle$  and then the pair  $(L_1, L_2, L_3)$ ,  $\xi^*$  does the trick. So finally, suppose  $\xi^* \cap \Pi$  is a plane  $\alpha$ . Note that, by assumption on  $\xi^*$ ,  $\alpha$  contains precisely one line  $T$  of  $T_p(\xi^*)$ . By (MM2) and  $w_{\xi^*} = 0$ ,  $\alpha$  meets  $\langle L_1, L_2 \rangle$  and  $\langle L_3, L_4 \rangle$  in  $p$  only. A dimension argument then implies that there is a unique plane  $\alpha_i$  through  $L_i$  that meets both  $\langle L_3, L_4 \rangle$  and  $\alpha$  in respective lines  $M_i$  and  $M'_i$ , for  $i = 1, 2$ . Clearly,  $M'_1 \neq M'_2$ , so we may assume that  $M'_1 \neq T$ . Since  $M'_1 \not\subseteq T_p(\xi^*)$ , it contains a unique point  $p' \in X_x \setminus \{p\}$ . Taking a point  $p_1$  on  $L_1 \setminus \{p\}$  and considering the line  $\langle p', p_1 \rangle$  and its intersection with  $M_1$ , we deduce as in the previous paragraph, that the plane  $\langle L_1, M_1 \rangle$  is singular, contradicting the fact that  $M'_1$  is not. This concludes Case 2 and the claim follows.

Henceforth, let  $L_1, L_2, L_3 \in \mathcal{L}_x$  be three lines containing  $p$  such that  $S := \langle L_1, L_2, L_3 \rangle$  has dimension 3 and such that there exists  $\xi \in \Xi_x$  with  $S \cap \xi = \{p\}$ . Since  $\max W_x \leq 2$  and by absence of singular 3-spaces, we obtain that, up to renumbering,  $\xi_1 := [L_1, L_2]$  and  $\xi_3 := [L_2, L_3]$  are members of  $\Xi_x$ . Since  $S \cap \xi = \{p\}$ , also  $\xi \cap \xi_i = \{p\}$  for  $i = 1, 3$ . Consequently the subspaces  $\langle \xi, \xi_1 \rangle$  and  $\langle \xi, \xi_3 \rangle$  are 6-dimensional, and recalling that  $\dim \langle X_x \rangle \leq 7$ , they share a 5-space  $\Sigma$  containing  $\langle \xi, L_2 \rangle$ . It follows that  $\Sigma$  meets  $\xi_i$  in a plane  $\pi_i$  containing  $L_2$  and hence at least one other singular line  $R_i$  of  $\xi_i$ , for  $i = 1, 3$ .

Suppose first that  $\langle R_1, R_3 \rangle$  is a 3-space (equivalently,  $R_1 \cap L_2 \neq R_3 \cap L_2$ ). Then  $\langle R_1, R_3 \rangle$  meets  $\xi$  in a line  $R$  containing  $p$ . Let  $r$  be any point of  $R \setminus \{p\}$  and consider the unique line  $R'$  through  $r$  meeting  $R_1$  and  $R_3$  non-trivially, say in points  $r_1$  and  $r_3$ , respectively. By (MM2),  $R'$  is singular. Note that  $r \neq p$  implies that  $r_1, r_3 \notin L_2$ . However,  $r_3$  is now collinear to the non-collinear points  $L_2 \cap R_3$  and  $r_1$  of  $X(\xi_1)$ , so  $r_3 \in \xi_1 \cap \xi_3 = L_2$ , a contradiction.

Therefore  $R_1 \cap L_2 = R_3 \cap L_2 =: y$ . Note that  $y \neq p$ , for otherwise  $R_1 = L_1$  and  $R_3 = L_3$ , violating the fact that  $\langle R_1, L_2, R_3, \xi \rangle = 5$ . Then the plane  $\langle R_1, R_3 \rangle$  meets  $\xi$  in a point  $z$ . Clearly  $z \neq p$  (otherwise  $R_3 \in \langle R_1, p \rangle \subseteq \xi_1$ ) and  $z \notin R_1 \cup R_3$  (since  $\xi \cap \xi_i = \{p\}$  and  $p \notin R_i$  for  $i = 1, 3$ ). Using (MM2) if  $\langle R_1, R_3 \rangle$  is non-singular, we obtain that  $z \in X(\xi)$  and hence  $\langle R_1, R_3 \rangle$  is singular anyway. If  $p$  and  $z$  are not collinear, then  $y$ , being collinear to both  $p$  and  $z$ , belongs to  $\xi$ , a contradiction. Hence  $\langle p, z \rangle$  is singular and as a consequence,  $\langle L_2, z \rangle$  is a singular plane intersecting the singular plane  $\langle R_1, R_3 \rangle$  in line. By  $\max W_x \leq 2$ , this yields a singular 3-space  $\langle R_1, L_2, R_3 \rangle$ , contradicting Lemma 7.4.

We conclude  $(X_x, \Xi_x)$  is an AVV of type 2 whose global index set  $W'$  has  $\max(W') = 1$ . Proposition 6.1 yields  $W' = \{1\}$  and hence  $W_x = \{0, 2\}$ . By Lemma 7.4, there are no singular planes in  $X_x$ . Recalling that  $(X_x, \mathcal{L}_x)$  is connected by Lemma 7.5, it now follows from Proposition 6.4 that  $(X_x, \mathcal{L}_x)$  is isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$  after all.  $\square$

*Proof of Proposition 7.1 in the case  $d = 4$ .* We already noted that, if  $W = \{2\}$ , the proposition follows from the Main Result of [18], and that we therefore may assume that there is a point  $x \in X$  with  $\min\{W_x\} < \max\{W_x\} = 2$ . By Lemma 7.7,  $(X_x, \mathcal{L}_x)$  is isomorphic to  $\mathcal{S}_{1,1,1}(\mathbb{K})$ . Noting that, for any  $y, z \in X_x$  at distance 3 from each other measured in  $(X_x, \mathcal{L}_x)$ , we have  $[y, z] \in \Xi_x$  is contained in  $\langle X_x \rangle$ , this contradicts Lemma 4.3.  $\square$

## 8 Case 3: For each $x \in X$ , either $W_x = \{0\}$ or $\max(W_x) \geq 3$

**Proposition 8.1.** *Let  $(X, \Xi)$  be an AVV of type  $d$  with global index set  $W$  such that, for each  $x \in X$ , either  $W_x = \{0\}$  or  $\max W_x \geq 3$ . Then  $W$  is a singleton  $\{w^*\}$  and one of the following occurs.*

- (i)  $w^* = 0$ ,  $d \in \{2^a \mid a \in \mathbb{N}\} \cup \{\infty\}$  and  $(X, \Xi)$  is the standard Veronese representation of a projective plane over a quadratic alternative division ring;
- (ii)  $w^* = 3$ ,  $d = 6$  and  $(X, \Xi)$  is the half spin variety  $\mathcal{D}_{5,5}(\mathbb{K})$ ;
- (iii)  $w^* = 4$ ,  $d = 8$  and  $(X, \Xi)$  is the Cartan variety  $\mathcal{E}_{6,1}(\mathbb{K})$ .

Again, we show this in a series of lemmas. Throughout, let  $w^*$  be the maximum of  $W$ , which is well defined as  $|W|$  is bounded above by  $\lceil \frac{d+1}{2} \rceil$ . If  $w^* = 0$ , then by the Main Result of [13], (i) of Proposition 8.1 holds. So we assume from now on that  $w^* \geq 3$  (and hence  $d \geq 6$ ).

**Lemma 8.2.** *Let  $x \in X$  be a point with  $\max(W_x) \geq 3$  and let  $p \in X$  be a point collinear to  $x$ . Then there exists  $\xi \in \Xi$  of index at least 3 going through  $p$  and not through  $x$ .*

*Proof.* Suppose for a contradiction that all members of  $\Xi$  through  $p$  of index at least 3 also contain  $x$ . First note that (MM1) assures that there is at least one member of  $\Xi$ , necessarily of index at least 1, through the singular line  $\langle p, x \rangle$ ; hence our assumptions imply  $\max W_p \geq 3$  and so there is at least one member of  $\Xi$  of index at least 3 through  $p$ .

By Corollary 3.12 and Lemma 5.2,  $(X_p, \Xi_p)$  is a pre-AVV. The line  $\langle p, x \rangle$  corresponds to a point  $q \in X_p$  which, by the previous paragraph, has the property that it is contained in all members of  $\Xi_p$  of index at least 2.

We claim that there is a pair  $\xi_1$  and  $\xi_2$  in  $\Xi_p$  of index at most 1 and index 1, respectively, intersecting each other in exactly one point. Let  $\xi^*$  be a member of  $\Xi_p$  of index at least 2 containing  $q$  (which exists by the above). Take a point  $p_1$  in  $X(\xi^*)$  not collinear to  $q$ . Then all members of  $\Xi_p$  through  $p_1$ , except for  $\xi^* = [p_1, q]$ , have index at most 1. There are three cases:

1. *There are  $\xi_1, \xi_2$  in  $\Xi_p \setminus \{\xi^*\}$  through  $p_1$  of index 0 and 1, respectively.* In this case, it is clear that  $\xi_1 \cap \xi_2 = \{p_1\}$ .
2. *All members of  $\Xi_p \setminus \{\xi^*\}$  through  $p_1$  have index 1.* By Lemma 4.6 applied in  $(X_{pp_1}, \Xi_{pp_1})$ , we find a pair of members of  $\Xi_p$  with index 1 intersecting in precisely  $p_1$ .
3. *All members of  $\Xi_p \setminus \{\xi^*\}$  through  $p_1$  have index 0.* Let  $\xi_1$  be any such quadric, and take  $r \in X(\xi_1) \setminus \{p_1\}$  not collinear to  $q$ . Let  $p_2$  be a point in  $X(\xi^*)$  collinear to  $q$  but not contained in  $[r, q]$  (in particular,  $r$  and  $p_2$  are not collinear). By (MM1), there is a  $\xi_2 \in \Xi_p$  through  $r$  and  $p_2$ , which does not contain  $q$  by choice of  $p_2$ , and hence has index at most 1. If  $\xi_2$  has index 1, then  $\xi_1$  and  $\xi_2$  satisfy our requirements, so suppose  $\xi_2$  has index 0. Applying Lemma 4.2 (note that  $d - 2 \geq 4$ ) on the triple  $\xi^*, \xi_1, \xi_2$  yields a singular line  $L$  meeting these three quadrics in three distinct points  $z, z_1$  and  $z_2$ , respectively, with  $z_2$  and  $p_2$  non-collinear. This implies that  $z \neq q$ : otherwise,  $p_2 \perp q = z \perp z_2$ , and hence  $q \in \xi_2$ , a contradiction. Hence we can take a line  $L'$  through  $z$  in  $X(\xi^*)$  collinear to neither  $q$ , nor  $z_1$ . The unique member  $\xi'_2$  of  $\Xi_p$  through  $L$  and  $L'$  then does not contain  $q$ ; hence it has index at most 1. As it contains the singular line  $L$ , it has precisely index 1. The pair  $\xi_1, \xi'_2$  qualifies.

This shows the claim. Let  $\xi_1, \xi_2$  be such members of  $\Xi_p$ , intersecting in a unique point  $p'$ . By Lemma 4.4, we may assume that there is a singular line  $\langle z_1, z_2 \rangle$  with  $z_1 \in X(\xi_1) \setminus p'^{\perp}$  and  $z_2 \in X(\xi_2) \setminus p'^{\perp}$ .

As  $\xi_2$  has index 1, we can take a line  $L$  through  $z_2$  in  $\xi_2$  that is not collinear to  $z_1$ . Then  $L$  and  $z_1$  determine a unique member  $\xi$  of  $\Xi_p$ . According to Lemma 4.5, there would be members of  $\Xi_p$  of index at least 2 not going through  $q$ , a contradiction. The lemma follows.  $\square$

**Lemma 8.3.** *Suppose  $x \in X$  has  $\max(W_x) \geq 3$ . Then either  $T_x \cap X \subseteq x^{\perp}$  or  $(X_x, \mathcal{L}_x)$  is isomorphic to one of the following:  $\mathcal{G}_{n,1}(\mathbb{K})$  for  $n \in \{4, 5\}$  or  $\mathcal{E}_{6,1}(\mathbb{K})$ .*

*Proof.* Suppose that  $T_x \cap X$  contains a point  $y$  not contained in  $x^\perp$ . Then, by Lemma 3.15,  $\min(W_x) \geq 2$ , in which case  $(X_x, \mathcal{L}_x)$  is a strong parapolar space of diameter 2 (cf. Corollary 3.12 and Lemma 3.5). If there are two members  $\xi_1, \xi_2 \in \Xi$  intersecting in  $x$  only, then  $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$  by (MM3) and Lemma 3.14 leads to a contradiction. So  $(X_x, \mathcal{L}_x)$  is  $(-1)$ -lacunary. Since  $\max(W_x) \geq 3$ , the only possibilities are, according to Fact 3.7,  $\mathcal{G}_{n,1}(\mathbb{K})$  for  $n \in \{4, 5\}$  or  $\mathcal{E}_{6,1}(\mathbb{K})$ .  $\square$

**Lemma 8.4.** *Let  $x$  be a point of  $X$  with  $\max(W_x) \geq 3$ . Then  $(X_x, \Xi_x)$  is an AVV of type  $d - 2$  with global index set  $W'_x$  and  $\dim\langle X_x \rangle \leq 2d - 1$ .*

*Proof.* By Lemma 8.3, we may assume that  $T_x \cap X \subseteq x^\perp$ , as otherwise  $(X_x, \mathcal{L}_x)$  is isomorphic to  $\mathcal{G}_{n,1}(\mathbb{K})$  for  $n \in \{4, 5\}$  or to  $\mathcal{E}_{6,1}(\mathbb{K})$ , which are AVVs indeed.

*Claim 1:* for each  $z \in X_x$ ,  $\dim T_z \leq 2d - 4$ .

We consider the situation in  $(X, \Xi)$ , in which  $z$  corresponds to a singular line  $L$  containing  $x$ . Let  $p$  be a point of  $L \setminus \{x\}$ . Since  $\max(W_x) \geq 3$ , Lemma 8.2 yields a member  $\xi \in \Xi$  of index at least 3 containing  $p$  and not containing  $x$ . The fact that  $T_x \cap X \subseteq x^\perp$  implies that  $T_x \cap X(\xi)$  is a singular subspace  $S$  of  $X(\xi)$ . Then  $S \subseteq T_x \cap \xi \subseteq T_p(\xi)$  and  $\dim(T_x \cap \xi) \leq d - w_\xi$  by Lemma 3.13. Consequently there is a subspace  $S'$  in  $T_p(\xi) \setminus T_x$  of dimension  $w_\xi - 1 \geq 2$ . Since  $S'$  is not contained in  $T_x$ , we obtain that  $\dim(T_p \cap T_x) \leq 2d - 3$ . The claim follows.

*Claim 2:*  $(X_x, \mathcal{L}_x)$  is connected. Let  $\xi$  be a member of  $\Xi$  through  $x$  with  $w_\xi \geq 3$ . Suppose for a contradiction that there is a point  $z \in X_x$  not contained in the connected component  $\mathcal{C}_\xi$  of  $\xi$  in  $(X_x, \mathcal{L}_x)$ . Then  $z^\perp \cap \xi = \emptyset$  and  $[z, p] \in \Xi_x$  has index 0 for each  $p \in X_x(\xi)$ . In exactly the same manner as in the proof of Lemma 7.5, we obtain  $T_z \cap X_x(\xi) = \emptyset$  and, by Lemma 5.1, this implies that  $z \in \mathcal{C}_\xi$  after all. This contradiction shows the claim.

By Lemma 3.10 and Claim 1,  $(X_x, \Xi_x)$  is a weak AVV of type  $d - 2$  with global index set  $W'_x$  and  $\dim\langle X_x \rangle \leq 2d - 1$ . We show that  $(X_x, \Xi_x)$  satisfies (MM1). Consider an arbitrary  $\xi \in \Xi_x$  of index at least 2. Let  $L$  be a line of  $\mathcal{L}_x$  intersecting  $\xi$  in a unique point  $p$ . Taking a singular line  $M$  in  $\xi \setminus L^\perp$ , we obtain that  $[L, M]$  is a member of  $\Xi_x$  containing  $L$ . Claim 2 now implies that (MM1) holds in  $(X_x, \Xi_x)$ . Proposition 5.2 implies that  $|\Xi_x| \geq 2$ . The lemma follows.  $\square$

We are ready to show that  $W$  has to be a singleton.

*Proof of Proposition 8.1.* We show this by induction on  $w^*$ . If  $w^* = 0$ , then clearly,  $W = \{0\}$  and the main result of [13] leads us to possibility (i) of Proposition 8.1. So assume  $w^* \geq 0$  and take an arbitrary  $x \in X$  with  $w^* = \max W_x$ . By Lemma 8.4, the residue  $(X_x, \Xi_x)$  is an AVV of type  $d - 2$  with global index set  $W'_x$  and with  $\dim\langle X_x \rangle \leq 2d - 1$ .

First, suppose that  $w^* = 3$ . Then  $(X_x, \Xi_x)$  has a point  $z$  with  $\max W'_z = 2$  and hence, by Proposition 7.1,  $(X_x, \Xi_x)$  is isomorphic to  $\mathcal{G}_{n,1}(\mathbb{K})$  for  $n \in \{4, 5\}$ ; in particular,  $d = 6$ . Since  $\dim\langle X_x \rangle \leq 2d - 1 = 11$  and  $\dim\langle \mathcal{G}_{5,1}(\mathbb{K}) \rangle = 14$ , we deduce that  $(X_x, \Xi_x)$  is isomorphic to  $\mathcal{G}_{4,1}(\mathbb{K})$  (which lives in dimension 9). Since the latter's diameter is 2, all members of  $\Xi_x$  have index 2 and consequently,  $W_x \subseteq \{0, 3\}$ . Suppose for a contradiction that there exists  $\xi \in \Xi$  with  $x \in \xi$  and  $w_\xi = 0$ . Then  $T_x(\xi)$  is a 6-space in  $T_x$  which has at least a 3-space  $\Pi$  in common with  $\langle X_x \rangle \subseteq T_x \setminus \{x\}$ . Proposition 2.1(ii) implies that there are points  $x_1, x_2 \in X_x$  such that  $\langle x_1, x_2 \rangle$  intersects  $\Pi$  non-trivially, and hence  $[x_1, x_2] \cap T_x(\xi)$  is non-empty, contradicting (MM2). We conclude that  $W_x = \{3\}$ . Now let  $y \in X \setminus \{x\}$  be arbitrary. By (MM1), there is a member of  $\Xi$  containing  $x$  and  $y$ , which necessarily has index 3 as  $W_x = \{3\}$ . Therefore,  $\max W_y = 3$  and we may apply the above arguments to  $y$  as well to obtain  $W_y = \{3\}$ . We conclude that  $W = \{w^*\} = \{3\}$  indeed. It follows from the main result of [18] that  $(X, \Xi)$  is isomorphic to  $\mathcal{D}_{5,5}(\mathbb{K})$ .

Next, suppose that  $w^* \geq 4$ . By induction, all members of  $\Xi_x$  have index  $w^* - 1 \geq 3$ , i.e.,  $W_x \subseteq \{0, w^*\}$ . In particular, the main result of [18] implies that  $(X_x, \Xi_x)$  is isomorphic to either  $\mathcal{D}_{5,5}(\mathbb{K})$  or  $\mathcal{E}_{6,1}(\mathbb{K})$ ;



in particular  $d = 8$  and  $w^* = 4$ . Since  $\dim\langle X_x \rangle \leq 2d - 1 = 1$  and  $\dim\langle \mathcal{E}_{6,1}(\mathbb{K}) \rangle = 26$ , we conclude that  $(X_x, \Xi_x)$  is isomorphic to  $\mathcal{D}_{5,5}(\mathbb{K})$  (and  $\dim(\mathcal{D}_{5,5}(\mathbb{K})) = 15$ ).

We show that  $0 \notin W_x$ . Indeed, suppose that there is a member  $\xi \in \Xi$  with  $x \in \xi$  and  $w_\xi = 0$ . Then  $T_x(\xi)$  is an 8-space in  $T_x$  sharing a 7-space  $\Pi$  with the 15-space  $\langle X_x \rangle \subseteq T_x$ . As above, Proposition 2.1(iii) leads to a contradiction. We conclude that  $W_x = \{w^*\}$ . Just like in the previous case, we deduce that  $W_y = \{w^*\}$  for any  $y \in X \setminus \{x\}$  as well, and hence  $W = \{w^*\} = \{4\}$ . The main result of [18] now yields that  $(X, \Xi)$  is isomorphic to  $\mathcal{E}_{6,1}(\mathbb{K})$ .  $\square$

## 9 Conclusion

**Proof of the Main Theorem.** Let  $(X, \Xi)$  be an AVV of type  $d$  with index set  $W$ .

- (1) If for some  $x \in X$ ,  $\max W_x = 1$ , then Proposition 6.1 implies that  $(X, \Xi)$  is split, and we have Case  $d = 2$  of Theorem 1.2.
- (2) If for some  $X \in X$ ,  $\max W_x = 2$ , then Proposition 7.1 implies that  $(X, \Xi)$  is split, and we have Case  $d = 4$  of Theorem 1.2.
- (0,  $\geq 3$ ) If for all  $x \in X$ ,  $\max W_x \geq 3$  or  $W_x = \{0\}$ , then by Proposition 8.1
  - (0) either  $W = \{0\}$  (and we have a Veronese cap), and we have the case  $d = 2^\ell$  (including  $\ell = 0$  giving rise to the case  $d = 1$ ) of Theorem 1.2,
  - (3) or  $W = \{3\}$ ,  $(X, \Xi)$  is split and we have the case  $d = 6$  of Theorem 1.2,
  - (4) or  $W = \{4\}$ ,  $(X, \Xi)$  is split and we have the case  $d = 6$  of Theorem 1.2.

This covers all cases and proves Theorem 1.2, in particular the Main Theorem.

## References

- [1] M. Aschbacher, The 27-dimensional module for E6, I. *Invent. Math.* **89** (1987), 159–195.
- [2] F. Buekenhout and A. Cohen, *Diagram geometries Related to classical groups and buildings*, EA Series of Modern Surveys in Mathematics **57**. Springer, Heidelberg, (2013), xiv+592 pp.
- [3] C. Chevalley, *The algebraic theory of spinors*. Columbia University Press, New York, 1954.
- [4] A. M. Cohen, *Point-line geometries related to buildings*. Handbook of Incidence Geometry, Buildings and Foundations, (ed. F. Buekenhout), Chapter 9, North-Holland, Amsterdam (1995). pp. 647–737.
- [5] B. Cooperstein, J. A. Thas & H. Van Maldeghem, Hermitian Veroneseans over finite fields. *Forum Math.* **16** (2004), 365–381.
- [6] A. De Schepper, J. Schillewaert, H. Van Maldeghem and M. Victoor, A geometric characterisation of Hjelmslev-Moufang planes, *submitted*.
- [7] A. De Schepper, J. Schillewaert, H. Van Maldeghem and M. Victoor, On exceptional Lie geometries, *submitted*.
- [8] A. De Schepper and H. Van Maldeghem, Veronese representation of Hjelmslev planes over Cayley-Dickson algebras, to appear in *Result. Math.*
- [9] J. W. P. Hirschfeld & J. A. Thas, *General Galois Geometries*. Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991.
- [10] L. Manivel, On spinor varieties and their secants. *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009), Paper 078, 22 pp.
- [11] F. Mazzocca & N. Melone, Caps and Veronese varieties in projective Galois spaces. *Discrete Math.* **48** (1984), 243–252.
- [12] O. Krauss, *Geometrische Charakterisierung von Veronesemannigfaltigkeiten*. Ph.D. thesis, Braunschweig, February 2014.

- [13] O. Krauss, J. Schillewaert and H. Van Maldeghem, Veronesean representations of Moufang planes. *Mich. Math. J.* **64** (2015), Issue 4, 819–847.
- [14] J. Schillewaert & H. Van Maldeghem, Hermitian Veronesean caps. *Springer Proceedings in Mathematics*, **12** (2012), 175–191.
- [15] J. Schillewaert & H. Van Maldeghem, Quadric Veronesean caps. *Bull. Belgian Math. Soc. Simon Stevin* **20** (2013), 19–25.
- [16] J. Schillewaert and H. Van Maldeghem, Projective planes over quadratic two-dimensional algebras. *Adv. Math.*, **262** (2014), 784–822.
- [17] J. Schillewaert and H. Van Maldeghem, A combinatorial characterization of the Lagrangian Grassmannian  $LG(3, 6)(\mathbb{K})$ . *Glasgow Math. J.*, **58** (2016), no. 2, 293–311
- [18] J. Schillewaert and H. Van Maldeghem, On the varieties of the second row of the split Freudenthal-Tits Magic Square. *Ann. Inst. Fourier*, **67** no. 6 (2017), 2265–2305
- [19] F. Severi, Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a suoi punti tripli apparenti, *Rend. Circ. Mat. Palermo* **15** (1901), 33–51.
- [20] E. E. Shult, *Points and Lines, Characterizing the Classical Geometries*. Universitext, Springer-Verlag, Berlin Heidelberg, 2011.
- [21] J. Tits, Sur la trinité et certains groupes qui s'en déduisent. *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60.
- [22] J. Tits, Sur certaines classes d'espaces homogènes de groupes de Lie. *Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8<sup>o</sup>* (2) **29** (3) (1955), 268 p.
- [23] H. Van Maldeghem & M. Victoor, Combinatorial and geometric constructions of spherical buildings. **In** *Surveys in Combinatorics 2019*, Cambridge University Press (ed. A. Lo et al.), *London Math. Soc. Lect. Notes Ser.* **456** (2019), 237–265.
- [24] F. Zak, Tangents and secants of algebraic varieties. *Translation of mathematical monographs*, AMS, 1983.