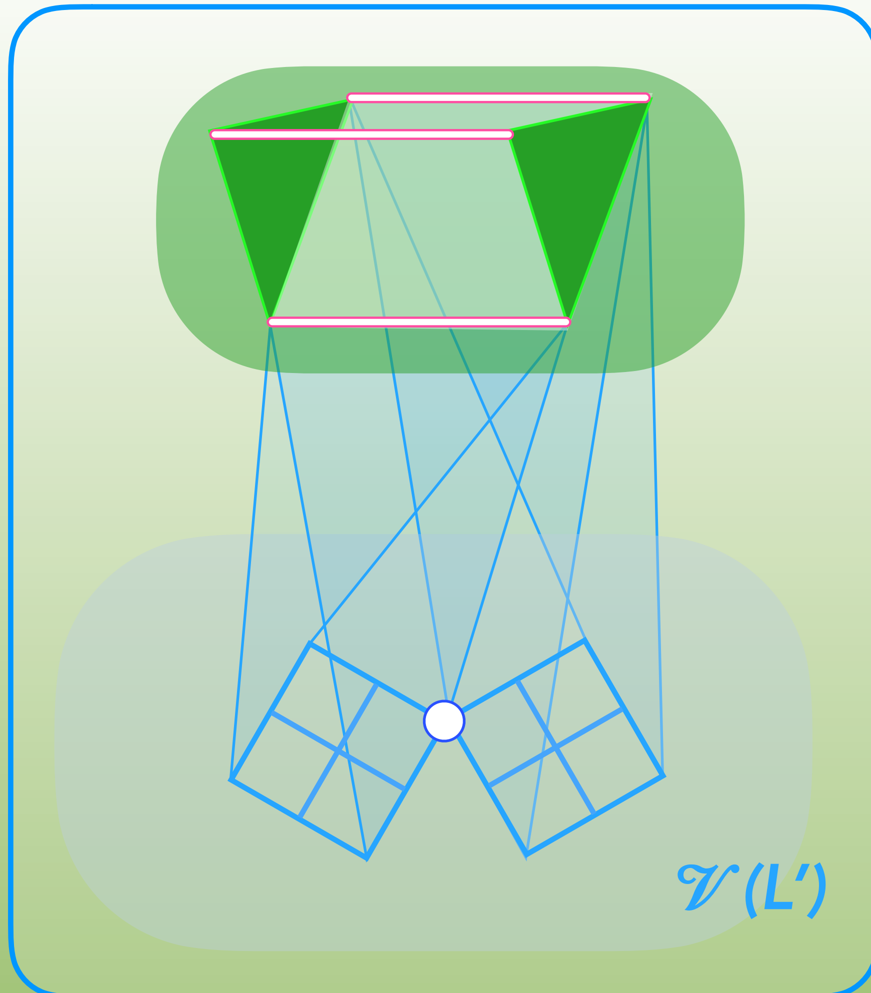
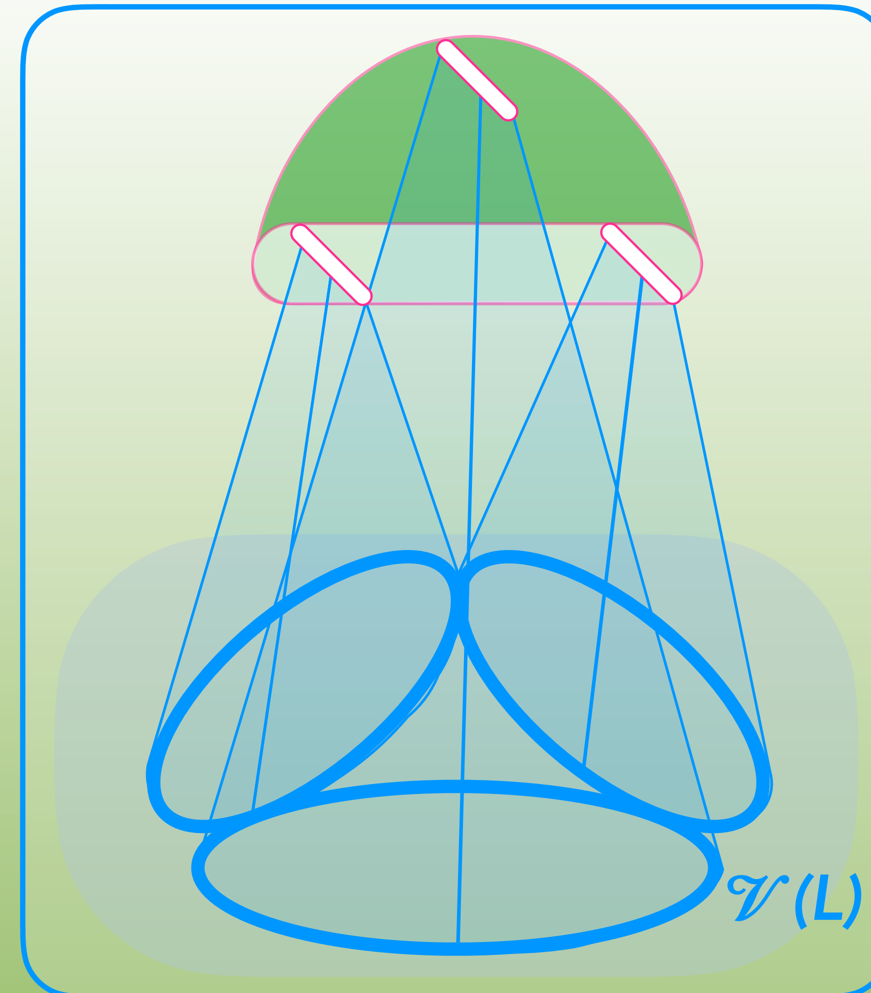




CHARACTERISING SINGULAR VERONESE VARIETIES

Buildings 2017

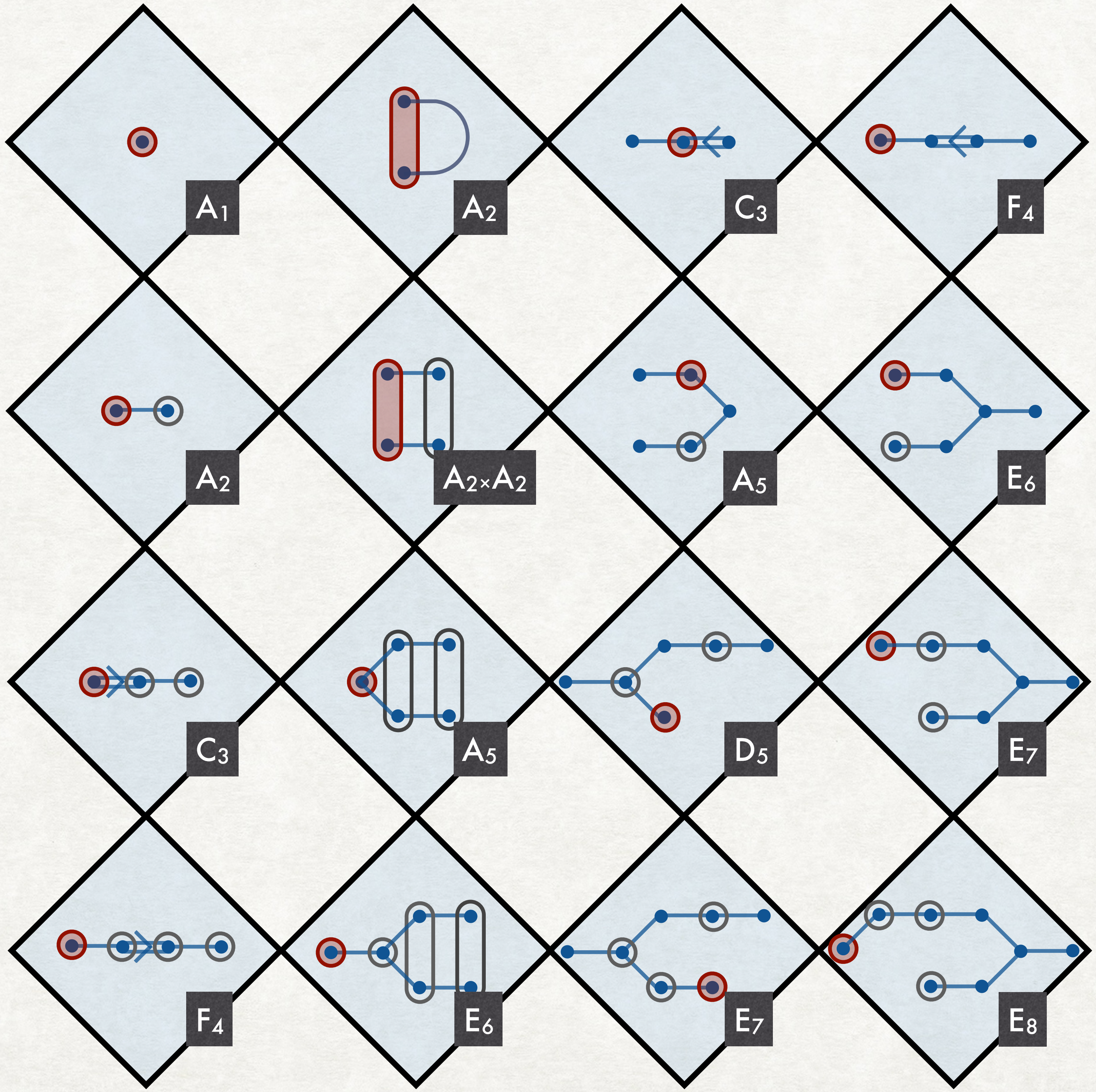
GHENT UNIVERSITY

ANNELEEN DE SCHEPPER

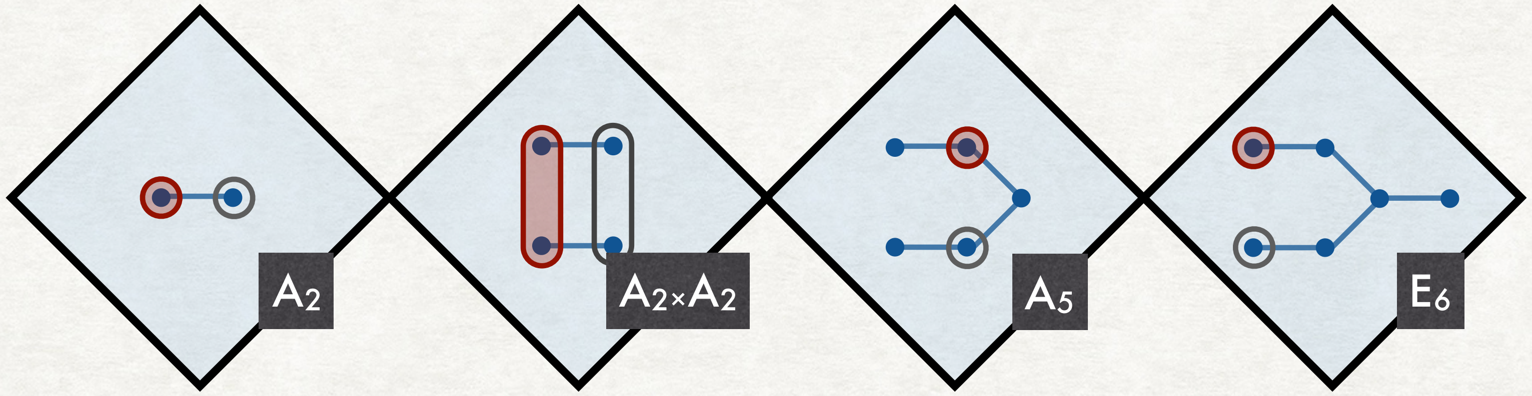
0

Origin

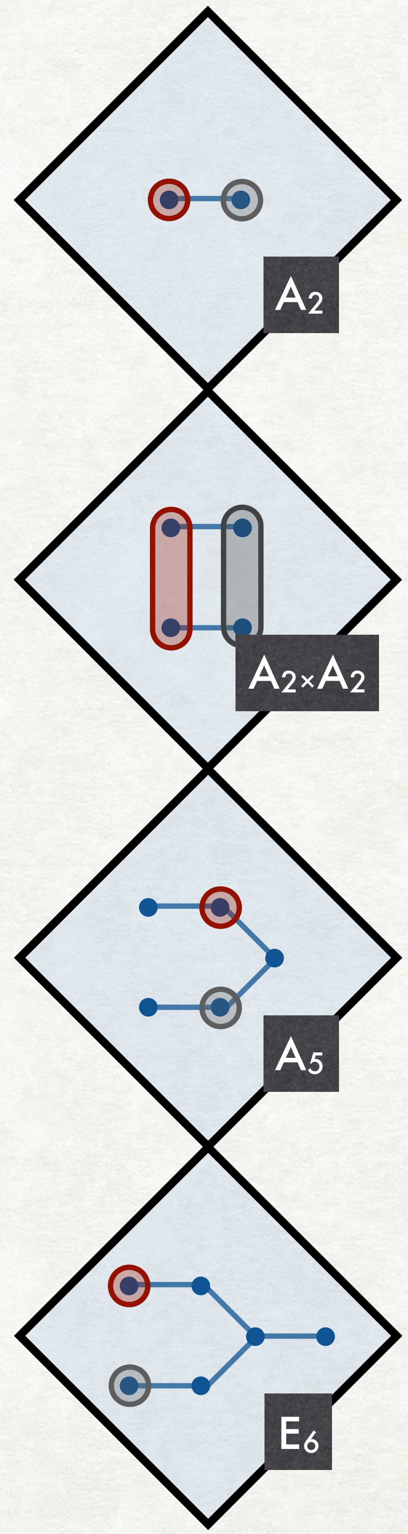
THE MAGIC SQUARE



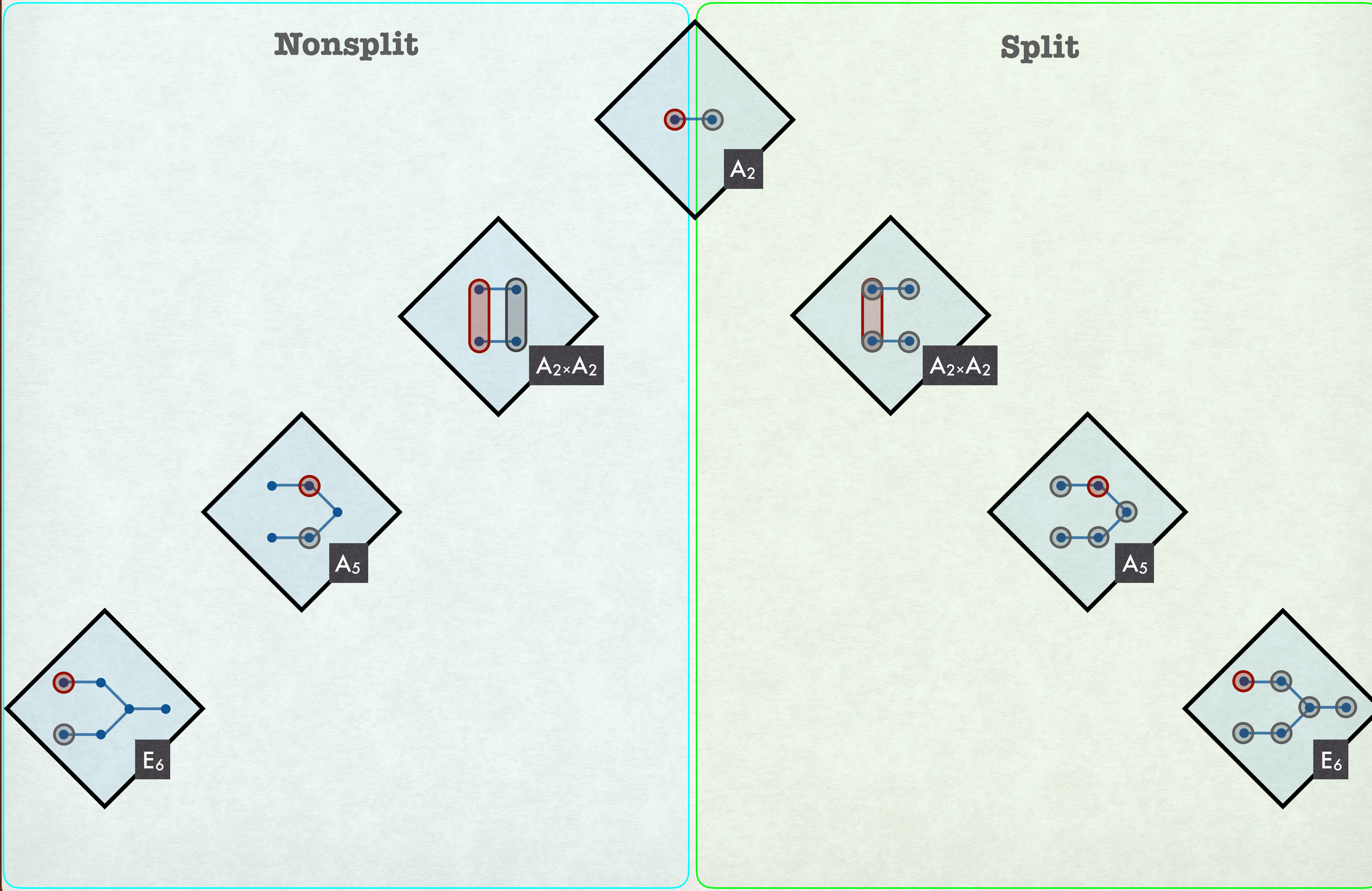
THE MAGIC SQUARE: 2ND ROW



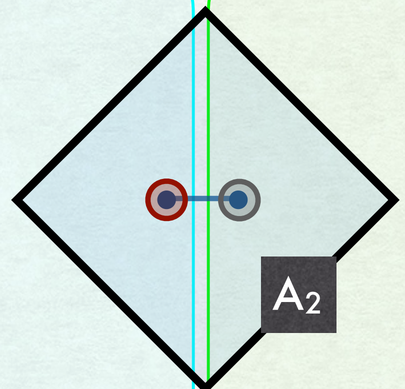
THE MAGIC SQUARE: 2ND ROW



THE MAGIC SQUARE: 2ND ROW



THE MAGIC SQUARE: 2ND ROW



Nonsplit

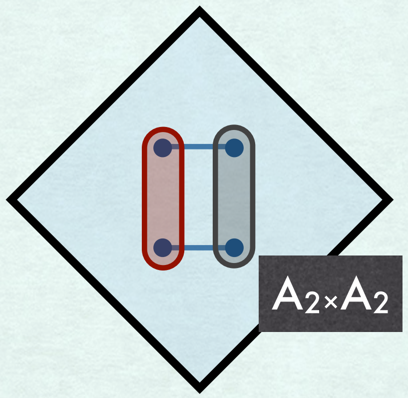
Moufang projective planes

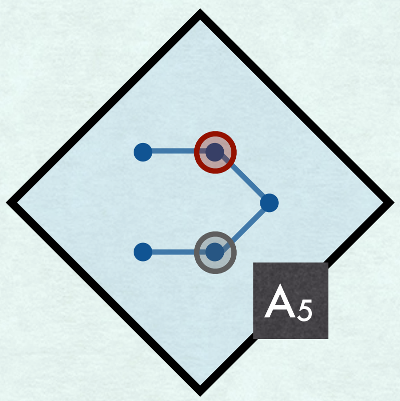
PG(2,K)

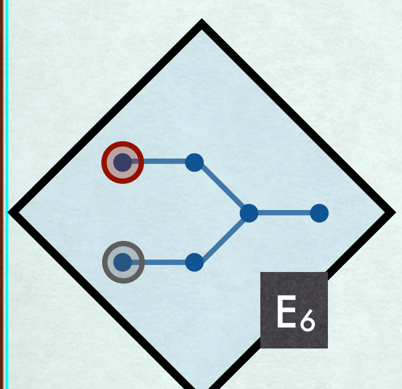
PG(2,L)

PG(2,H)

PG(2,O)







Split

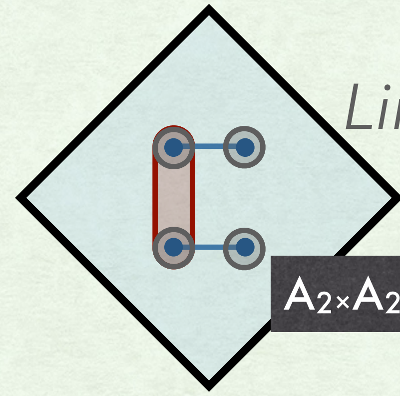
Severi varieties

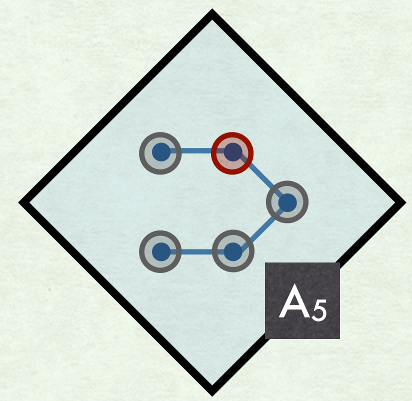
PG(2,K)

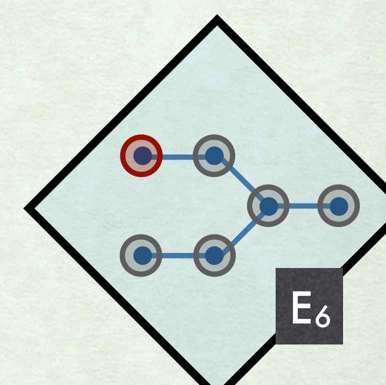
Segre variety $S_{2,2}(K)$

Line Grassmannian of $A_5(K)$

$E_{6,1}(K)$ *variety*







THE MAGIC SQUARE: 2ND ROW

Nonsplit

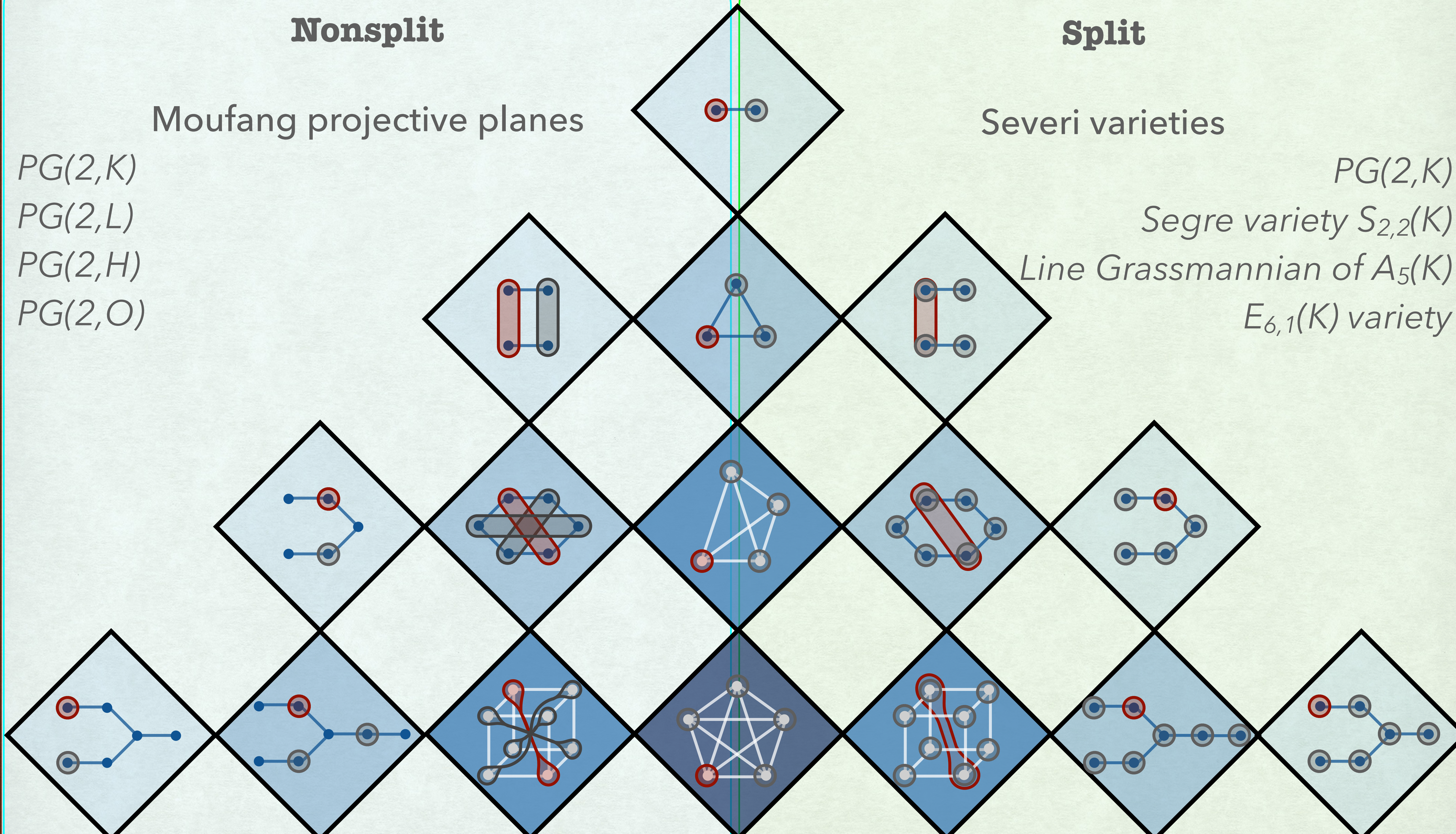
Moufang projective planes

PG(2,K)
PG(2,L)
PG(2,H)
PG(2,O)

Split

Severi varieties

PG(2,K)
Segre variety $S_{2,2}(K)$
Line Grassmannian of $A_5(K)$
$E_{6,1}(K)$ variety



THE MAGIC SQUARE: 2ND ROW

Nonsplit

Moufang projective planes

PG(2,K)

PG(2,L)

PG(2,H)

PG(2,O)

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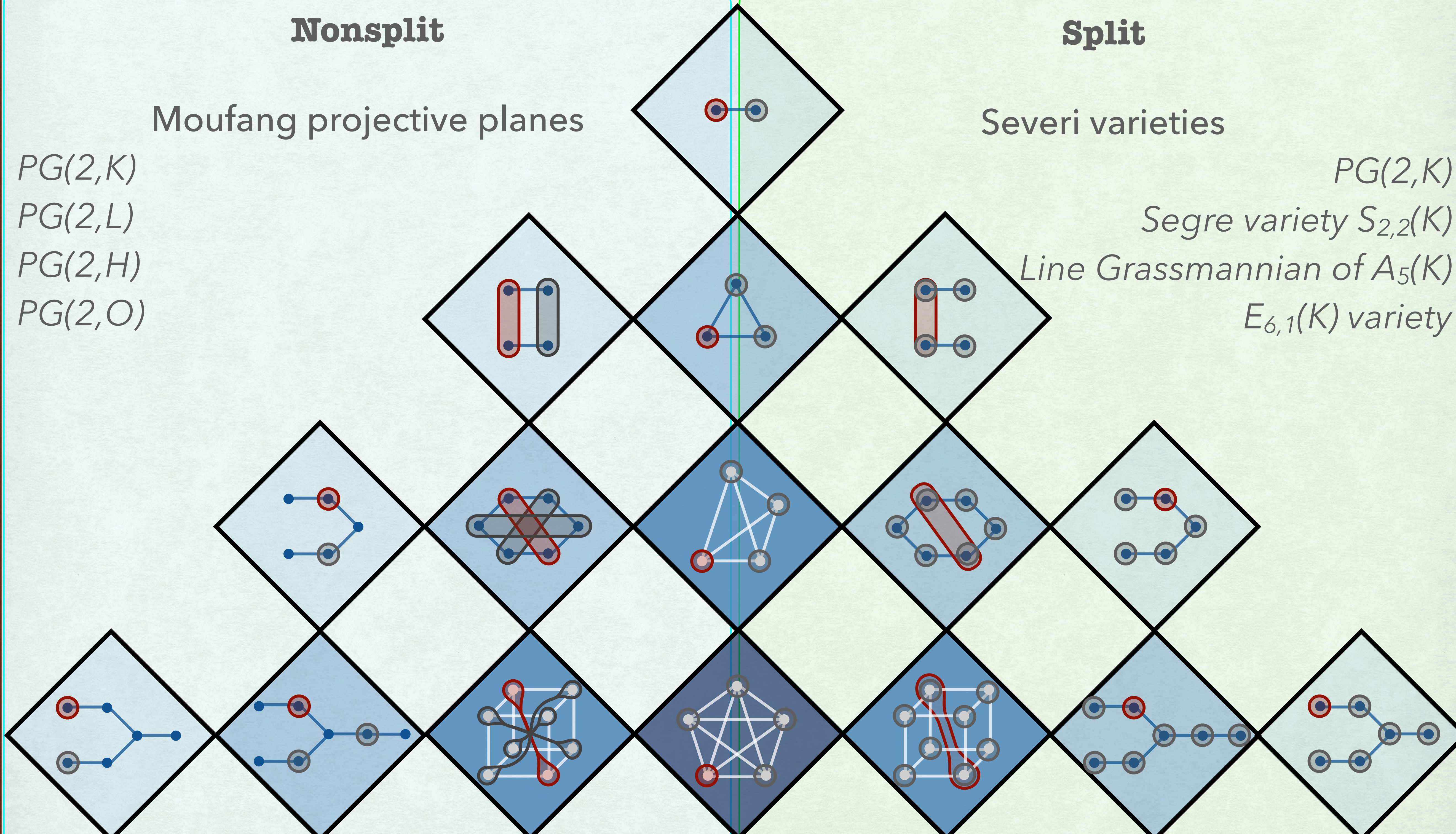
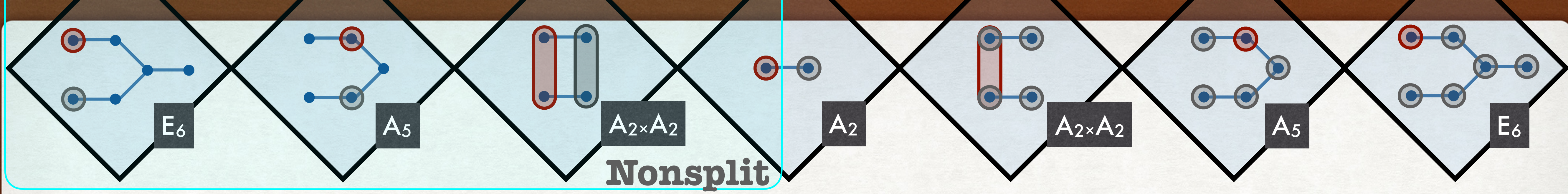


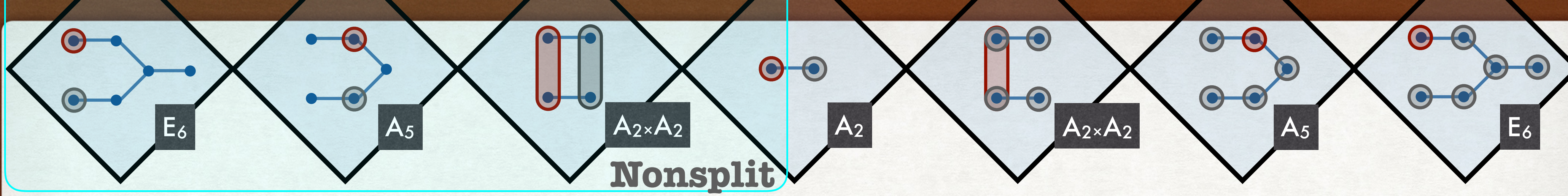
set of points and quadrics + some axioms

1

Axiomatisation



Axiomatic description

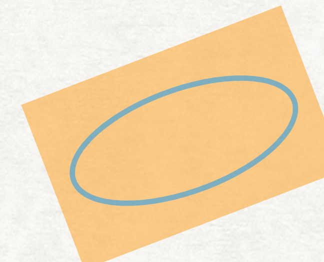


Axiomatic description

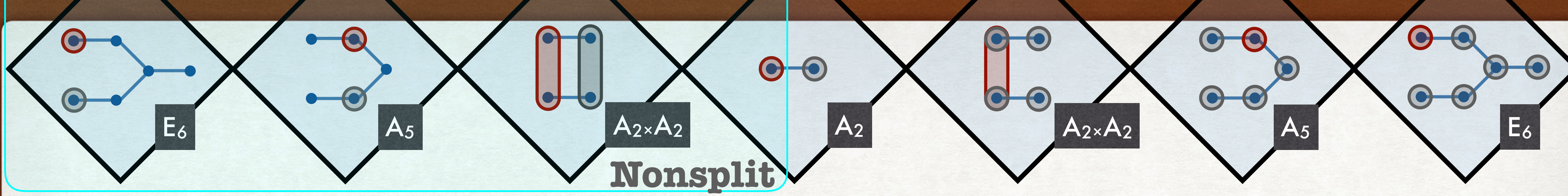
K field, kar(K) ≠ 2 (for simplicity)

X points spanning PG(N,K)

Ξ d-spaces ξ in PG(N,K) s.th. ξ ∩ X is:



a quadric of *minimal* *Witt index*

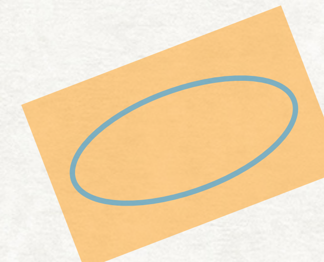


Axiomatic description

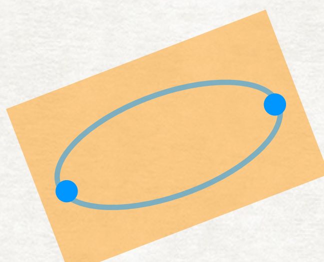
K field, kar(K) $\neq$ 2 (for simplicity)

X — points spanning PG(N,K)

Ξ — d-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

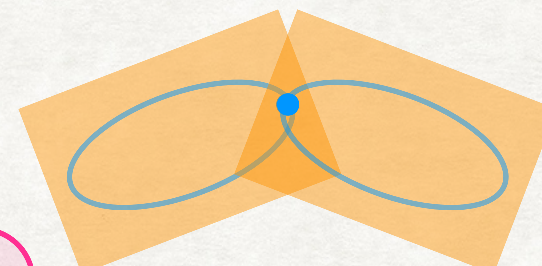


quadric $Q^{min}(d,K)$ of minimal Witt index



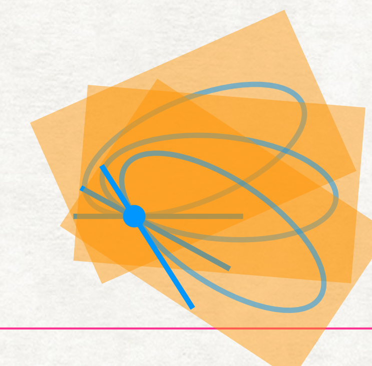
MM1

each two **points** of **X** belong to a **[d]** of **Ξ**



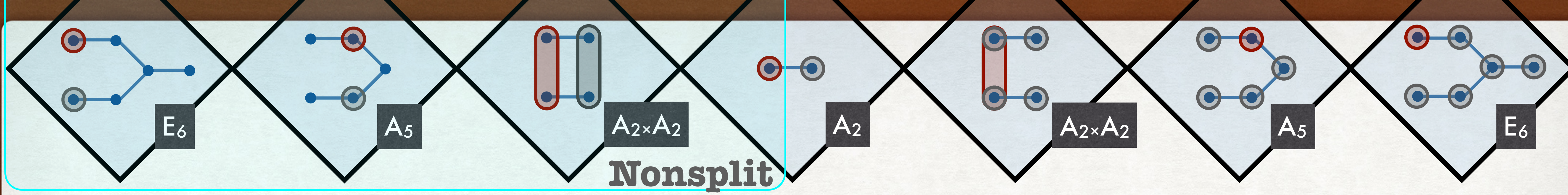
MM2

two **[d]s** of **Ξ** intersect in **points** of **X**



MM3

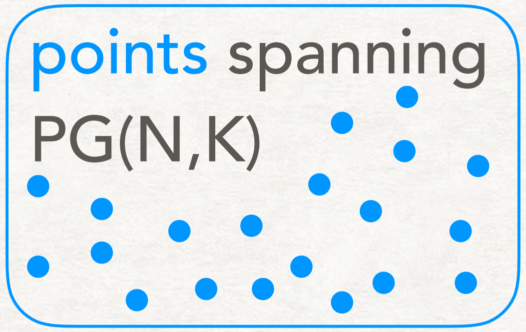
the tangent space of a **point** of **X** is contained in a [2(d-1)]



Axiomatic description

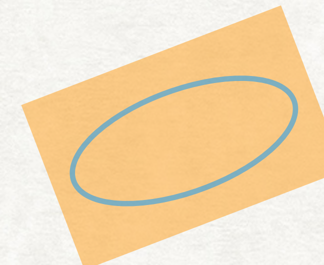
K field, kar(K) ≠ 2 (for simplicity)

X

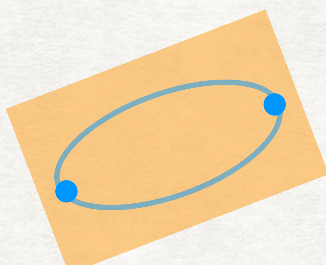


Ξ

d-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

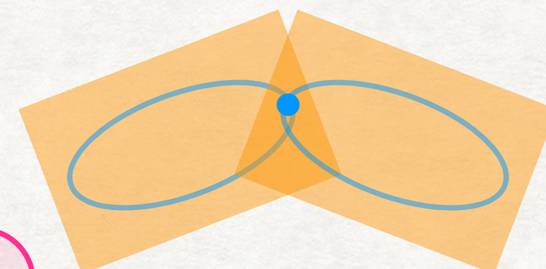


quadric $Q^{min}(d,K)$ of minimal Witt index



MM1

each two **points** of **X** belong to a [d] of Ξ



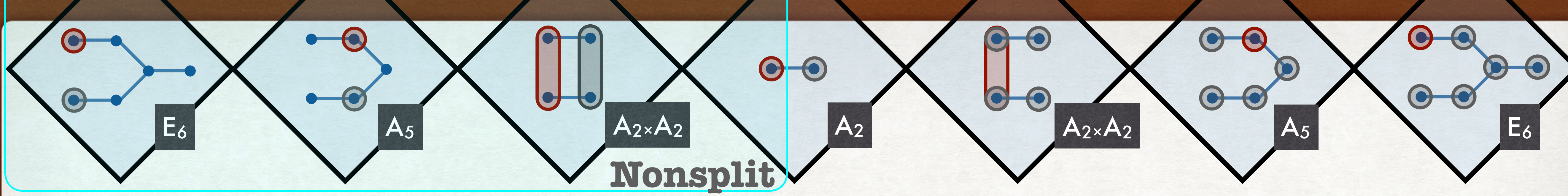
MM2

two [d]s of Ξ intersect in **points** of **X**

MM3

the tangent space of a **point** of **X** is contained in a [2(d-1)]

The pair (X, Ξ) together with MM1, MM2 and MM3 is called a **Mazzocca Melone (MM) set with quadrics of minimal Witt index**

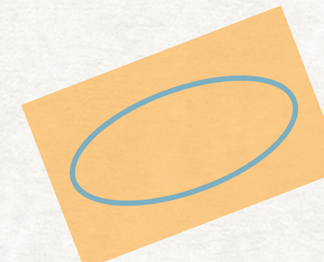


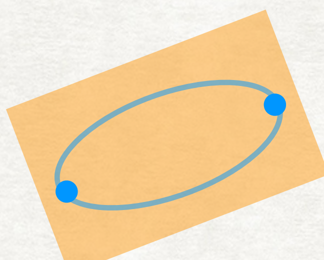
Axiomatic description

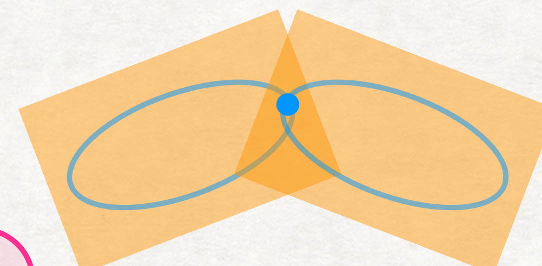
K field, kar(K) ≠ 2 (for simplicity)

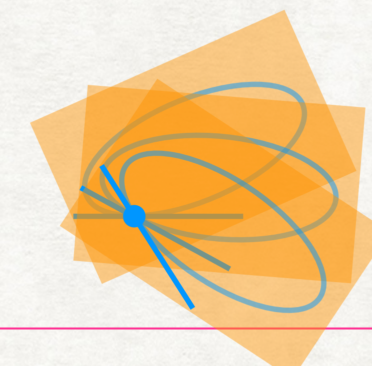
X — points spanning PG(N,K)

Ξ — d-spaces ξ in PG(N,K) s.th. ξ ∩ X is: *quadric $Q^{min}(d,K)$ of minimal Witt index*









MM1: each two points of X belong to a [d] of Ξ

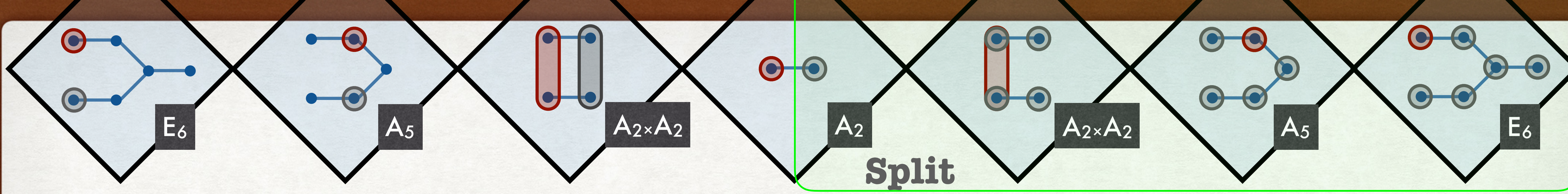
MM2: two [d]s of Ξ intersect in points of X

MM3: the tangent space of a point of X is contained in a [2(d-1)]

Schillewaert, Van Maldeghem, Krauss (2015)

For any field K, d ∈ {2,3,5,9} and, per d, (X, Ξ) is projectively unique.

d	2	3	5	9
(X, Ξ) isomorphic to	PG(2,K)	PG(2,L)	PG(2,H)	PG(2,O)
geometry in PG(N,K)	$\mathscr{V}$(K) (5)	$\mathscr{V}$(L) (8)	$\mathscr{V}$(H) (14)	$\mathscr{V}$(O) (26)



Axiomatic description

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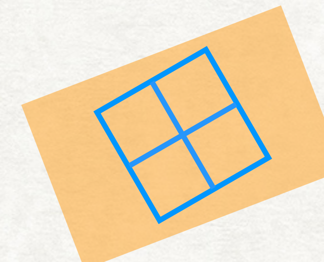
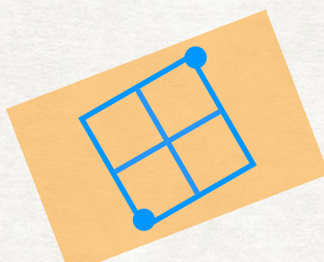
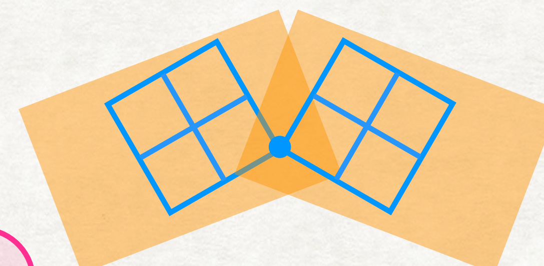
Ξ — d-spaces ξ in PG(N,K) s.th. ξ ∩ X is:



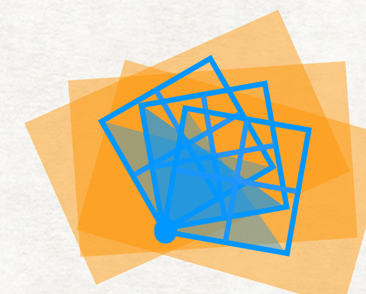
quadric $Q^{max}(d,K)$ of maximal Witt index



MM1: each two **points** of **X** belong to a **[d]** of **Ξ**

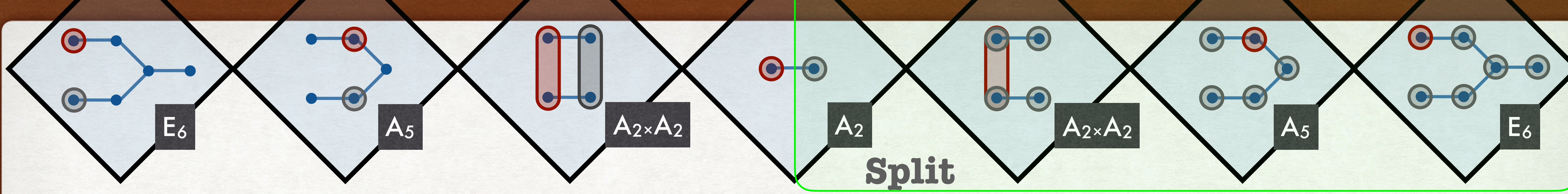


MM2: two **[d]s** of **Ξ** intersect in **points** of **X**



MM3: the tangent space of a **point** of **X** is contained in a [2(d-1)]

The pair (**X**, **Ξ**) together with MM1, MM2 and MM3 is called a **Mazzocca Melone (MM) set with quadrics of maximal Witt index**



Axiomatic description

K field, kar(K) ≠ 2 (for simplicity)

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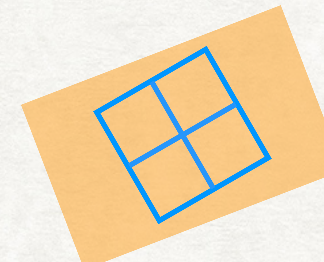

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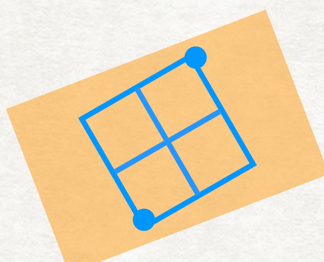

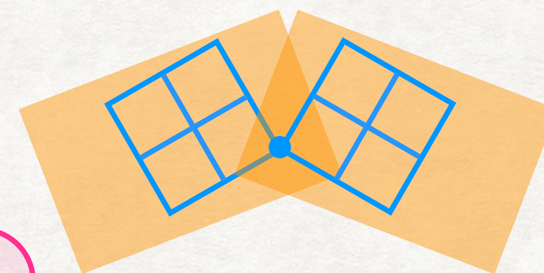

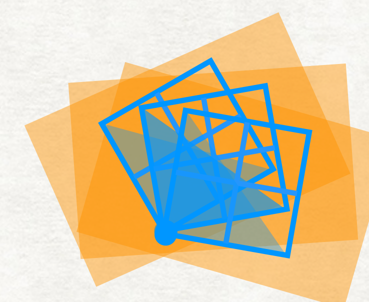

MM1 each two **points** of **X** belong to a **[d]** of **Ξ**

MM2 two **[d]s** of **Ξ** intersect in **points** of **X**

MM3 the tangent space of a **point** of **X** is contained in a [2(d-1)]

Schillewaert, Van Maldeghem (2015)

For any field K, if N > 3d +1, d ∈ {2,3,5,9} and, per d, (**X**, **Ξ**) is projectively unique.

d	2	3	5	9
(X, Ξ) isomorphic to	PG(2,K)	$A_2 \times A_2(K)$	$A_{5,2}(K)$	$E_{6,1}(K)$
geometry in PG(N,K)	$\mathscr{V}(K)'$ (5)	$\mathscr{V}(L)'$ (8)	$\mathscr{V}(H)'$ (14)	$\mathscr{V}(O)'$ (26)

MM SETS WITH OTHER QUADRICS

Axiomatic description

X: points spanning PG(N,K)

Ξ: d-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

some quadric

MM1

each two **points** of **X** belong to a **[d]** of **Ξ**

MM2

two **[d]s** of **Ξ** intersect in **points** of **X**

MM3

the tangent space of a **point** of **X** is contained in a [2(d-1)]

MM SETS WITH OTHER QUADRICS

Axiomatic description

X points spanning PG(N,K)

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MM1

each two points of X belong to a [d] of Ξ

MM2

two [d]s of Ξ intersect in points of X

MM3

the tangent space of a point of X is contained in a [2(d-1)]

Conjecture:
There are no MM sets with quadrics of intermediate Witt index

MM SETS WITH OTHER QUADRICS

Axiomatic description

X points spanning PG(N,K)

Ξ d-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

some quadric

MM1

each two **points** of **X** belong to a **[d]** of **Ξ**

MM2

two **[d]s** of **Ξ** intersect in **points** of **X**

MM3

the tangent space of a **point** of **X** is contained in a [2(d-1)]

Yet

There are MM sets with singular quadrics

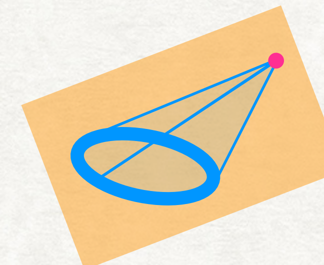
SINGULAR MM SETS: A FIRST EXAMPLE

Axiomatic description

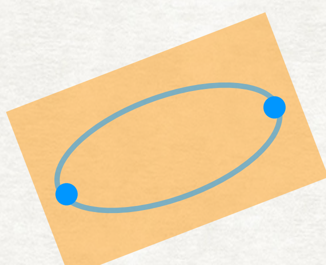
(2,0)-tube

X: points spanning PG(N,K)

Ξ: 3-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

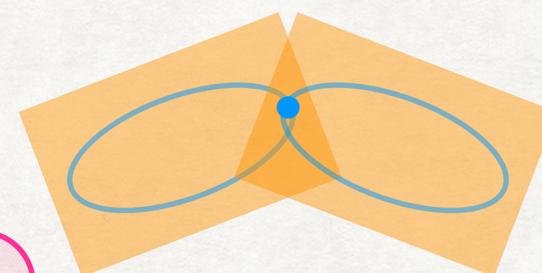


a point-cone over $Q^{min}(2,K)$; without vertex



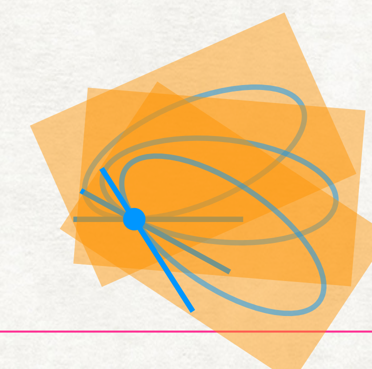
MM1

each two **points** of **X** belong to a [3] of **Ξ**



MM2

two [3]s of **Ξ** intersect in **points** of **X**



MM3

the tangent space of a **point** of **X** is contained in a [2(3-1)]

SINGULAR MM SETS: A FIRST EXAMPLE

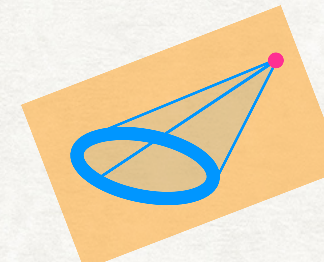
Axiomatic description

(2,0)-tube

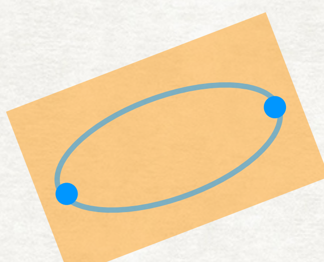
X — points spanning PG(N,K)

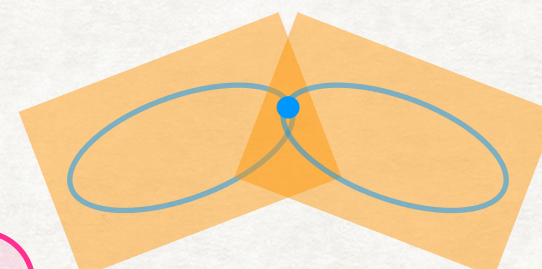
Y — vertices

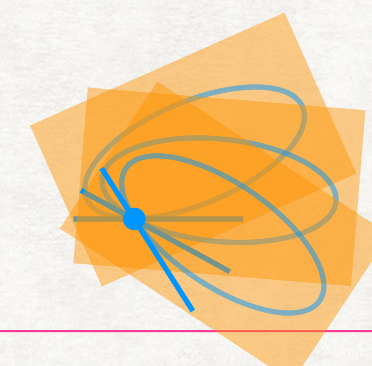
Ξ — 3-spaces ξ in PG(N,K) s.th. ξ ∩ X is:



a point-cone over $Q^{min}(2,K)$; <u>without</u> vertex







MM1: each two points of X belong to a [3] of Ξ

MM2: two [3]s of Ξ intersect in points of X

MM3: the tangent space of a point of X is contained in a [2(3-1)]

SINGULAR MM SETS: A FIRST EXAMPLE

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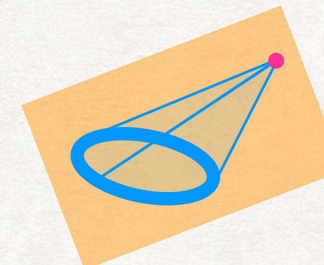
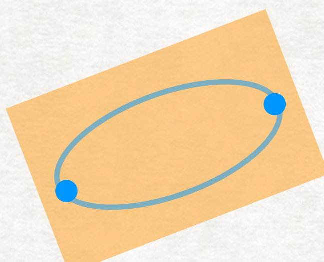
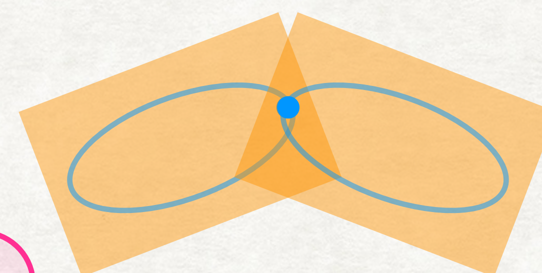
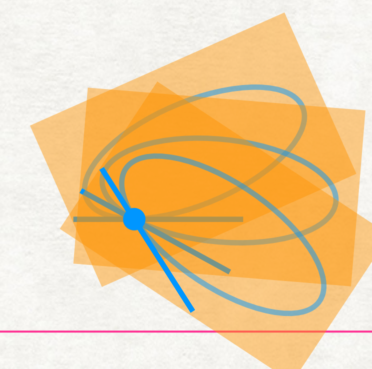


a point-cone over $Q^{min}(2,K)$; without vertex







MM1

each two **points** of **X** belong to a [3] of **Ξ**

MM2′

two [3]s of **Ξ** intersect in **points** of **X**∪**Y** but never in Y only

MM3

the tangent space of a **point** of **X** is contained in a [2(3-1)]

SINGULAR MM SETS: A FIRST EXAMPLE

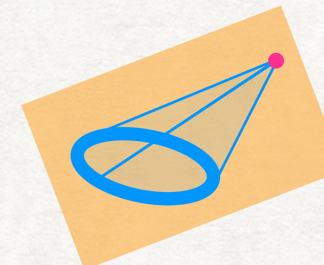
Axiomatic description

(2,0)-tube

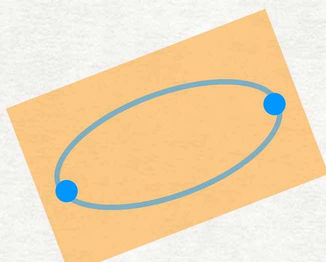
X points spanning PG(N,K)

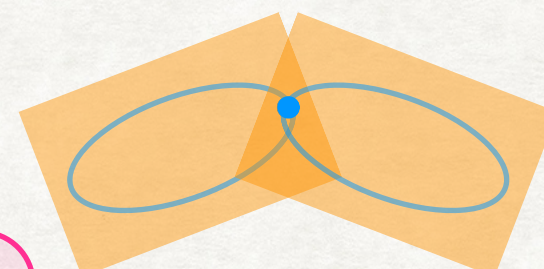
Y vertices

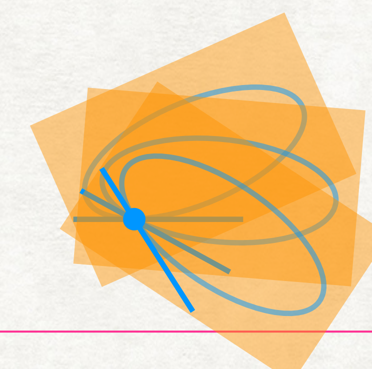
Ξ 3-spaces ξ in PG(N,K) s.th. ξ ∩ X is:



a point-cone over $Q^{min}(2,K)$; without vertex







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The pair (**X**, **Ξ**) together with MM1, MM2′ and MM3 is called a singular MM-set with (2,0)-tubes.

SINGULAR MM SETS: A FIRST EXAMPLE

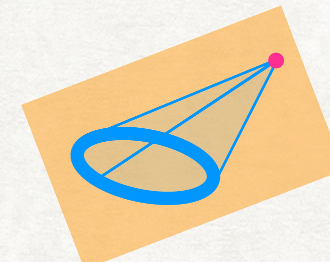
Axiomatic description

(2,0)-tube

X — points spanning PG(N,K)

Y — vertices

Ξ — 3-spaces ξ in PG(N,K) s.th. ξ ∩ X is:



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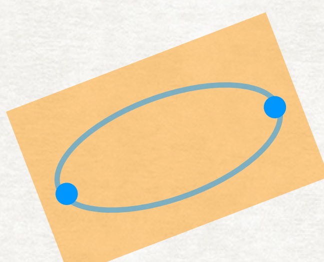

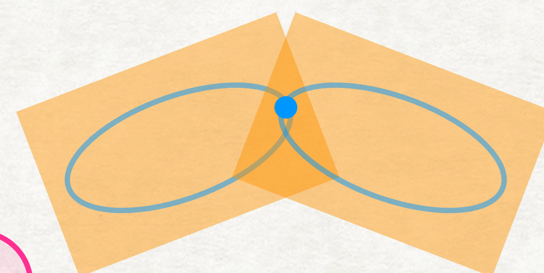

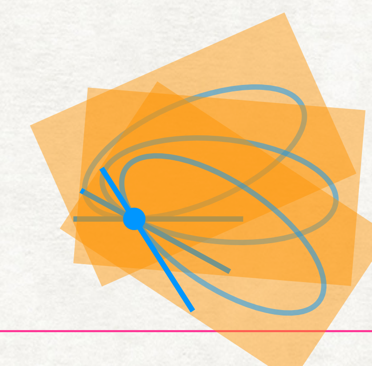

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MM2

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MM3

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Schillewaert, Van Maldeghem (2015)
If nontrivial, (**X**, Ξ) is projectively unique and isomorphic to a Hjelmslevian projective plane.

SINGULAR MM SETS: A FIRST EXAMPLE

Schillewaert, Van Maldeghem (2015)

If nontrivial, (X, Ξ) is <u>projectively unique</u> and <u>isomorphic</u> to a Hjelmslevian projective plane.

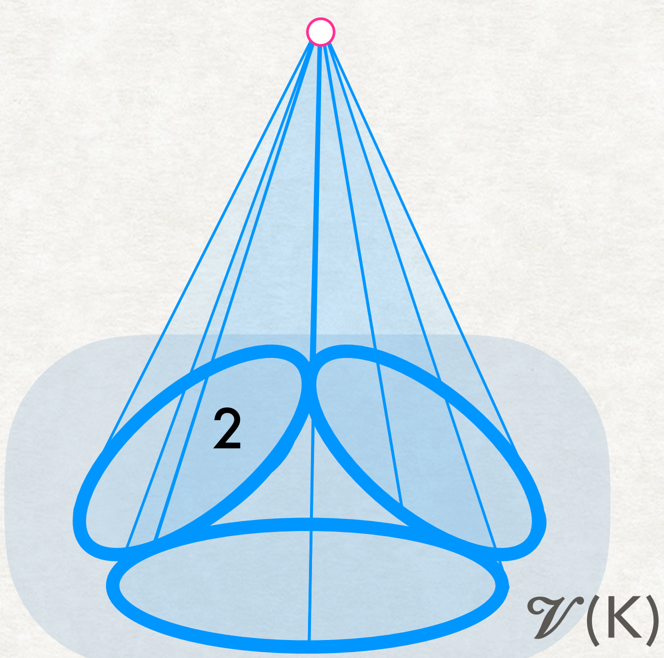
Trivial:

(X, Ξ) is a **<u>cone</u>** with vertex a point and base $\mathscr{V}(K)$

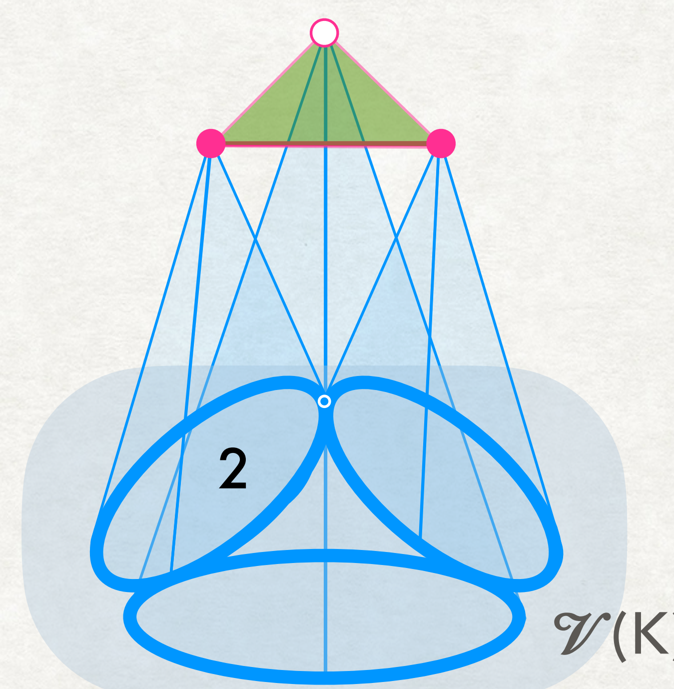
A Hjelmslevian projective plane:

(X, Ξ) is **<u>something</u>** with vertices in a plane and base $\mathscr{V}(K)$

(to be continued)



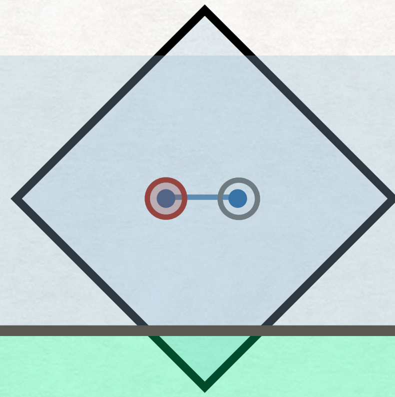
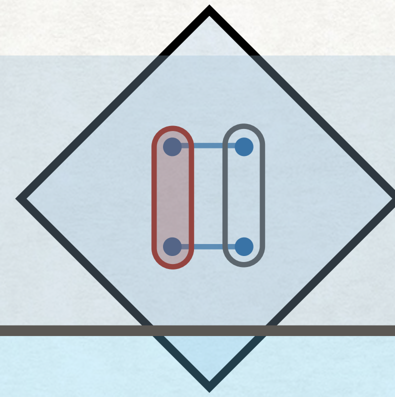
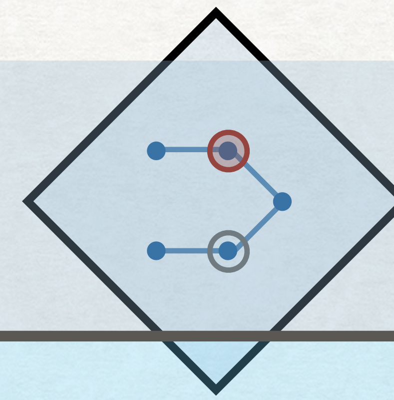
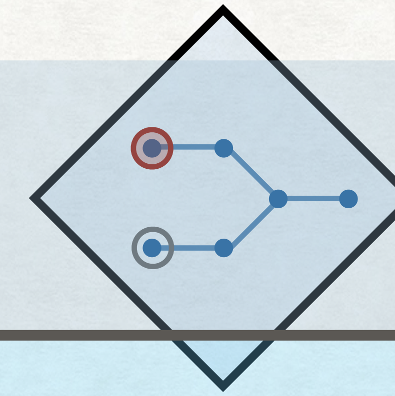
(MM set with $Q^{min}(2,K)$s)



(MM set with $Q^{min}(2,K)$s)

WHY DOES THIS WORK?

Algebraic explanation.

type				
nonsplit	**PG(2,K)** projective plane, field K	**PG(2,L)** projective plane quadr. div. ext. L	**PG(2,H)** projective plane H quat. div. alg.	**PG(2,O)** projective plane O oct. div. alg.
split		**$(A_2 \times A_2)(K)$** Segre variety $S_{2,2}$	**$A_{5,2}(K)$** line Grassmannian	**$E_{6,1}(K)$**

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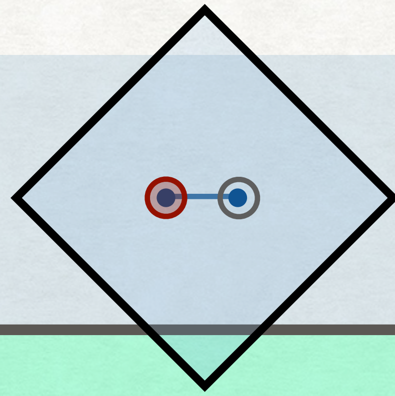
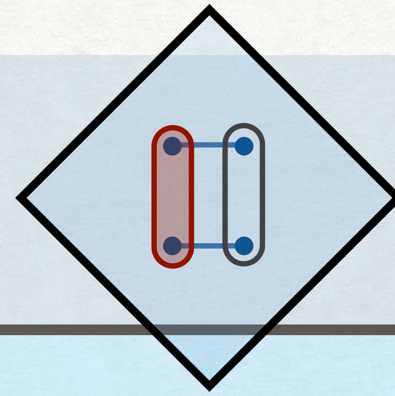
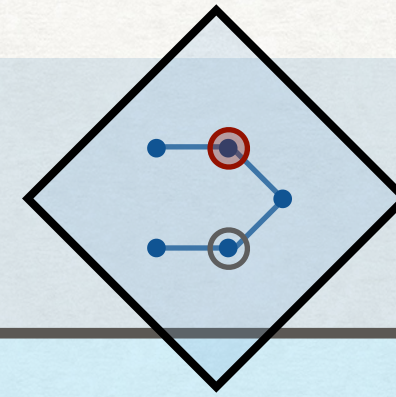
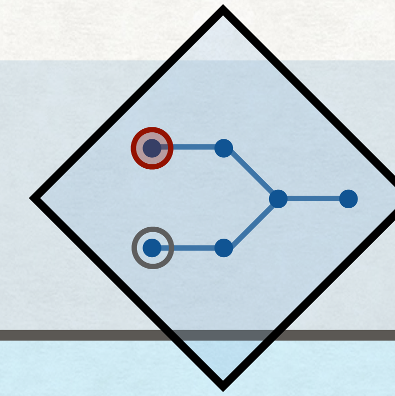
Algebraic explanation.

type				
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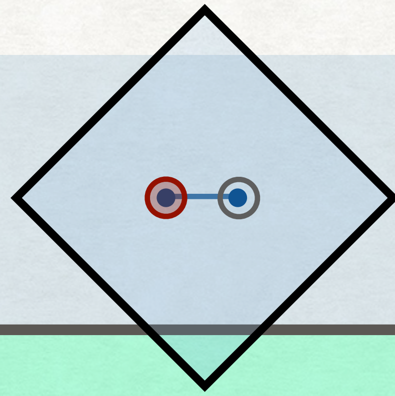
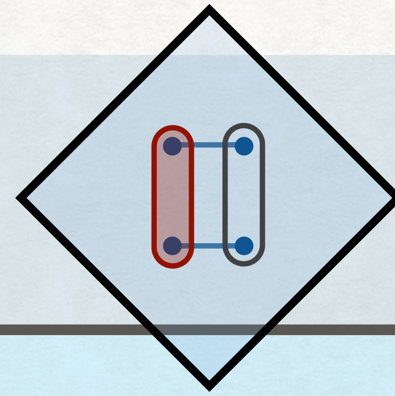
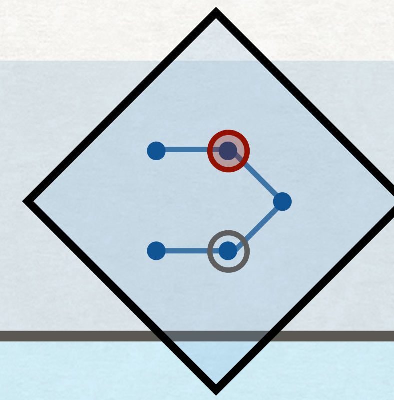
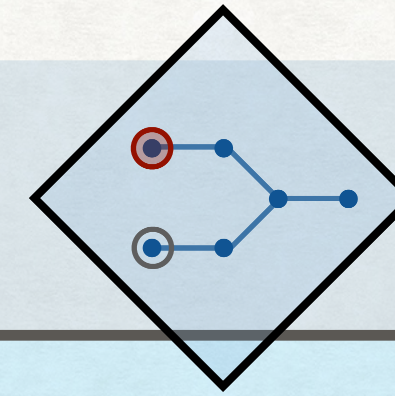
type				
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These are Cayley-Dickson algebras.

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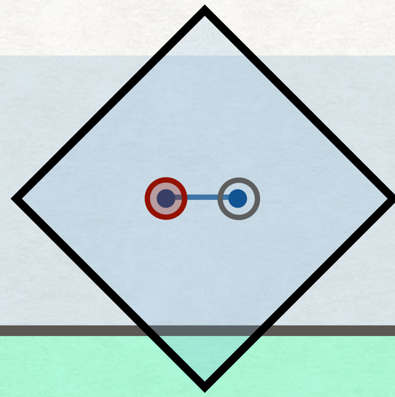
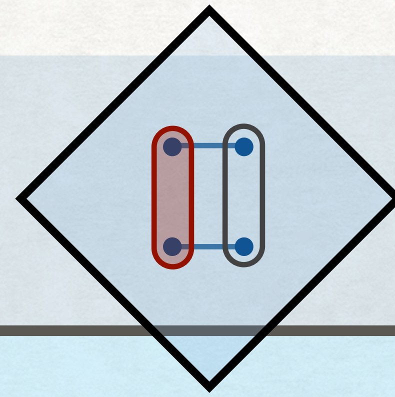
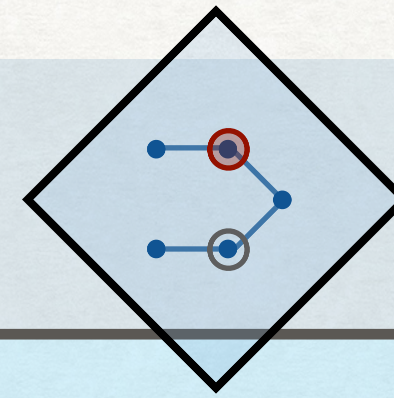
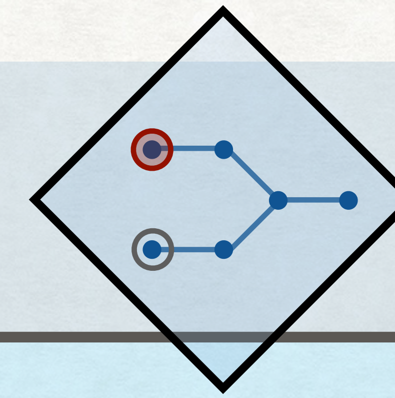
type				
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These are Cayley-Dickson algebras.

The Hjelmslevian projective plane is a proj. remoteness plane over the **dual numbers over K**, which can also be seen as a Cayley-Dickson algebra.

2

Cayley Dickson algebras

THE CAYLEY-DICKSON PROCESS

Let $\mathbb{K}$ be a field with kar(K) ≠ 2 (for simplicity)

THE CAYLEY-DICKSON PROCESS

Let K be a field with kar(K) ≠ 2 (for simplicity)

Algebra A	Involution $x \mapsto \underline{x}$
K	$\underline{x} = x$

→→→

$\zeta \in K \setminus \{0\}$

→→→

L	$(a,b) +_L (c,d)$	$(a,b) \cdot_L (c,d)$	$\underline{(a,b)}$
K × K	$(a+c, b+d)$	$(ac + \zeta d\underline{b}, \underline{a}d + cb)$	$(\underline{a},-b)$

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L comes with a **norm function**

$$n_L : L \to L : (a,b) \mapsto (a,b) \cdot_L \underline{(a,\ b)}$$

$(a,b) \cdot_L \underline{(a,\ b)}$

$= (a\underline{a} - \zeta b\underline{b},\ 0)$

$= (n_K(a) - \zeta n_K(b), 0)$

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L division algebra

$\zeta \notin n_K(K) = K^2$

$n_L((a,b)) = a^2 - \zeta b^2$

n_L *anisotropic*

L split algebra

$\zeta = s^2$ $(s \in K\backslash\{0\})$

$n_L((a,b)) = (a - sb)(a + sb)$

n_L *splits*

THE GENERALISED CAYLEY-DICKSON PROCESS

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THE GENERALISED CAYLEY-DICKSON PROCESS

Let K be a **field** with $\mathrm{kar}(K) \neq 2$ (for simplicity)

Algebra A	Involution $x \mapsto \underline{x}$
K	$\underline{x} = x$

→→→

$\zeta = 0$

→→→

L	$(a,b) +_L (c,d)$	$(a,b) \cdot_L (c,d)$	$\underline{(a,b)}$
$K \times K$	$(a+c, b+d)$	$(ac + \zeta d\underline{b}, \underline{a}d + cb)$	$(\underline{a}, -b)$

L comes with a degenerate **norm function**

$$n_L : L \to K : (a,b) \mapsto n_K(a) - 0 n_K(b)$$

Now $(a,b) \neq (0,0)$ **invertible** $\iff n_L((a,b)) \neq 0 \iff n_K(a) \neq \zeta n_K(b) \iff n_K(ab^{-1}) \neq \zeta$

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n_L *splits*

THE GENERALISED CAYLEY-DICKSON PROCESS

Let $\mathbb{K}$ be a field with kar(K) ≠ 2 (for simplicity)

Algebra A	Involution $x \mapsto \underline{x}$
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→→→ $\zeta = 0$ →→→

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This yields **two possibilities** for the algebra $\mathbb{L}$:

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n_L *anisotropic*

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n_L *splits*

THE GENERALISED CAYLEY-DICKSON PROCESS

Let K be a **field** with kar(K) $\neq 2$ (for simplicity)

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K	$\underline{x} = x$

$\rightarrow\rightarrow\rightarrow$

$\zeta = 0$

$\rightarrow\rightarrow\rightarrow$

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$K \times K$	$(a+c, b+d)$	$(ac + \zeta d\underline{b}, \underline{a}d + cb)$	$(\underline{a}, -b)$

L comes with a degenerate **norm function**

$$n_L : L \rightarrow K : (a,b) \mapsto n_K(a) - 0 n_K(b)$$

Now $(a,b) \neq (0,0)$ **invertible** $\iff n_L((a,b)) \neq 0 \iff n_K(a) \neq 0$

This yields **three possibilities** for the algebra L:

L division algebra

$\zeta \notin n_K(K) = K^2$

$n_L((a,b)) = a^2 - \zeta b^2$

n_L *anisotropic*

L singular algebra

$\zeta = 0$

$n_L((a,b)) = a^2$

n_L degenerate

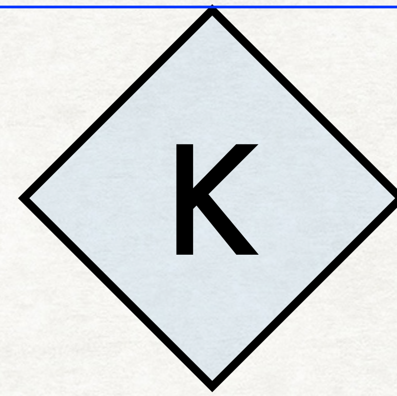
L split algebra

$\zeta = s^2$ $(s \in K \backslash \{0\})$

$n_L((a,b)) = (a - sb)(a + sb)$

n_L *splits*

THE GENERALISED CAYLEY-DICKSON PROCESS



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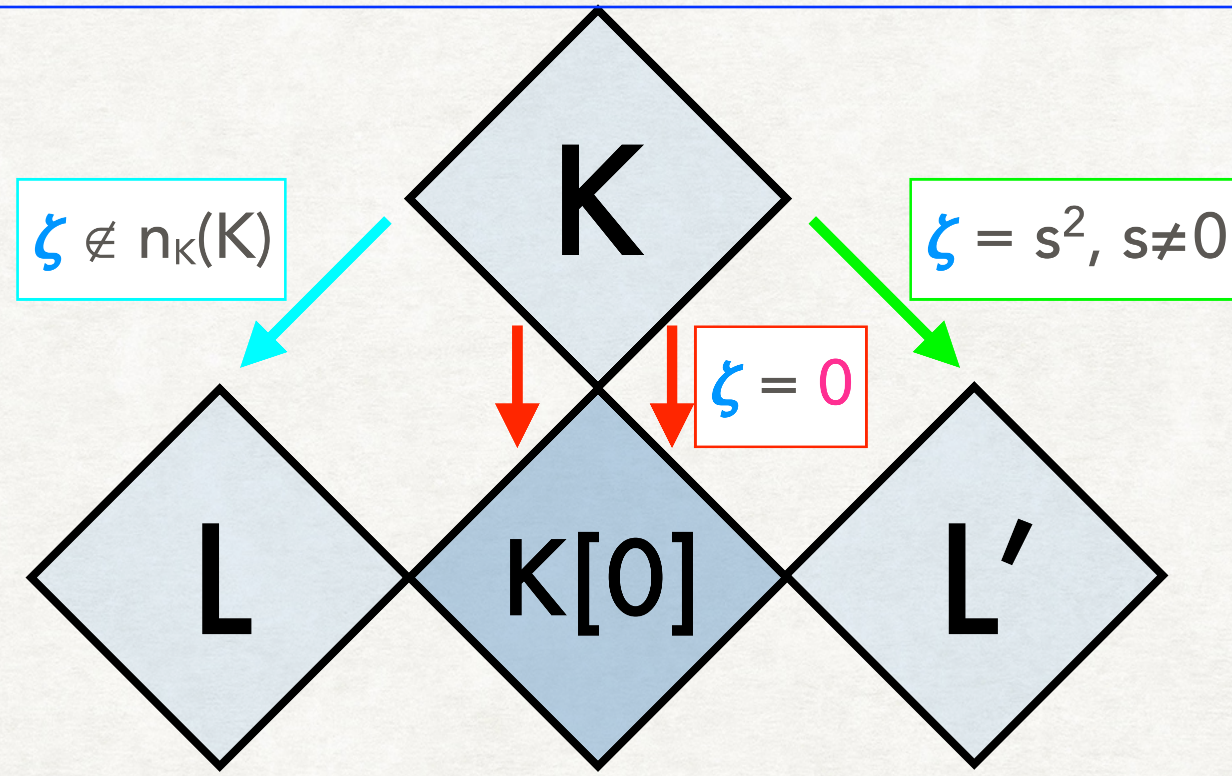

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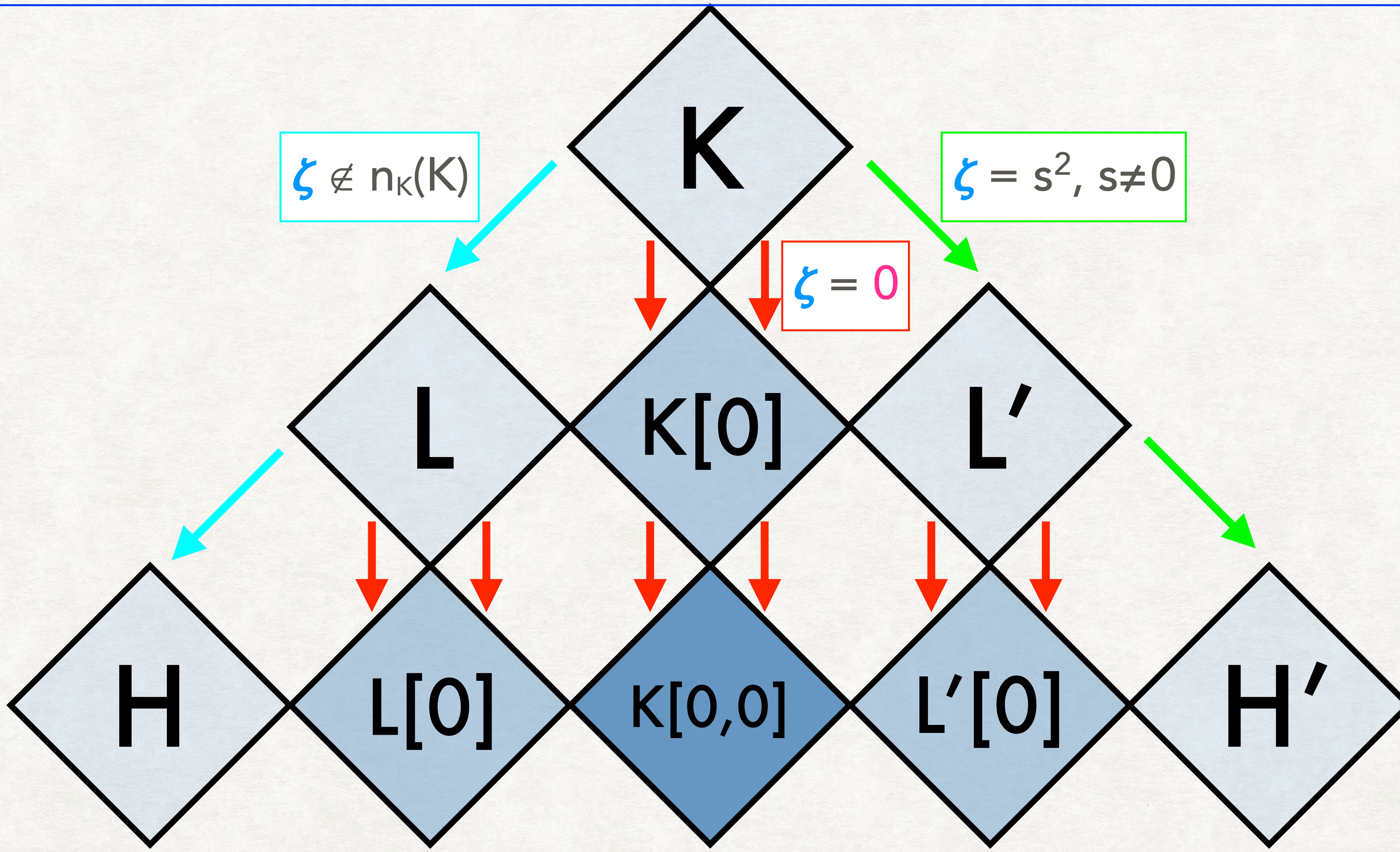

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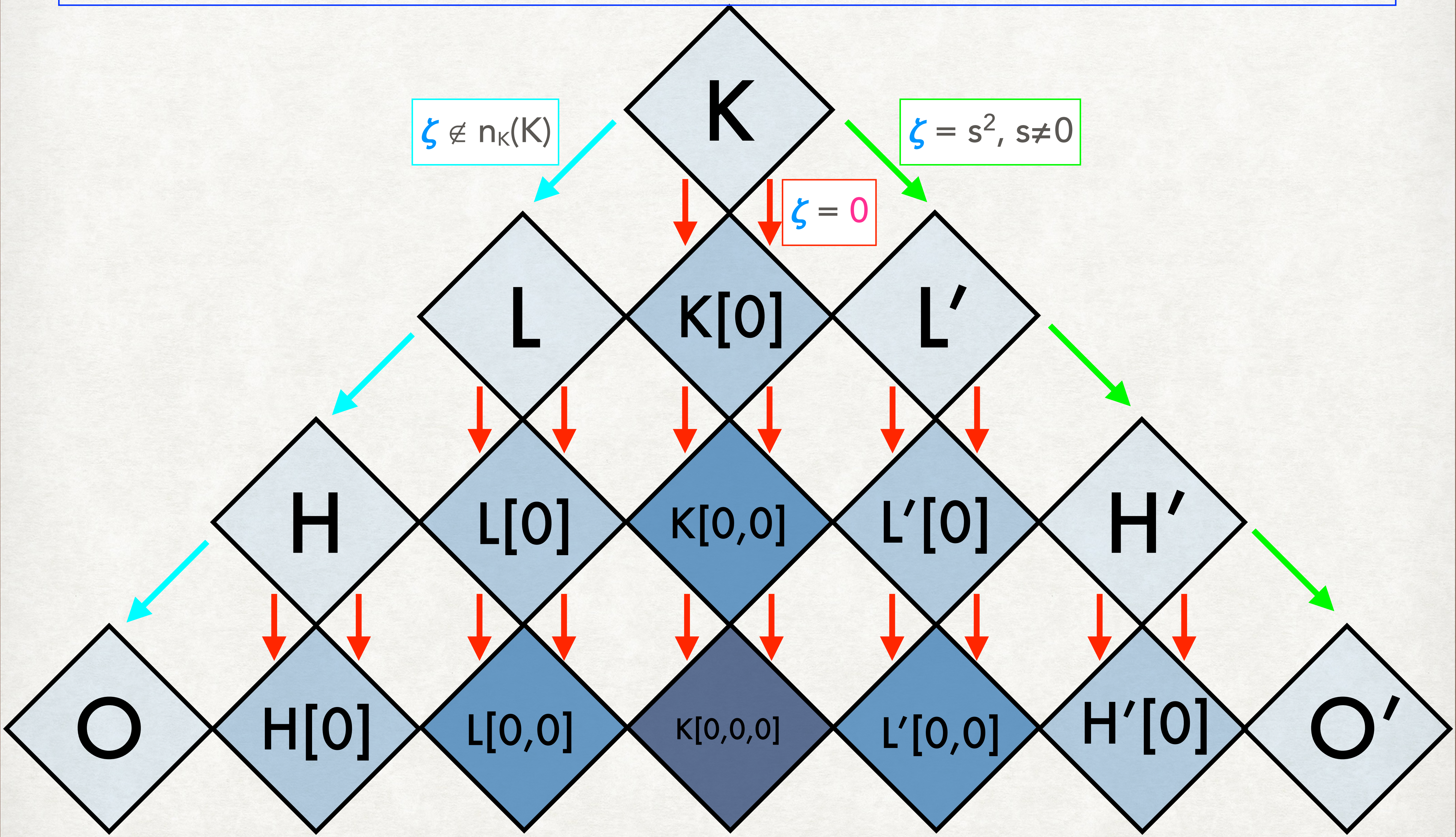
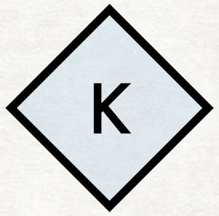


3

Veronese varieties

CD ALGEBRA ➜ VERONESE VAR

Let $\mathbb{K}$ be a field. The Veronese variety $\mathscr{V}(\mathbb{K})$ is defined as follows



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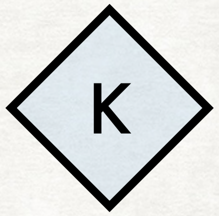

ρ: PG(2,K) → PG(5,K): $(x,y,z) \mapsto (x^2, y^2, z^2; yz, zx, xy)$

point → **point**

line → **conic in a plane ($Q^{min}(2,K)$)**

$(0,y,z) \mapsto (0, y^2, z^2; yz, 0, 0)$ satisfies $X_1X_2=X_3^2$, $X_0=X_4=X_5=0$

CD ALGEBRA → VERONESE VAR

Let K be a field. The Veronese variety $\mathscr{V}(K)$ is defined as follows

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The variety (X,Ξ) = (im(points),im(lines)) satisfies MM1 MM2 MM3

i.e., $\mathscr{V}(K)$ is a MM set with $Q^{min}(2,K)$s

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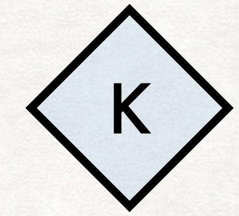

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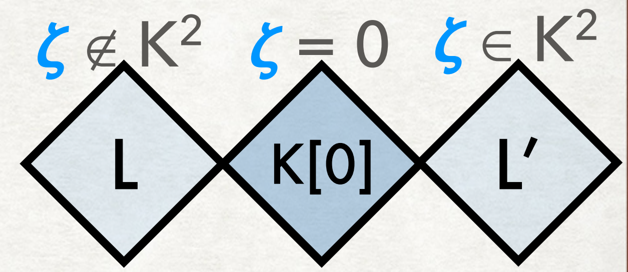

Similarly, for R =CD(K,ζ) we have the Veronese variety $\mathscr{V}(\mathbb{R})$

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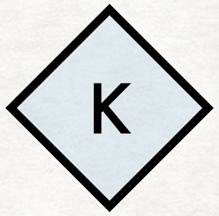

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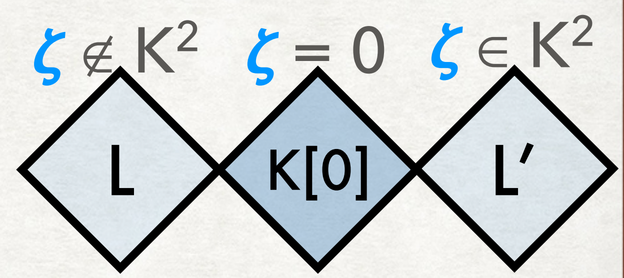

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➜ rewrite ρ, using that $x\underline{x} = x^2 = n(x)$ for $x \in K$

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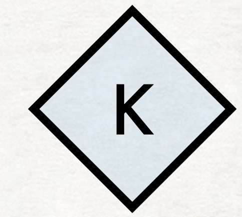

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point → **point**

line → **conic in a plane ($Q^{min}(2,K)$)**

$(0,y,z) \mapsto (0,\ y\underline{y},\ z\underline{z};\ y\underline{z},\ 0,\ 0)$ satisfies $X_1X_2=n(X_3)$, $X_0=X_4=X_5=0$

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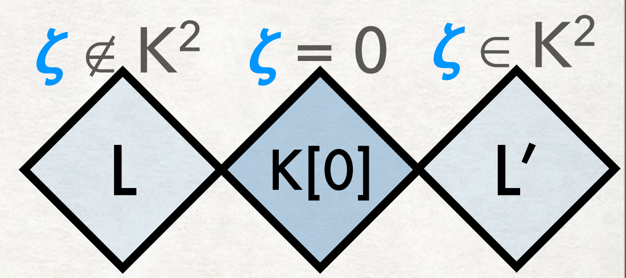

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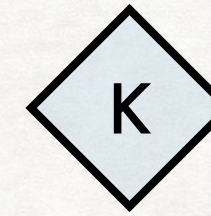

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point → **point**

line → **conic in a plane ($Q^{min}(2,K)$)**

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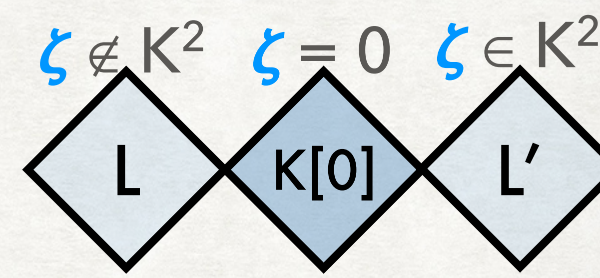


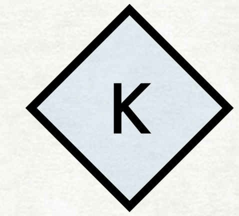
Similarly, for $\mathbb{R}$ =CD(K,ζ) we have the Veronese variety $\mathscr{V}(\mathbb{R})$

ρ: PG(2,R) → PG(8,K): $(x,y,z) \mapsto (x\underline{x},\ y\underline{y},\ z\underline{z};\quad y\underline{z}\quad,\quad z\underline{x}\quad,\quad x\underline{y}\quad)$

$X_0 \quad X_1 \quad X_2 \quad (X_3, X_4)\ (X_5, X_6)\ (X_7, X_8)$

CD ALGEBRA ➜ VERONESE VAR

Let K be a field. The Veronese variety $\mathscr{V}(K)$ is defined as follows



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point → **point**

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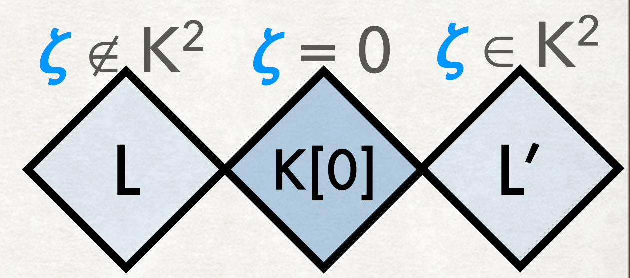

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X_0 X_1 X_2 (X_3, X_4) (X_5, X_6) (X_7, X_8)

Warning: if R = L′ or K[0], there is no projective plane over it.

CD ALGEBRA ➜ VERONESE VAR

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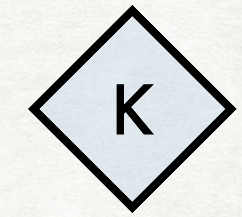

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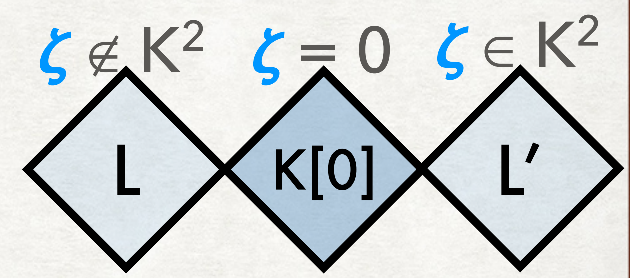
point → **point**

line → **conic in a plane ($Q^{min}(2,K)$)**

$(0,y,z) \mapsto (0, y\underline{y}, z\underline{z}; y\underline{z}, 0, 0)$ satisfies $X_1X_2=n(X_3)$, $X_0=X_4=X_5=0$



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X_0 X_1 X_2 (X_3, X_4) (X_5, X_6) (X_7, X_8)

Warning: if R = L′ or K[0], there is no projective plane over it.

➜ take a ring geometry G(2,R) instead:

- points : {(x,y,z)R* | x, y, z ∈ R & (x,y,z)r = 0 for r ∈ R implies r = 0}
- lines : {R*[a,b,c] | a, b, c ∈ R & r[a,b,c] = 0 for r ∈ R implies r = 0}
- incidence: $ax + by + cz = 0$

If R = L, then G(2,L)=PG(2,L)

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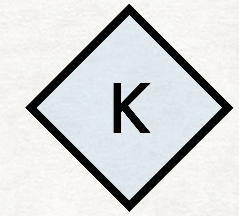

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point → **point**

line → **conic in a plane ($Q^{min}(2,K)$)**

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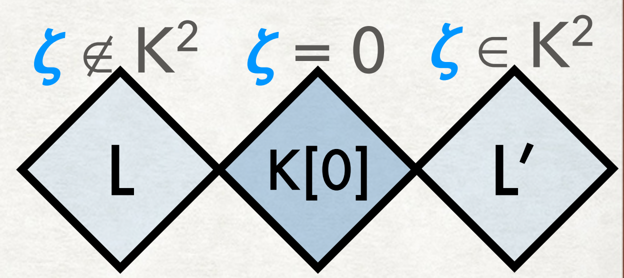

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X_0 X_1 X_2 (X_3, X_4) (X_5, X_6) (X_7, X_8)

CD ALGEBRA ➜ VERONESE VAR

Let K be a field. The Veronese variety $\mathscr{V}(\mathbb{K})$ is defined as follows

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point → **point**

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$\zeta \notin K^2$ $\quad \zeta = 0$ $\quad \zeta \in K^2$

L $\quad$ K[0] $\quad$ L′

Similarly, for $\mathbb{R}$ =CD(K,ζ) we have the Veronese variety $\mathscr{V}(\mathbb{R})$

$$\rho: G(2,R) \to PG(8,K): (x,y,z) \mapsto (x\underline{x}, y\underline{y}, z\underline{z}; \; y\underline{z} \;,\; z\underline{x} \;,\; x\underline{y} \;)$$

$X_0 \quad X_1 \quad X_2 \quad (X_3, X_4) \; (X_5, X_6) \; (X_7, X_8)$

$(0,y,z) \mapsto (0, y\underline{y}, z\underline{z}; y\underline{z}, 0, 0)$ satisfies $X_1X_2 = \mathbf{n}(X_3, X_4) = X_3^2 - \zeta X_4^2$

L	K[0]	L′
$Q^{min}(3,K)$	cone in PG(3,K) point – $Q^{min}(2,K)$	$Q^{max}(3,K)$

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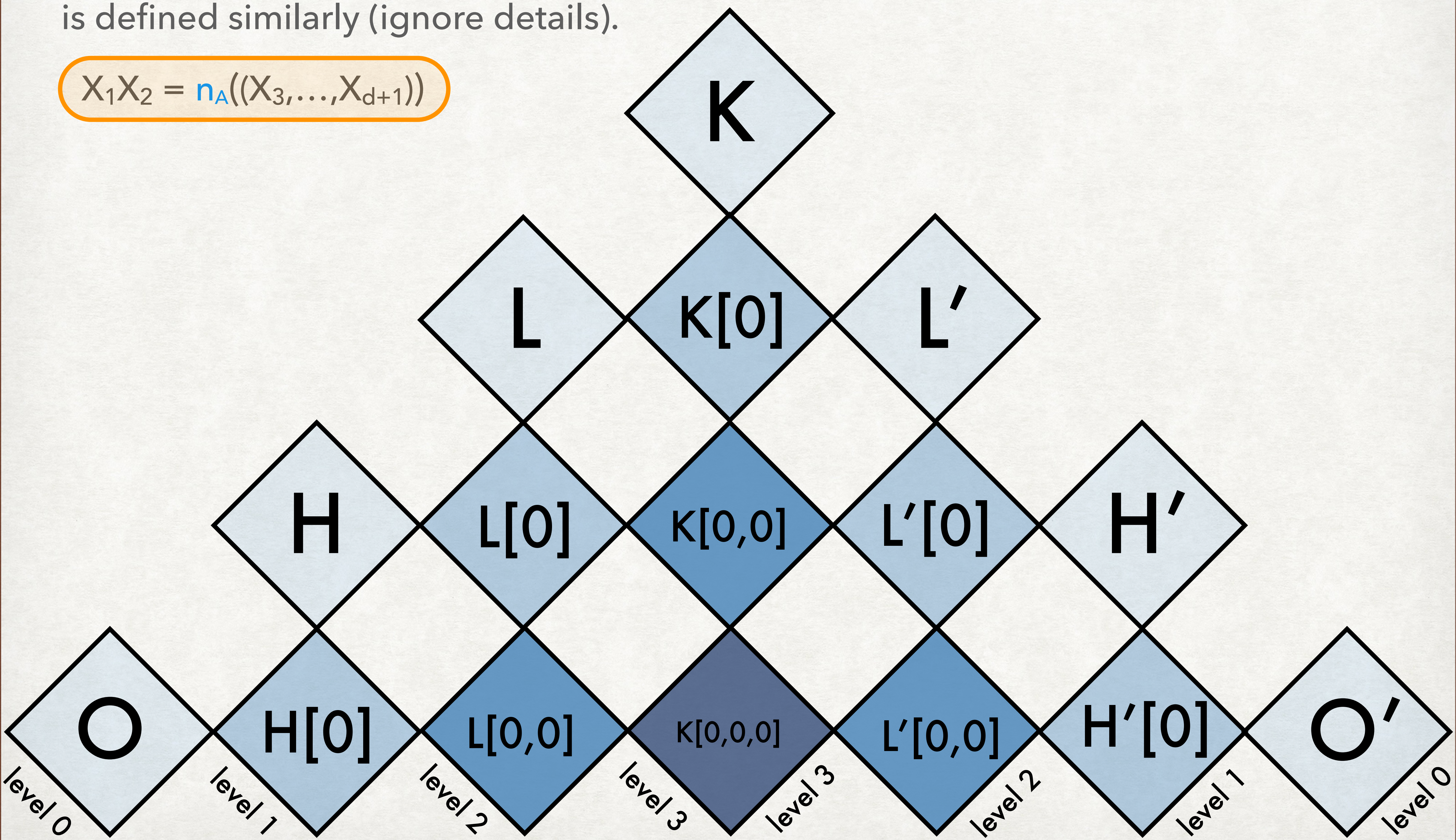
L	K[0]	L′
$Q^{min}(3,K)$	cone in PG(3,K) point – $Q^{min}(2,K)$	$Q^{max}(3,K)$

Again, (im(points),im(lines)) satisfies the MM axioms so $\mathscr{V}(\mathbb{R})$ is an MM set.

CD ALGEBRA ➜ VERONESE VAR

Let A be a Cayley Dickson algebra with dim(A/K) = d. The Veronese variety $\mathscr{V}(A)$ is defined similarly (ignore details).

$X_1X_2 = n_A((X_3,\ldots,X_{d+1}))$



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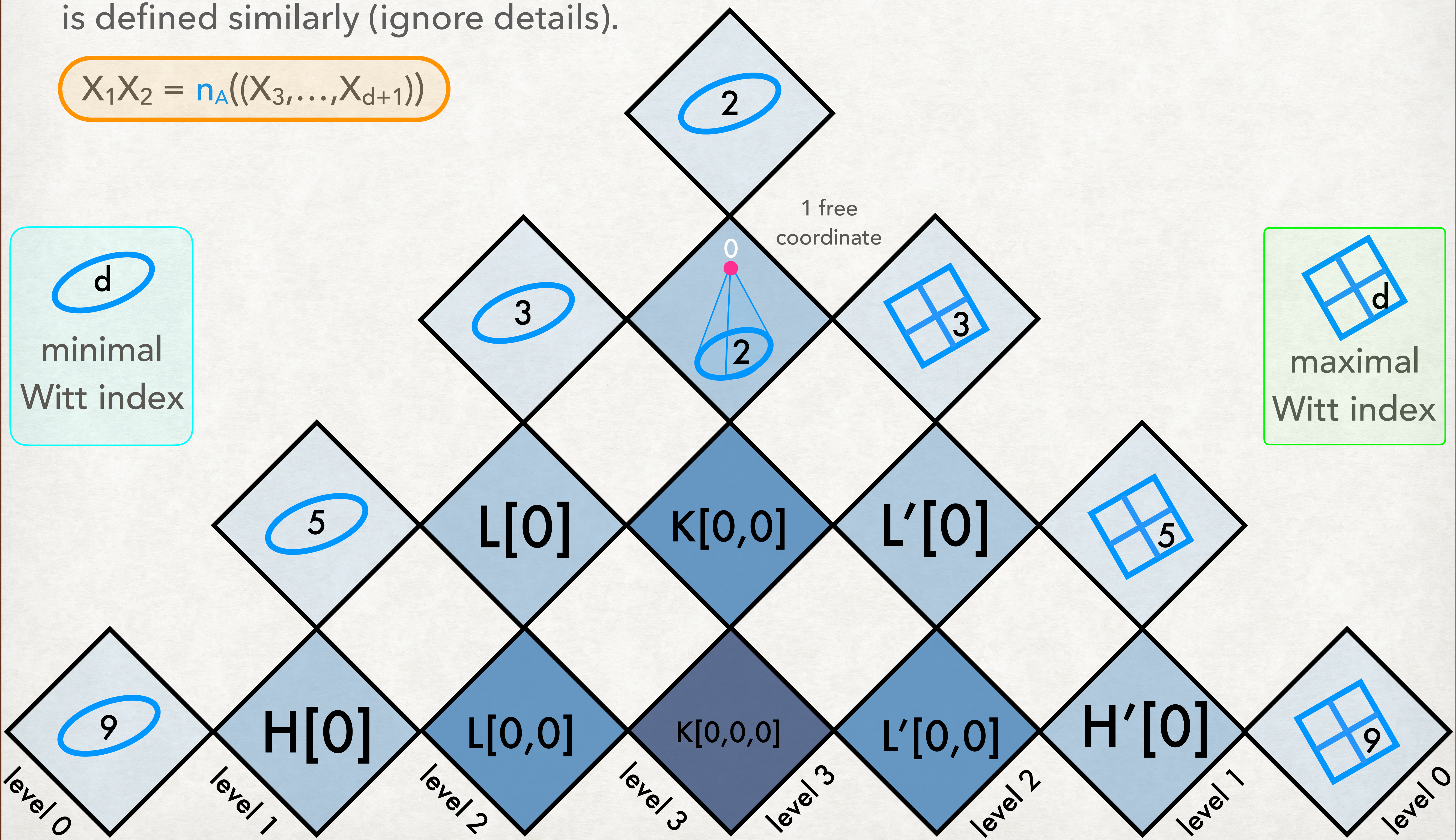

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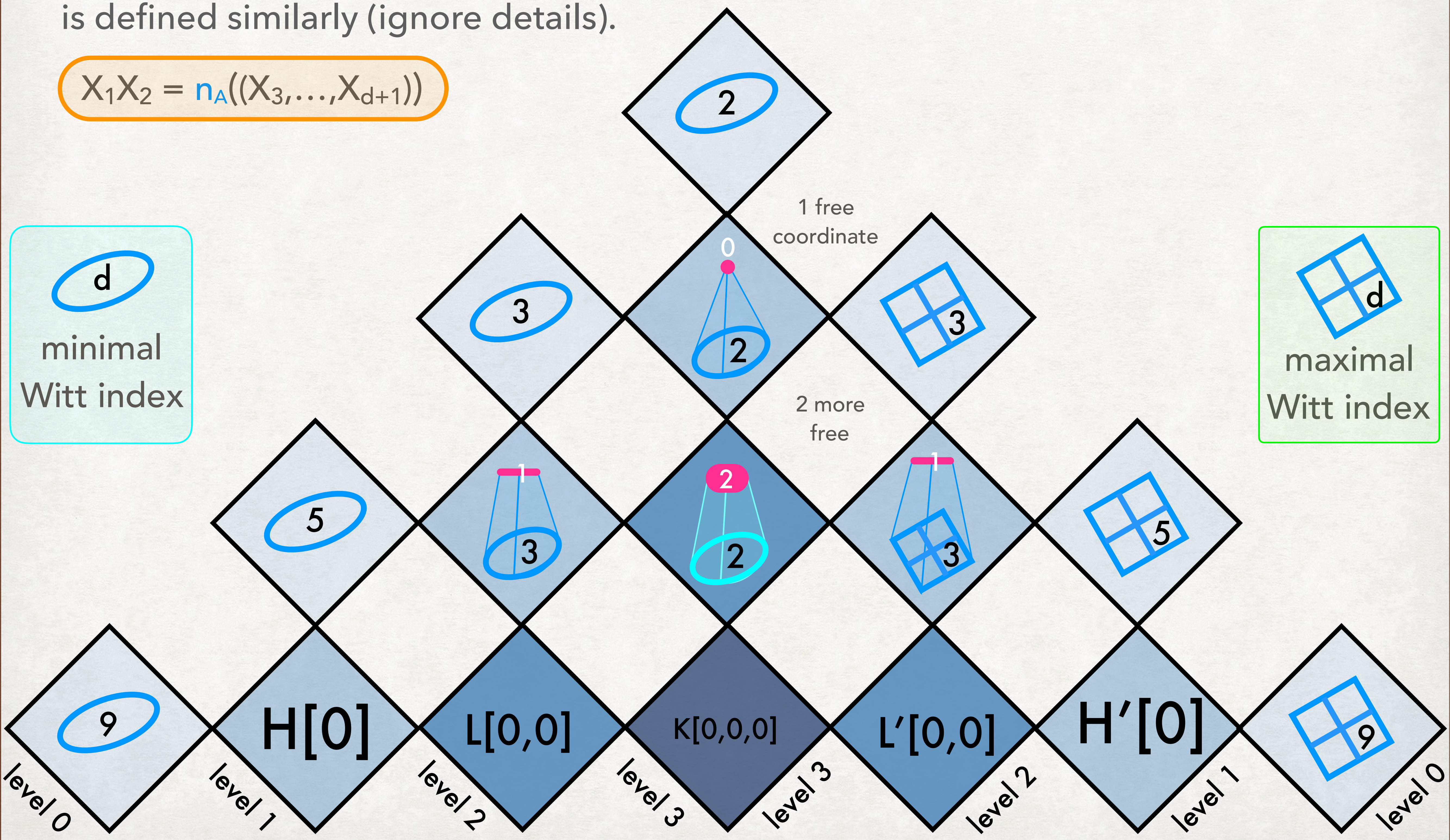

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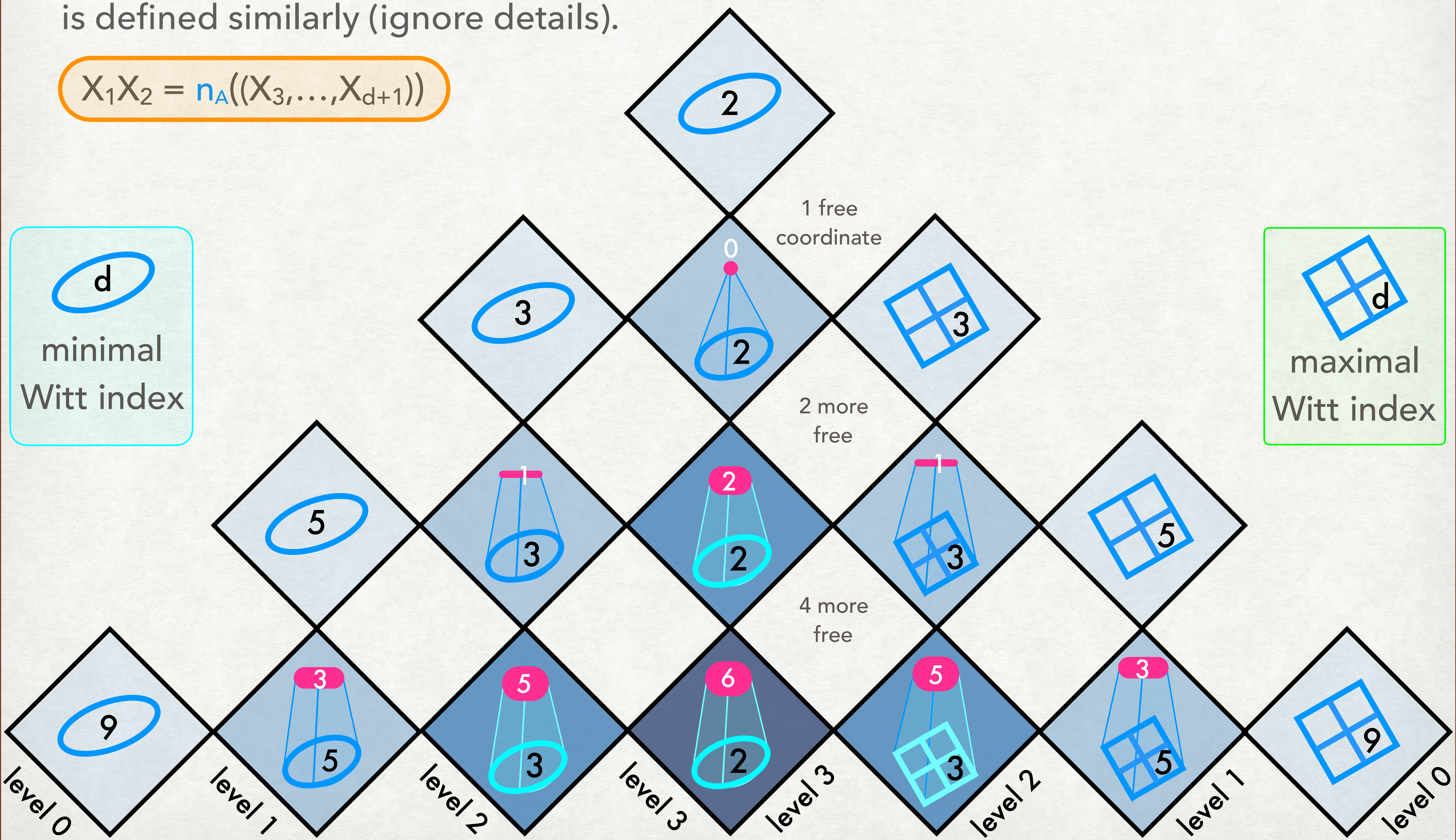

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✓ MM sets with

✓ MM sets with

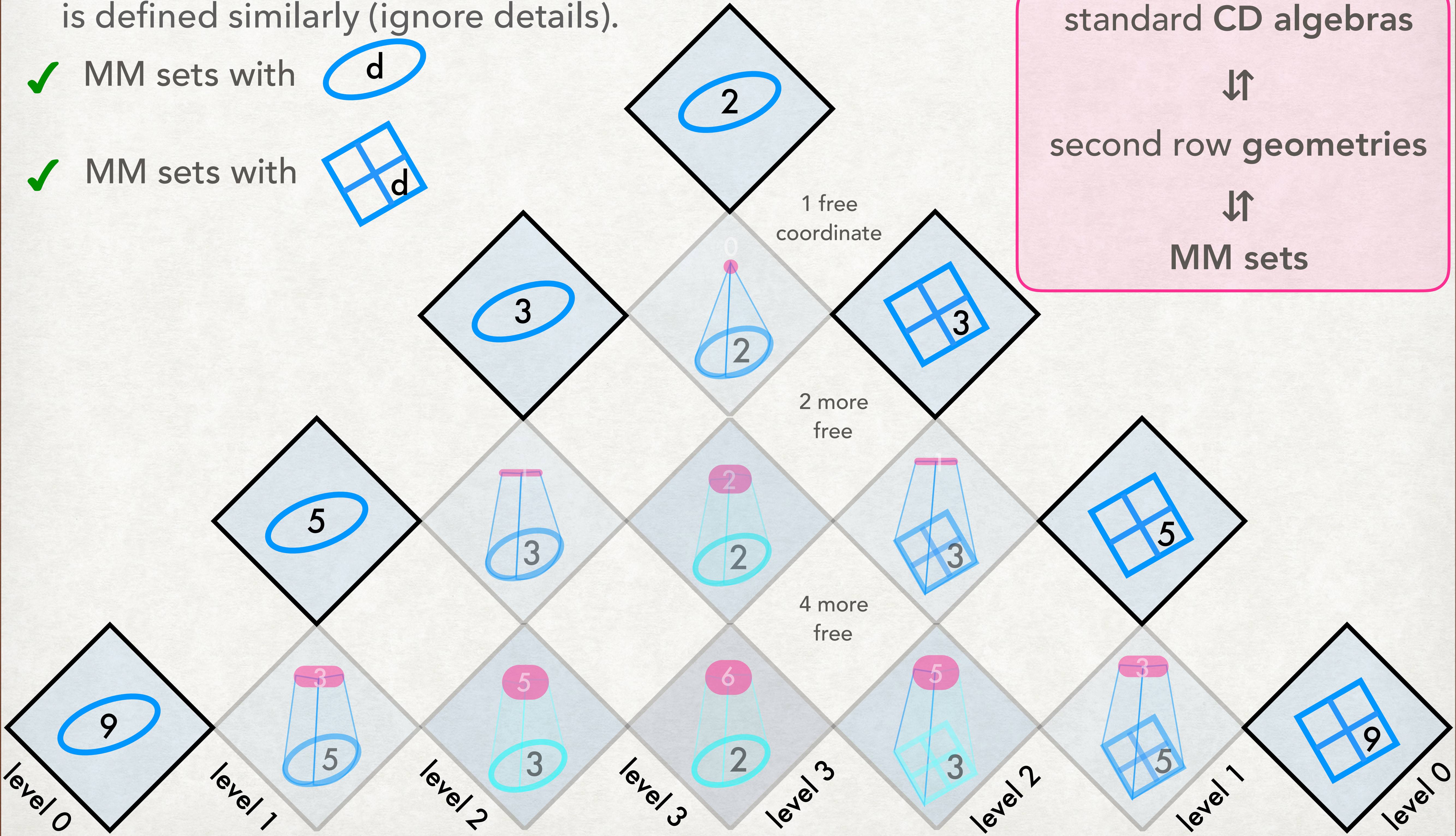
standard **CD algebras**

⇅

second row **geometries**

⇅

MM sets



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- ✓ MM sets with
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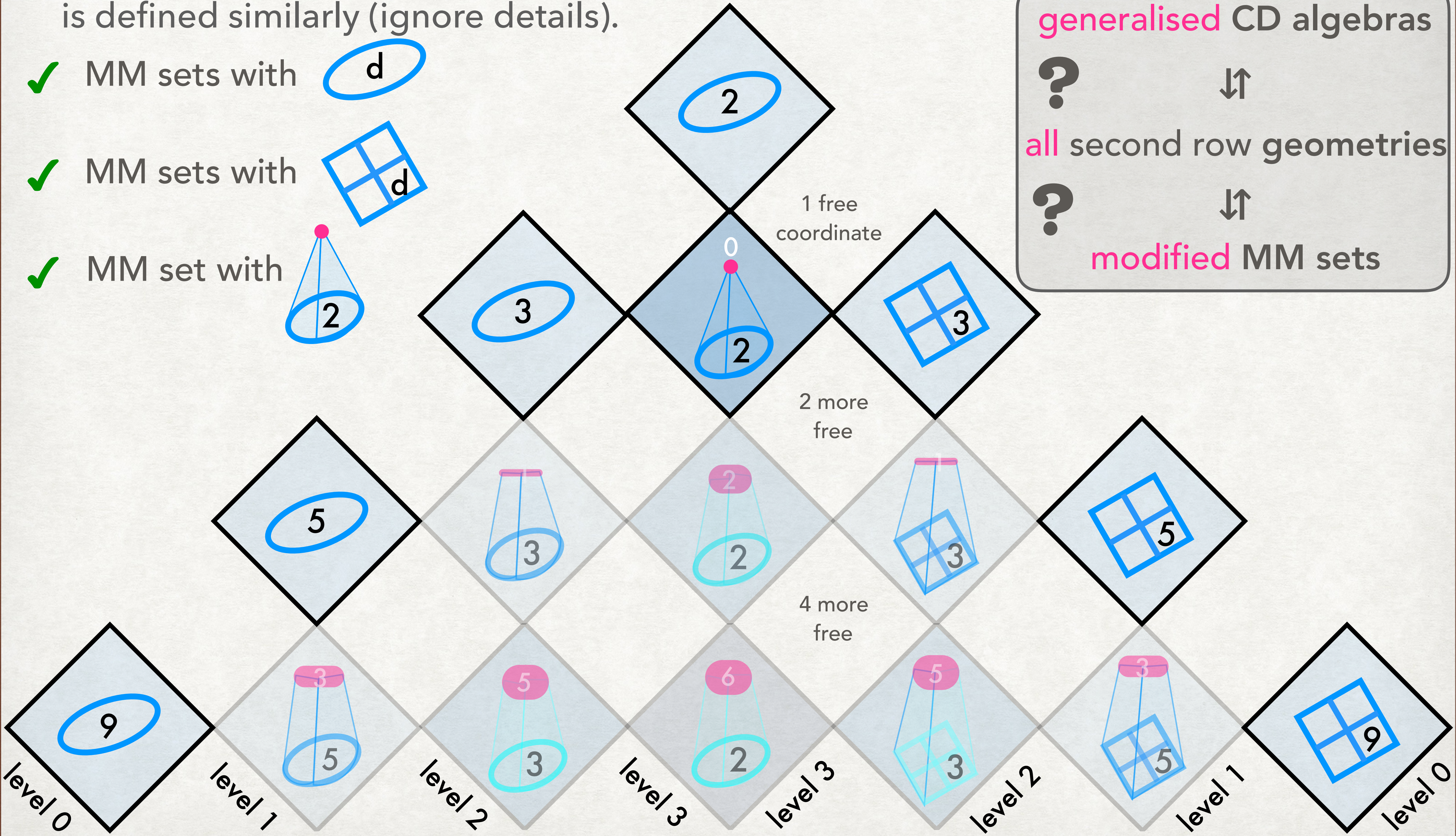
- ✓ MM set with



generalised CD algebras

? ⇅

all second row geometries

? ⇅

modified MM sets

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LEVEL 1

- ☛ MM set with ((d,v) general)
- ☛ MM set with ((d,v) general)

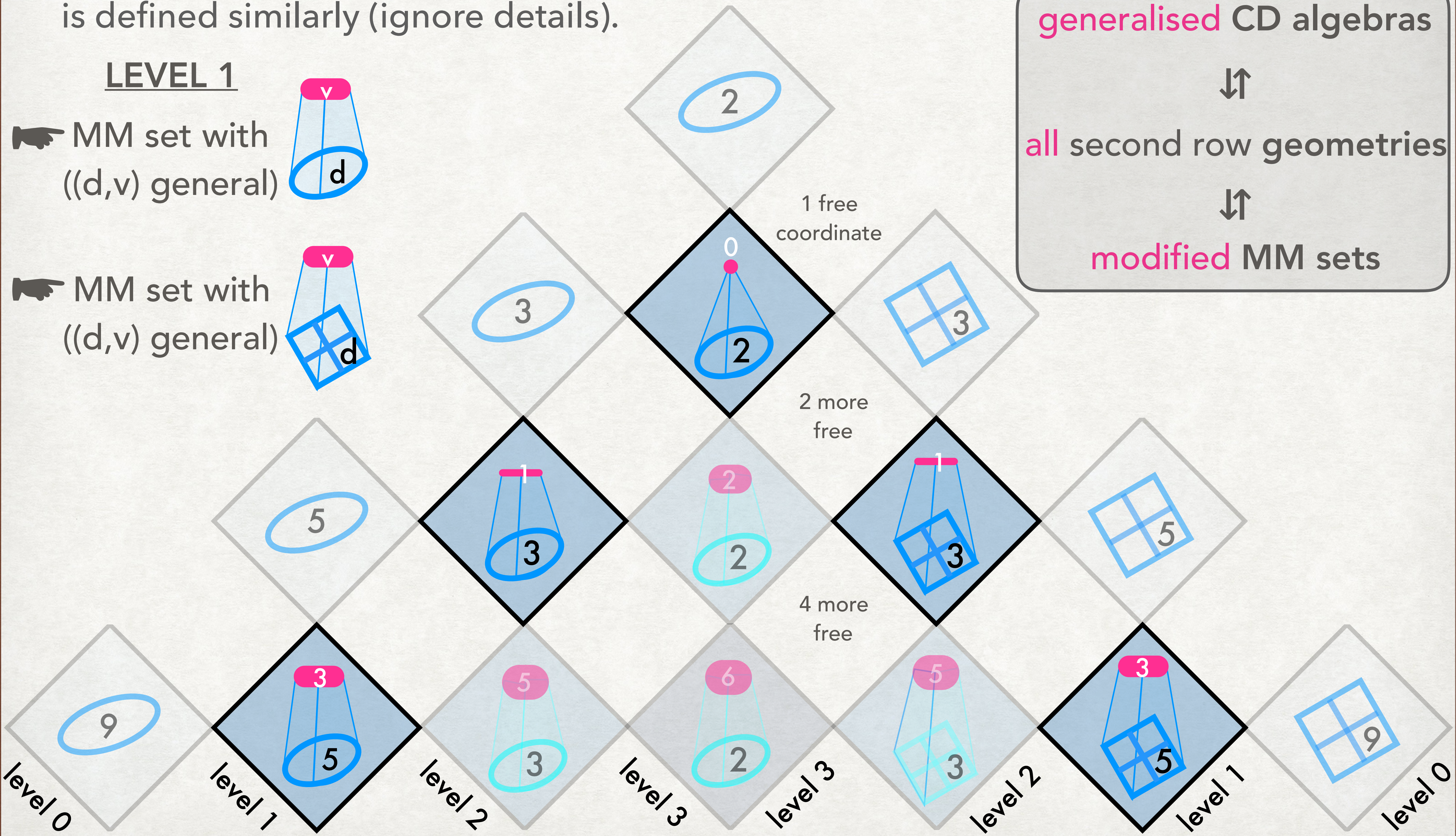
generalised CD algebras

⥮

all second row geometries

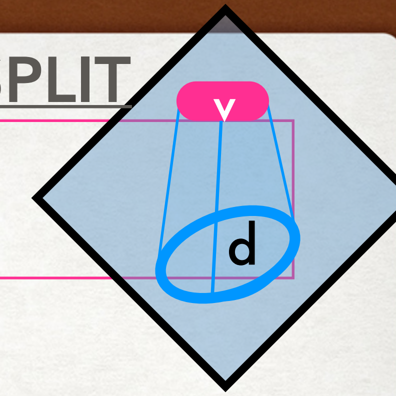
⥮

modified MM sets



4

Results



MM SETS WITH (D,V)-TUBES

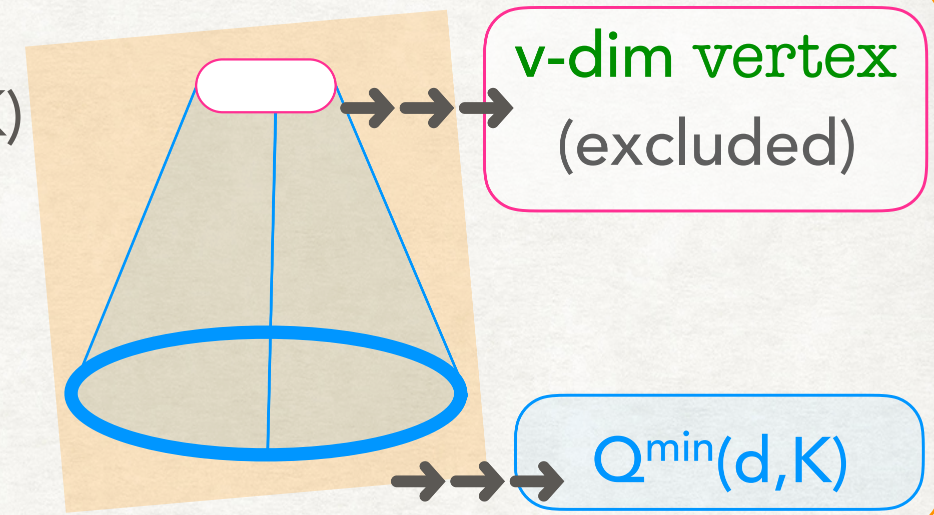
Axiomatic description

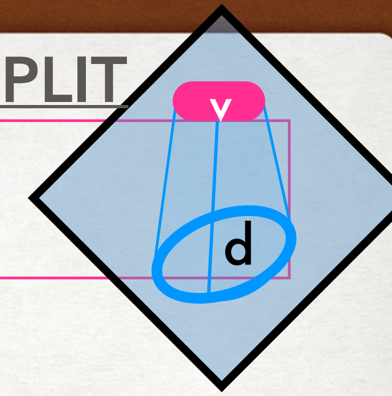
X — points spanning PG(N,K)

Y — vertices

Ξ — d'-spaces ξ in PG(N,K) s.th. ξ ∩ X is: (d,v)-tube

(d'=d+v+1)





MM SETS WITH (D,V)-TUBES

Axiomatic description

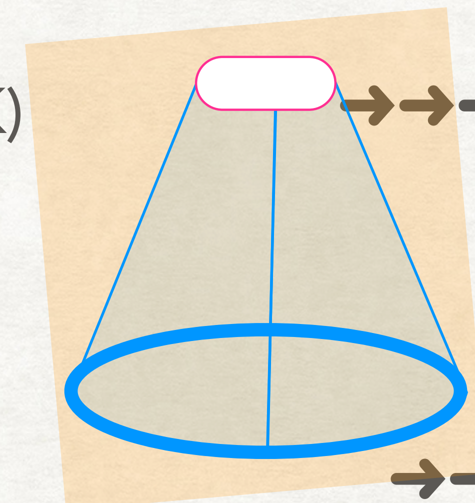
X points spanning PG(N,K)

Y vertices

Ξ d′-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

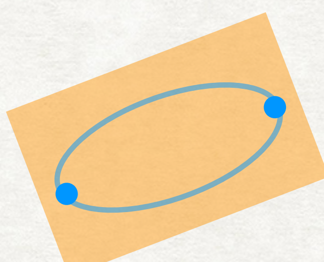
(d′=d+v+1)

(d,v)-tube



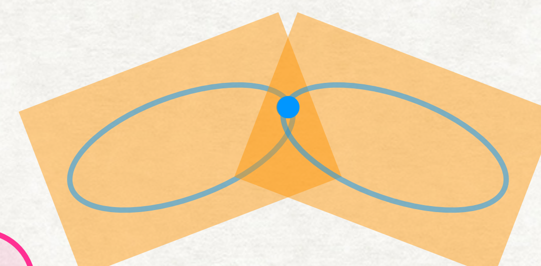
v-dim vertex (excluded)

$Q^{min}(d,K)$



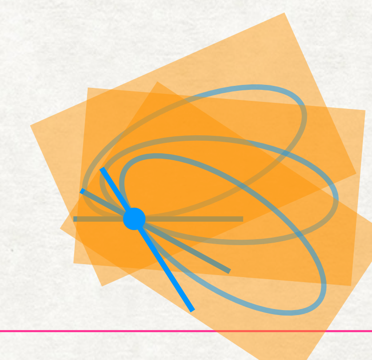
MM1

each two points of X belong to a [d′] of Ξ



MM2′

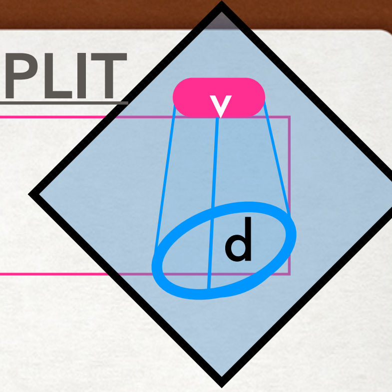
two [d′]s of Ξ intersect in points of X∪Y but never in Y only



MM3

the tangent space of a point of X is contained in a [2(d′-1)]

MM SETS WITH (D,V)-TUBES



Axiomatic description

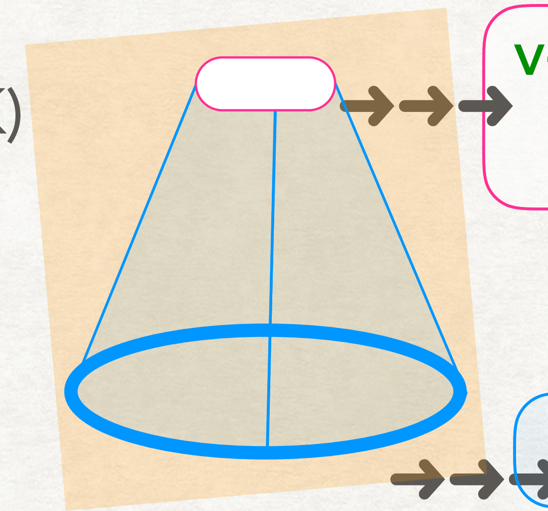
X points spanning PG(N,K)

Y vertices

Ξ d′-spaces ξ in PG(N,K) s.th. ξ ∩ X is:

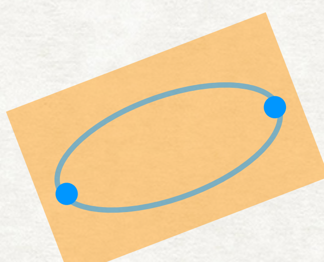
(d,v)-tube

$(d' = d + v + 1)$



v-dim vertex (excluded)

$Q^{min}(d,K)$



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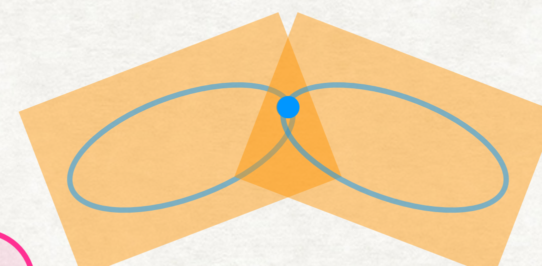
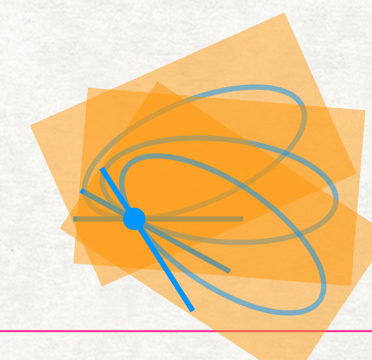


MM2′

two [d′]s of Ξ intersect in points of X∪Y **but** never in Y only

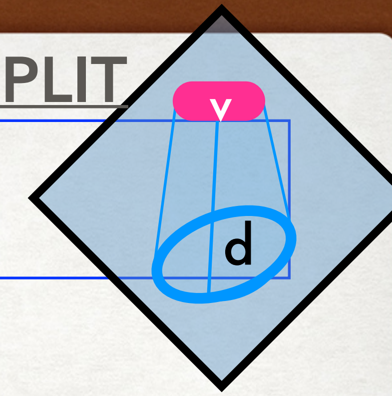


MM3

the tangent space of a point of X is contained in a [2(d′-1)]

The pair (X, Ξ) together with MM1, MM2′ and MM3 is called a singular MM-set with (d,v)-tubes.

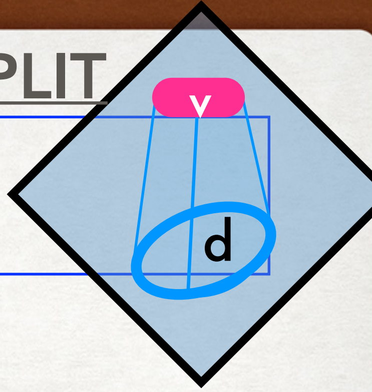
MM SETS WITH (D,V)-TUBES: RESULTS



Case 1: the vertex is only a point (v=0)

For any field K, let (**X**, **Ξ**) be a singular MM-set with (d,**0**)-tubes.

MM SETS WITH (D,V)-TUBES: RESULTS

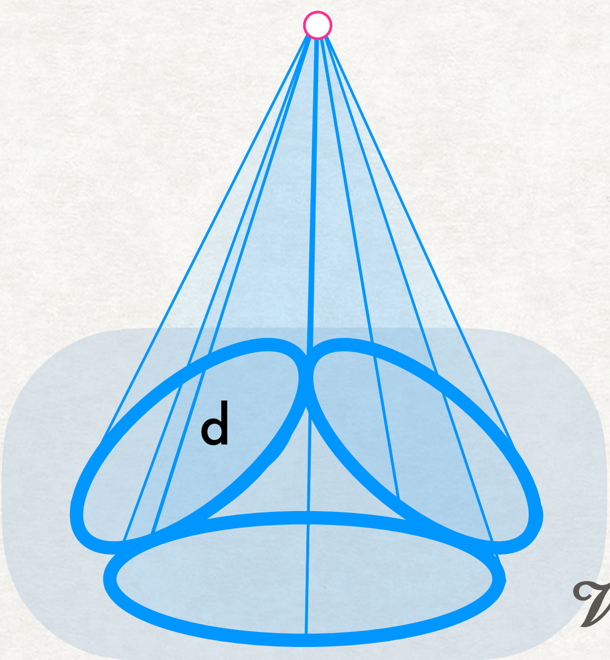


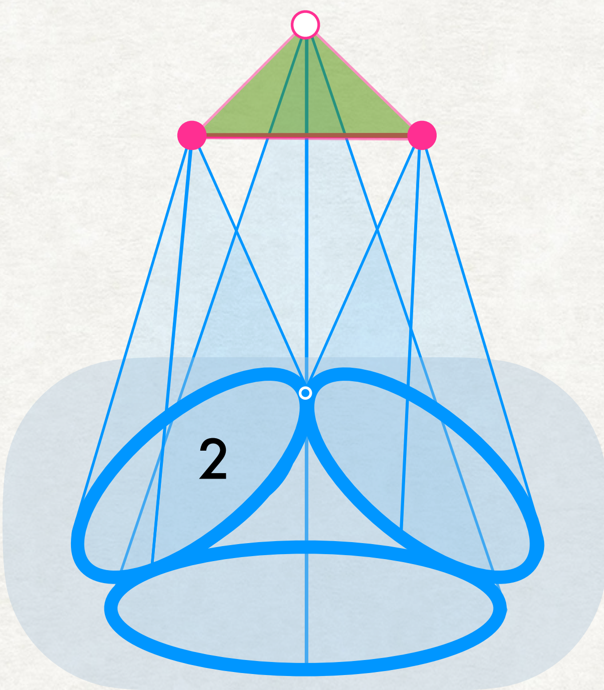
Case 1: the vertex is only a point (v=0)

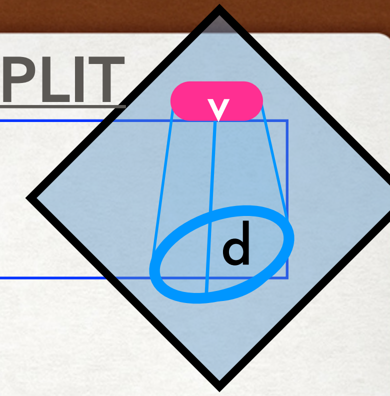
For any field K, let (X, Ξ) be a singular MM-set with (d,0)-tubes.

d=2

Schillewaert, Van Maldeghem (2015)
If nontrivial, (X, Ξ) is projectively unique and isomorphic to a Hjelmslevian projective plane.







MM SETS WITH (D,V)-TUBES: RESULTS

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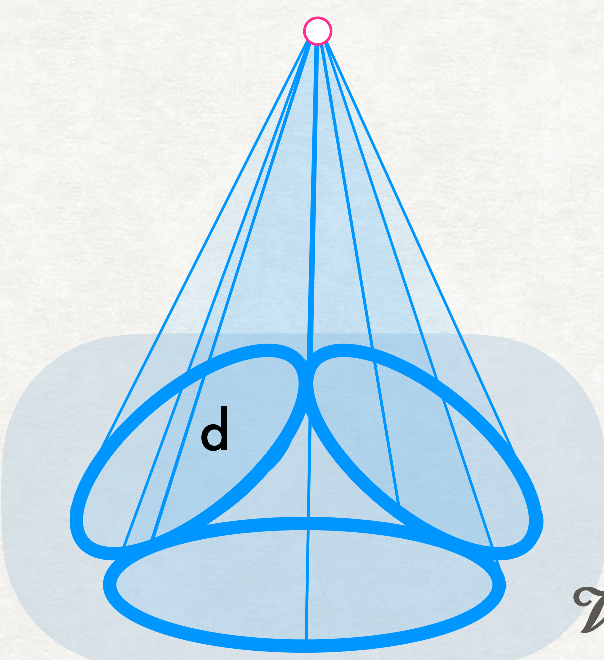
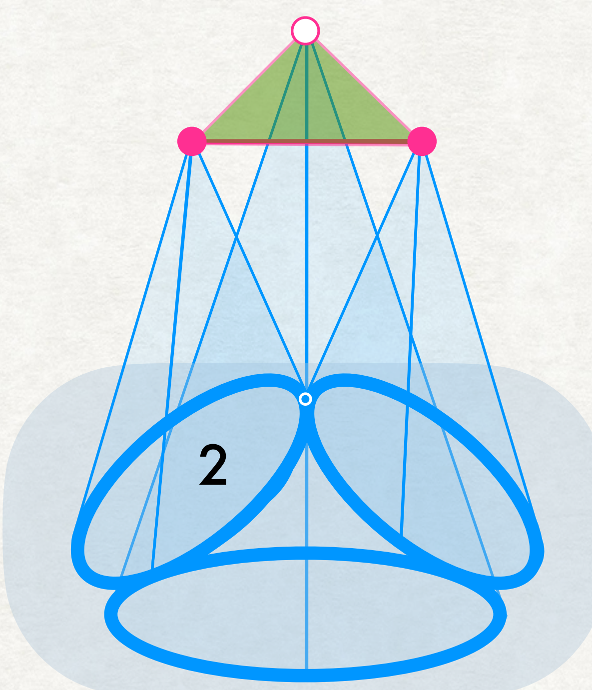




d>2

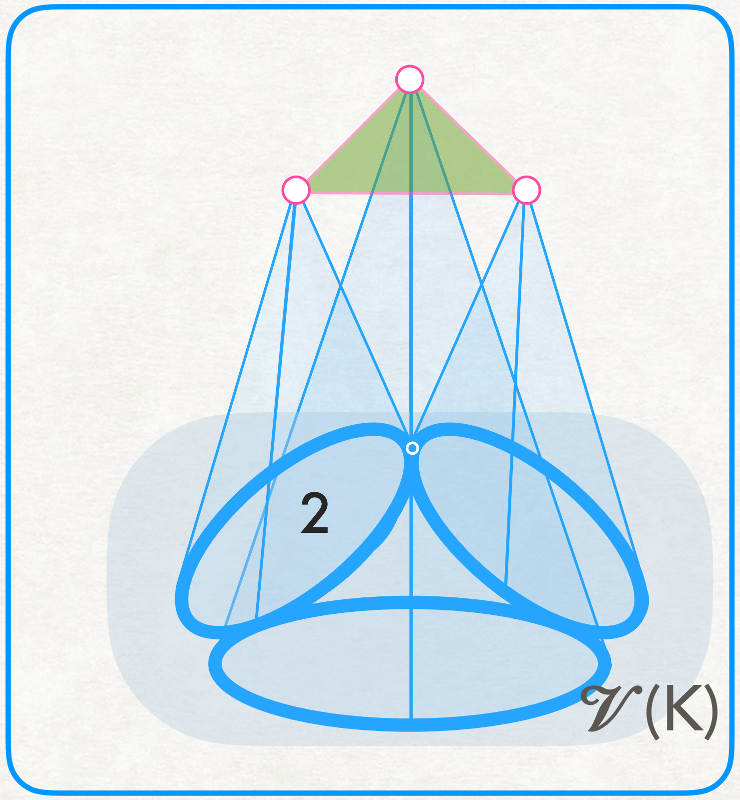
ADS, Van Maldeghem (2017)

(X, Ξ) is always trivial.

HJELMSLEVIAN PROJECTIVE PLANES

A Hjelmslevian projective plane:

(X, Ξ) is <u>something</u> with vertices in a plane and base an MM set with $Q^{min}(2,K)$s



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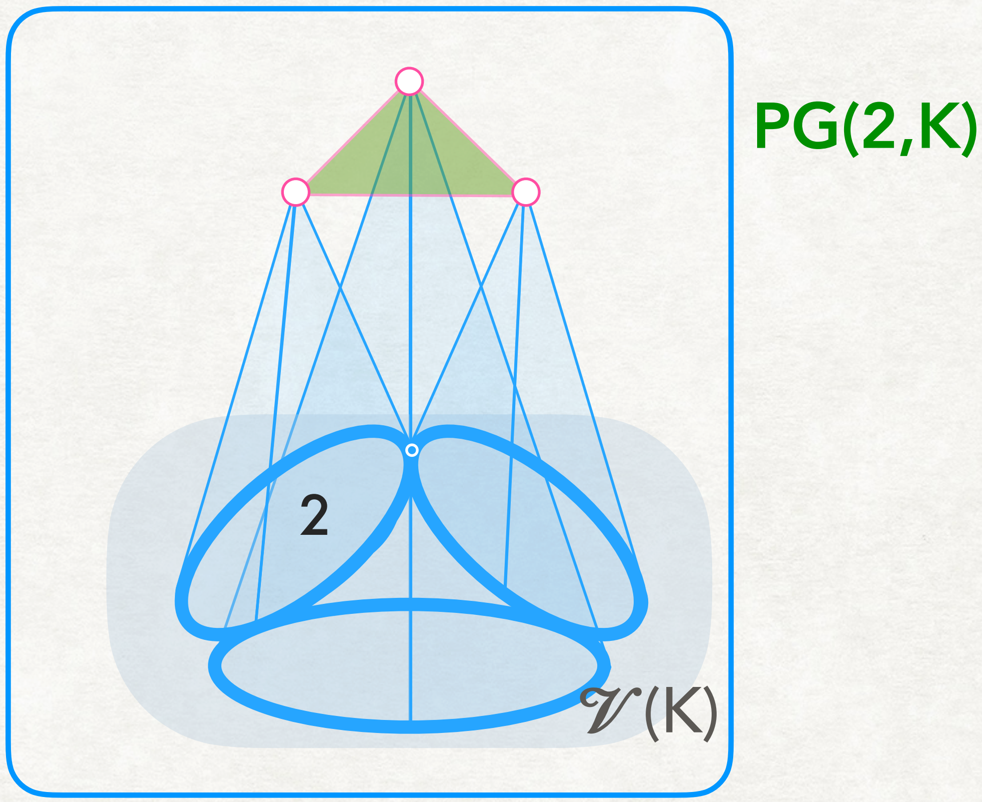

The vertices form a projective plane over K.

HJELMSLEVIAN PROJECTIVE PLANES

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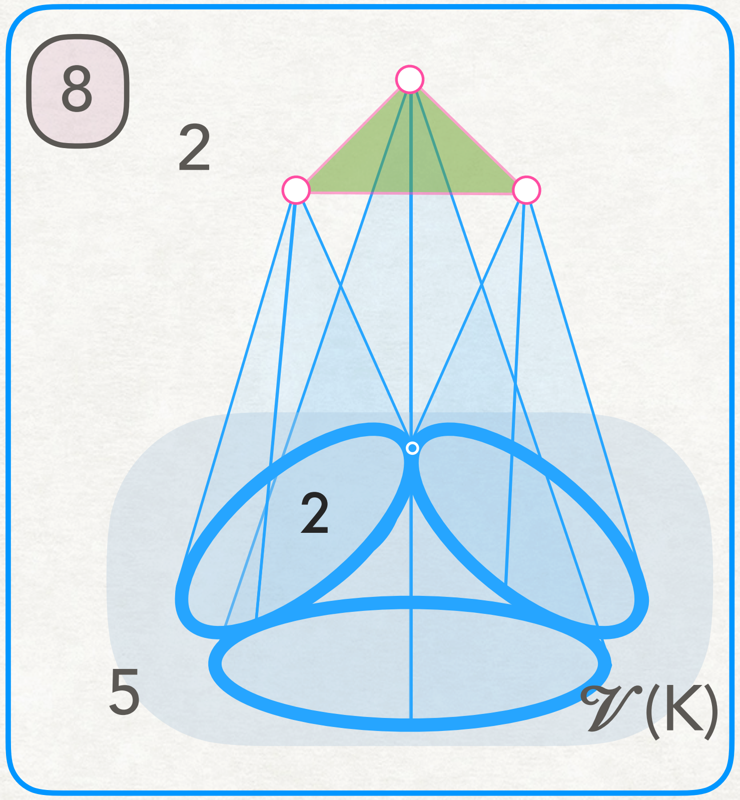


PG(2,K)

𝒱(K)

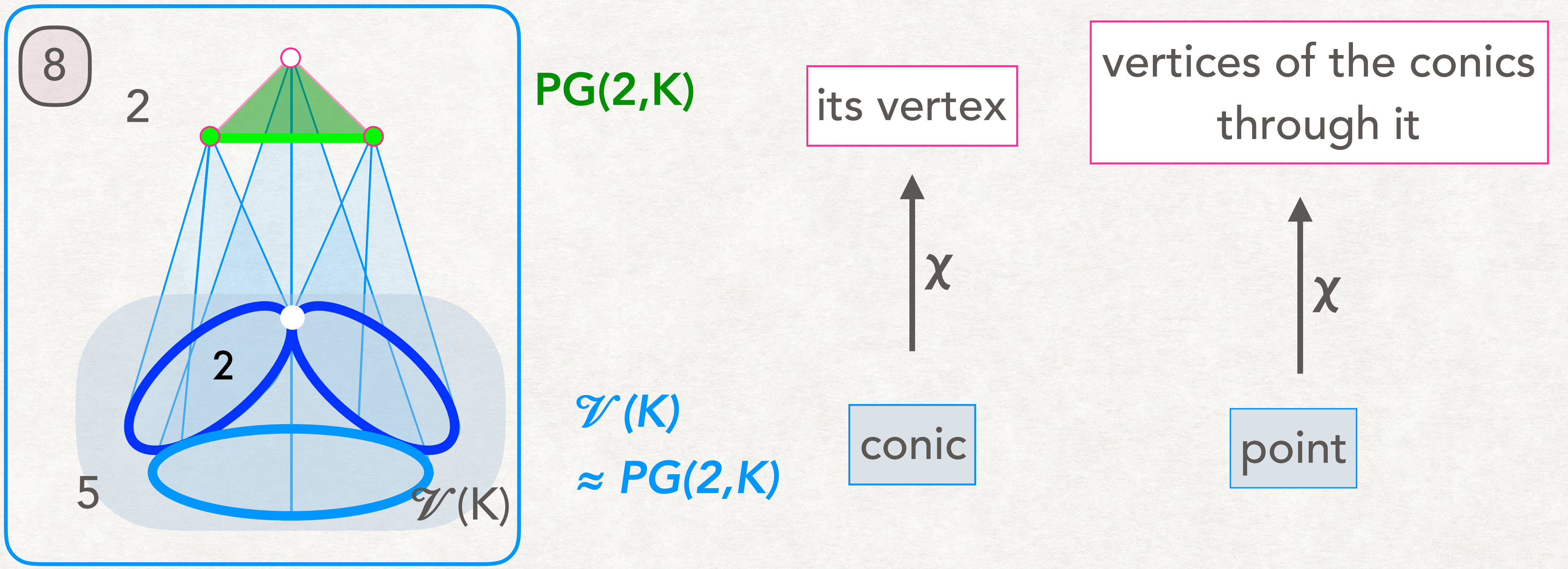
The **vertices** form a projective plane over K.

In a complementary subspace, the **points of X** form the Veronese variety ***𝒱(K)***.

HJELMSLEVIAN PROJECTIVE PLANES

A Hjelmslevian projective plane:

(X, Ξ) is <u>something</u> with vertices in a plane and base an MM set with $Q^{min}(2,K)$s



The **vertices** form a projective plane over K.

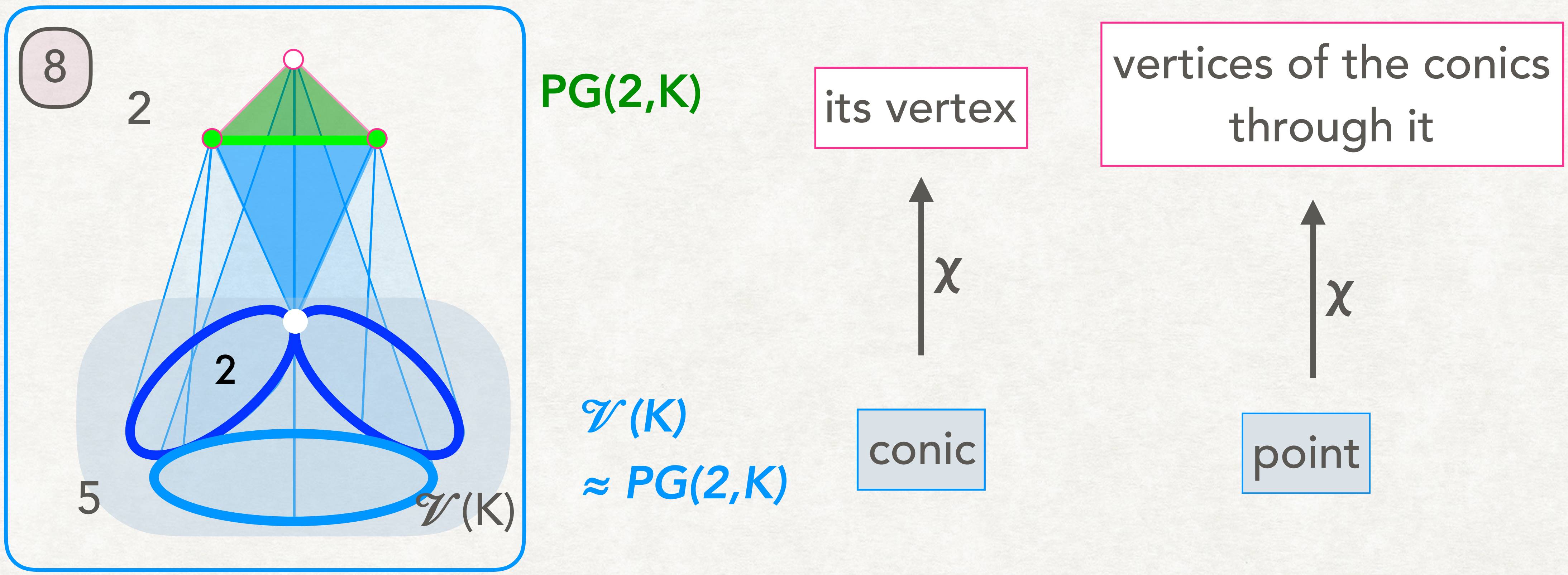
In a complementary subspace, the **points of X** form the Veronese variety $\mathscr{V}(K)$.

The mapping χ is a **linear duality** between $\mathscr{V}(K)$ and PG(2,K).

HJELMSLEVIAN PROJECTIVE PLANES

A Hjelmslevian projective plane:

(X, Ξ) is <u>something</u> with vertices in a plane and base an MM set with $Q^{min}(2,K)$s



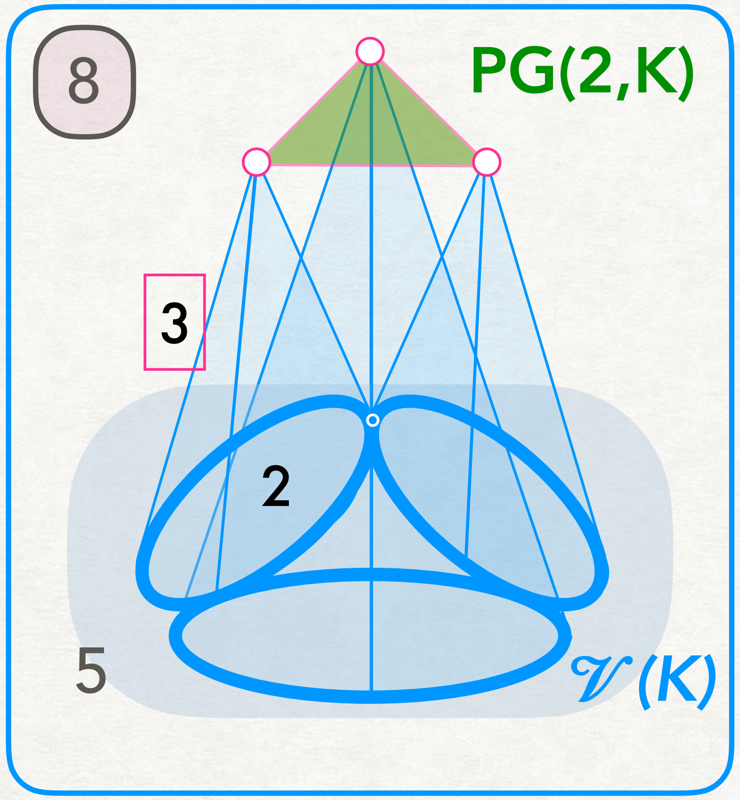
The **vertices** form a projective plane over K.

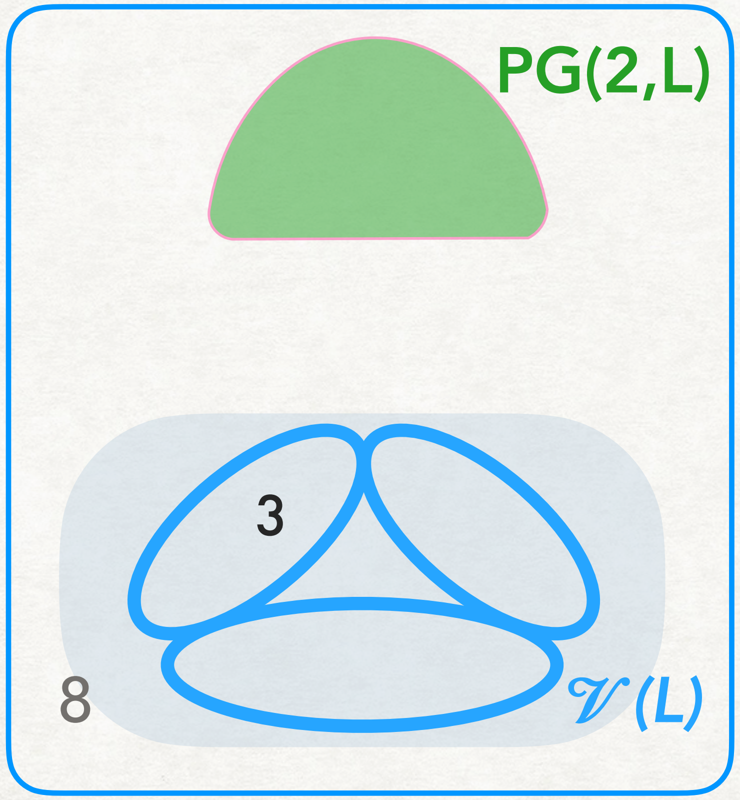
In a complementary subspace, the **points of X** form the Veronese variety *𝒱(K)*.

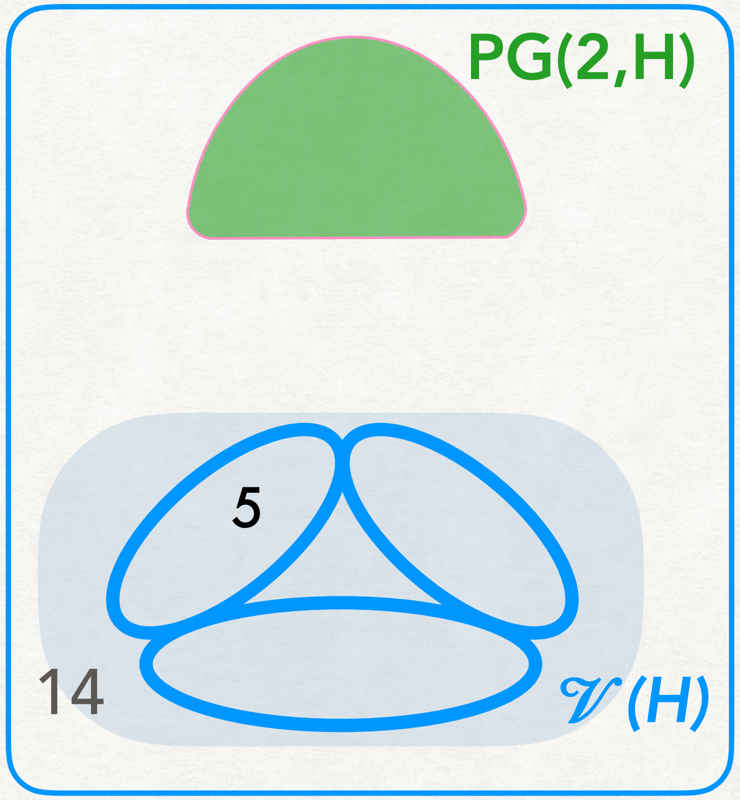
The mapping χ is a **linear duality** between *𝒱(K)* and **PG(2,K)**.

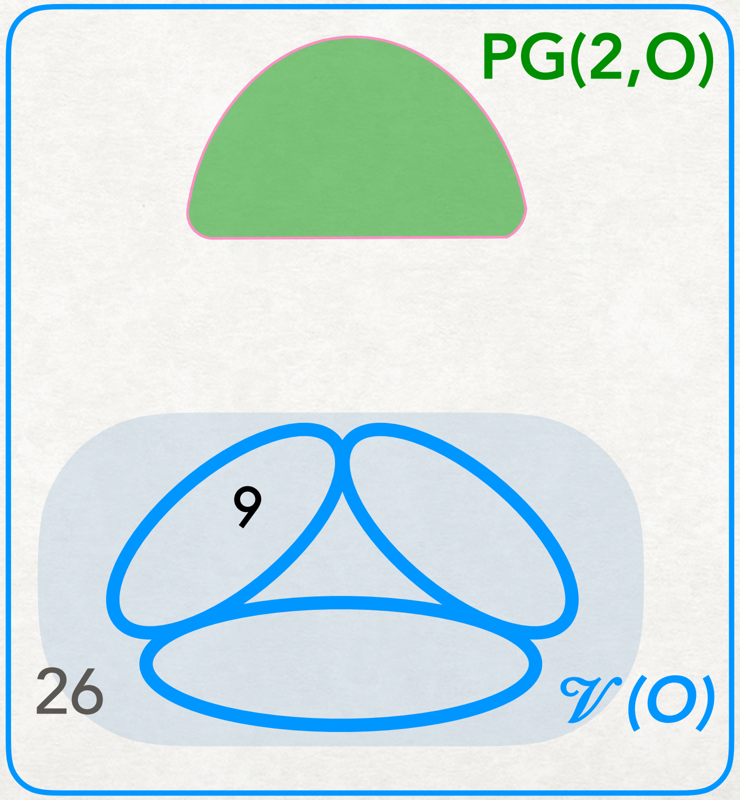
The union of the **affine planes** xχ(x)\χ(x), with x in *𝒱(K)*, equals ***X.***

A SIMILAR CONSTRUCTION









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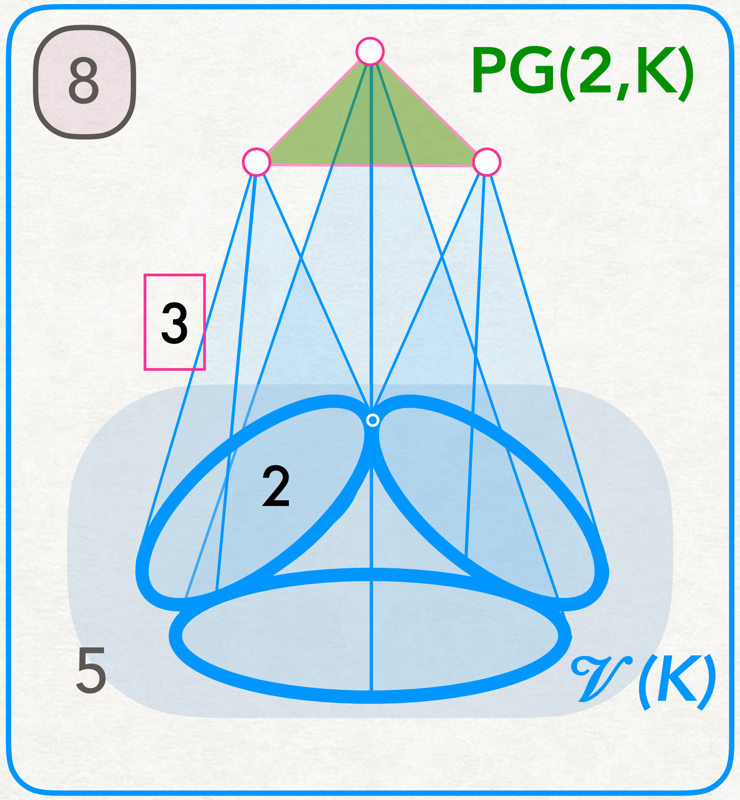
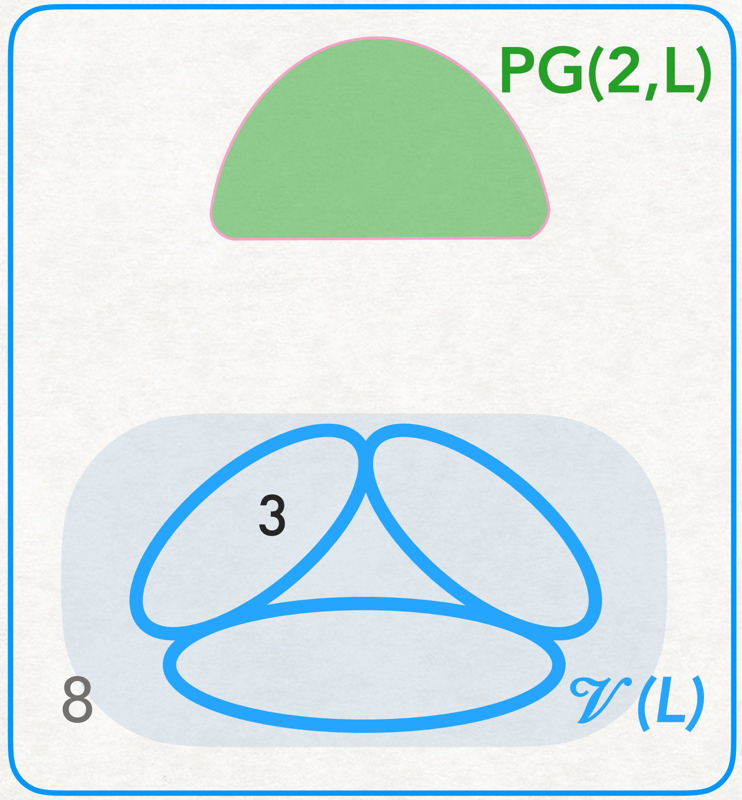
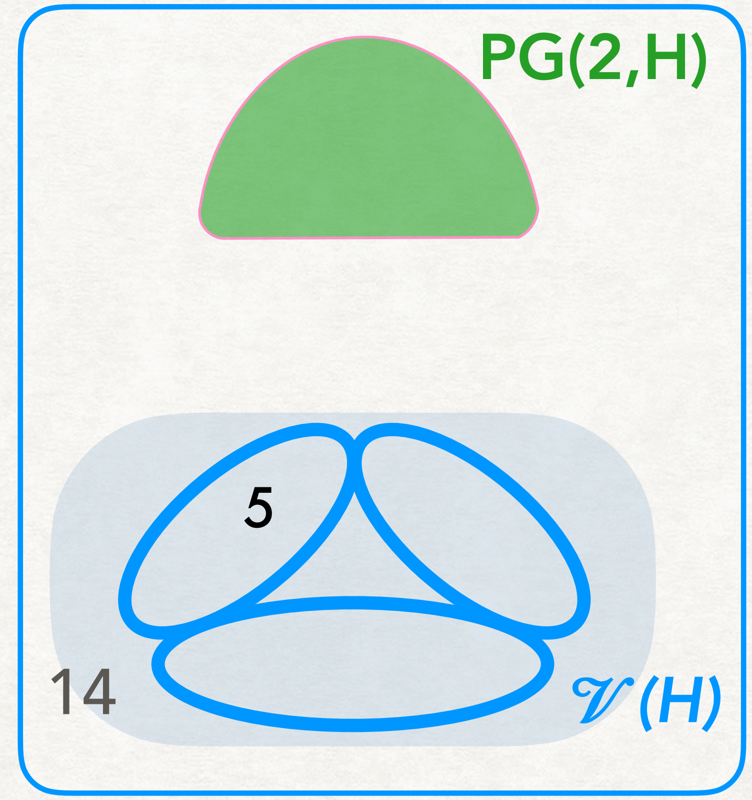
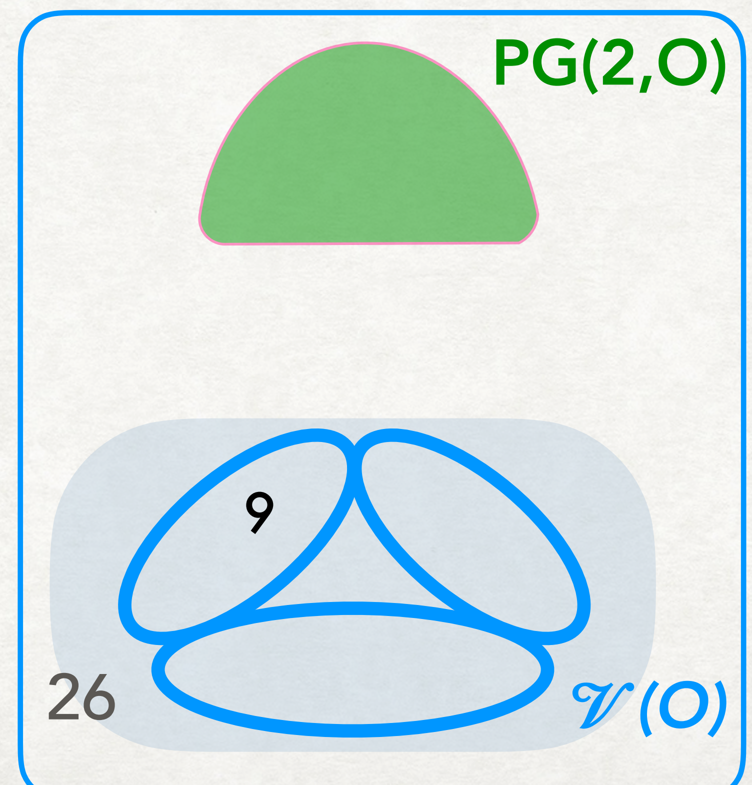






dim quadric	total dim
2	5
3	8
5	14
9	26
$d=2^a+1$	3d-1



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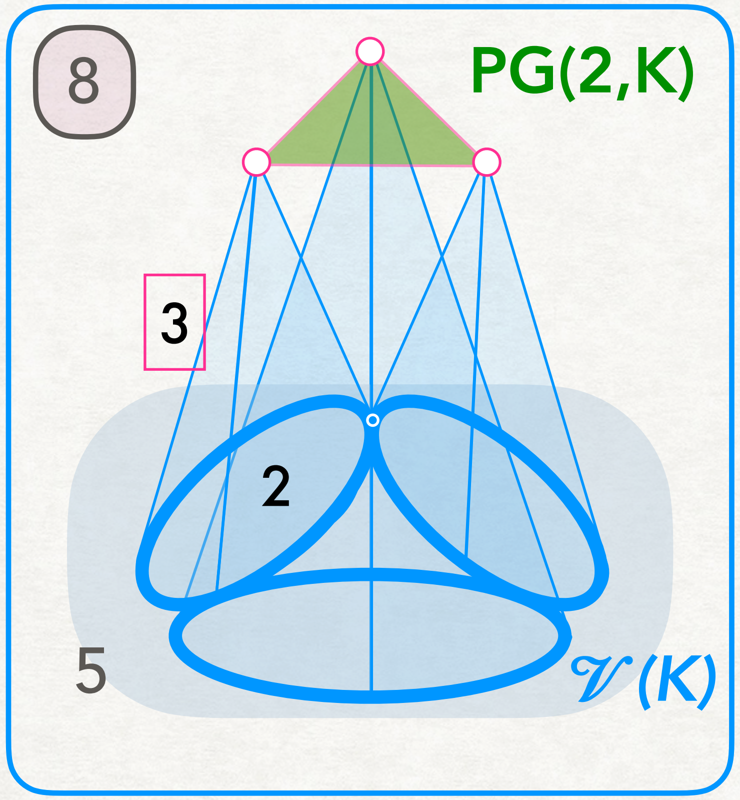

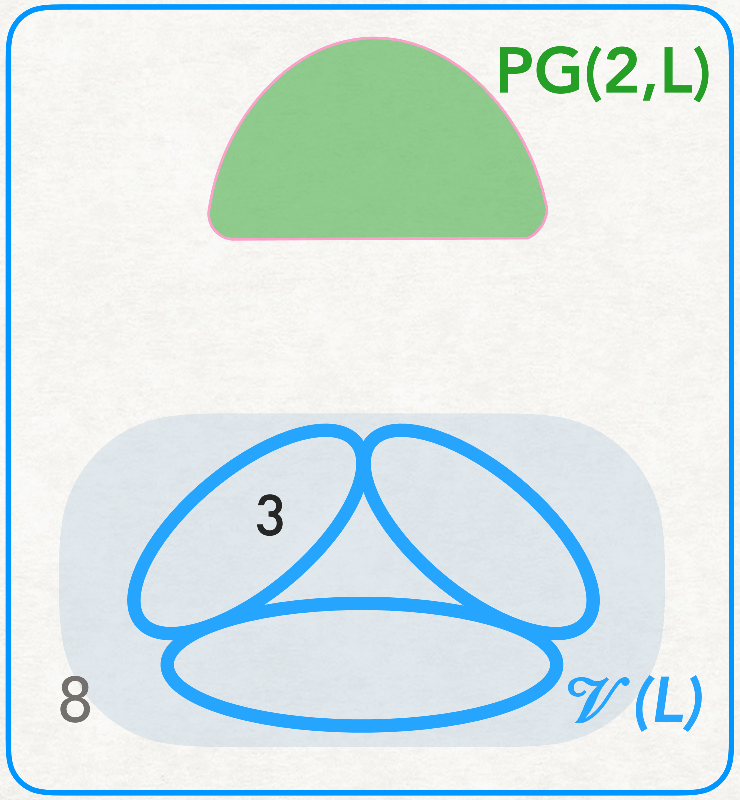

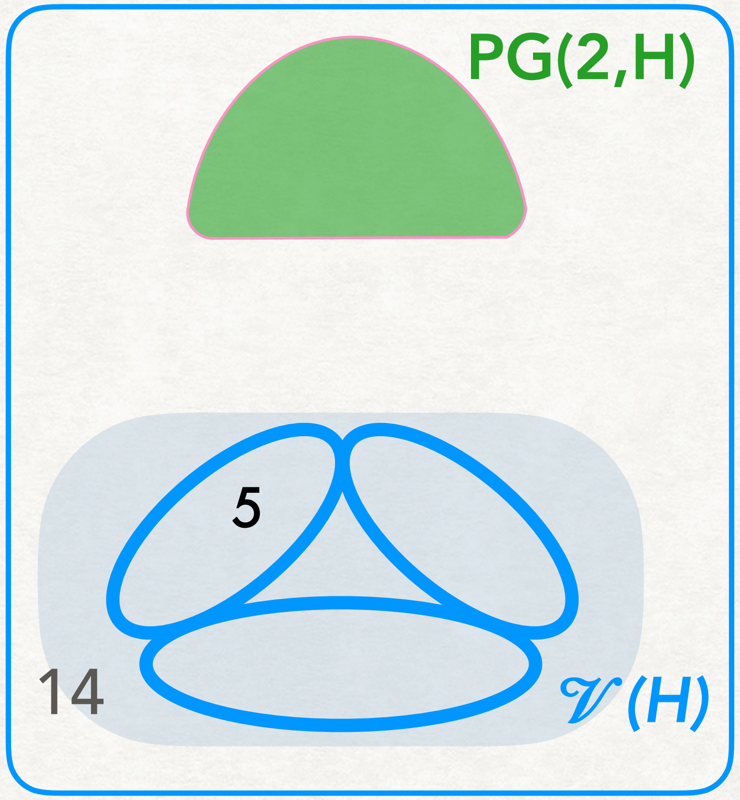

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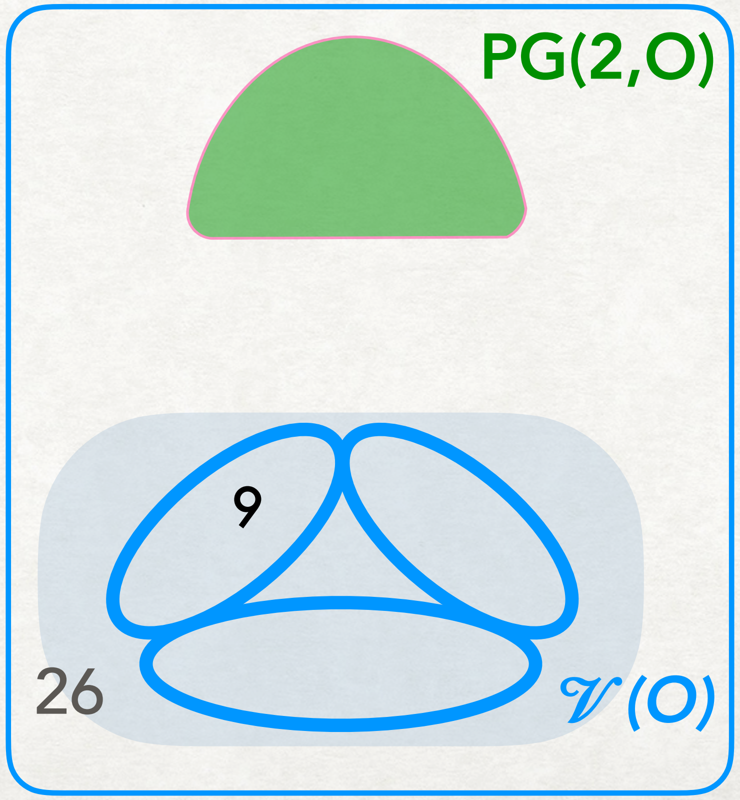

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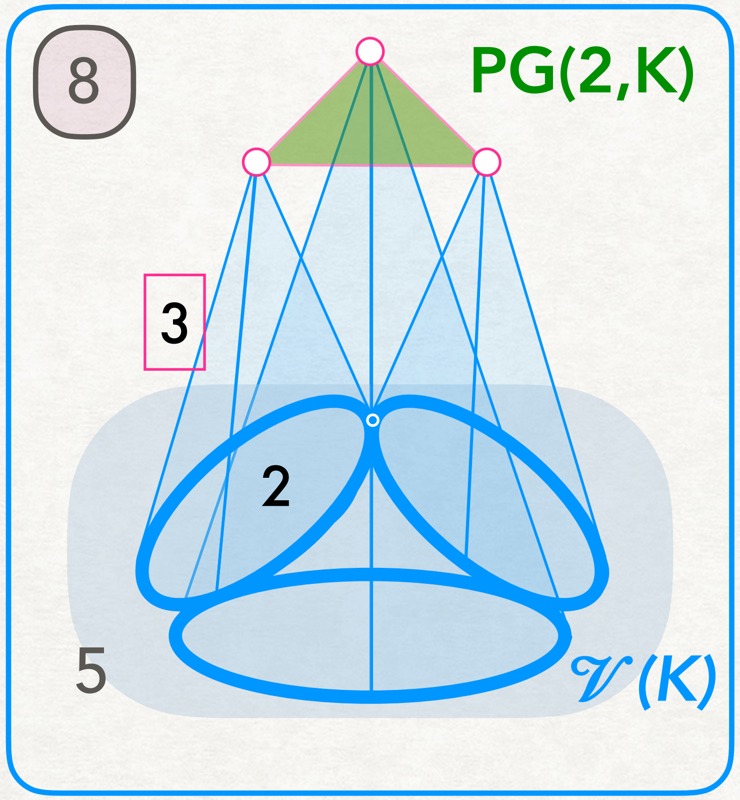

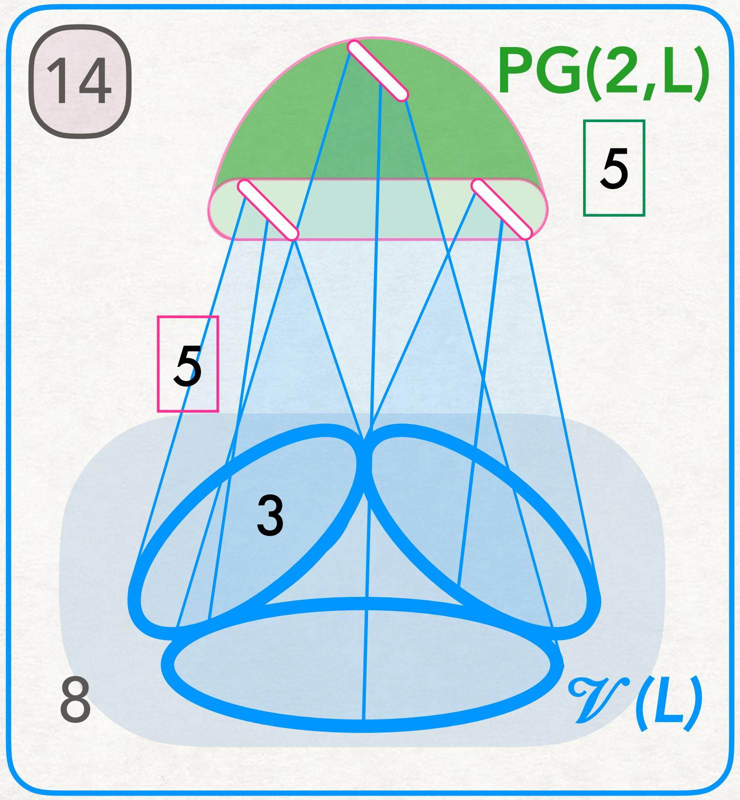

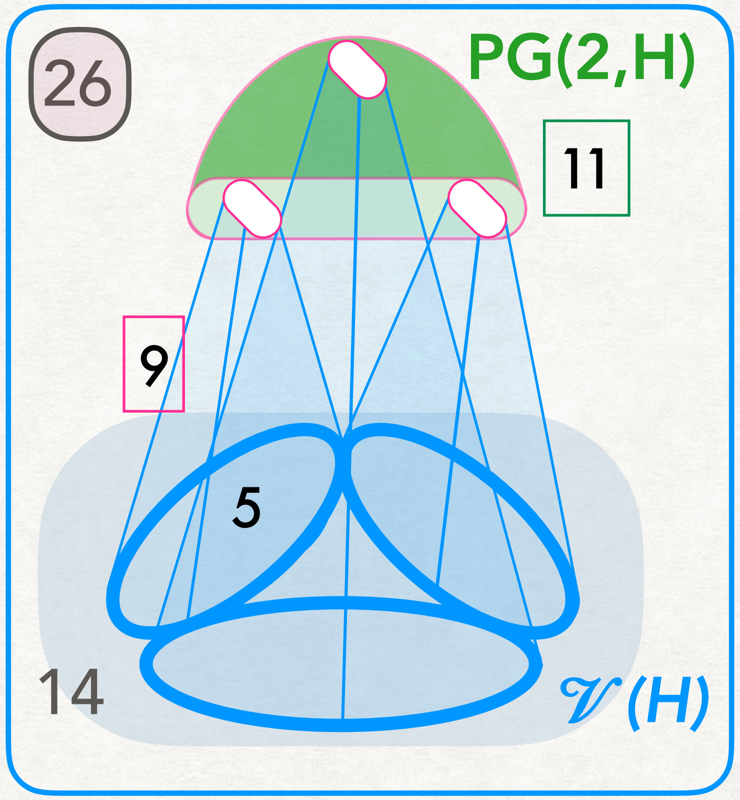

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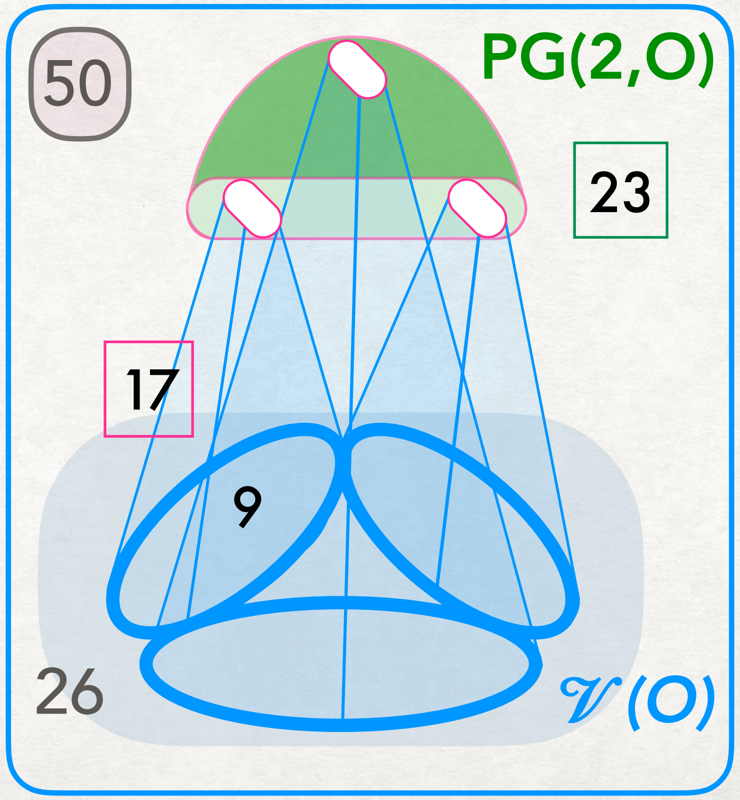

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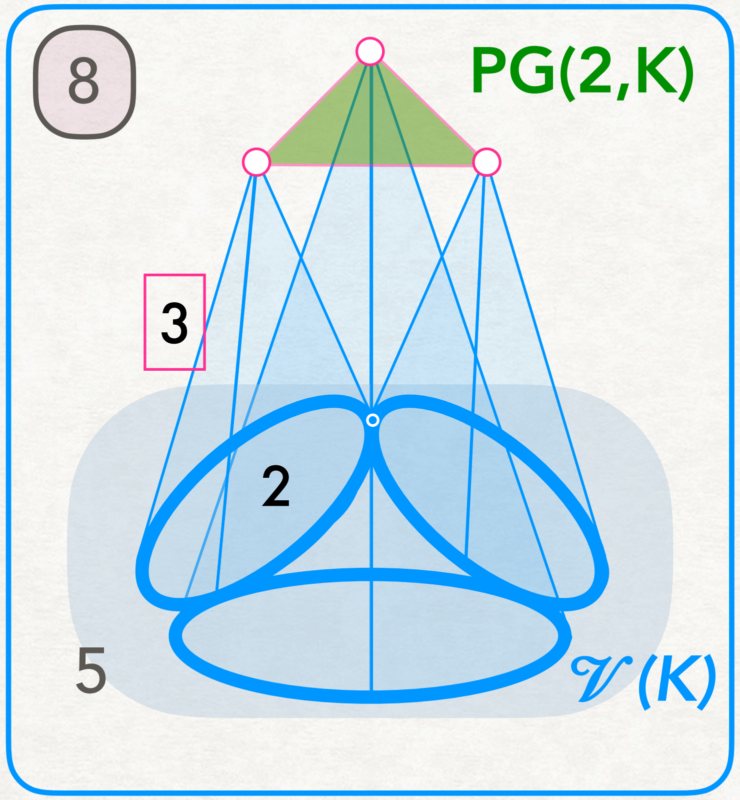
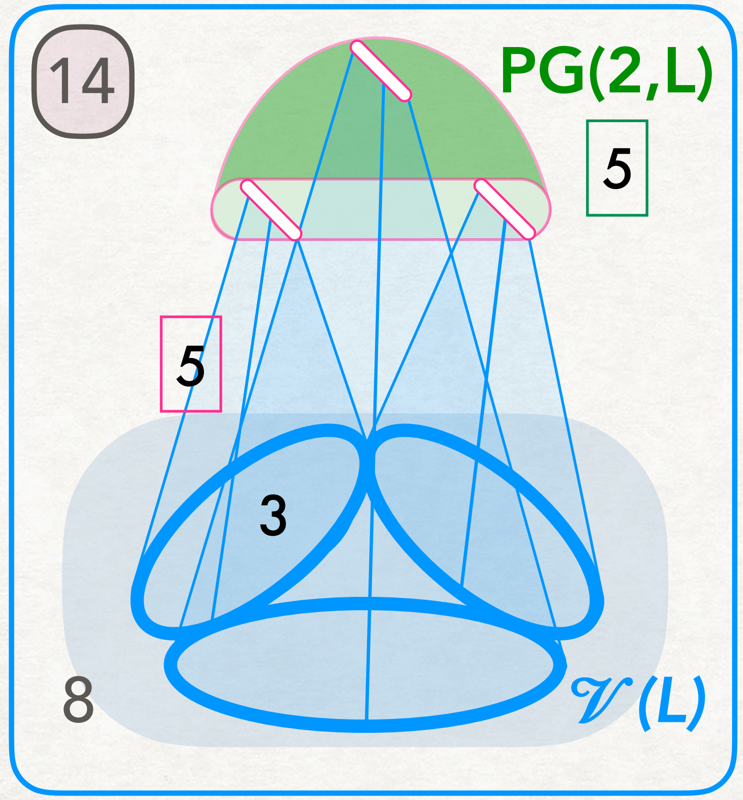
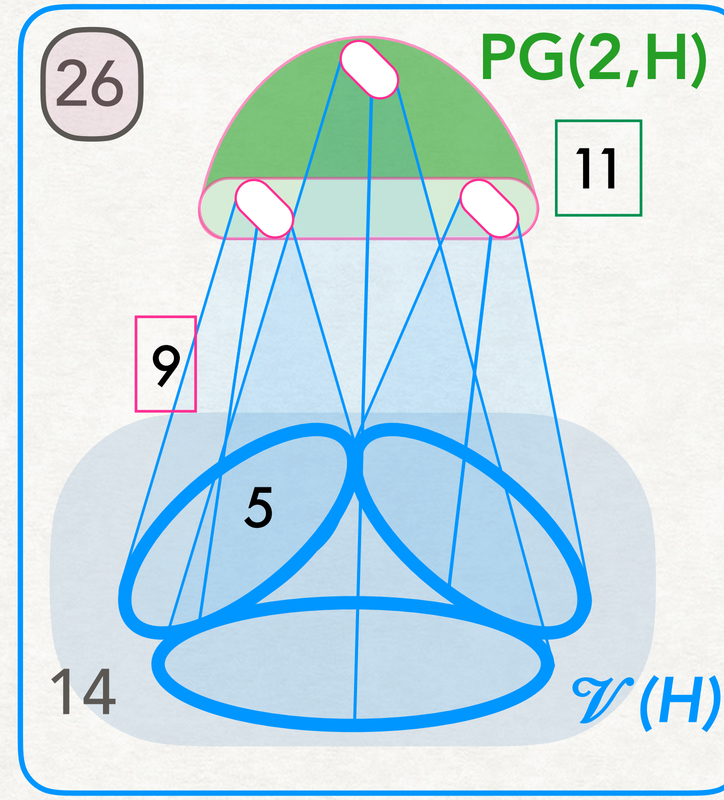
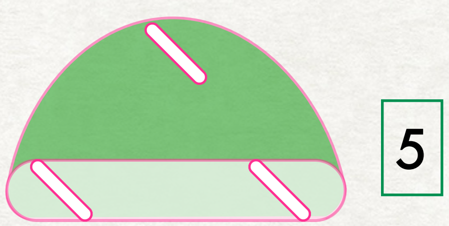






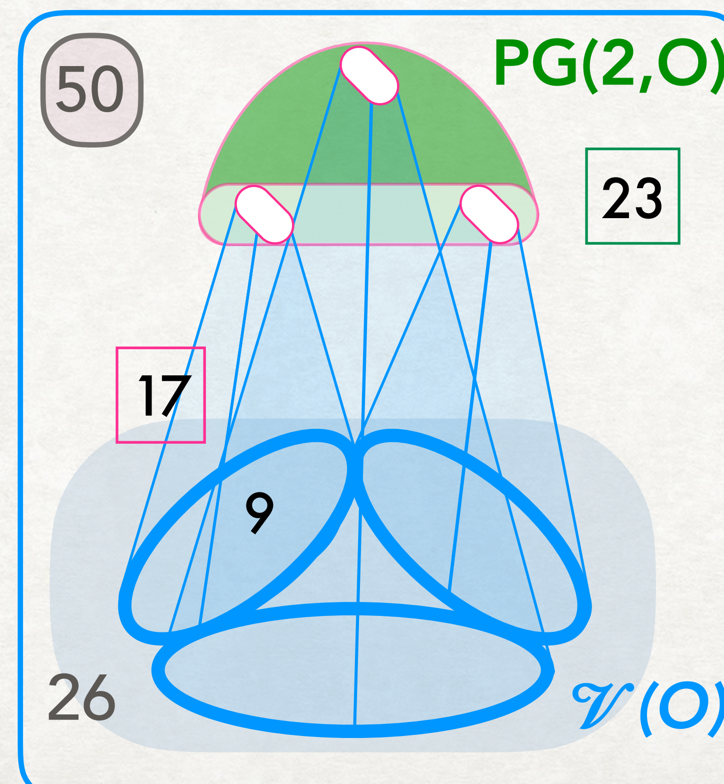


Why isomorphic to PG(2,L)?

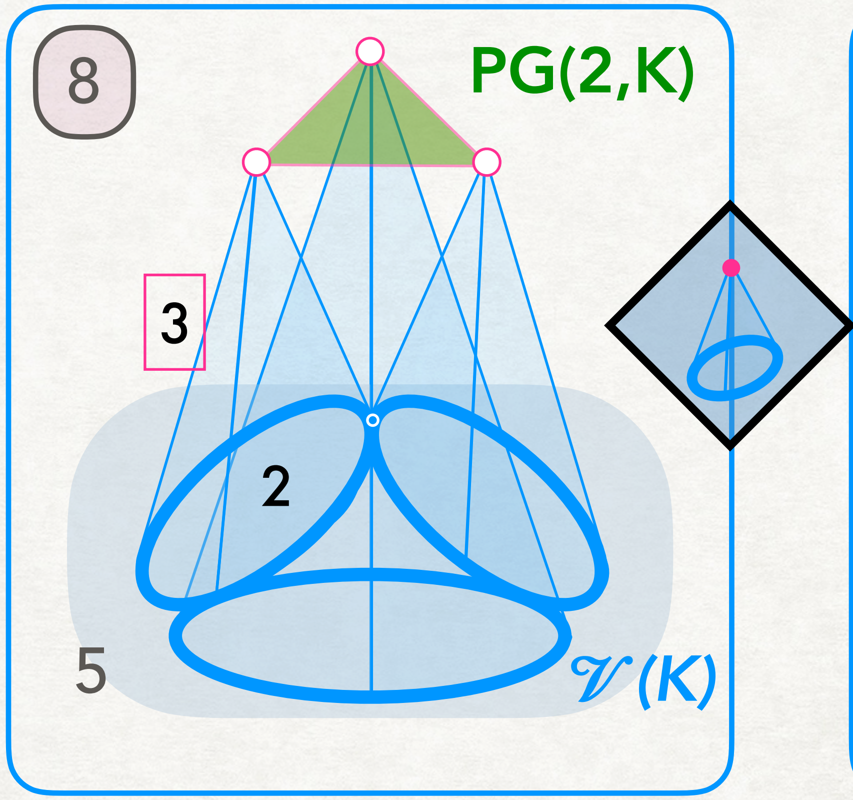
PG(2,L) — V(3,L) — V(6,K) — PG(5,K)

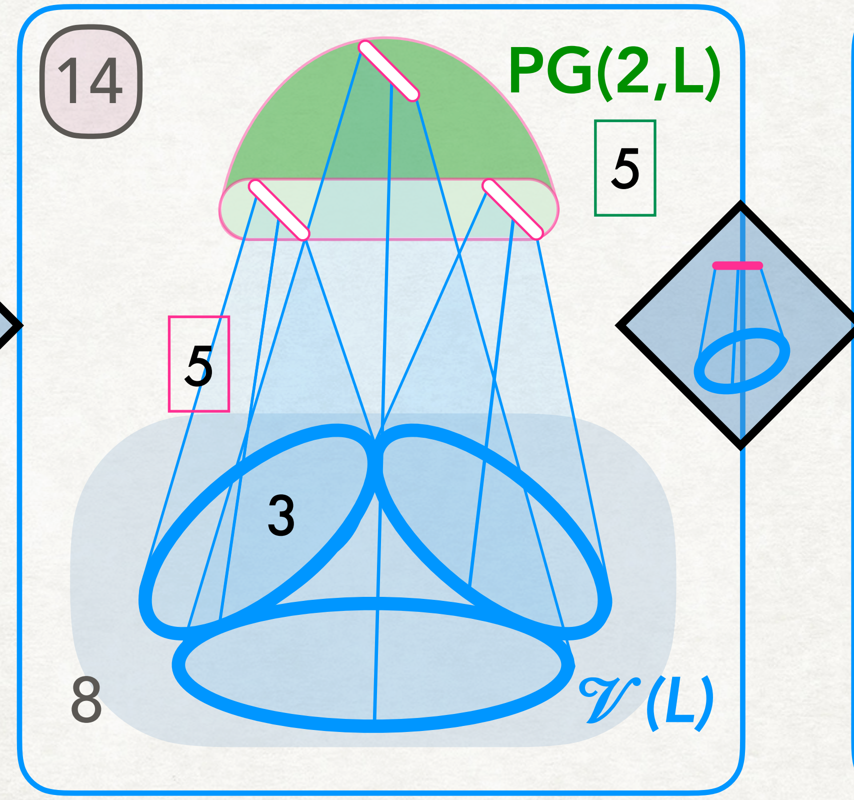
point — vector line — vector plane — line

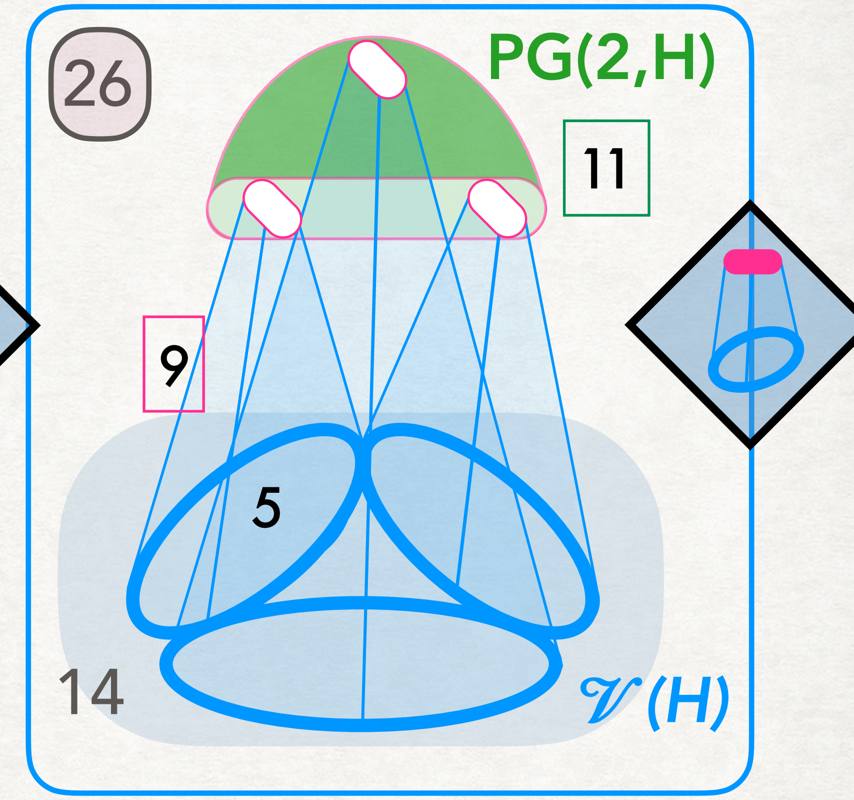
line — regular line-spread in 3-space



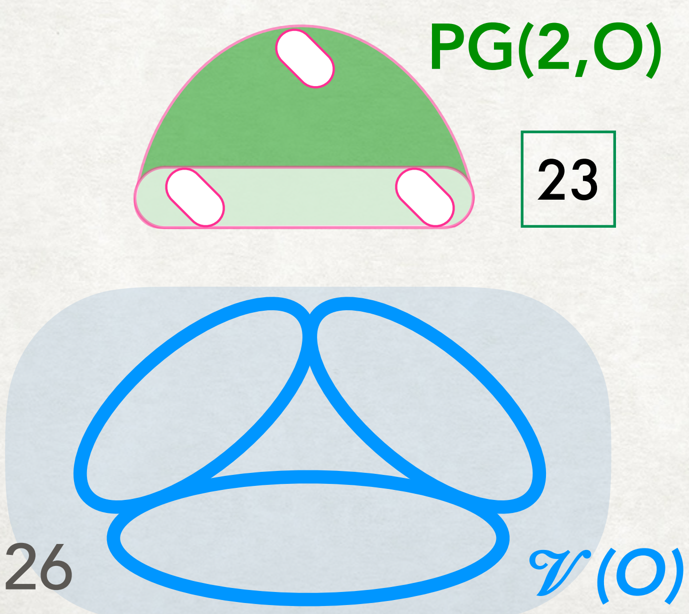
A SIMILAR CONSTRUCTION





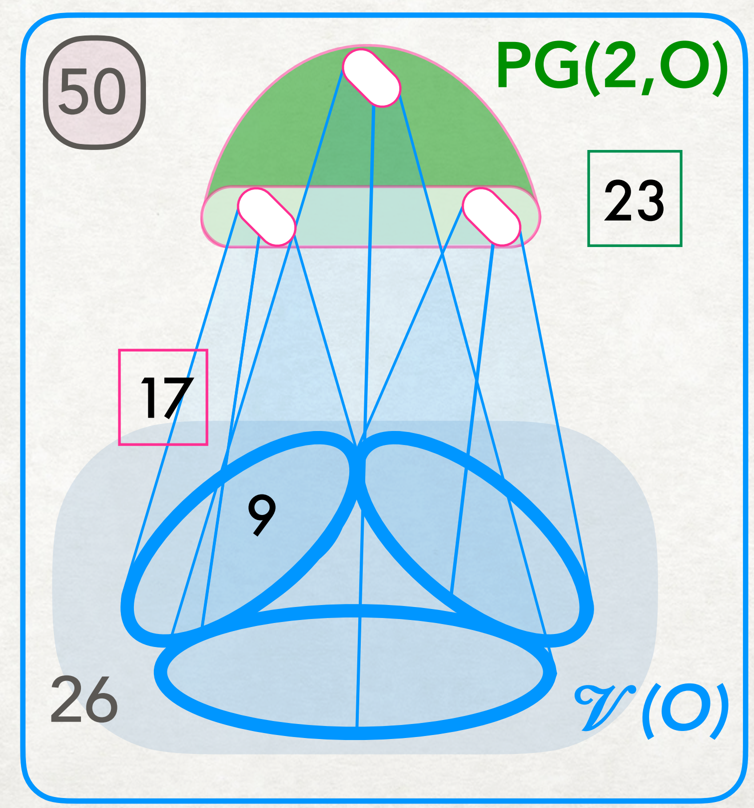


What is wrong with the last one?



The regular 7-spread defines a Desarguesian plane.

𝒱(O) is a representation of a non-Desarguesian plane.



MM SETS WITH (D,V)-TUBES: RESULTS

Case 2: the vertex is higher dimensional ($v > 0$)

For any field K, let (X, Ξ) be a singular MM-set with (d,v)-tubes.

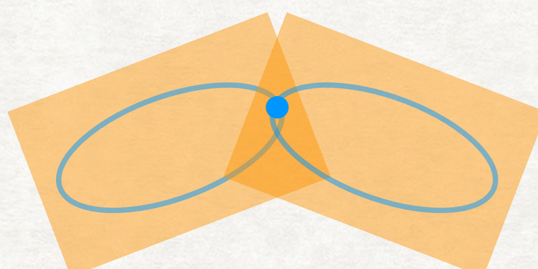
MM SETS WITH (D,V)-TUBES: RESULTS

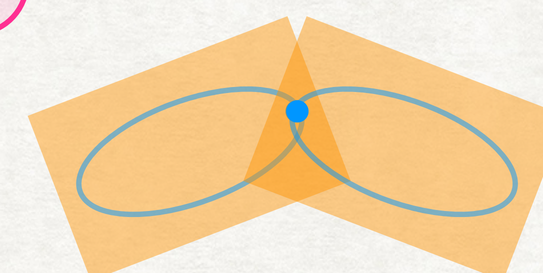
Case 2: the vertex is higher dimensional (v > 0)

For any field K, let (X, Ξ) be a singular MM-set with (d,v)-tubes.

We need to change MM2′

MM2′ ➜ MM2*





MM2′	MM2*
two [d′]s of Ξ intersect in **points** of X∪Y but never in Y only	two [d′]s of Ξ intersect in **points** of X∪Y and always contain a point of X

MM SETS WITH (D,V)-TUBES: RESULTS

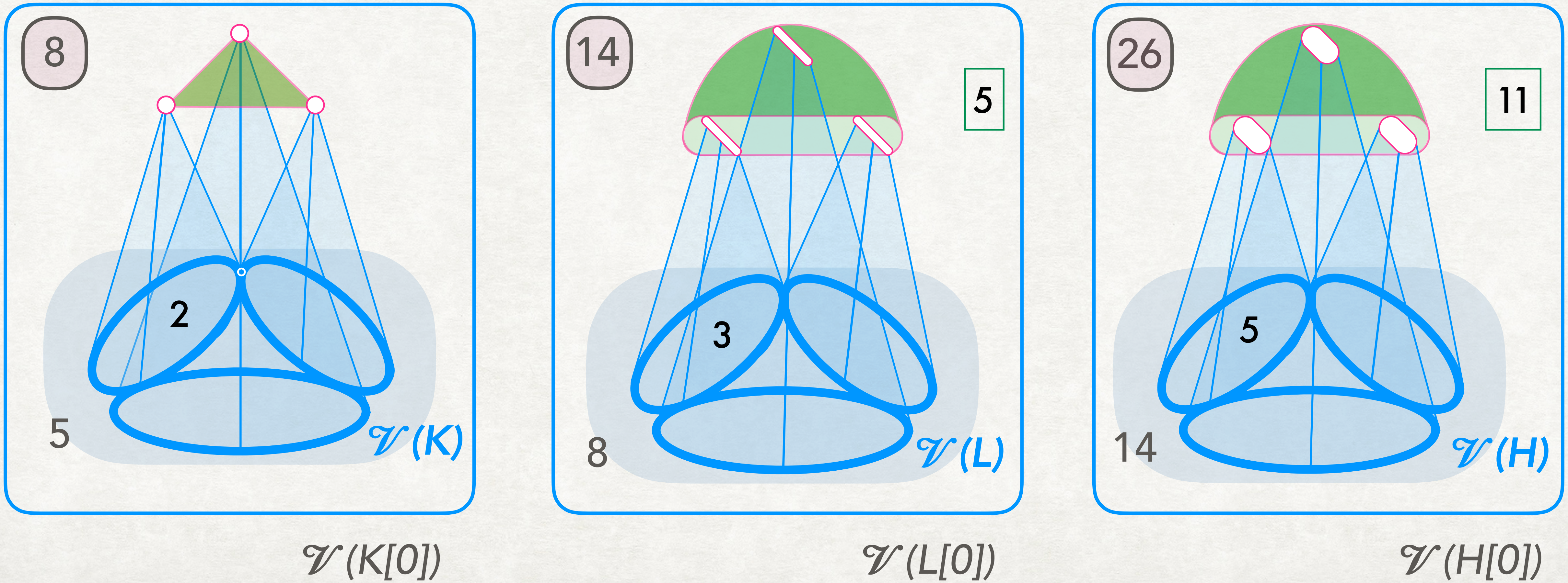
Case 2: the vertex is higher dimensional (v > 0)

For any field K, let (**X**, **Ξ**) be a singular MM-set with (d,v)-tubes.

With MM1, MM2* and MM3 we obtain:

ADS, Van Maldeghem (2017)

If nontrivial, (**X**, **Ξ**) is <u>projectively unique</u> and <u>isomorphic</u> to a Hjelmslevian projective plane:



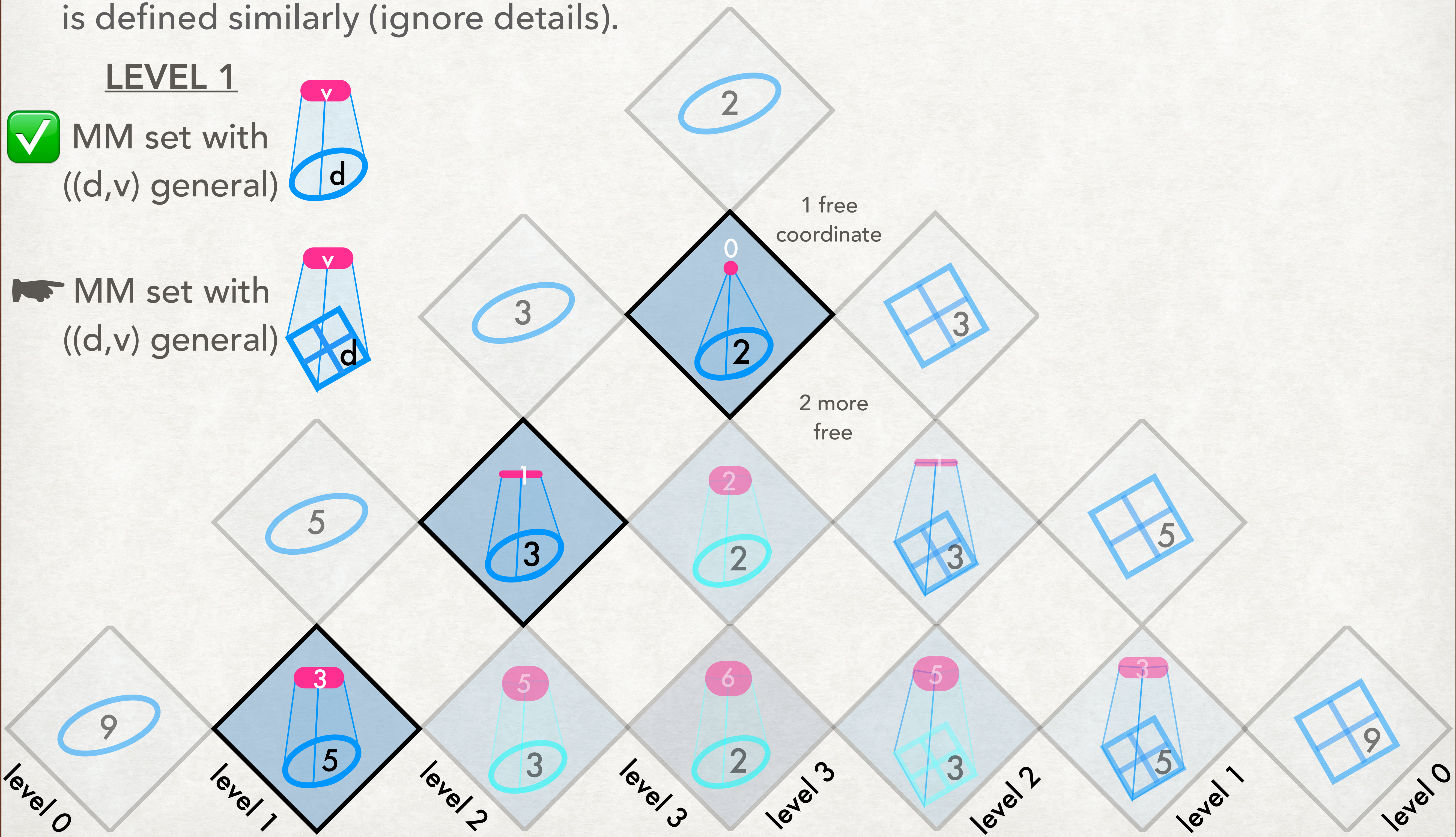
CD ALGEBRA ➜ VERONESE VAR

Let A be a Cayley Dickson algebra with dim(A/K) = d. The Veronese variety $\mathscr{V}(A)$ is defined similarly (ignore details).

LEVEL 1

✅ MM set with ((d,v) general)

☛ MM set with ((d,v) general)



CD ALGEBRA → VERONESE VAR

Let A be a Cayley Dickson algebra with dim(A/K) = d. The Veronese variety $\mathscr{V}(A)$ is defined similarly (ignore details).

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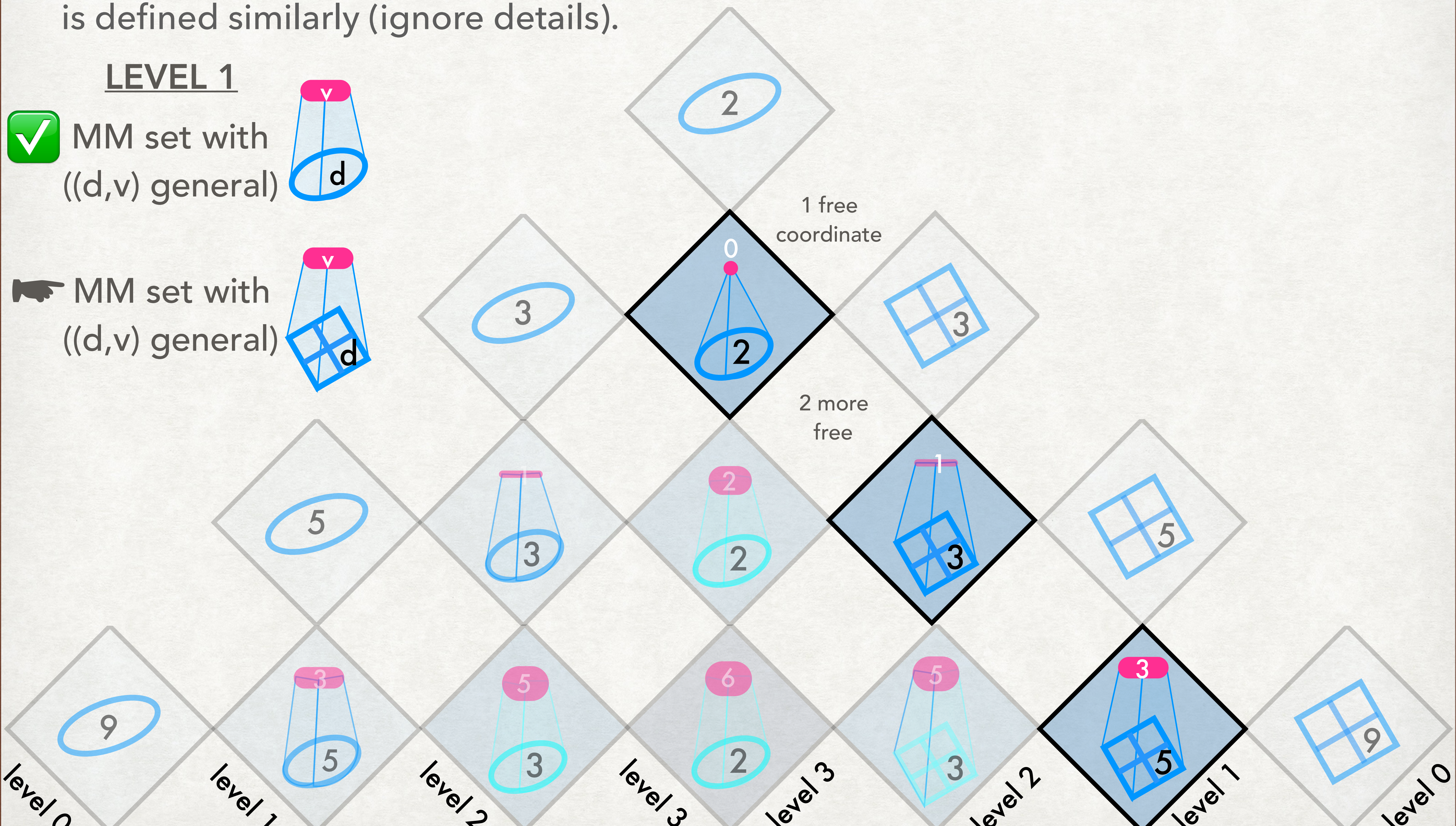
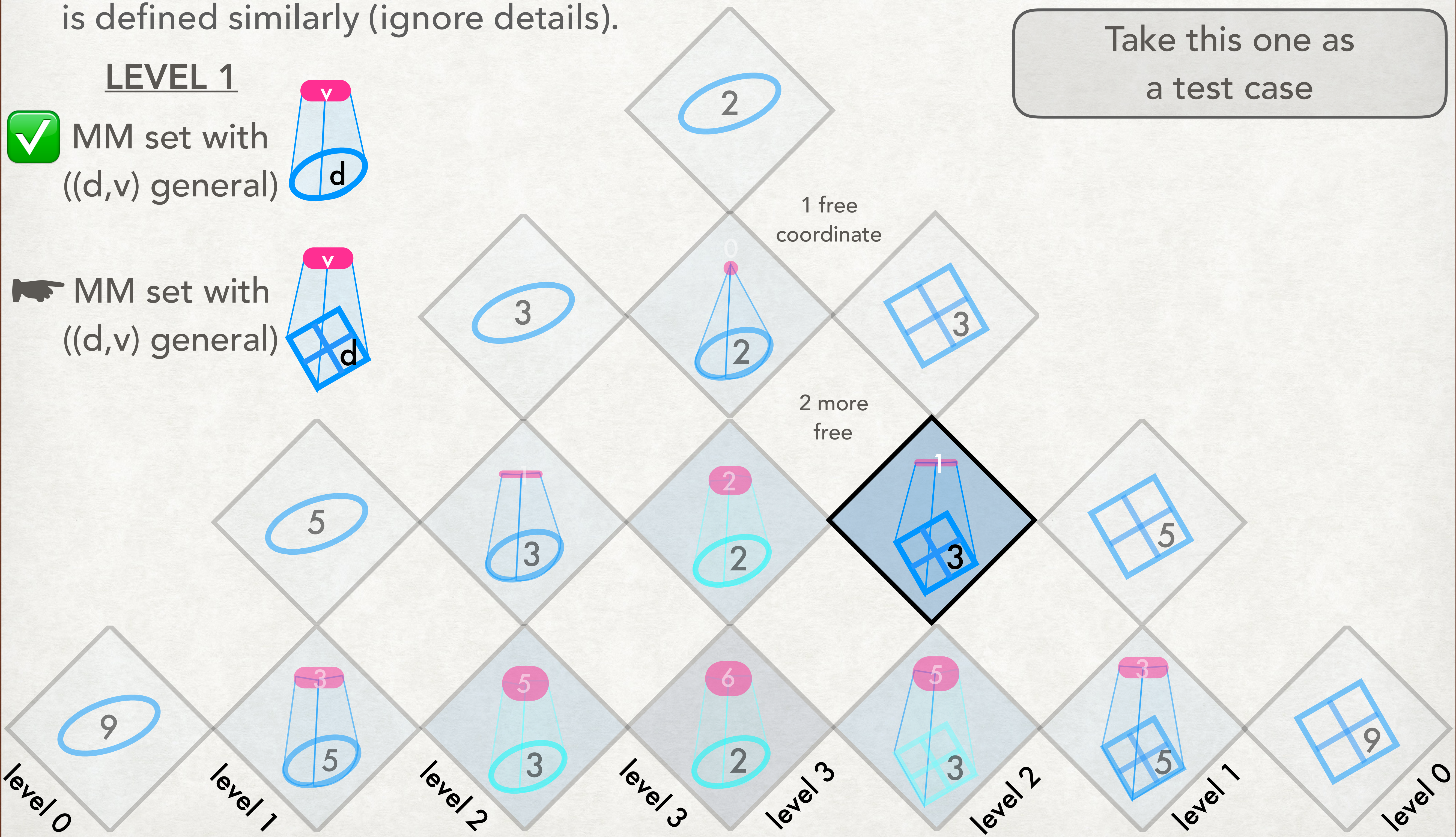
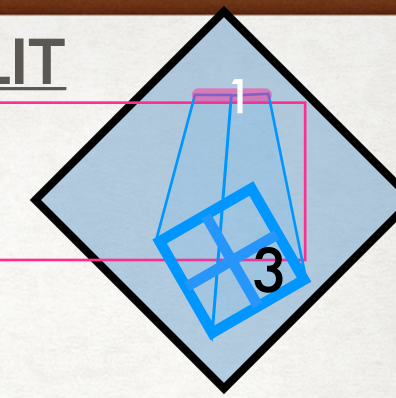


CD ALGEBRA ➔ VERONESE VAR

Let A be a Cayley Dickson algebra with dim(A/K) = d. The Veronese variety $\mathscr{V}(A)$ is defined similarly (ignore details).

Take this one as a test case





MM SETS WITH (3,1)-SYMPS

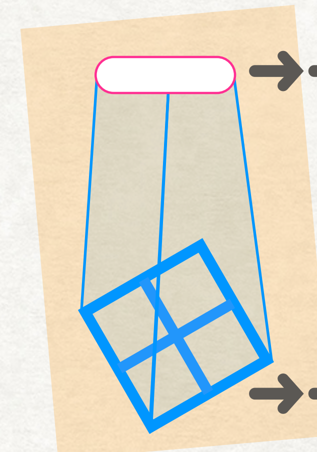
Axiomatic description

X points spanning PG(14,K)

Y vertices

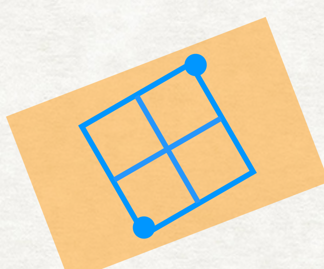
Ξ 5-spaces ξ in PG(14,K) s.th. ξ ∩ X is:

(3,1)-symp



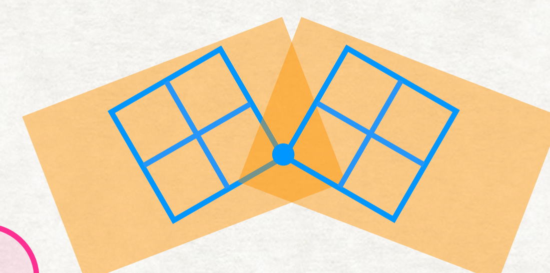
1-dim vertex (excluded)

$Q^{max}(3,K)$



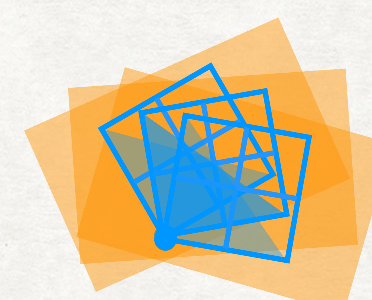
MM1

each two points of X belong to a [5] of Ξ



MM2

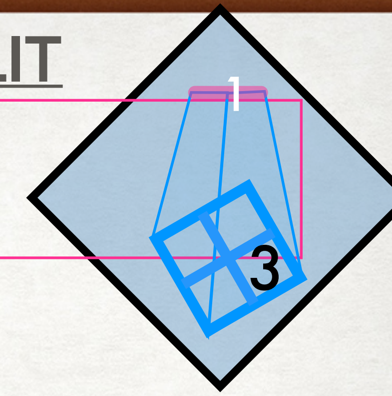
two [5]s of Ξ intersect in points of X∪Y but never in Y only



MM3

the tangent space of a point of X is contained in a [2(5-1)]

MM SETS WITH (3,1)-SYMPS



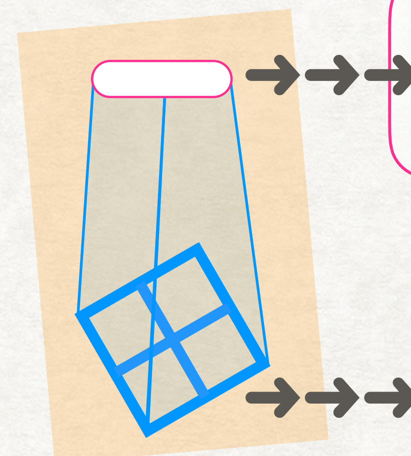
Axiomatic description

(3,1)-symp

X — points spanning PG(14,K)

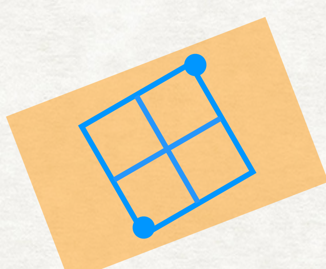
Y — vertices

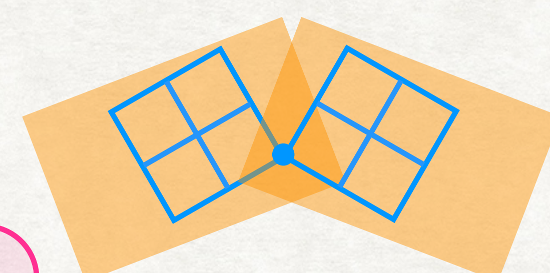
Ξ — 5-spaces ξ in PG(14,K) s.th. ξ ∩ X is:

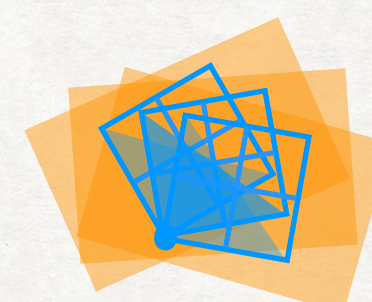


1-dim vertex (excluded)

$Q^{max}(3,K)$







MM1: each two points of X belong to a [5] of Ξ

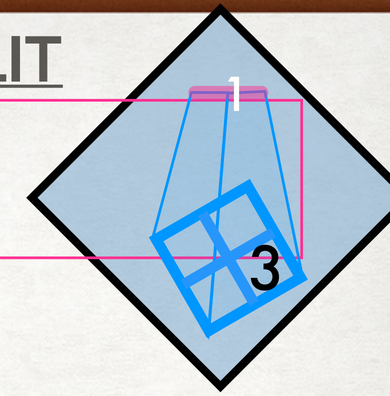
MM2: two [5]s of Ξ intersect in points of X∪Y but never in Y only

MM3: the tangent space of a point of X is contained in a [2(5-1)]

Surprise: The Veronese variety $\mathscr{V}$(L'[0]) does not satisfy axioms MM1 and MM2!

LEVEL 1 SPLIT

MM SETS WITH (3,1)-SYMPS



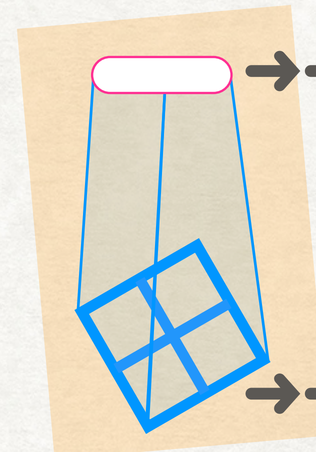
Axiomatic description

(3,1)-symp

X — points spanning PG(14,K)

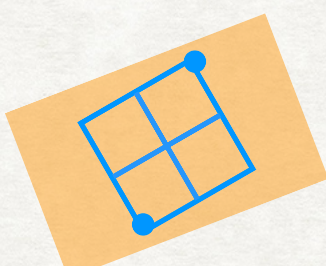
Y — vertices

Ξ — 5-spaces ξ in PG(14,K) s.th. ξ ∩ X is:



→→→ 1-dim vertex (excluded)

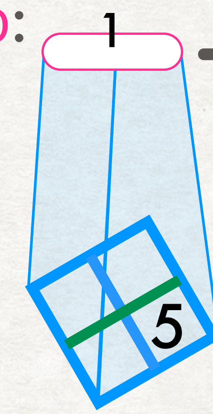
→→→ $Q^{max}(3,K)$



MM1

each two points of X belong to a [5] of Ξ

Yet, each two points not belonging to a [5] of Ξ, belong to a supersymp:

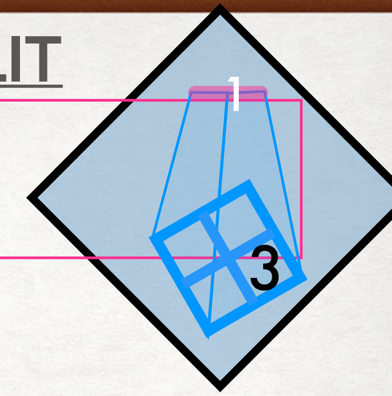


→ 1-dim vertex (excl.)

→ $Q^{max}(5,K)$
1 MSS `missing'

Surprise: The Veronese variety $\mathscr{V}$(L'[0]) does not satisfy axioms MM1 and MM2!

LEVEL 1 SPLIT



MM SETS WITH (3,1)-SYMPS

Axiomatic description

X — points spanning PG(14,K)

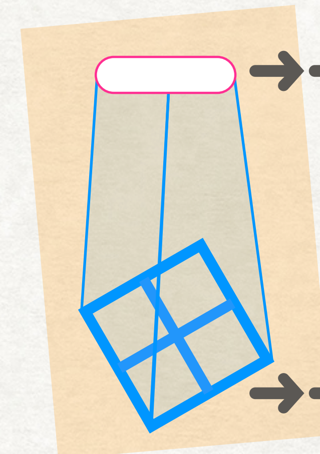
Y — vertices

Ξ —

(3,1)-symp

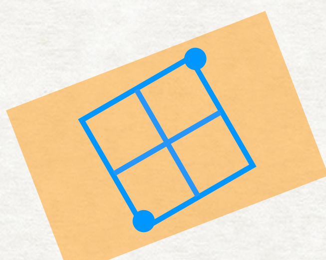
5-spaces ξ in PG(14,K)
s.th. ξ ∩ X is:

7-spaces ξ′ in PG(14,K)
s.th. ξ′ ∩ X is
a supersymp



→→→ 1-dim vertex (excluded)

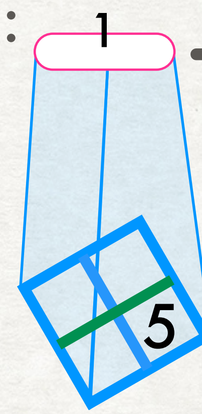
→→→ $Q^{max}(3,K)$



MM1

each two **points** of **X** belong to a [5] of Ξ

Yet, each two **points** not belonging to a [5] of Ξ, belong to a supersymp:

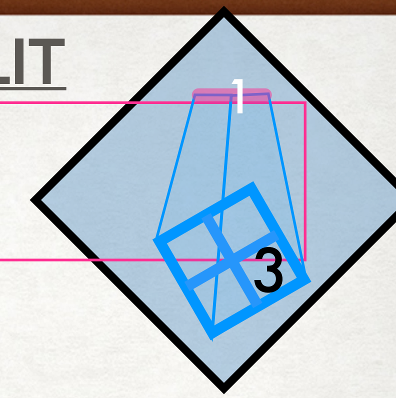


→ 1-dim vertex (excl.)

→ $Q^{max}(5,K)$
1 MSS `missing'

Surprise: The Veronese variety $\mathscr{V}$(L’[0]) does not satisfy axioms MM1 and MM2!

MM SETS WITH (3,1)-SYMPS



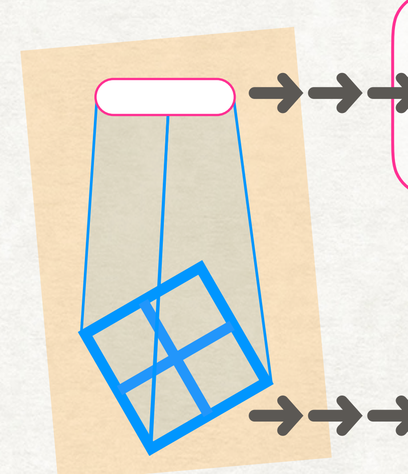
Axiomatic description

(3,1)-symp

X — points spanning PG(14,K)

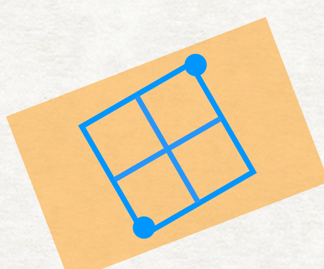
Y — vertices

Ξ — 5-spaces ξ in PG(14,K) s.th. ξ ∩ X is: 7-spaces ξ' in PG(14,K) s.th. ξ' ∩ X is a supersymp



1-dim vertex (excluded)

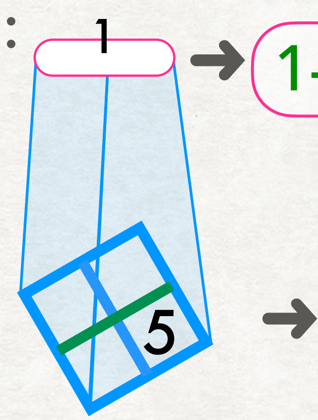
$Q^{max}(3,K)$



MM1

each two **points** of **X** belong to a **member** of **Ξ**

Yet, each two **points** not belonging to a [5] of Ξ, belong to a supersymp:

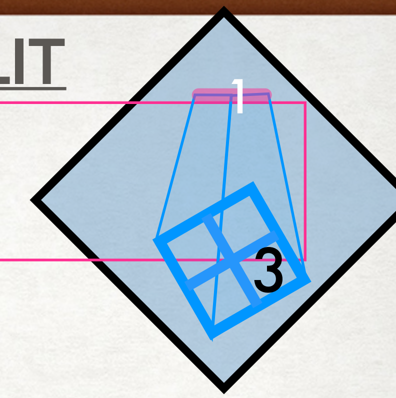


1-dim vertex (excl.)

$Q^{max}(5,K)$
1 MSS `missing'

Surprise: The Veronese variety $\mathscr{V}$(L'[0]) does not satisfy axioms MM1 and MM2!

MM SETS WITH (3,1)-SYMPS



Axiomatic description

X — points spanning PG(14,K)

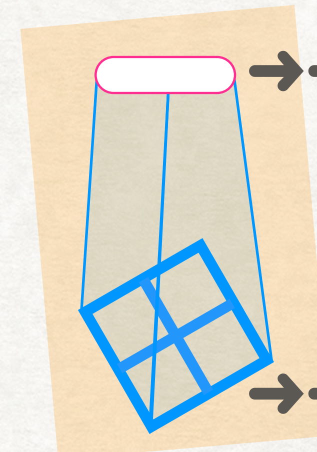
Y — vertices

(3,1)-symp

Ξ

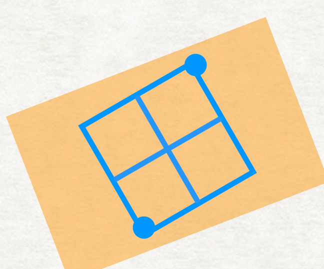
5-spaces ξ in PG(14,K) s.th. ξ ∩ X is:

7-spaces ξ' in PG(14,K) s.th. ξ' ∩ X is a supersymp



→→→ 1-dim vertex (excluded)

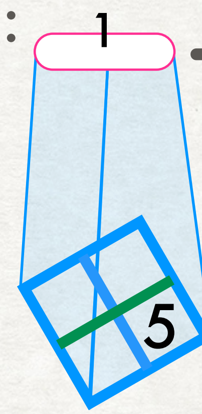
→→→ $Q^{max}(3,K)$



MM1

each two **points** of **X** belong to a **member** of **Ξ**

Yet, each two **points** not belonging to a **[5]** of **Ξ**, belong to a supersymp:

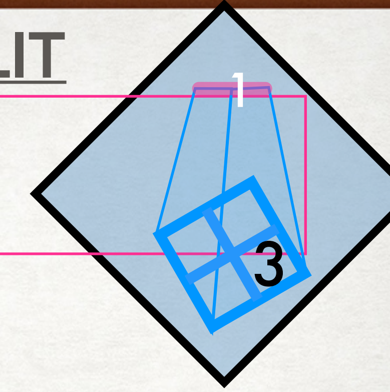


→ 1-dim vertex (excl.)

Dually, there are also superpoints.

→ $Q^{max}(5,K)$ 1 MSS `missing'

Surprise: The Veronese variety $\mathscr{V}$(L'[0]) does not satisfy axioms MM1 and MM2!



MM SETS WITH (3,1)-SYMPS

Axiomatic description

X — points spanning PG(14,K)

Y — vertices

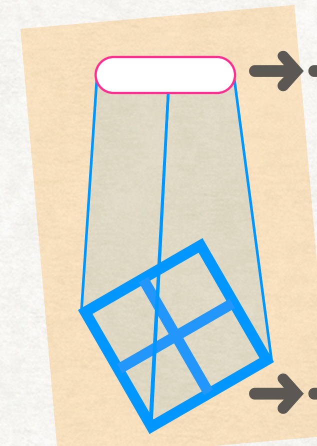
super-points

Ξ

(3,1)-symp

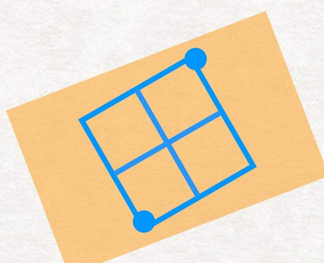
5-spaces ξ in PG(14,K) s.th. ξ ∩ X is:

7-spaces ξ' in PG(14,K) s.th. ξ' ∩ X is a supersymp



→ 1-dim vertex (excluded)

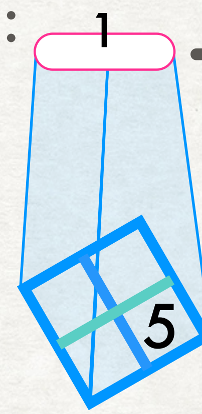
→ $Q^{max}(3,K)$



MM1

each two **points** of **X** belong to a **member** of **Ξ**

Yet, each two **points** not belonging to a **[5]** of **Ξ**, belong to a supersymp:

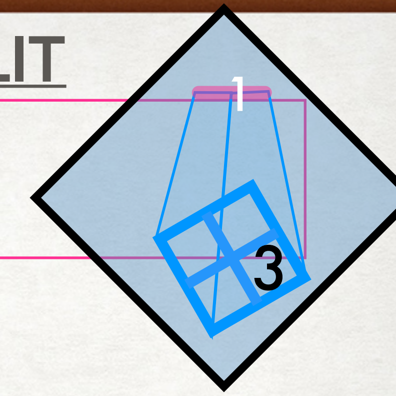


→ 1-dim vertex (excl.)

→ $Q^{max}(5,K)$ 1 MSS `missing'

Dually, there are also superpoints.

Surprise: The Veronese variety $\mathscr{V}$(L'[0]) does not satisfy axioms MM1 and MM2!



MM SETS WITH (3,1)-SYMPS

Axiomatic description

X — points spanning PG(14,K)

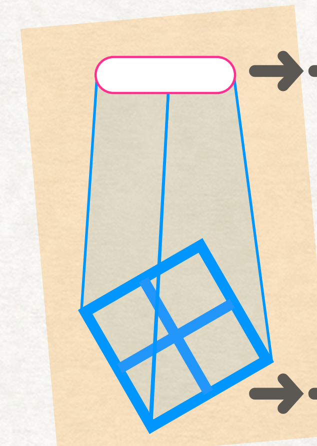
Y — vertices

super-points

Ξ — 5-spaces ξ in PG(14,K) s.th. ξ ∩ X is:

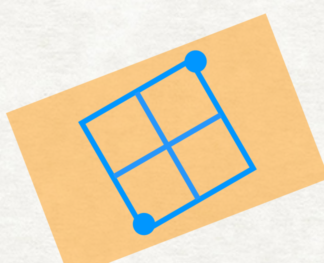
7-spaces ξ′ in PG(14,K) s.th. ξ′ ∩ X is a supersymp

(3,1)-symp



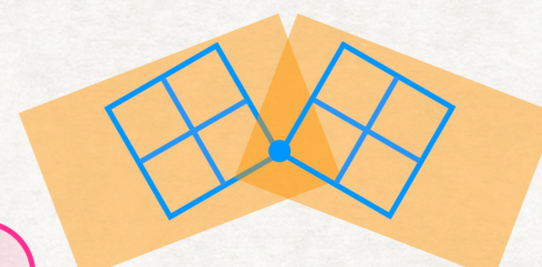
1-dim vertex (excluded)

hyp. quadric in PG(3,K)



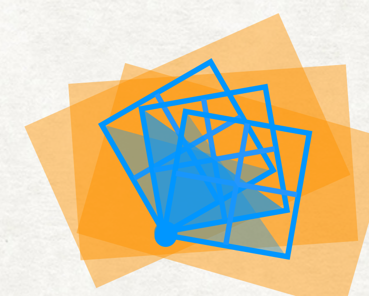
MM1′

each two **points** of **X** belong to a **member** of **Ξ**



MM2

two [5]s of Ξ intersect in **points** of **X**∪**Y** **but** never in Y only

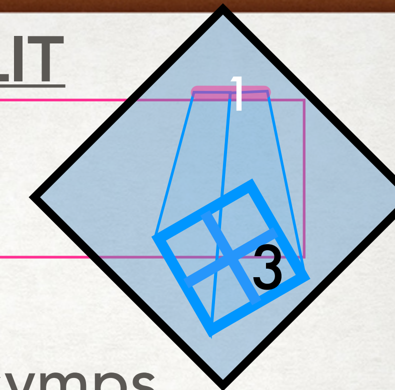


MM3

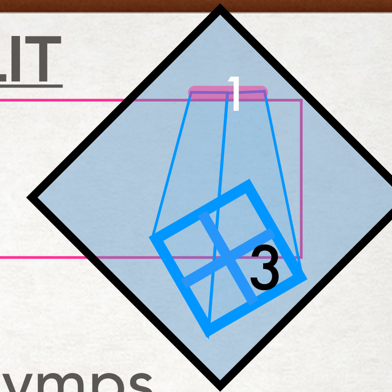
the tangent space of a **point** of **X** is contained in a [2(5-1)]

Together with the **superpoints and -symps**, axioms MM1, MM2 and MM3 are satisfied.

MM SETS WITH (3,1)-SYMPS: RESULT



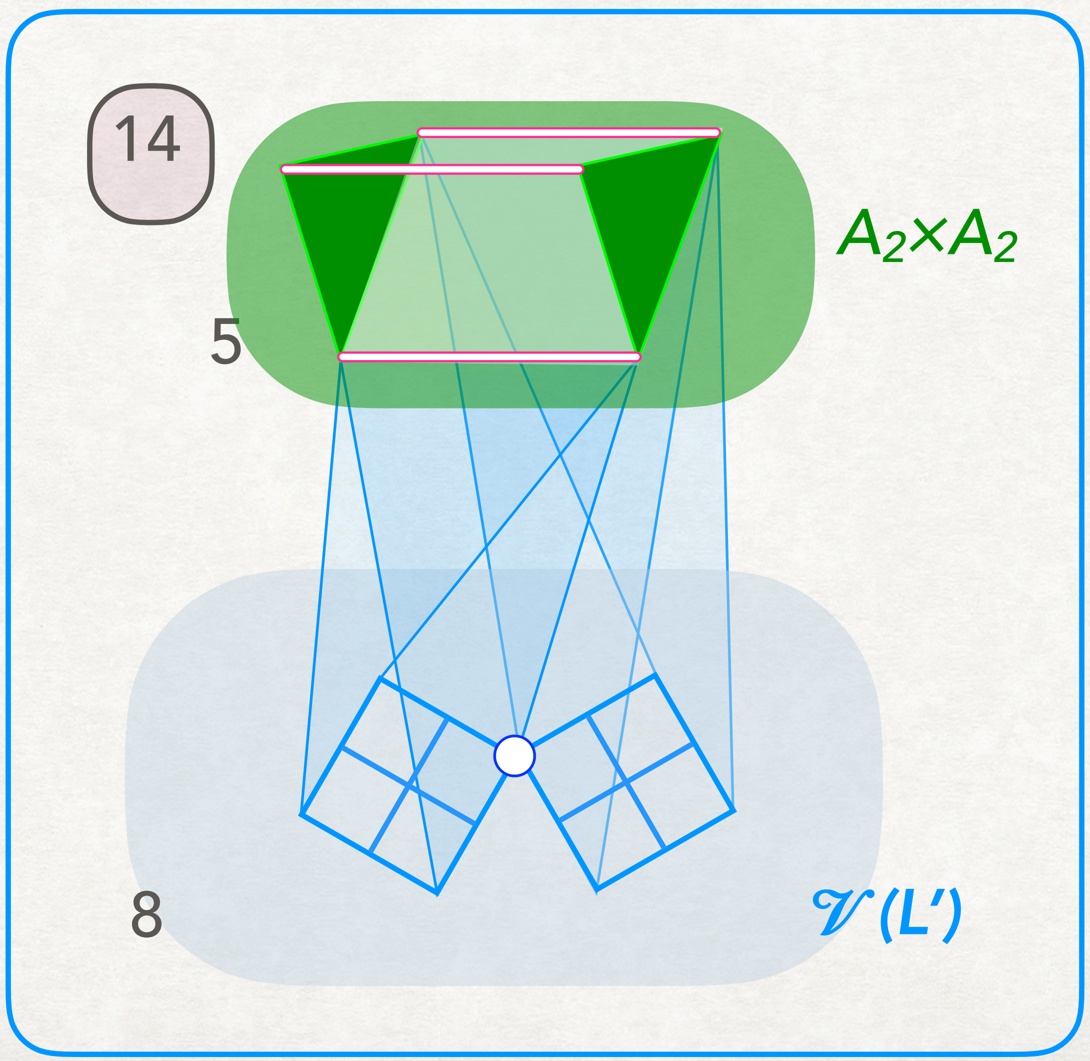
For any field K, let (**X**, **Ξ**) be a singular MM-set with (3,1)-symps and supersymps.



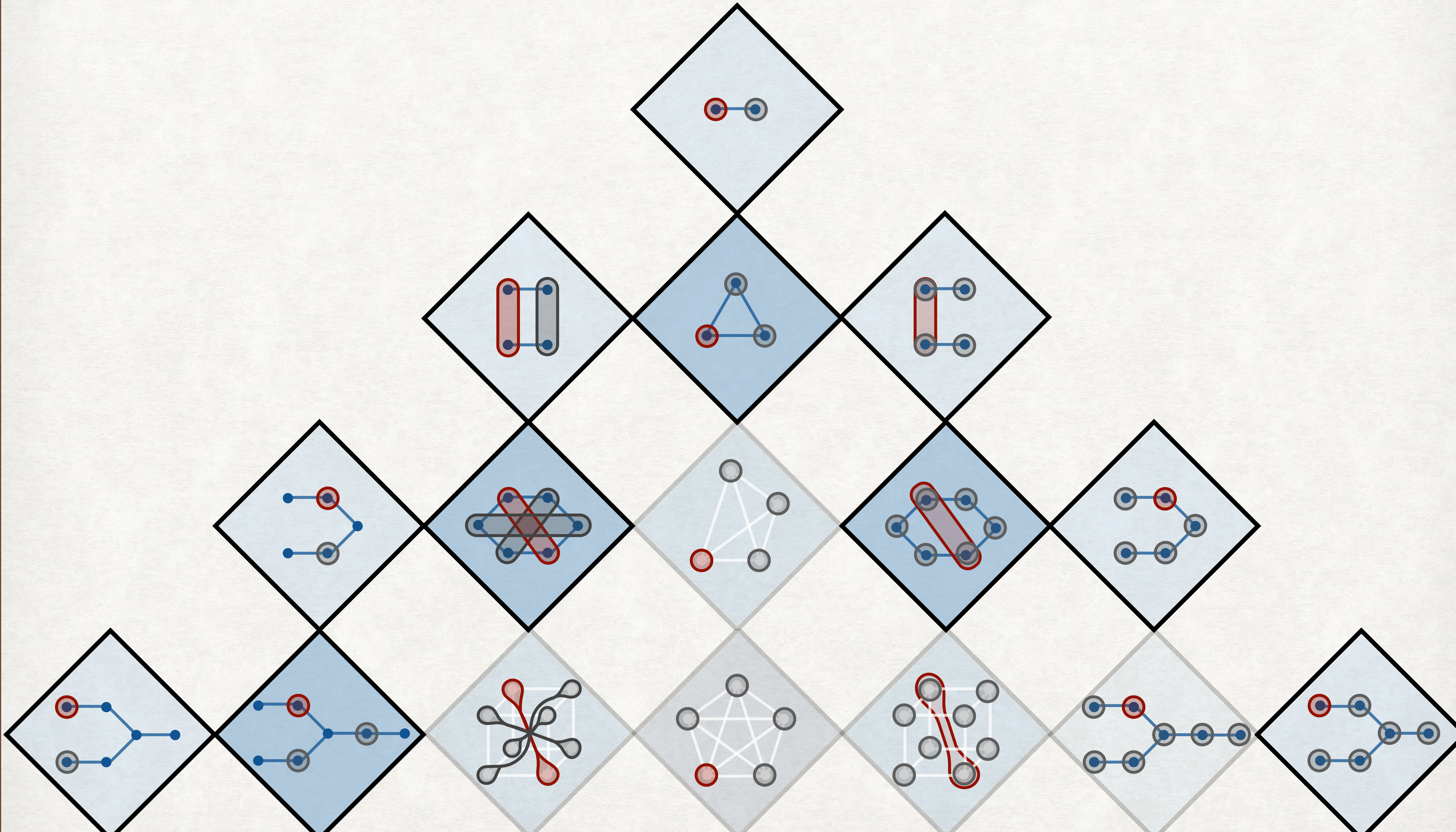
MM SETS WITH (3,1)-SYMPS: RESULT

For any field K, let (**X**, **Ξ**) be a singular MM-set with (3,1)-symps and supersymps.

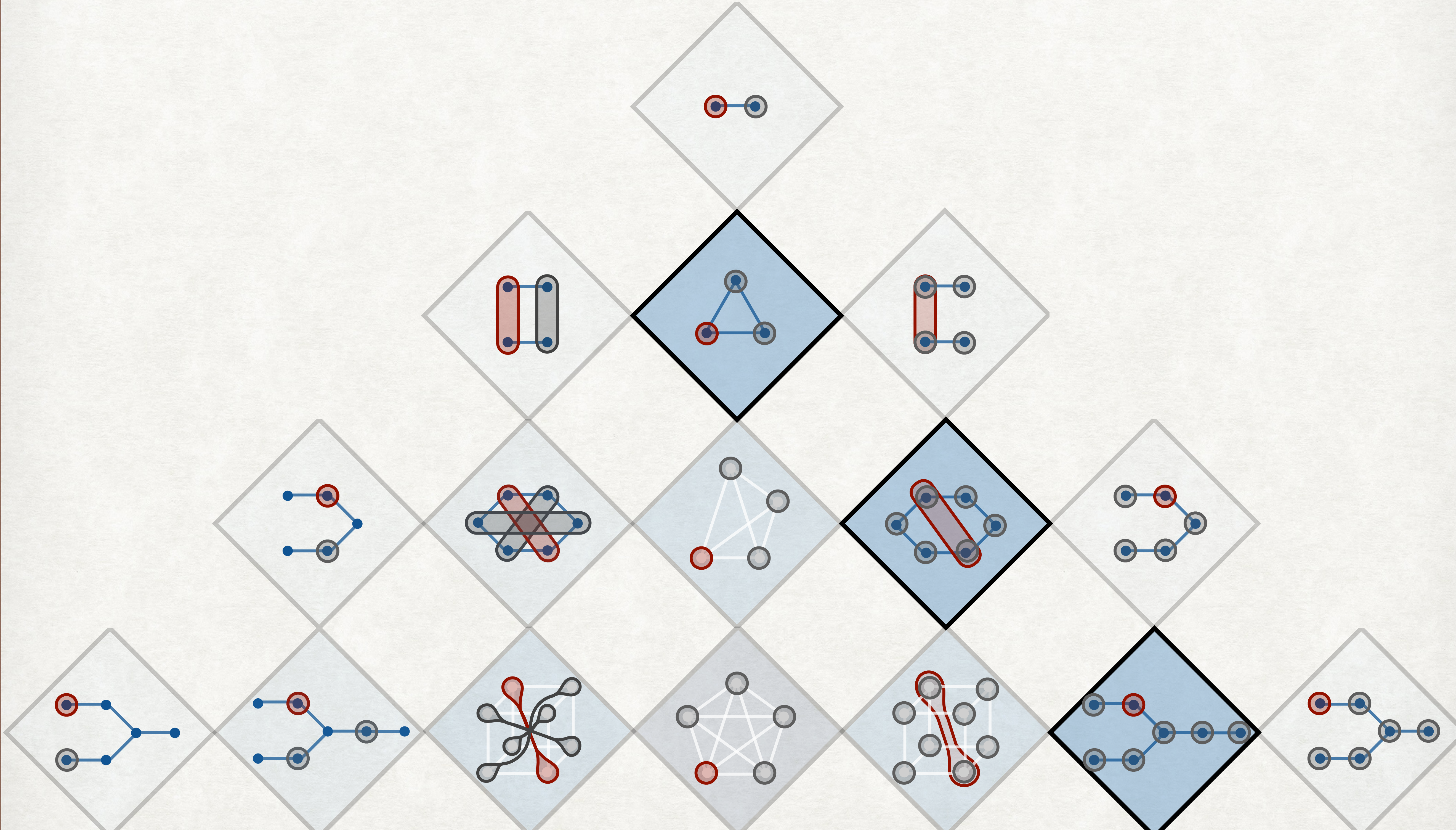
d=3 v=1	**ADS, Van Maldeghem (2017)** If nontrivial, (**X**, **Ξ**) is projectively unique and hence isomorphic to $\mathscr{V}$(L′[0])



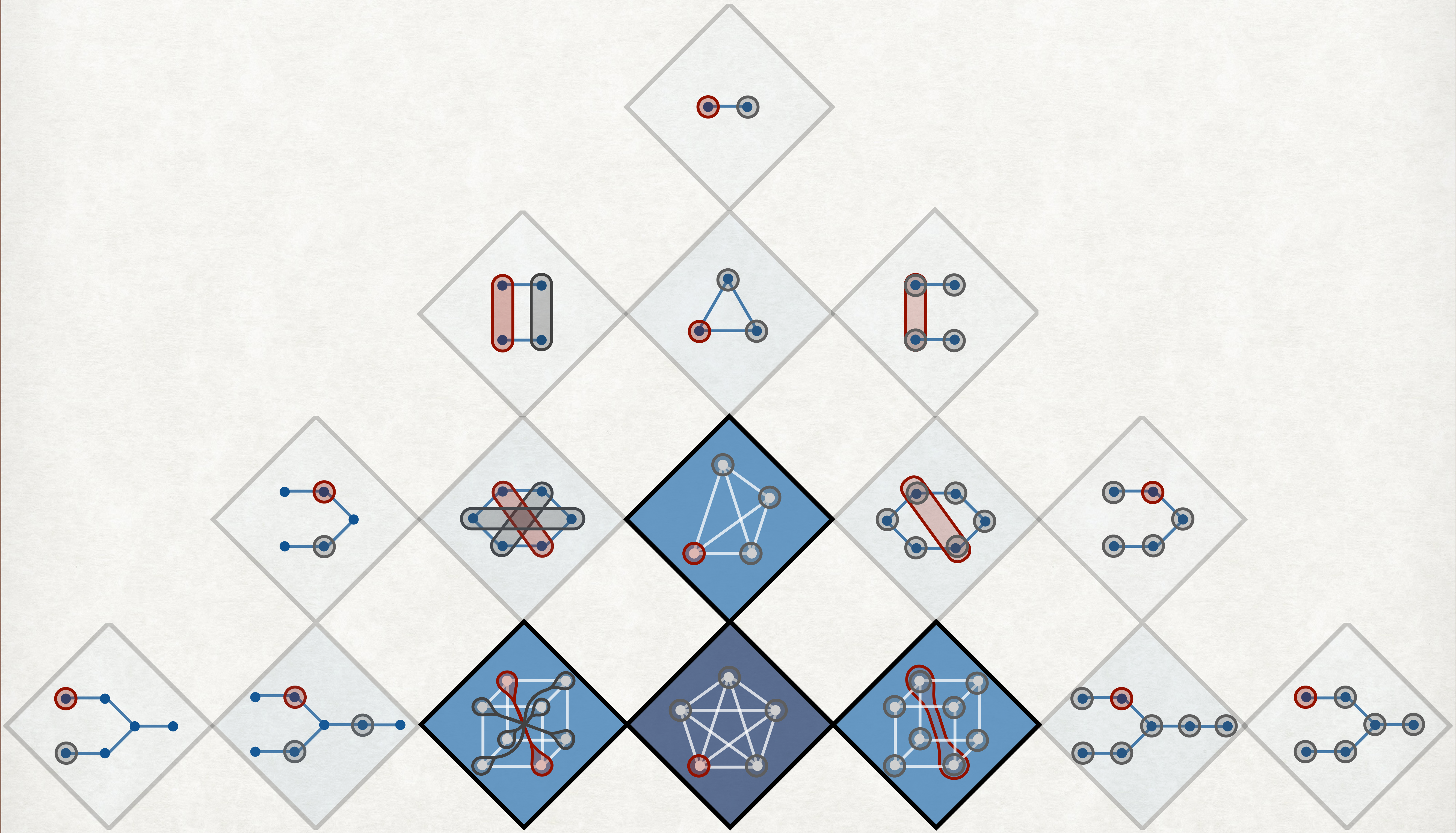
FINAL OVERVIEW



FINAL OVERVIEW



FINAL OVERVIEW



THANKS FOR YOUR ATTENTION