Buildings — What's The Point?

Antwerp Algebra Colloquium — March 5 — Anneleen De Schepper
I. Buildings: a definition

II. Point-line geometries related to buildings

III. Parapolar spaces

IV. Partial classification results
I. Buildings: a definition
DEFINITION

A building is a pair $\Delta = (\mathcal{C}, \delta)$

such that:

1. \[ \delta(c, d) = 1 \iff c = d \]

2+3. if $\delta(e, d) = w$ and $\delta(d, c) = s$ then $\delta(e, c) \in \{w, ws\}$

there is a unique chamber in $P_s(d)$ closest to $e$

and at least one further away

(w.r.t. length function on $(W, S)$)

4. thickness: each $s$-panel contains $\geq 3$ chambers.

If instead each $s$-panel 2 chambers: “apartment"

- The rank of $\Delta$ is $|S|$.  
- $\Delta$ is called spherical if $W$ is finite.

consequences of (2+3):

- each $s$-panel contains $\geq 2$ chambers
- if $c, c'$ in $P_s(d)$ then $\delta(c, c') = s$ if $c \neq c'$

$\delta(c, d) = s$ so $\delta(c, c') \in \{s, s^2 = 1\}$, ok by (1)
A **building** is a pair $\Delta = (C, \delta)$ where $C$ is a set and $\delta: C \times C \to W$ is a **distance function**, with gated s-panels, rank $n = |S|$, spherical if $W$ finite.

The thick irreducible spherical buildings of rank at least 3 have the following Coxeter diagrams:

- **$A_n$**
  - \[1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow n-1 \rightarrow n\]
- **$B_n$**
  - \[1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow n-1 \rightarrow n\]
- **$C_n$**
  - \[1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow n-1 \rightarrow n\]
- **$D_n$**
  - \[1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow n-2 \rightarrow n-1 \rightarrow n\]
- **$F_4$**
  - \[
    \begin{array}{c}
    1 \\
    2 \\
    3 \\
    4 \\
    n-1 \\
    n
    \end{array}
  \]
- **$E_6$**
  - \[
    \begin{array}{c}
    1 \\
    2 \\
    3 \\
    4 \\
    5 \\
    6 \\
    7
    \end{array}
  \]
- **$E_7$**
  - \[
    \begin{array}{c}
    1 \\
    2 \\
    3 \\
    4 \\
    5 \\
    6 \\
    7
    \end{array}
  \]
- **$E_8$**
  - \[
    \begin{array}{c}
    1 \\
    2 \\
    3 \\
    4 \\
    5 \\
    6 \\
    7 \\
    8
    \end{array}
  \]
**EXAMPLE OF A THIN BUILDING OF TYPE $A_3$**

Take $W = S_4$, the symmetric group on $\{1,2,3,4\}$

and $S = \{s_1, s_2, s_3\}$ with $s_1=(12)$, $s_2=(23)$ and $s_3=(34)$

$\mathcal{C} := \{\text{permutations of } \{1,2,3,4\}\}$

with $\delta(c,d) = w \Leftrightarrow c = w(d)$ (so (1) is ok)

<table>
<thead>
<tr>
<th>expression in $W$ is reduced</th>
<th>e.g. $s_3 s_1 s_2 s_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>path in graph of this type is a shortest path</td>
<td>e.g. from A to B</td>
</tr>
<tr>
<td>lengths correspond</td>
<td>e.g. 4</td>
</tr>
</tbody>
</table>

$s$-panel = edge with color $s = s$-residue

- $\{s_1, s_2\}$-residue: red-purple hexagons
- $\{s_2, s_3\}$-residue: purple-orange hexagons
- $\{s_1, s_3\}$-residue: red-orange squares

all residues are gated
II. Point-line geometries associated to buildings
2. POINT-LINE GEOMETRIES ASSOCIATED TO BUILDINGS

EXAMPLE

Take $W = S_4$, the symmetric group on $\{1,2,3,4\}$
and $S = \{s_1, s_2, s_3\}$ with $s_1=(12)$ $s_2=(23)$ and $s_3=(34)$

Define the following point-line geometry

points: $\{s_2, s_3\}$-residues, so purple-orange hex.

lines: $s_1$-panels, so red edges

point and line are incident if they share a chamber

two lines incident with the same point set are equal

→ we can also take $\{s_1,s_3\}$-residue as lines
This is a (thin) projective space of dimension 3:
- each two points determine a unique line
- two intersecting lines determine a projective plane
2. POINT-LINE GEOMETRIES ASSOCIATED TO BUILDINGS

EXAMPLE

Take $W = S_4$, the symmetric group on $\{1,2,3,4\}$ and $S = \{s_1, s_2, s_3\}$ with $s_1=(12)$ $s_2=(23)$ and $s_3=(34)$

Define the following point-line geometry

- **points**: $\{s_1, s_3\}$-residues, so red-orange squares
- **lines**: $s_2$-panels, so purple edges

Point and line are **incident** if they share a chamber

$$ab \sim cd \quad \text{iff} \quad |\{a,b,c,d\}|=3$$
**EXAMPLE**

Take $W = S_4$, the symmetric group on $\{1,2,3,4\}$ and $S = \{s_1, s_2, s_3\}$ with $s_1=(12)$ $s_2=(23)$ and $s_3=(34)$

\[
A_3 \\
\text{1} \text{2} \text{3} \\
\text{3} \text{2} \text{1} \\
\text{3} \text{3} \text{1} \\
\text{s}_1 \text{s}_2 \text{s}_3
\]

**Define the following point-line geometry**

- **points**: $\{s_1, s_3\}$-residues, so **red-orange** squares
- **lines**: $s_2$-panels, so **purple** edges

**point** and **line** are **incident** if they share a **chamber**

- $ab \text{--} cd$ **iff** $|\{a,b,c,d\}| = 3$

From this back to chamber system:

A **chamber** is a maximal flag: $\{\text{point}, \text{line}\}$ with point **inc** line

This is a (thin) **polar space** of rank 3: 1-or-all axiom
2. POINT-LINE GEOMETRIES ASSOCIATED TO BUILDINGS

GRASSMANNIANS OF SPHERICAL BUILDINGS

Take any thick irreducible spherical building $\Delta$ of rank $n \geq 3$ of type $X_n$, let $k$ be any type ($k \in S$).

**Definition**

The $k$-Grassmannian $X_{n,k}$ of $\Delta$ is the following point-line geometry:

- **Points**: the residues of cotype $k$
- **Lines**: the $k$-panels
- **Incidence**: a point and a line are incident if they share a chamber

**Projective spaces**

- $A_{n,1}$
- $A_{n,n}$

**Polar spaces**

- $B_{n,1}$
- $C_{n,1}$
- $D_{n,1}$
- $A_{3,2} = D_{3,1}$

**Parapolar spaces**

- $A_{n,k}$, $1 < k < n$
- $B_{n,k}$, $k > 1$
- $C_{n,k}$, $k > 1$
- $D_{n,k}$, $k > 1$
- $E_{6,k}$
- $E_{7,k}$
- $E_{8,k}$
III. Parapolar spaces
**PARAPOLAR SPACES**

**Definition**

A **polar space** is a **point-line geometry** such that

- 1-or-all axiom

- 3 non-degeneracy axioms
  (each line \( \geq 3 \) points, empty radical, finite rank)

The **rank** is \( r \) if \( r-1 = \dim \) (maximal singular subspace)

**Examples:**

isotropic vectors of a quadratic/hermitian form

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**Definition**

A **parapolar space** is a **connected point-line geometry** such that

(i) if \( p,q \) are **points** at distance 2:
  - convex closure of \( \{ p,q \} \) is a **polar space** (a symp)
  or
  - there is a unique path between them

(ii) each **line** is contained in a symp

(iii) no symp contains all **points** (not a polar space itself)

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Thin “examples”: 2 **points** per **line** yet they have most characteristic features

thin polar space of rank 3

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distance 3

**special pair**

**symp**

**type F\(_{4,1}\)**
**PARAPOLAR SPACES**

Prominent examples: k-Grassmannian of spherical buildings

- read off the symps: maximal polar subdiagram with points

![Diagram](image)

- remark: in general, symps can be of different kinds and ranks

Some other examples: direct products

Example: direct product of two projective spaces

abstractly: \( A_{n,1}(*) \times A_{m,1}(*) \)

if embeddable in projective space: Segre variety \( S_{m,n}(K) \)

Parapolar space: connected **point-line** geometry

- points at dist. 2:

- each line \( \subset \) symp

- not polar
IV. Partial classification results of parapolar spaces
No classification of parapolar spaces in general, despite
- so many building-related examples
- projective spaces and polar spaces of rank $\geq 3$ being classified

Aim: find additional properties such that there are
- many parapolar spaces associated to (exceptional) spherical buildings satisfying them
- not too many other parapolar spaces satisfy them

Several such results by a.o. Buekenhout, Cohen, Cooperstein, Kasikova, Shult — 20 years time nothing — ADS, Van Maldeghem, Victoor, Schillewaert

Example of such a property:

“The intersection of two symps is never empty.”

Theorem (ADS, HVM, MV, JS; 2020+):
A parapolar space in which each two symps have a non-empty intersection, and in which there are no special pairs if there is a symp of rank 2, is one of the following:

- $A_{1,1}(\ast) \times A_{2,1}(\ast)$
- $A_{2,1}(\ast) \times A_{2,1}(\ast)$
- $A_{4,2}(L)$
- $A_{5,2}(L)$
- $E_{6,1}(K)$
Theorem (ADS, HVM, MV, JS; 2020+):
A parapolar space in which each two symps have a non-empty intersection, and in which there are no special pairs if there is a symp of rank 2, is one of the following:

Remark: the intersection of two symps is always a (projective) singular subspace (possibly empty)

(i) if p, q are points at distance 2:
- convex closure of \{p, q\} is a polar space (a symp)
  or
  - there is a unique path between them

→ the intersection of two symps has a well-defined dimension (which is -1 if it is empty)

A more general such property:

“The intersection of two symps never has dimension k.”

where k is an integer with k ≥ -1 and such that each symp contains k-spaces as singular subspaces.
Theorem (ADS, HVM, MV, JS; 2020): The parapolar spaces such that

- two symps never intersect in exactly a k-space
- each symp contains k-spaces
- there are no special pairs IF there is a symp of rank 2 (only needed when k < 2)

are classified. If each symp contains (k+2)-spaces, and if locally connected, then:

- k=-1
- k=0
- k=1
- k=2
- k=3
- k=4

Surprisingly, a large part of the Freudenthal-Tits magic square turns up!
Thanks!