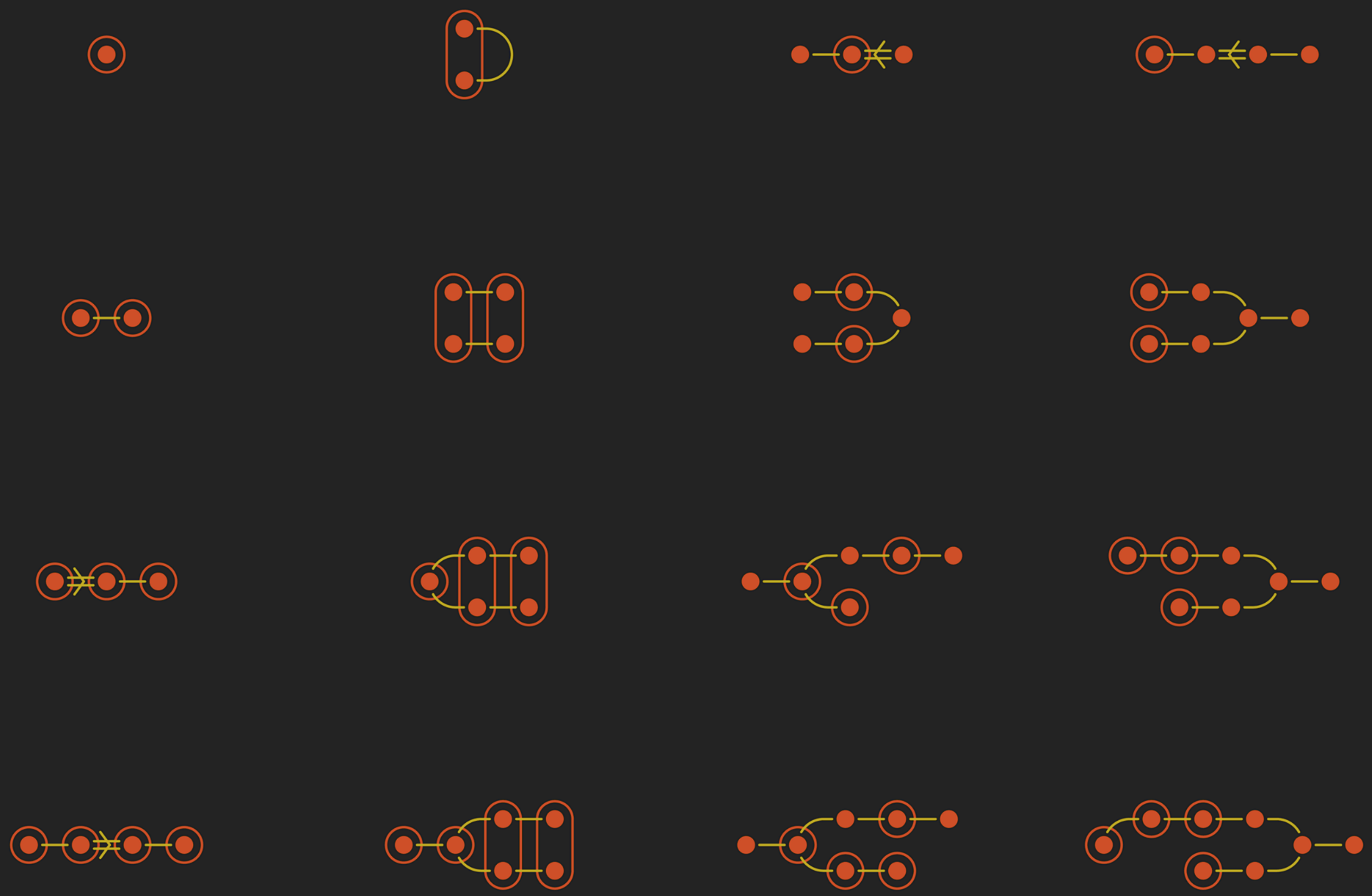
Buildings — What's The Point?



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Antwerp Algebra Colloquium — March 5 — Anneleen De Schepper

OVERVIEW

I. Buildings: a definition

II. Point-line geometries related to buildings

III. Parapolar spaces

IV. Partial classification results

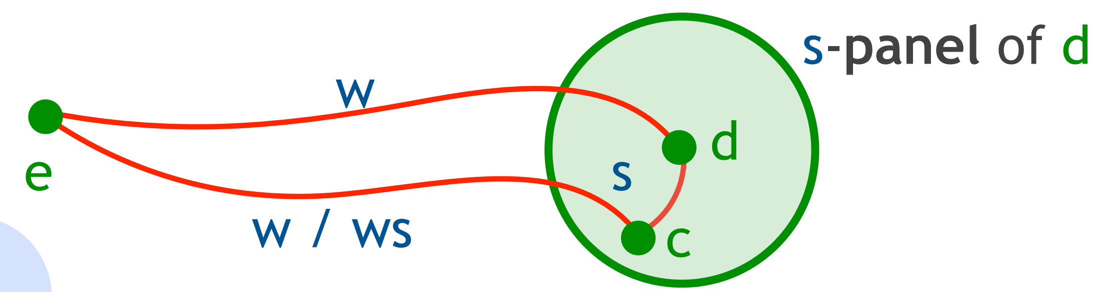
I. Buildings: a definition

DEFINITION

A building is a pair $\Delta = (\mathscr{C}, \delta)$ — where $\mathscr{C}$ is a set, $\delta : \mathscr{C} \times \mathscr{C} \to W$ is a distance function, and W is a Coxeter group, say with generator set S —

such that:

(1) $\delta(c,d) = 1_W \Leftrightarrow c = d$



(2+3) if $\delta(e,d) = w$ and $\delta(d,c) = s$ then $\delta(e,c) \in \{w, ws\}$

there is a **unique** chamber in $P_s(d)$ closest to e and at least one further away

(w.r.t. length function on (W,S))

$P_s(d) = \{d\} \cup \{c \in \mathscr{C} : \delta(c,d) = s\}$

"panels are gated"

(4) **thickness**: each s-panel contains ≥3 chambers.

If instead each s-panel 2 chambers: "**apartment**"

- The **rank** of Δ is |S|.
- Δ is called **spherical** if W is finite.

consequences of **(2+3)**:

- each s-panel contains ≥2 chambers
- if c, c' in $P_s(d)$ then $\delta(c,c') = s$ if $c \neq c'$

$\delta(c,d) = s$ so $\delta(c,c') \in \{s, s^2=1\}$, ok by (1)

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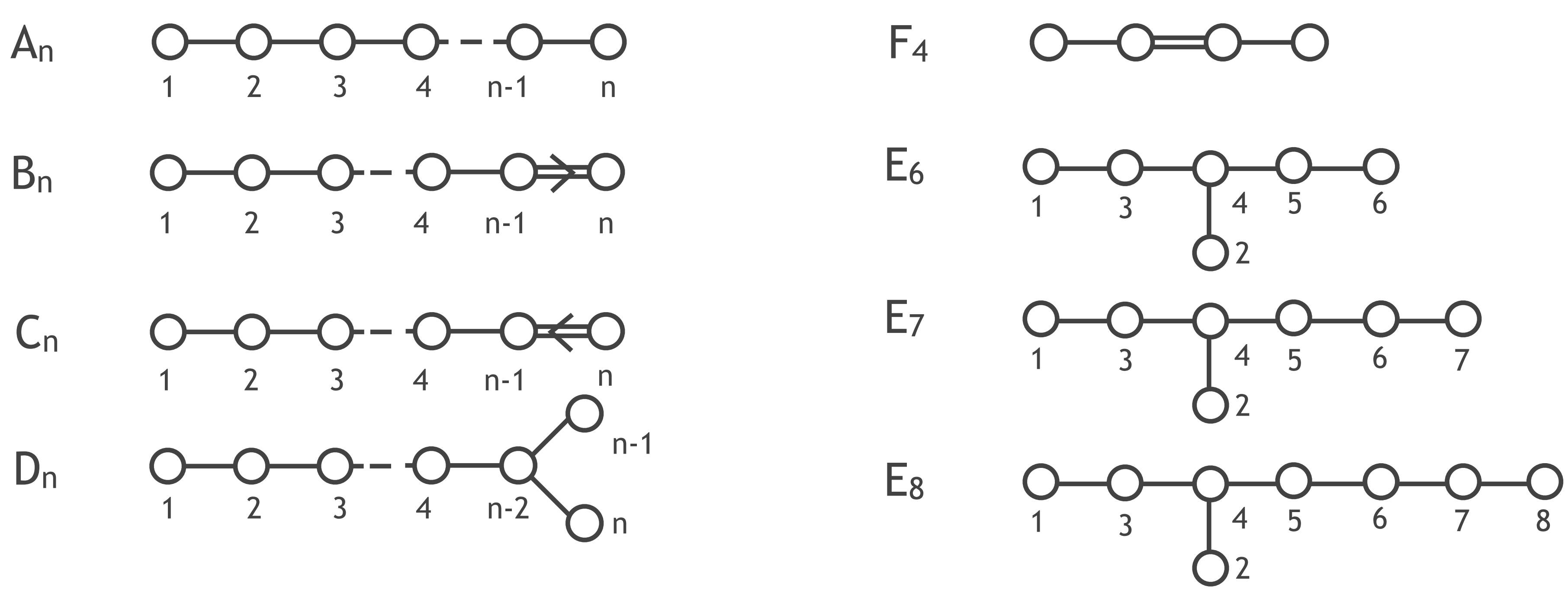
SPHERICAL BUILDINGS

A <u>building</u> is a pair $\Delta = (\mathscr{C}, \delta)$ where $\mathscr{C}$ is a **set** and $\delta: \mathscr{C} \times \mathscr{C} \to W$ is a **distance function**,

with gated s-panels, rank n = |S|, spherical if W finite.

The **thick irreducible spherical buildings of rank at least 3** have the following Coxeter diagrams:

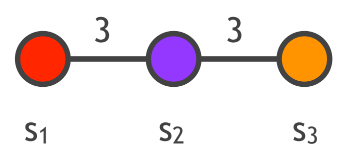


EXAMPLE OF A THIN BUILDING OF TYPE A_3

Take $W = S_4$, the symmetric group on {1,2,3,4}

and $S = \{s_1, s_2, s_3\}$ with $s_1=(12)$, $s_2=(23)$ and $s_3=(34)$

A_3



$\mathscr{C}$:= {permutations of {1,2,3,4}}

with $\delta(c,d) = w \Leftrightarrow c = w(d)$ (so **(1)** is ok)

expression in W is **reduced**	e.g. $s_3s_1s_2s_1$
path in graph of this type is a **shortest path**	e.g. from A to B
lengths correspond	e.g. 4

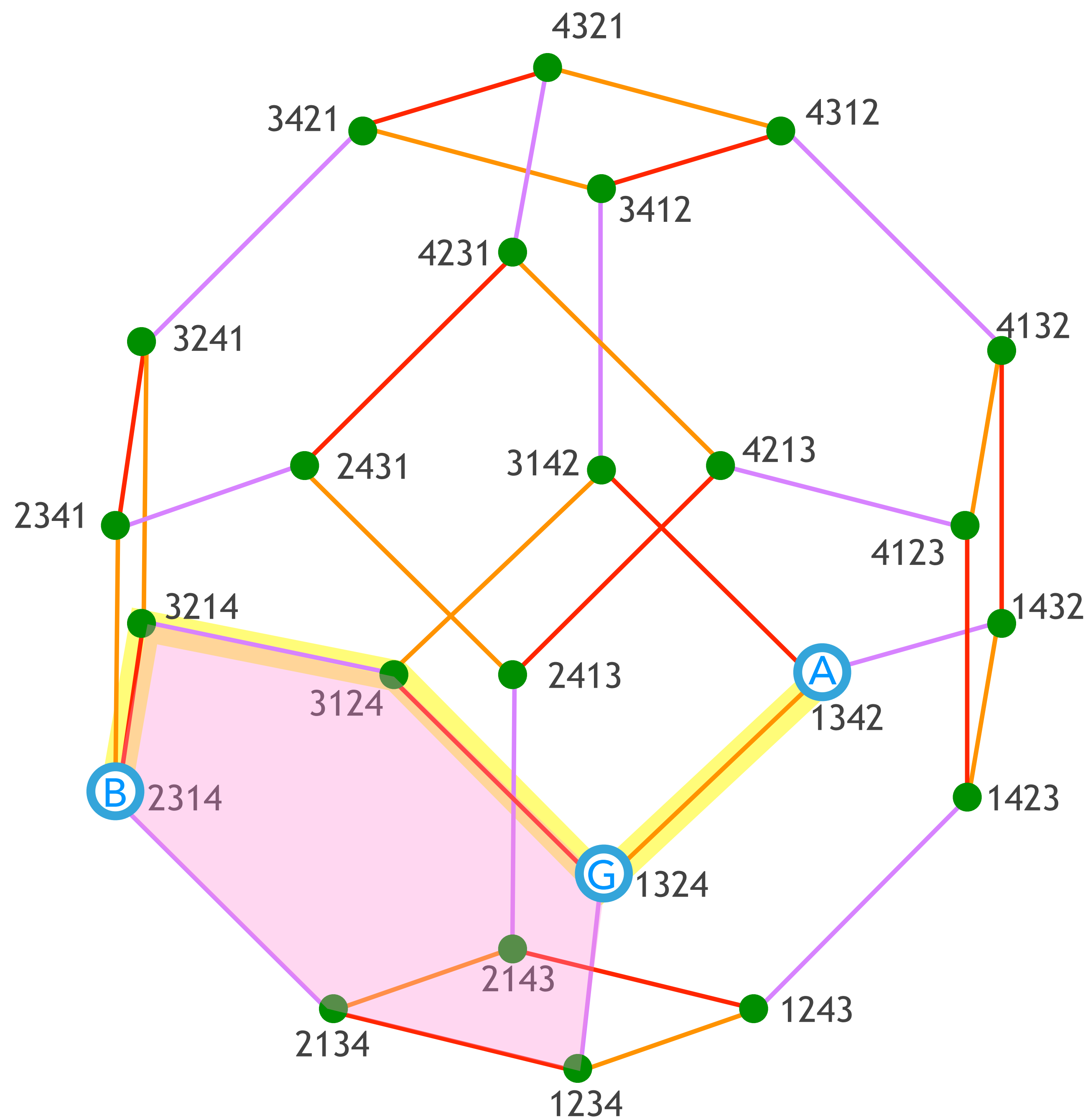
s-**panel** = **edge** with color s = s-residue

{s_1,s_2}-residue : red-purple hexagons

{s_2,s_3}-residue : purple-orange hexagons

{s_1,s_3}-residue : red-orange squares

all residues are gated



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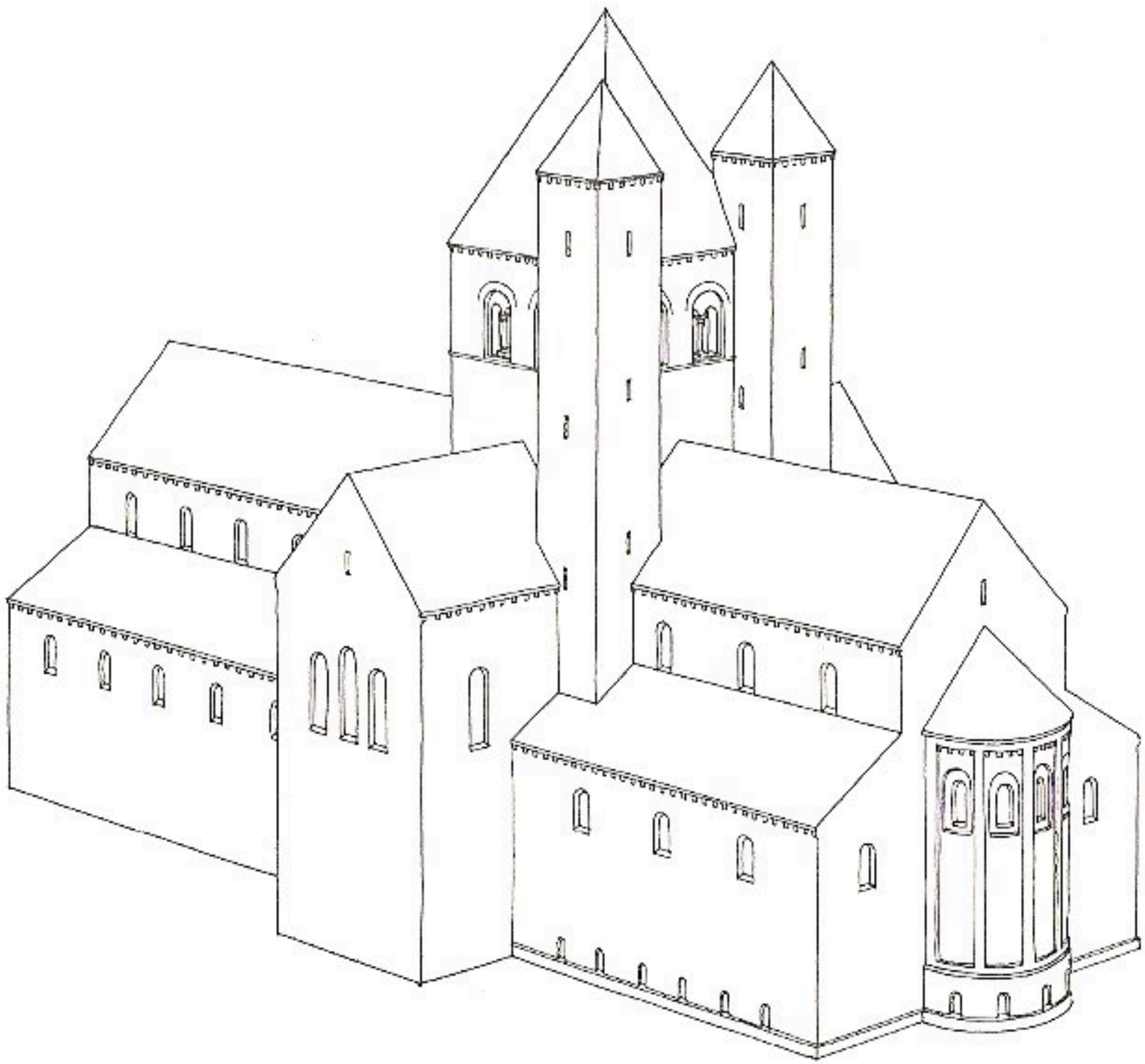
II. Point-line geometries associated to buildings



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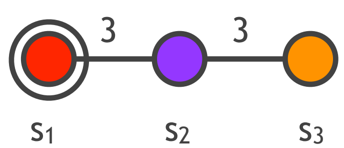
A_3 

Define the following point-line geometry

points: $\{s_2, s_3\}$-residues, so purple-orange hex.

lines: s_1-panels, so red edges

point and line are **incident** if they share a chamber

two lines incident with the same point set are equal

➜ we can also take $\{s_1,s_3\}$-residue as lines

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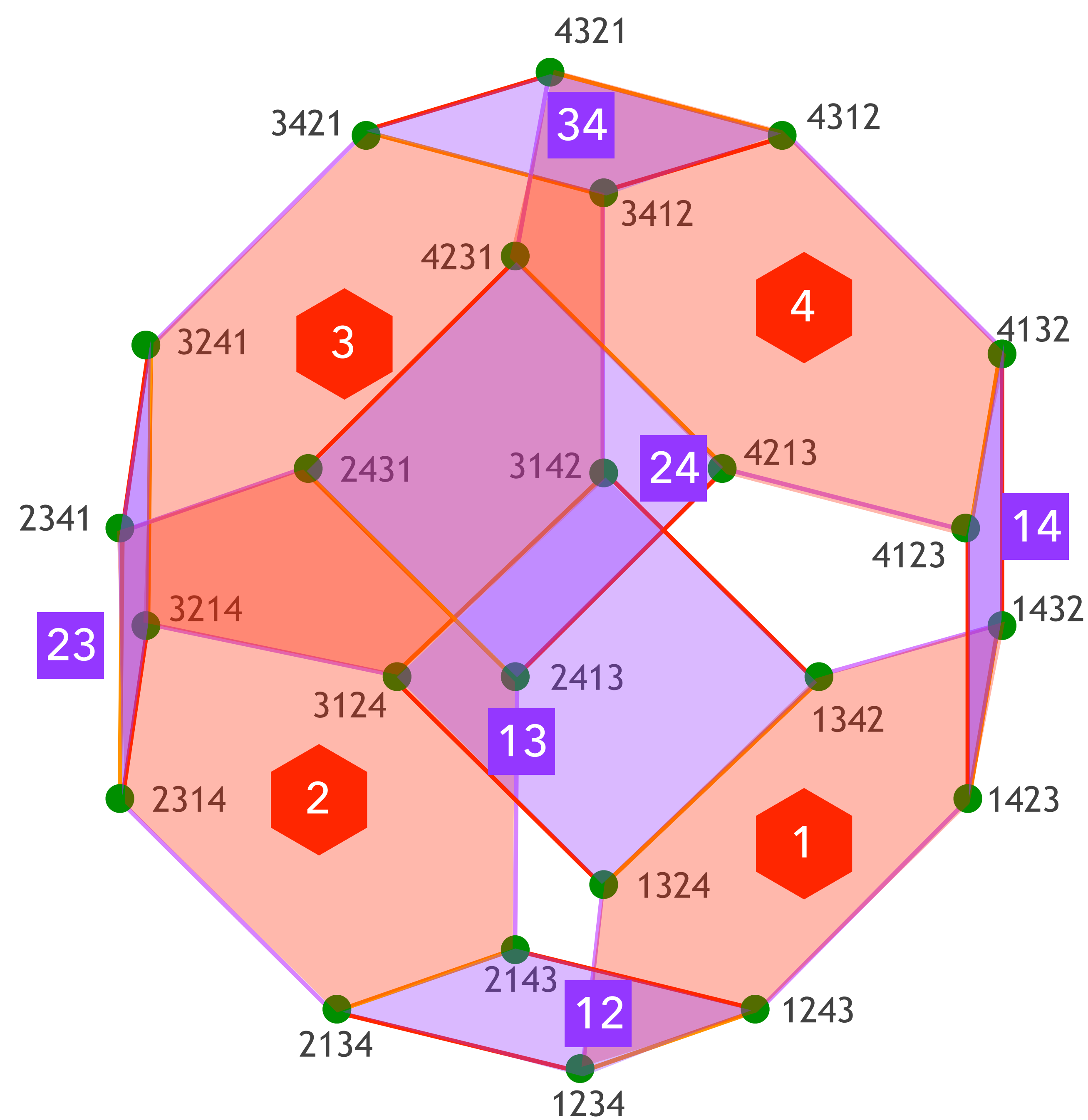

EXAMPLE

Take W = S_4, the symmetric group on {1,2,3,4}

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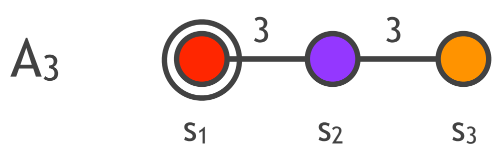

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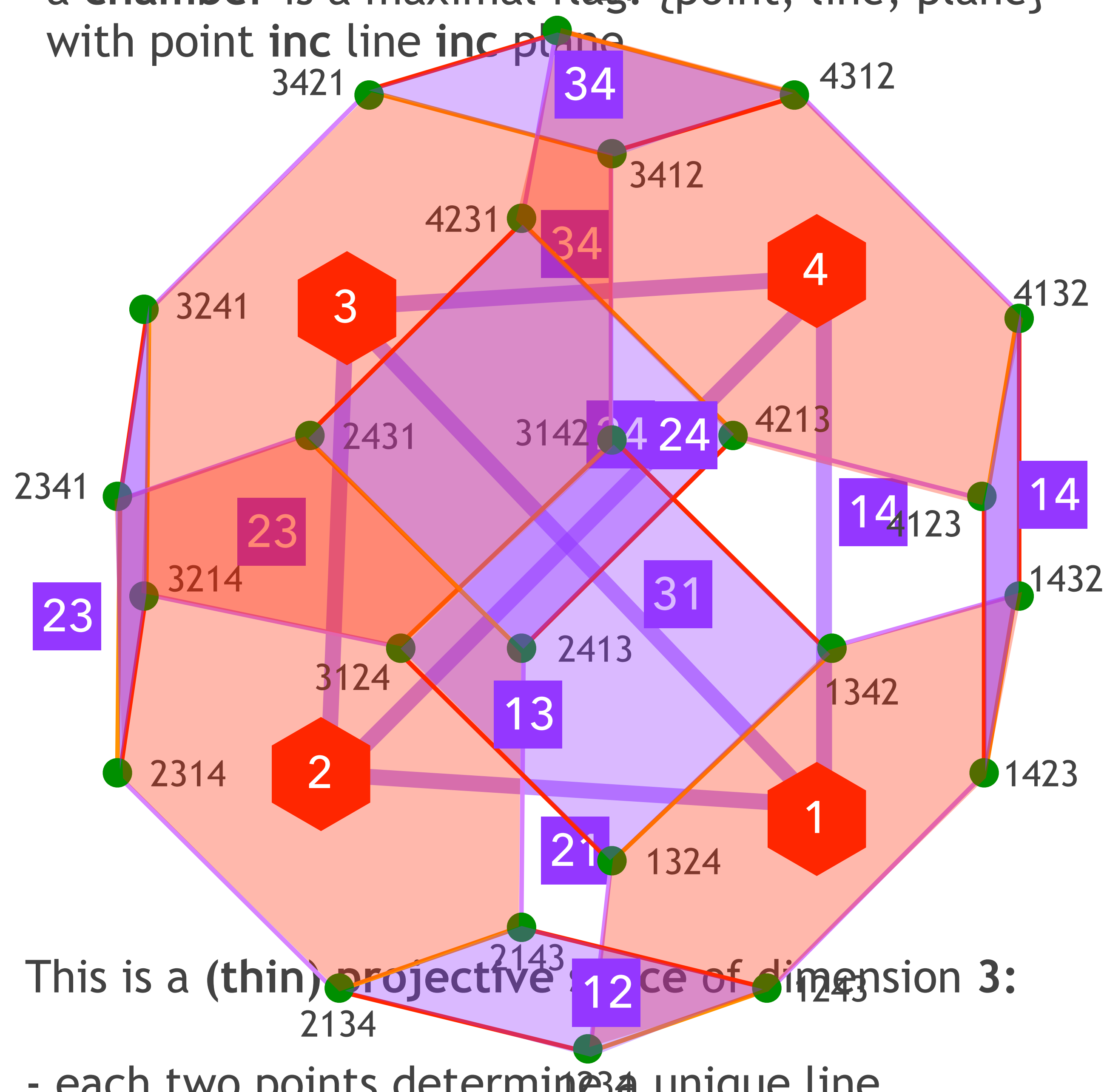
From this back to the chamber system:

a **chamber** is a maximal flag: {point, line, plane} with point **inc** line **inc** plane



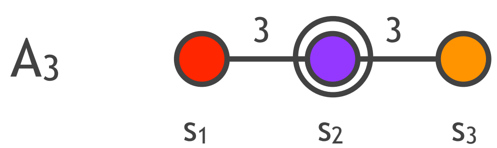
This is a **(thin) projective space of dimension 3:**

- each two points determine a unique line
- two intersecting lines determine a projective plane

EXAMPLE

Take $W = S_4$, the symmetric group on {1,2,3,4}

and $S = \{s_1, s_2, s_3\}$ with $s_1=(12)$ $s_2=(23)$ and $s_3=(34)$

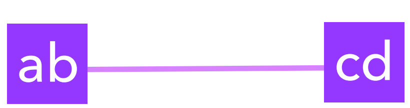


Define the following point-line geometry

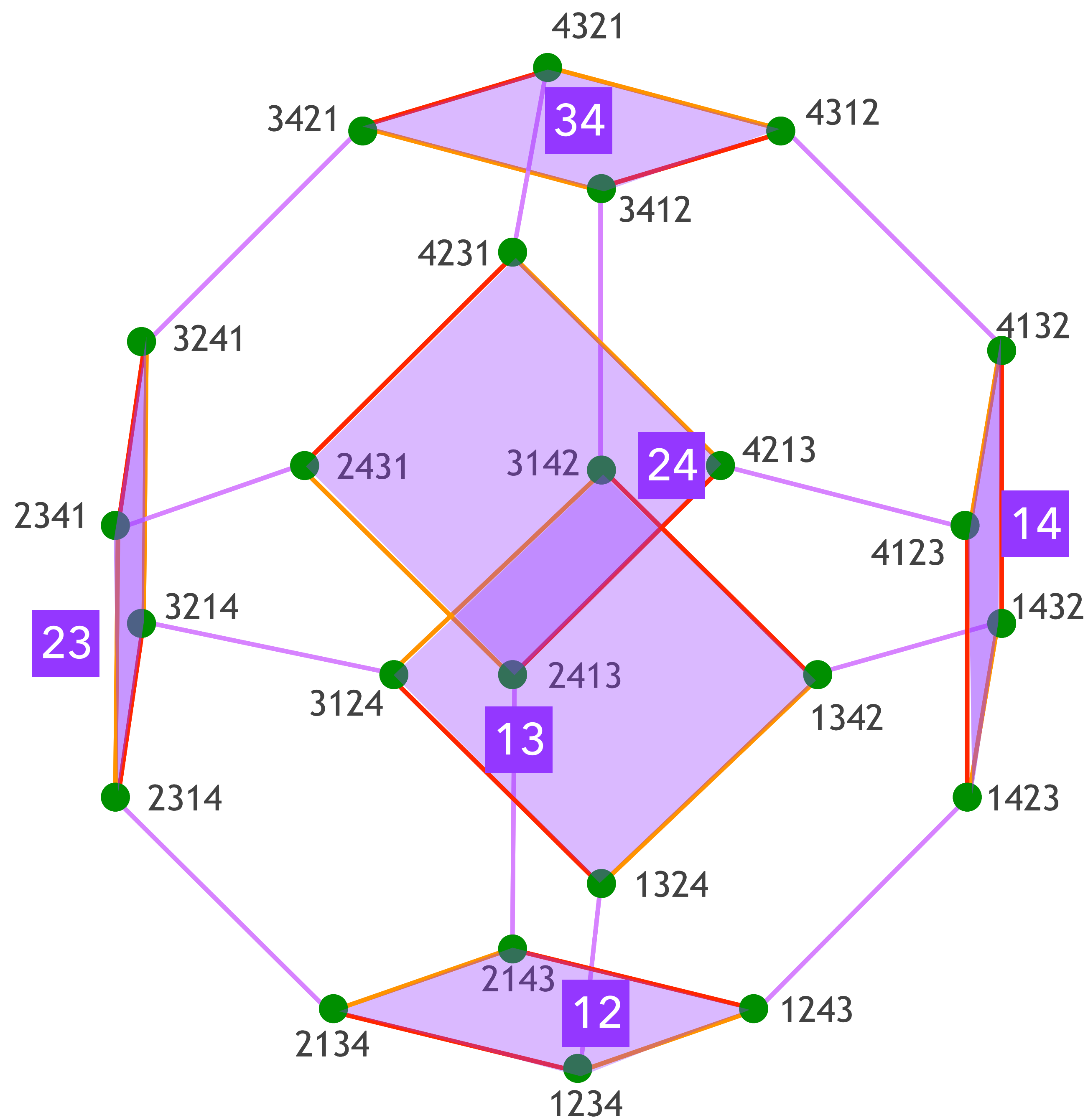
points: $\{s_1, s_3\}$-residues, so red-orange squares

lines: s_2-panels, so purple edges

point and line are **incident** if they share a chamber



iff |{a,b,c,d}|=3



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EXAMPLE

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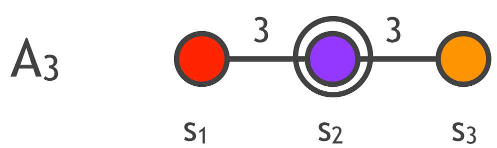

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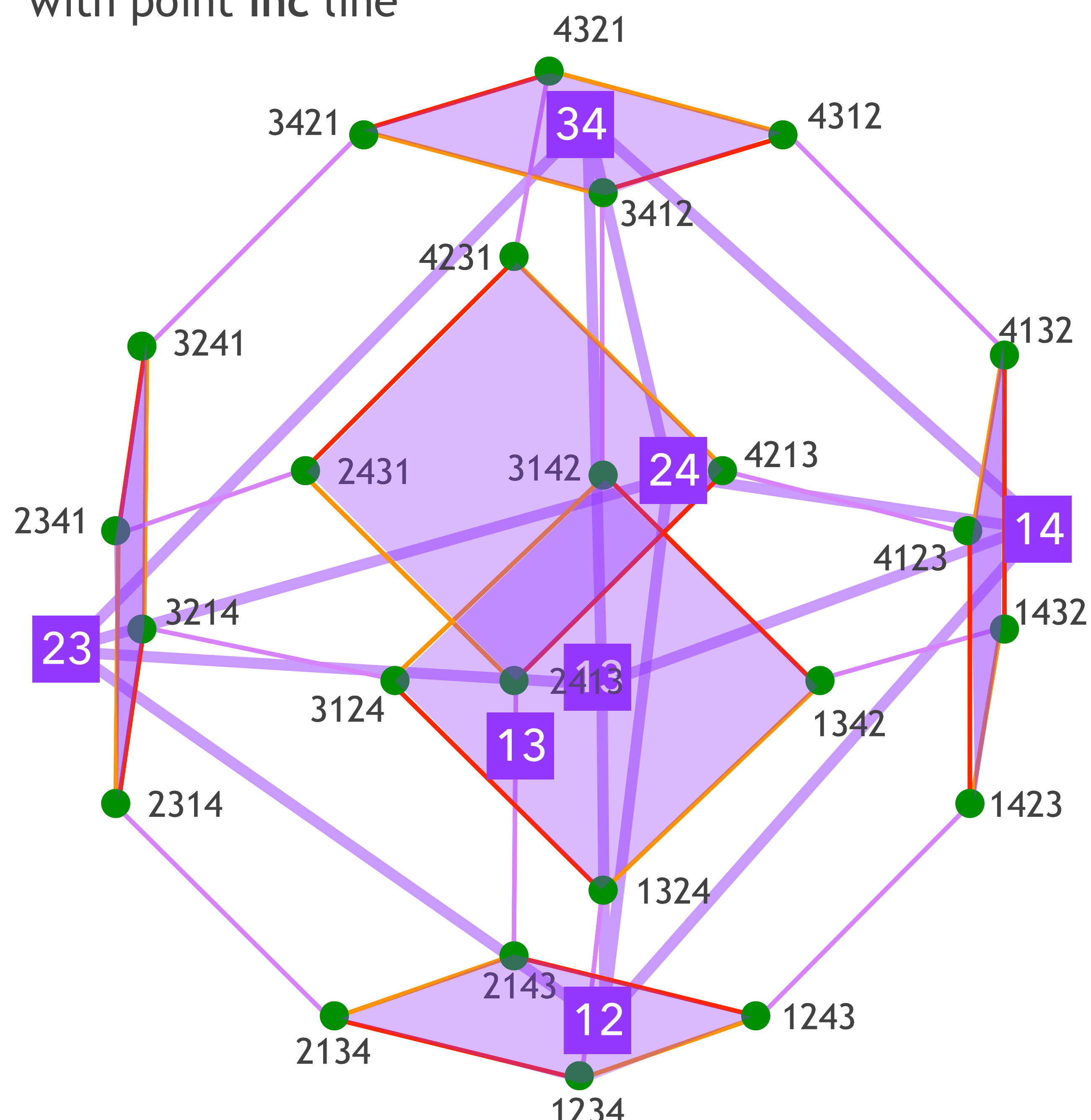
From this back to chamber system:

a **chamber** is a maximal flag: {point, line} with point **inc** line



This is a **(thin) polar space** of rank **3**: 1-or-all axiom

GRASSMANNIANS OF SPHERICAL BUILDINGS

Take any thick irreducible spherical building Δ of rank $n \geq 3$ of type $\mathbf{X_n}$, let k be **any** type ($k \in S$).

Definition

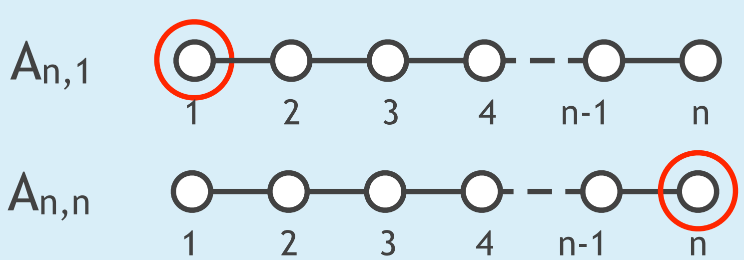
The **k-Grassmannian $X_{n,k}$ of** Δ is the following point-line geometry :

➜ the points are the residues of cotype k

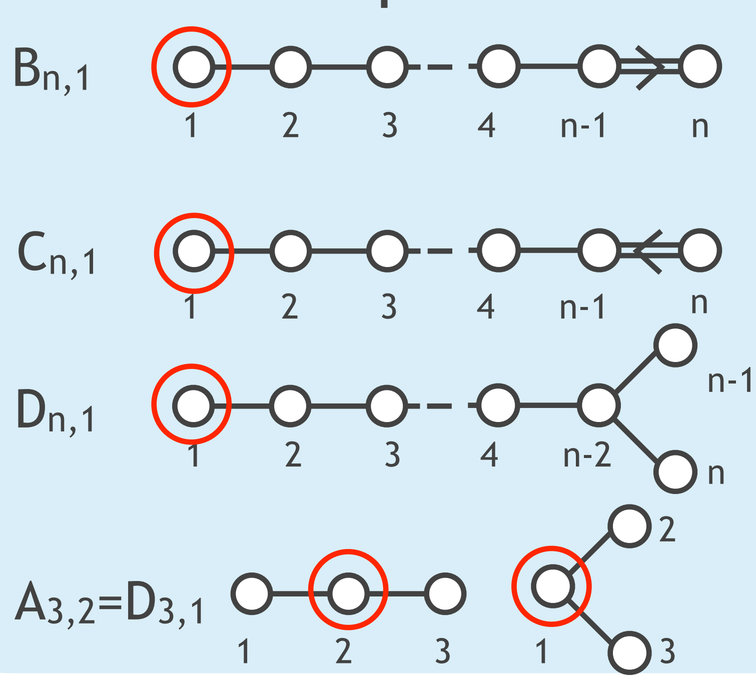
➜ the lines are the k-panels

➜ a point and a line are **incident** if they share a chamber

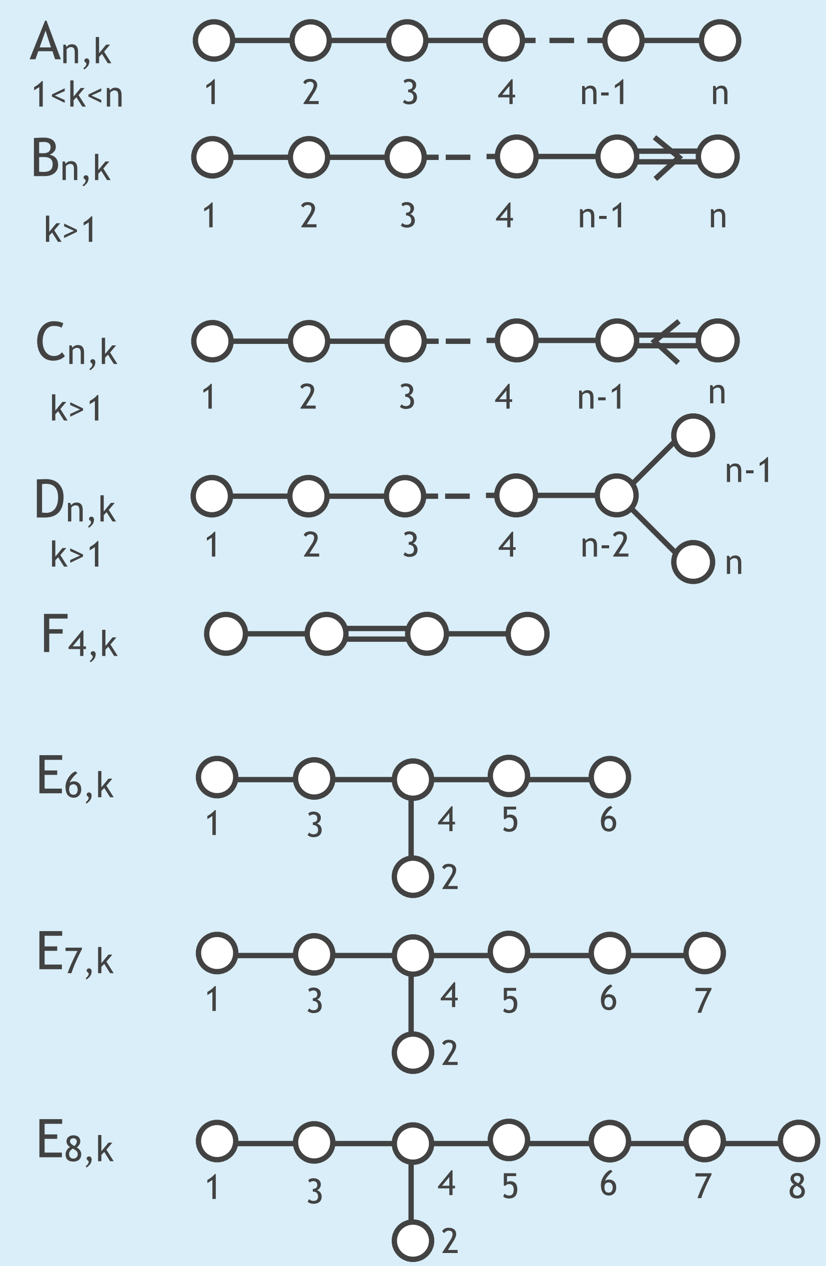
Projective spaces



Polar spaces



Parapolar spaces



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III. Parapolar spaces

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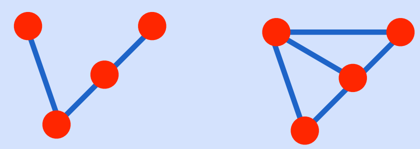
PARAPOLAR SPACES

Definition
A polar space is a **point-line geometry** such that

- 1-or-all axiom



- 3 non-degeneracy axioms
  (each line ≥ 3 points, empty radical, finite rank)

The rank is **r** if **r**-1 = dim (maximal singular subspace)

Examples:
isotropic vectors of a quadratic/hermitian form

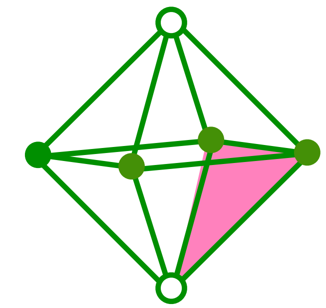
Definition
A parapolar space is a **connected point-line geometry** such that

(i) if p,q are points at distance 2:
- convex closure of {p,q} is a polar space (a *symp*)
or
- there is a unique path between them

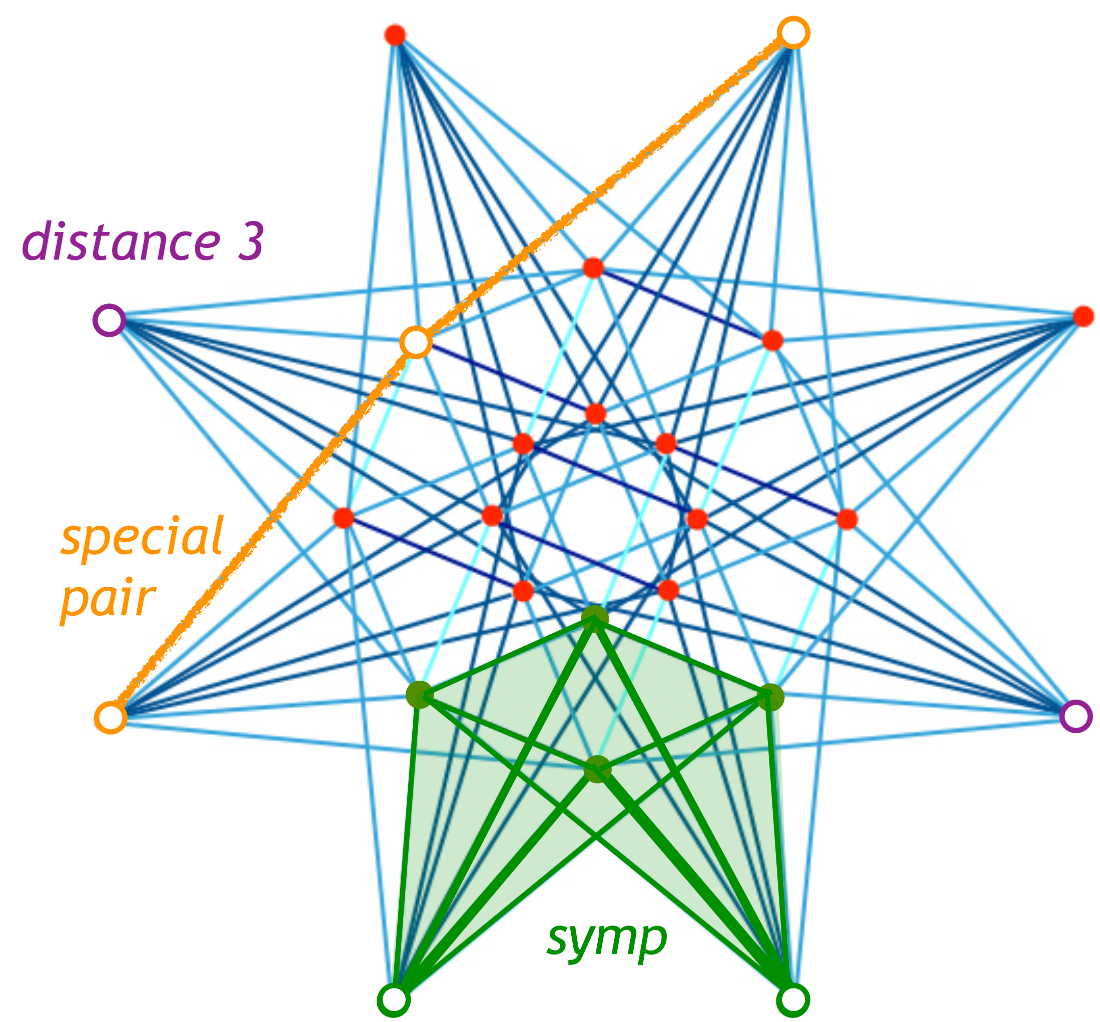
(ii) each line is contained in a symp

(iii) no symp contains all points (not a polar space itself)

Thin "examples": 2 points per line
yet they have most **characteristic features**



thin polar space of rank 3



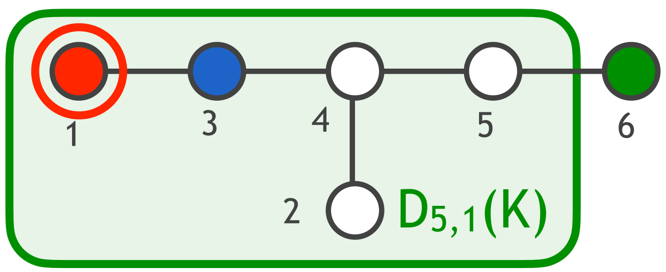
type $F_{4,1}$

PARAPOLAR SPACES

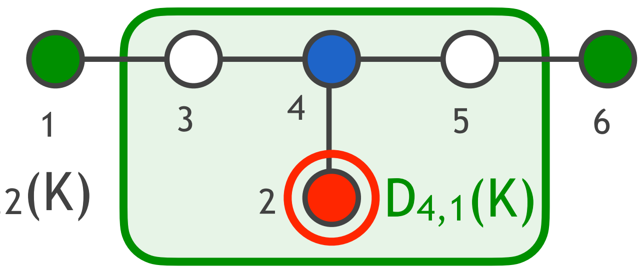
Prominent examples: k-Grassmannian of spherical buildings

- read off the symps: maximal polar subdiagram with points

$E_{6,1}(K)$



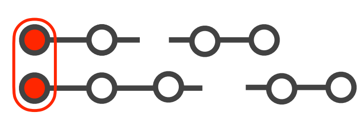
$E_{6,2}(K)$



- **remark**: in general, symps can be of different kinds and ranks

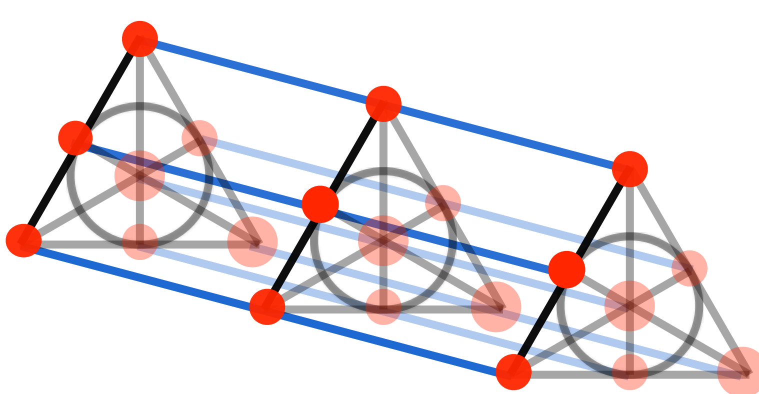
Some other examples: direct products

Example: direct product of two projective spaces

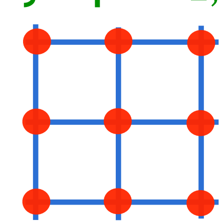
abstractly: $A_{n,1}(*) \times A_{m,1}(*)$ 

if embeddable in projective space: *Segre variety $S_{m,n}(K)$*

$S_{1,2}(F_2)$



symp: $D_{2,1}$



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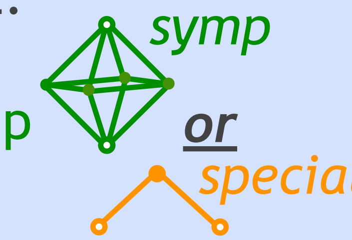
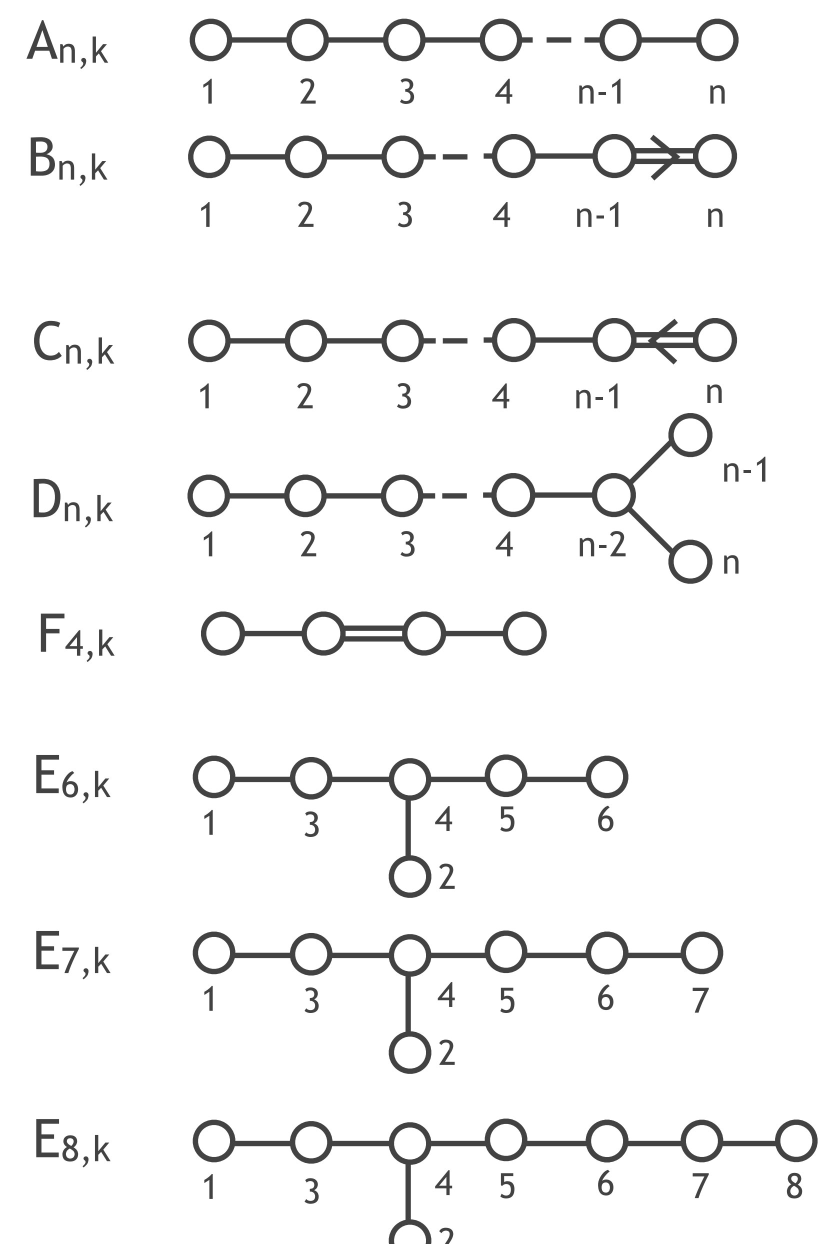
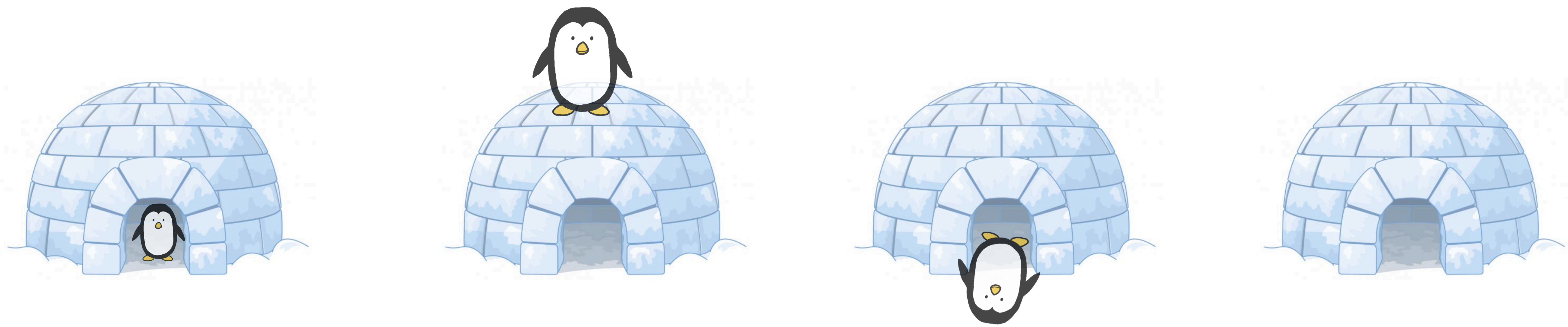
Parapolar space:
connected point-line geometry

- points at dist. 2:
- each line ⊂ symp
- not polar





IV. Partial classification results of parapolar spaces



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PARAPOLAR SPACES

No classification of parapolar spaces in general, despite

- so many building-related examples
- projective spaces and polar spaces of rank ≥3 being classified

Aim: find additional properties such that there are

- many parapolar spaces associated to (exceptional) spherical buildings satisfying them
- not too many other parapolar spaces satisfy them

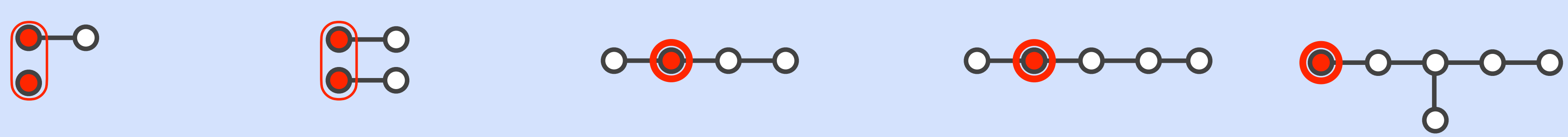
Several such results by a.o. Buekenhout, Cohen, Cooperstein, Kasikova, Shult — 20 years time nothing — ADS, Van Maldeghem, Victoor, Schillewaert

Example of such a property:

*"The intersection of two symps is **never empty**."*

Theorem (ADS, HVM, MV, JS; 2020+):

A parapolar space in which each two symps have a non-empty intersection, and in which there are no special pairs <u>IF</u> there is a symp of rank 2, is one of the following:



$A_{1,1}(*) \times A_{2,1}(*)$ $A_{2,1}(*) \times A_{2,1}(*)$ $A_{4,2}(L)$ $A_{5,2}(L)$ $E_{6,1}(K)$

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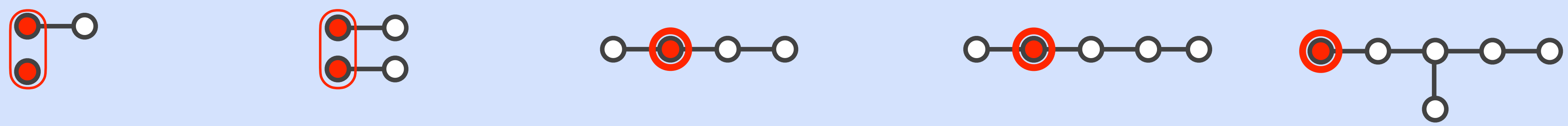
PARAPOLAR SPACES

Theorem (ADS, HVM, MV, JS; 2020+):

A parapolar space in which each two symps have a non-empty intersection, and in which there are no special pairs <u>**if**</u> there is a symp of rank 2, is one of the following:

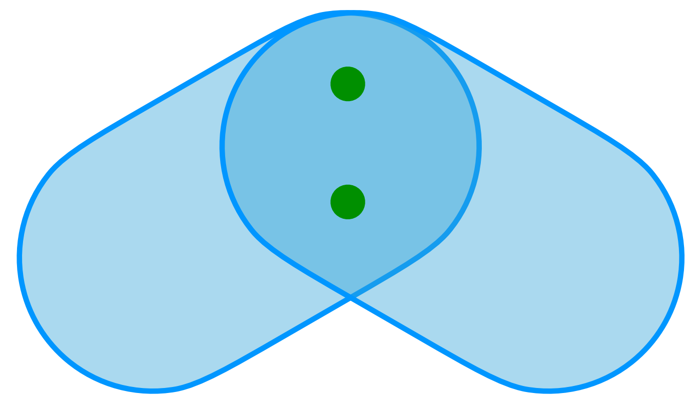


Remark: the intersection of two symps is always a (projective) singular subspace (possibly empty)

(i) if p,q are points at distance 2:
- convex closure of {p,q} is a <u>polar space</u> (a *symp*)

<u>***or***</u>

~~- there is a unique path between them~~



➜ the intersection of two symps has a well-defined dimension (which is -1 if it is empty)

A more general such property:

*"The intersection of two symps **never has dimension k.**"*

where k is an integer with k≥-1 and such that each symp contains k-spaces as singular subspaces.

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PARAPOLAR SPACES

Theorem (ADS, HVM, MV, JS; 2020): The parapolar spaces such that

- **two symps never intersect in exactly a k-space**
- each symp contains **k**-spaces
- there are no special pairs <u>**IF**</u> there is a symp of rank 2 (only needed when k < 2)

are **classified**. If each symp contains **(k+2)**-spaces, and if locally connected, then:

k=-1	k=0	k=1	k=2	k=3	k=4
$A_{1,1}(*) \times A_{2,1}(*)$	$A_{4,2}(L)$	$D_{5,5}(K)$	$E_{6,1}(K)$	$E_{7,7}(K)$	$E_{8,8}(K)$
$A_{4,2}(L)$	$D_{5,5}(K)$	$E_{6,1}(K)$	$E_{7,7}(K)$	$E_{8,8}(K)$	
$E_{6,1}(K)$	$E_{7,7}(K)$	$E_{8,8}(K)$			
$A_{5,2}(L)$	$D_{6,6}(K)$	$E_{7,1}(K)$			
$A_{2,1}(*) \times A_{2,1}(*)$	$A_{5,3}(L)$	$E_{6,2}(K)$			

Surprisingly, a large part of the Freudenthal-Tits magic square turns up!

Thanks!