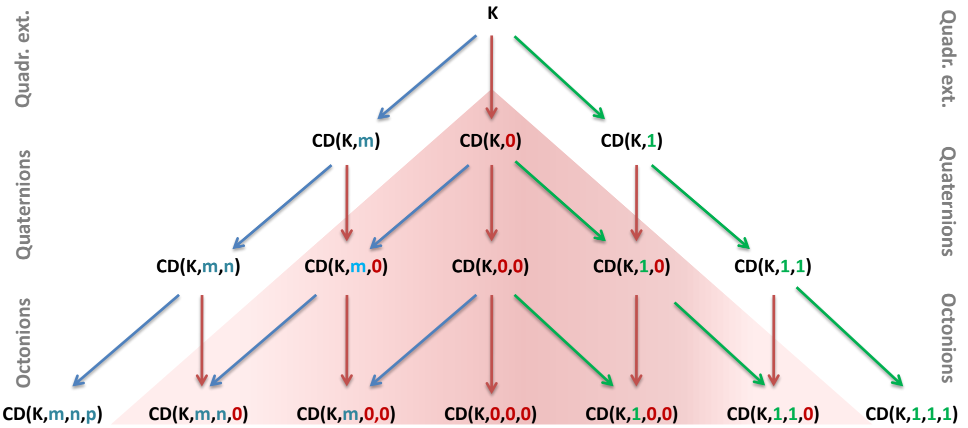

Degenerate Cayley–Dickson algebras

Anneleen De Schepper

DMV 2015

Alternative Cayley-Dickson algebras

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- (split) quadratic extension
- (split) quaternions
- (split) octonions

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1. extend CD-process by allowing **degenerate versions**
   $\rightarrow$ 9 extra alternative CD-algebras

2. define **generalized Veronese varieties** associated to them
   $\rightarrow$ projective remoteness planes

PART I

Extending the Cayley-Dickson process

The Cayley-Dickson Doubling Process

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- a (trival) involution $j_{\mathrm{K}} : \mathrm{K} \to \mathrm{K} : a \mapsto \bar{a} = a$
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Successive applications yield **alternative algebras**

$$\begin{aligned} \mathsf{L} &= \mathsf{CD}(\mathsf{K}, \zeta), \zeta \in \mathsf{K} && \text{(involution non-trivial)} \\ \mathsf{H} &= \mathsf{CD}(\mathsf{L}, \zeta'), \zeta' \in \mathsf{L} && \text{(no longer commutative)} \\ \mathsf{O} &= \mathsf{CD}(\mathsf{H}, \zeta''), \zeta'' \in \mathsf{H} && \text{(no longer associative)} \end{aligned}$$

Properties of $\mathsf{CD}(\mathsf{K}, \zeta)$

Primitive element

$\mathsf{L} = \mathsf{CD}(\mathsf{K}, \zeta) = \mathsf{K} \oplus s\mathsf{K}$ with $s = (0, 1)$, $s^2 = \zeta$

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Quadratic algebra

$\forall \mathbf{x} \in \mathsf{CD}(\mathsf{K},\zeta) : \mathbf{x}^2 + t(x)\mathbf{x} + n(x) = 0$

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Example $\mathrm{CD}(\mathbb{R}, \zeta)$

ζ	**elements**	**CD-algebra**	**n(a, b)**
-1	$a + ib$ with $i^2 = -1$	$\mathbb{C}$	$a^2 + b^2$
1	$a + jb$ with $j^2 = 1$	$\mathbb{C}'$	$a^2 - b^2$
0	$a + tb$ with $t^2 = 0$	$\mathrm{DN}(\mathbb{R}) := \mathbb{R}[t]/(t^2)$	a^2

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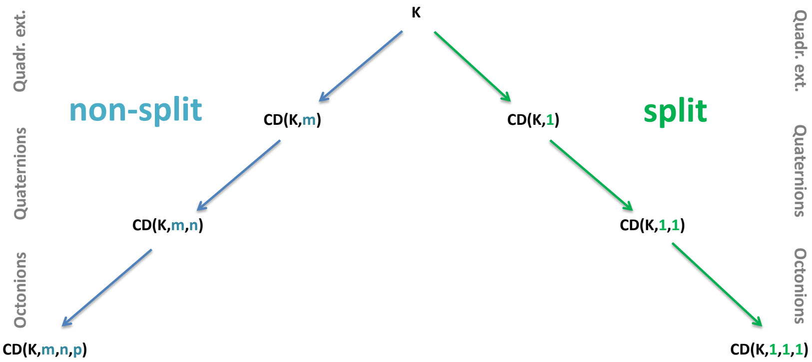
In general $\mathrm{CD}(K,\zeta)$, m a non-square, $\mathrm{kar}K \neq 2$

ζ	**elements**	**CD-algebra**	$\mathbf{n(a,b)}$
m	$a + ib$ with $i^2 = m$	$\mathbb{L} = \mathbb{K}[x]/(x^2+1)$	$a^2 + mb^2$
1	$a + jb$ with $j^2 = 1$	$\mathbb{L}' = \mathbb{K}[x]/(x^2-1)$	$(a+b)(a-b)$
0	$a + tb$ with $t^2 = 0$	$\mathrm{DN}(\mathbb{K}) := \mathbb{K}[t]/(t^2)$	a^2

(Degenerate) Cayley-Dickson algebras (kar$K \neq 2$)



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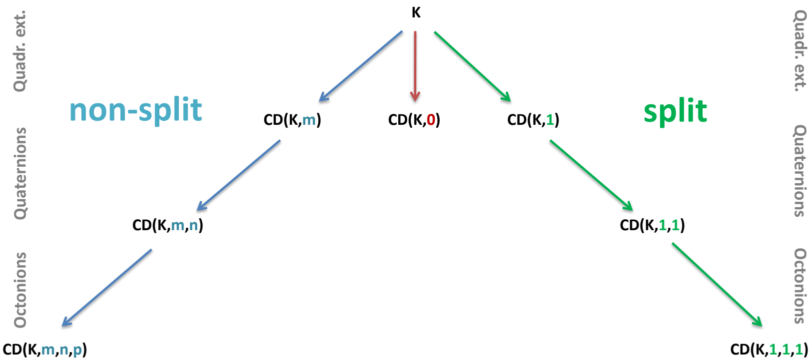

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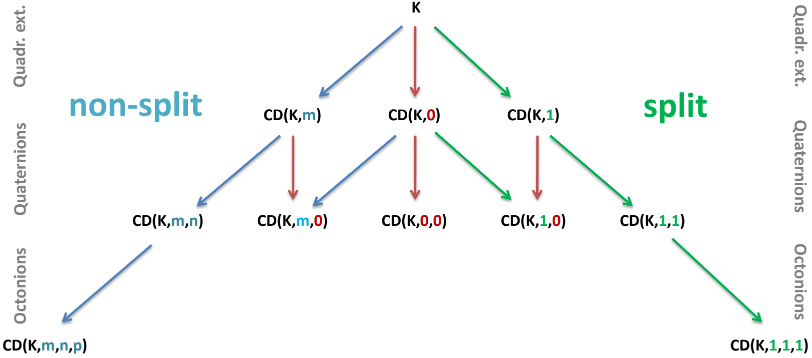

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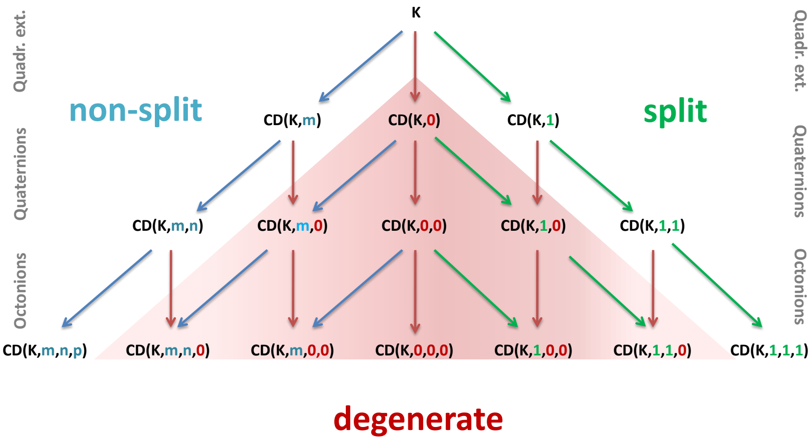

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CD(K, 1) ≅ DN(K) (if kar K $= 2$)

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$\rightarrow$ the CD-process yields no split extension

First step in CDDP in case $\operatorname{kar} K = 2$

Normal procedure

- $\zeta \notin \square \rightarrow$ inseparable extension with a trivial involution
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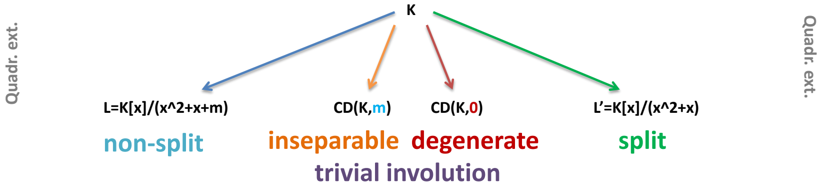
Remarks

- The split one is isomorphic to $K \times K : (a, b) \mapsto (a + b, a)$
- If kar $K \neq 2$ then $K[x]/(x^2 + ax + b) \cong \mathrm{CD}(K, v)$ with $v = u^2 - b$ and $u = \frac{a}{2}$.

(Degenerate) Cayley-Dickson algebras in characteristic 2



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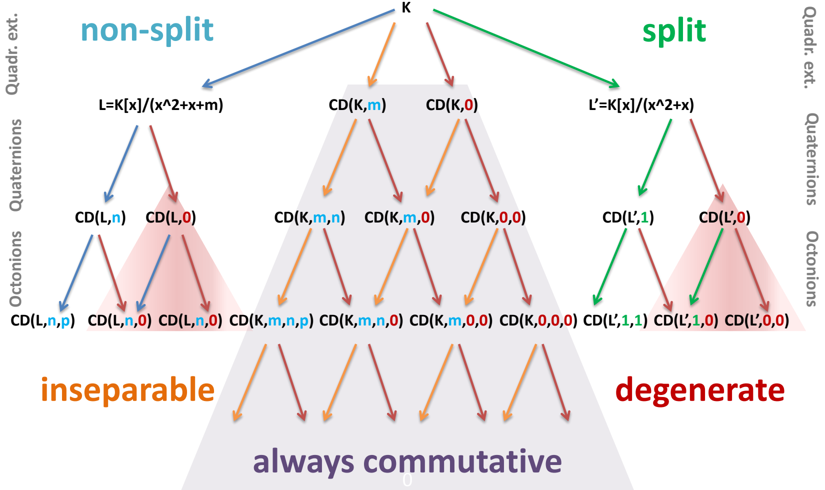

PART II

(Generalized) Veronese Varieties

after JS and HVM

Veronesean varieties

Representation of $\mathbf{PG(2,L)}$, L a non-split quadratic extension of K

points $(x,y,z) \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{y} & \bar{z} \end{pmatrix} = \begin{pmatrix} x\bar{x} & x\bar{y} & x\bar{z} \\ y\bar{x} & y\bar{y} & y\bar{z} \\ z\bar{x} & z\bar{y} & z\bar{z} \end{pmatrix}$

Veronesean varieties

Representation of **PG(2, L)** , L a non-split quadratic extension of K

points $(x, y, z) \mapsto (x\bar{x}, y\bar{y}, z\bar{z}; y\bar{z}, z\bar{x}, x\bar{y})$, **points** of PG(8, K)

Veronesean varieties

Representation of $\mathbf{PG(2,L)}$, L a non-split quadratic extension of K

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then it is isomorphic to $\mathrm{PG}(2,\mathrm{L})$ and proj. equivalent to HV(K).

Generalized Veronesean varieties

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$\mathcal{I}$ $ax + by + cz = 0$

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e. g. : $x = 0 \mapsto (0, y\bar{y}, z\bar{z}; y\bar{z}, 0, 0)$, put $Z_3 = X_3 + sX_4$:

$$X_1X_2 = Z_3\bar{Z}_3 = \begin{cases} X_3^2 - X_4^2 & (\zeta = 1) \rightarrow Q^+(3, \mathrm{K}) \\ X_3^2 & (\zeta = 0) \rightarrow pQ(2, \mathrm{K}) \end{cases}$$

All points given by these equations are contained in $\sigma(\mathcal{P})$, except for p, which is called the **radical** Q^R of the quadric Q.

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This time, the corresponding geometries are no projective planes, but **projective remoteness planes**.

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If these axioms and their duals holds, then we have a **projective remoteness plane.**

The Segre Variety

$\mathrm{PG}(2, \mathrm{K}) \times \mathrm{PG}(2, \mathrm{K})$ is a **projective remoteness plane**

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e. g. : $((0, v, w), (0, y, z)) \mapsto (0, 0, 0, 0, vy, vz, 0, wy, wz)$,
a quadric in $\mathrm{PG}(3, \mathbb{R})$:

$$X_4 X_8 = X_5 X_7 \to Q^+(3, \mathrm{K})$$

Projective Hjelmslev plane of level 2

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Vero(CD(K, 0)) is a PH_2(DN(K))

Recall

points $(x, y, z) \mapsto (x\bar{x}, y\bar{y}, z\bar{z}; y\bar{z}, z\bar{x}, x\bar{y}) \in \mathrm{PG}(8, \mathrm{K})$

a line $(0, y, z) \mapsto (0, y\bar{y}, z\bar{z}; y\bar{z}, 0, 0)$, a cone over

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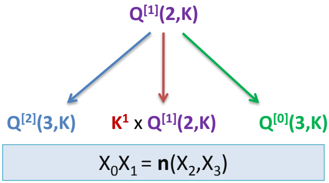
This is the **quadratic Veronesean**, isomorphic to PG(2, K).

(Generalized) Mazzocca-Melone Sets

$Q^{[1]}(2,K)$

$$X_0X_1 = \mathbf{n}(X_2)$$

(Generalized) Mazzocca-Melone Sets



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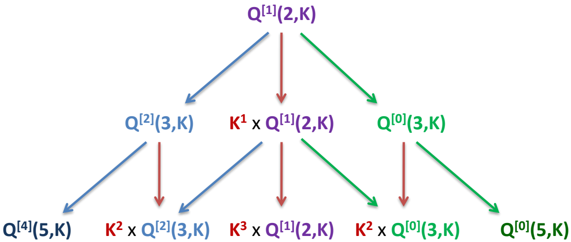

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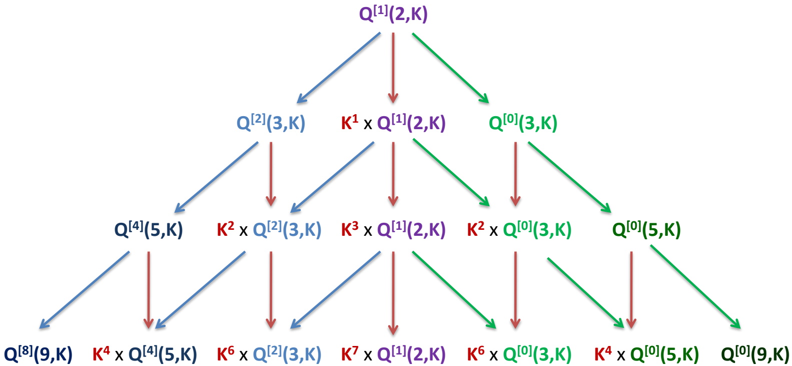
(Generalized) Mazzocca-Melone Sets



$$X_0X_1 = \mathbf{n}(X_2,X_3,X_4,\ X_5,X_6,X_7,\ X_8,X_9)$$

Witt index 1 … **… maximal Witt index**

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Conjecture

These are *all* projective remoteness planes over quadratic alternative algebras.

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The corresponding (generalized) Veronesean varieties are the *only* (generalized) MM-sets.

Thank you!