UNIVERSIDAD DE SEVILLA

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Change of Stability and Branching of Periodic Orbits in Reversible Systems

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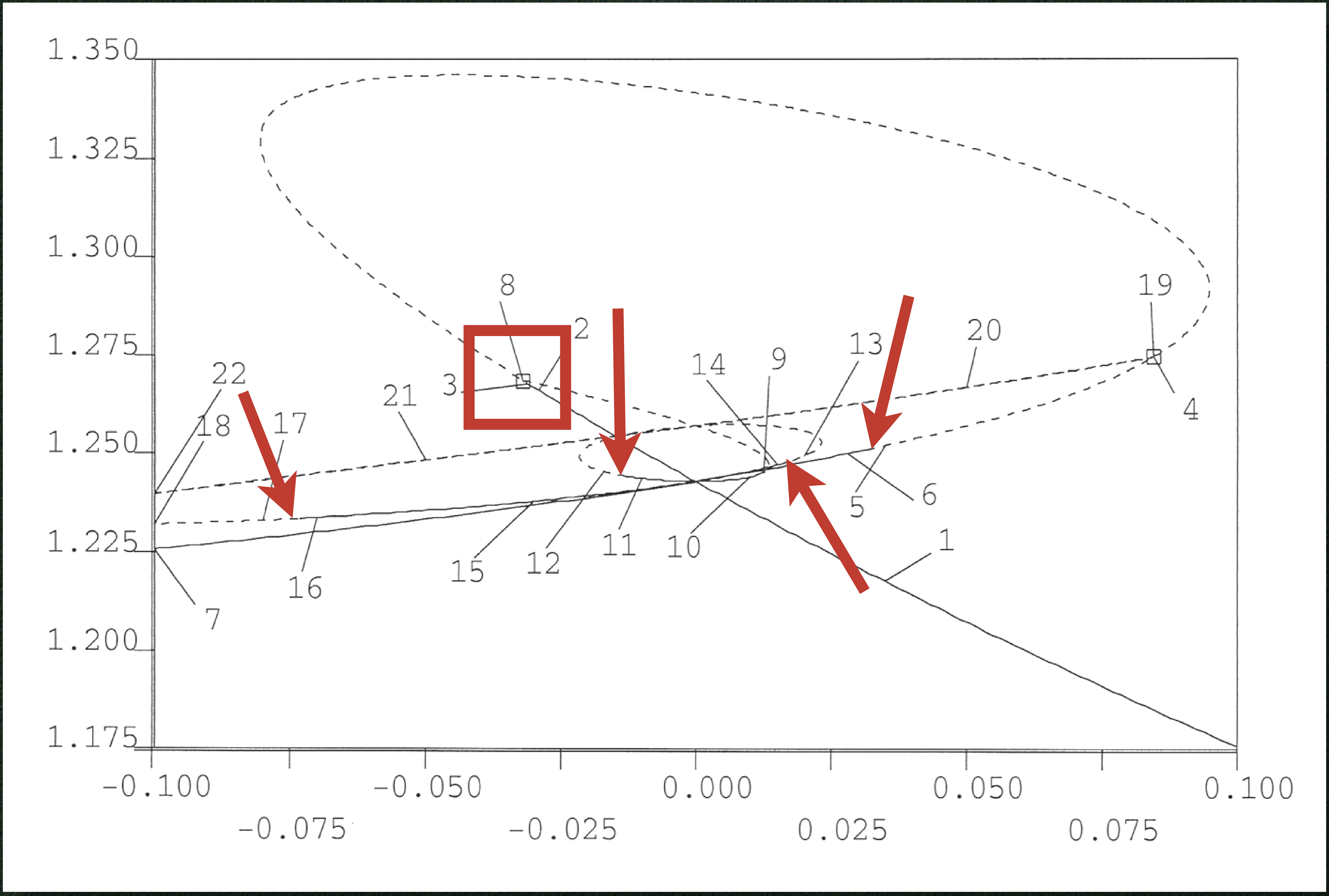
A folk theorem
from bifurcation theory…

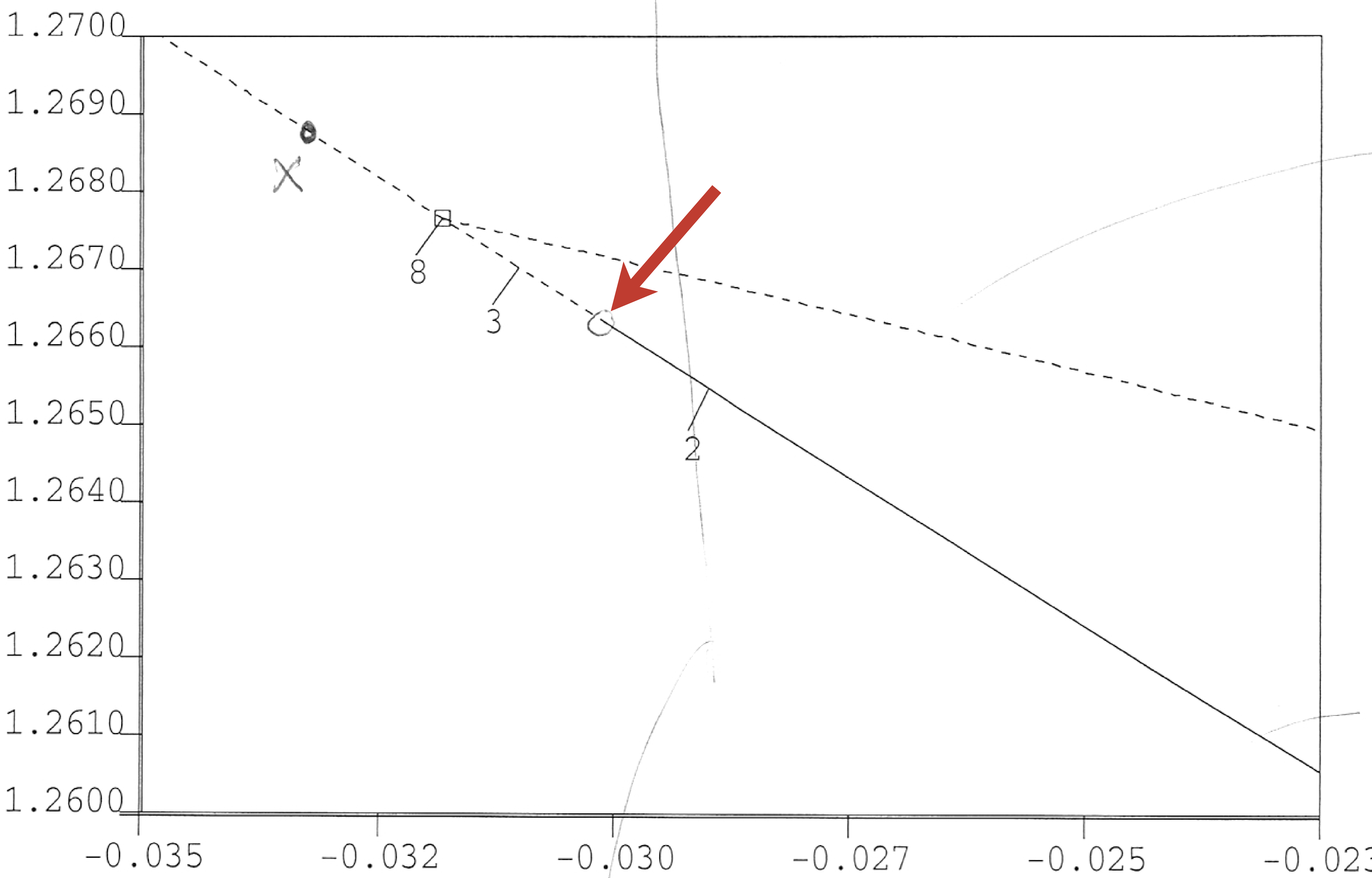
Change of Stability

implies

Bifurcation



Numerical calculations by Sebius Doedel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8
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How can this be explained?

The system studied by Sebius forms a special case of

Discrete Non-Linear Schroedinger Equations

DNLSE

The system is Hamiltonian:

$$\dot{\psi}_k = i\,\frac{\partial H}{\partial \bar{\psi}_k}, \qquad k \in \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z},$$

with

$$\begin{aligned} H &= H(\psi_k, \bar{\psi}_k) \\ &= \sum_{k=1}^{N} \Big(|\psi_k - \psi_{k+1}|^2 \\ &\qquad - |\psi_k|^4 - \delta_k |\psi_k|^2 \Big) \,. \end{aligned}$$

Explicitly the equations take the form:

$$\begin{aligned}\dot{\psi}_k &= -i\Big[(\psi_{k-1} + \psi_{k+1} - 2\psi_k) \\ &\qquad + 2|\psi_k|^2\psi_k + \delta_k\psi_k\Big],\end{aligned}$$

with

$$\psi_k \in \mathbb{C}$$

and

$$k \in \mathbb{Z}_N.$$

The system is equivariant with respect to the S^1-action defined by

$$(\theta, \psi_k) \mapsto e^{i\theta}\psi_k, \quad \forall\theta \in S^1, \forall k.$$

There is a corresponding first integral (Noether's theorem):

$$F = F(\psi_k, \bar{\psi}_k) = \sum_{k=1}^{N} |\psi_k|^2.$$

This S^1-equivariance allows us to consider **relative equilibria**, that is, special periodic solutions of the form

$$\psi_k(t) = e^{i\omega t}\widehat{\psi}_k, \qquad (\omega \in \mathbb{R},\ \widehat{\psi}_k \in \mathbb{C},\ k \in \mathbb{Z}_N).$$

The problem then reduces to a set of algebraic equations, namely:

$$(\omega + \delta_k + 2|\widehat{\psi}_k|^2)\widehat{\psi}_k + (\widehat{\psi}_{k-1} + \widehat{\psi}_{k+1} - 2\widehat{\psi}_k) = 0, \qquad (k \in \mathbb{Z}_N).$$

The system is also **reversible**, with reversor given by

$$R(\psi_1, \psi_2, \ldots, \psi_N) = (\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_N).$$

A special case

When

$$\delta_k = \begin{cases} 0 & \text{for } k \neq m \pmod{N}, \\ \delta & \text{for } k = m \pmod{N}, \end{cases}$$

then there is an additional $\mathbb{Z}_2$-equivariance, generated by

$$S(\psi_1, \ldots, \psi_{m-1}, \psi_m, \psi_{m+1}, \ldots, \psi_N)$$
$$= (\psi_{2m-1}, \ldots, \psi_{m+1}, \psi_m, \psi_{m-1}, \ldots, \psi_{2m}).$$

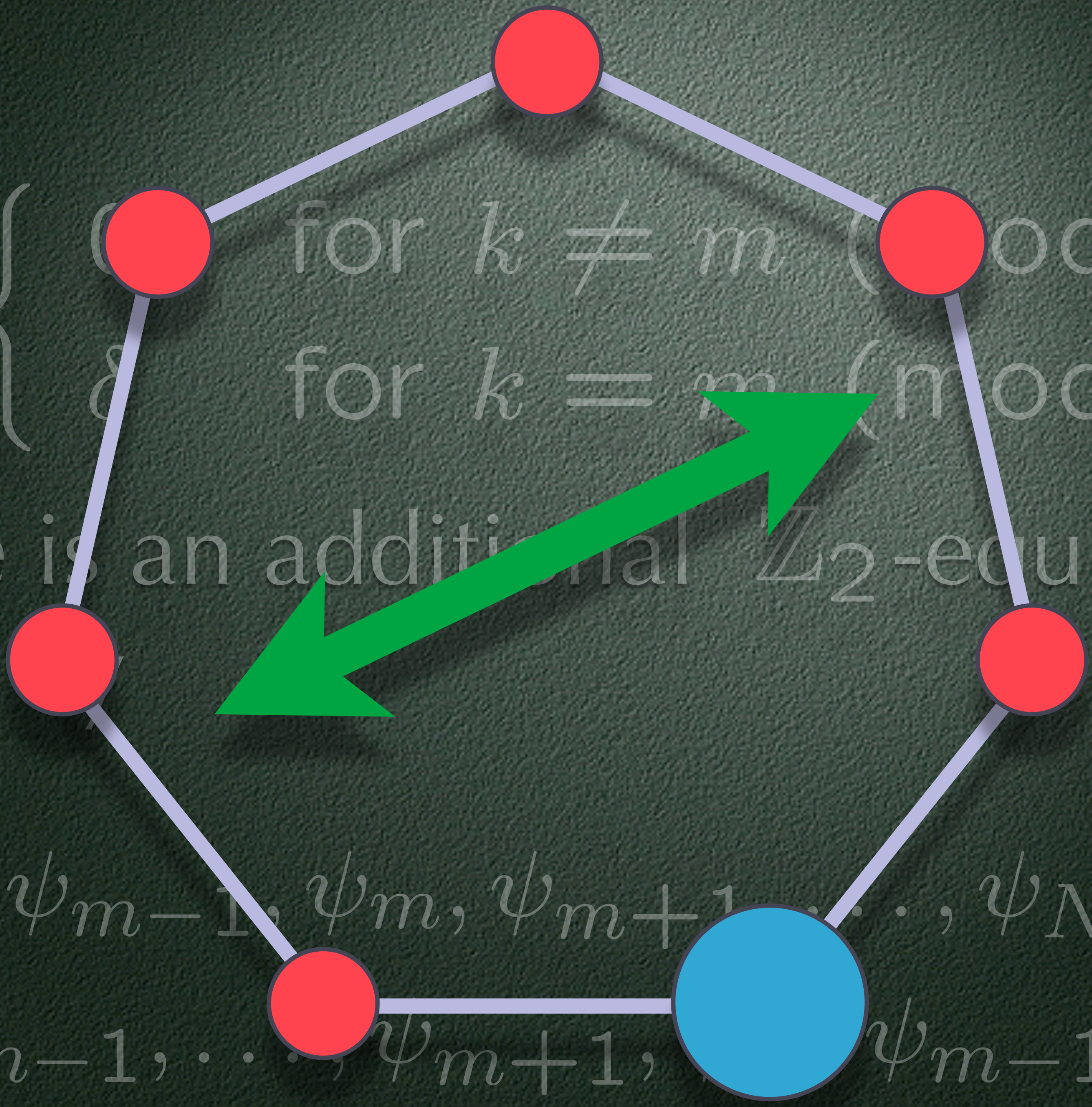
A special case

When

$$\delta_k = \begin{cases} 0 & \text{for } k \neq m \pmod{N}, \\ \delta & \text{for } k = m \pmod{N}, \end{cases}$$

then there is an additional $\mathbb{Z}_2$-equivariance, generated by

$$S(\psi_1,\ldots,\psi_{m-1},\psi_m,\psi_{m+1},\ldots,\psi_N) = (\psi_{2m-1},\ldots,\psi_{m+1},\psi_m,\psi_{m-1},\ldots,\psi_{2m}).$$



Symmetric Periodic Orbits in Reversible Systems

Consider a (smooth) $2N$-dimensional system

$$\dot{x} = X(x),$$

with flow $\tilde{x}(t, x)$.

Suppose there is a linear operator R such that

$$R^2 = Id, \qquad \text{dim}(\text{Fix}(R)) = N,$$

and

$$X(Rx) = -RX(x).$$

Then

$$\tilde{x}(t, Rx) = R\tilde{x}(-t, x).$$

A solution $\tilde{x}(t,x)$ is **symmetric** (or more precisely: R-**symmetric**) if the corresponding orbit

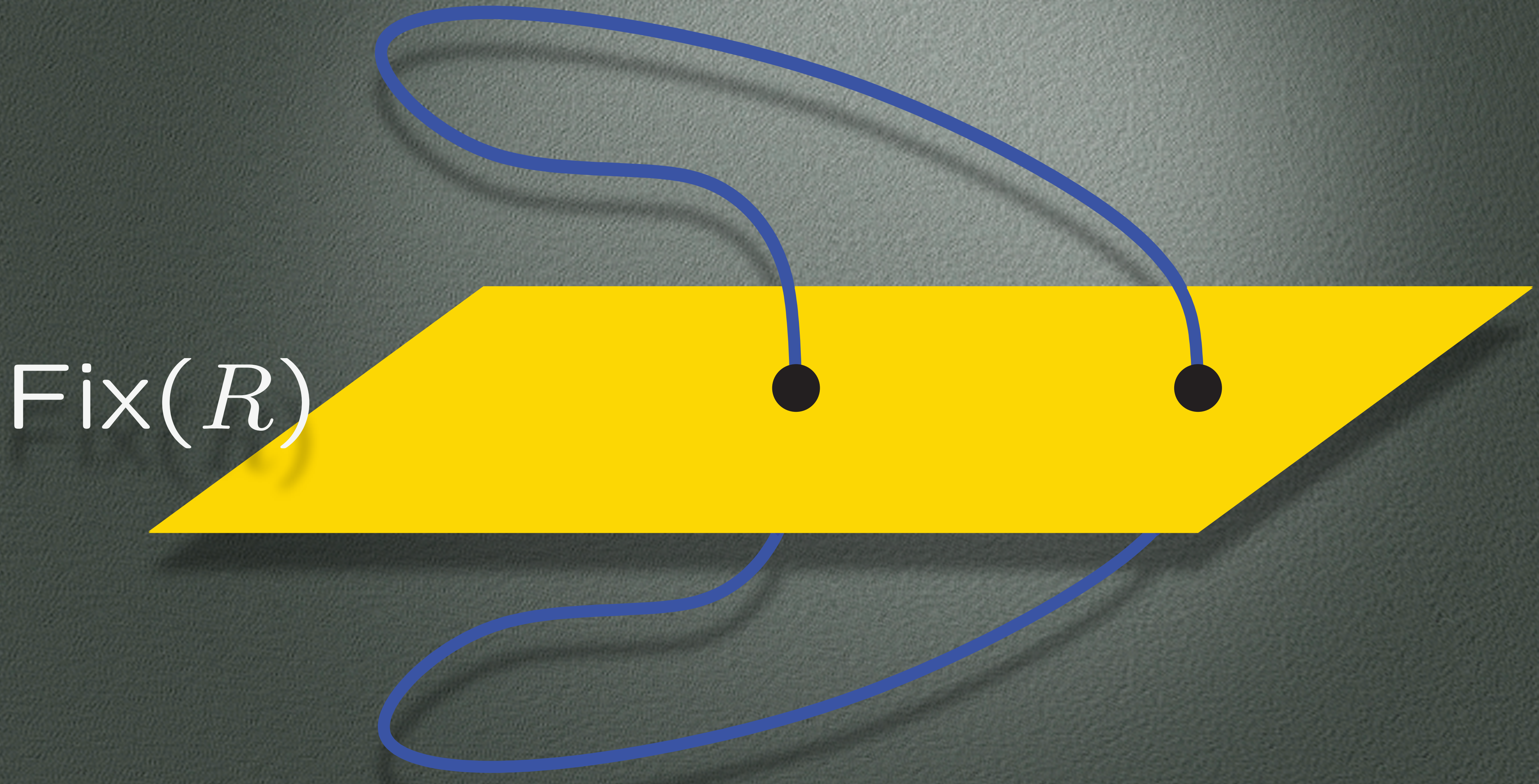
$$\gamma = \{\tilde{x}(t,x) \mid t \in \mathbb{R}\}$$

is R-invariant:

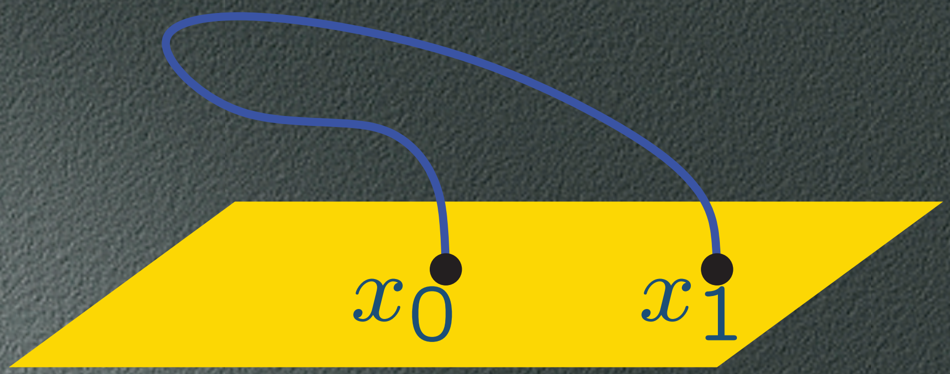
$$R(\gamma) = \gamma.$$

A solution is symmetric if and only if its orbit has at least one intersection point with Fix(R).

A non-equilibrium solution is **symmetric and periodic** if and only if its orbit has exactly two intersection points with Fix(R); the shortest time between these two intersections is one half of the minimal period.


Fix(R)

Now suppose that $x_0 \in \mathsf{Fix}(R)$ generates such symmetric periodic solution, with minimal period $T_0 > 0$, and let $x_1 = \tilde{x}(T_0/2, x_0) \in \mathsf{Fix}(R)$.



Then

$$x_1 \in \tilde{x}(T_0/2, \mathsf{Fix}(R)) \cap \mathsf{Fix}(R).$$

Both $\tilde{x}(T_0/2, \mathsf{Fix}(R))$ and $\mathsf{Fix}(R)$ are N-dimensional.

If the intersection of these two N-dimensional submanifolds is **transversal** at x_1 then the intersection point persists when we replace T_0 by any nearby T, i.e. we obtain a one-parameter family of symmetric periodic orbits parametrized by the period.

Even if $\tilde{x}(T_0/2, \mathrm{Fix}(R))$ and $\mathrm{fix}(R)$ do not intersect transversally at x_1 it may still be that the $(N+1)$-dimensional manifold $\{\tilde{x}(t,x) \mid t \in \mathbb{R}, x \in \mathrm{Fix}(R)\}$ and the N-dimensional manifold $\mathrm{Fix}(R)$ intersect transversally at x_1.

Also in this case we have a one-dimensional family of symmetric periodic orbits; only, this time the family can not be parametrized by the period.

Conclusion

Typically in reversible systems symmetric periodic orbits belong to one-parameter families of such symmetric periodic orbits.

Is it possible to have along such one-parameter family of symmetric periodic orbits a change of stability without any further branching?

Characteristic multipliers of symmetric periodic orbits

The monodromy matrix M_0 of a symmetric periodic orbit $\tilde{x}(t, x_0)$ (with $x_0 \in \mathrm{Fix}(R)$ and $x_1 = \tilde{x}(T_0/2, x_0) \in \mathrm{Fix}(R)$) satisfies the relation

$$RM_0R = M_0^{-1}.$$

As a consequence, if $\mu \in \mathbb{C}$ is a multiplier, then so are μ^{-1}, $\bar{\mu}$ and $\bar{\mu}^{-1}$.

Characteristic multipliers of symmetric periodic orbits

- $\mu = 1$ is always a multiplier, with even algebraic multiplicity ≥ 2
- if $\mu = -1$ is a multiplier, then its algebraic multiplicity is even
- all other multipliers come in pairs $\{\mu, \bar{\mu}\}$ with $|\mu| = 1$, or in quadruples $\{\mu, \bar{\mu}, \mu^{-1}, \bar{\mu}^{-1}\}$ with $|\mu| \neq 1$

Characteristic multipliers of symmetric periodic orbits

The same rules apply to the eigenvalues of $DP(x_0)$, where P is any Poincaré-map at x_0, except that $\mu = 1$ is always an eigenvalue, with odd multiplicity $m_a \geq 1$.

A change of stability is only possible (and will typically happen) if $m_a \geq 3$.

Now suppose that — as in our DNLSE example — the system has also a first integral $H : \mathbb{R}^{2N} \to \mathbb{R}$.

Let $\Sigma = X(x_0)^\perp$ and consider the corresponding Poincaré-map $P : x_0 + \Sigma \to x_0 + \Sigma$. Let H_Σ be the restriction of H to $x_0 + \Sigma$. Then

$$H_\Sigma(P(x)) = H_\Sigma(x), \quad \forall x \in x_0 + \Sigma.$$

Differentiating this relation at $x_0 = P(x_0)$ gives us

$$DH_\Sigma(x_0) \cdot DP(x_0) = DH_\Sigma(x_0),$$

in other words:

$$\mathrm{Im}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big) \subset \nabla H_\Sigma(x_0)^\perp.$$

From the other side, if we assume that the symmetric periodic solution $\tilde{x}(t, x_0)$ belongs to a one-parameter family of such solutions, then P must have a one-dimensional curve of fixed points passing through x_0:

$$P(\hat{x}(s)) = \hat{x}(s), \quad \text{with } \hat{x}(s) \in x_0 + \Sigma \text{ and } \hat{x}(0) = x_0.$$

Differentiating at $s = 0$ shows that

$$DP(x_0) \cdot \hat{x}'(0) = \hat{x}'(0),$$

that is:

$$\hat{x}'(0) \in \mathrm{Ker}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big).$$

So, our assumptions imply

$$\widehat{x}'(0) \in \mathrm{Ker}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big)$$

and

$$\mathrm{Im}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big) \subset \nabla H_\Sigma(x_0)^\perp.$$

We assume that both $\widehat{x}'(0)$ and $\nabla H_\Sigma(x_0)$ are non-zero.

If moreover 1 is a **simple eigenvalue** of $DP(x_0)$ then we have also that

$$\Sigma = \mathrm{Ker}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big) \oplus \mathrm{Im}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big).$$

Therefore, if 1 is a simple eigenvalue of $DP(x_0)$:

$$\mathrm{Im}\Big(DP(x_0) - \mathrm{Id}_\Sigma\Big) = \nabla H_\Sigma(x_0)^\perp$$

and

$$DH(x_0) \cdot \widehat{x}'(0) \neq 0.$$

In particular, if

$$DH(x_0) \cdot \widehat{x}'(0) = 0,$$

— for example when H reaches a local maximum or minimum along the given branch of symmetric periodic orbits — then we have necessarily

$$m_a \geq 3,$$

and typically there will be a change of stability.

A change of stability without branching!

Of course we could also look at it this way...

However, in our example, there is no reason why at each of the indicated transition points the Hamiltonian should reach a maximum or a minimum!

Therefore, in order to obtain some more convincing results, we have to make a more detailed analysis of the Poincaré-map.

We start with a symmetric periodic orbit in a reversible system.

We assume that the linearization of a Poincaré-map at this symmetric periodic orbit has 1 as an eigenvalue with algebraic multiplicity $m_a = 3$.

We then explore the different possibilities...

Let the symmetric periodic orbit be given by

$$\gamma_0 = \{\tilde{x}(t, x_0) \mid t \in \mathbb{R}\},$$

with $x_0 \in \mathrm{Fix}(R)$, and with minimal period $T_0 > 0$.

To define a Poincaré map we consider at x_0 a section

$$x_0 + \Sigma$$

transversal to γ_0, and such that the subspace Σ of $\mathbb{R}^{2N}$ is R-invariant. For example, we can assume that R is orthogonal, and take $\Sigma = X(x_0)^{\perp}$.

Since $X(x_0) \in \mathrm{Fix}(-R)$ this implies that

$$\dim\left(\mathrm{Fix}(R) \cap \Sigma\right) = N.$$

We define the Poincaré-map $P : \Sigma \to \Sigma$ by

$$P(y) := \tilde{x}(\tau(y), x_0 + y) - x_0,$$

where $\tau : \Sigma \to \mathbb{R}$ is the (unique) function which is smooth near the origin and such that

$$\tilde{x}(\tau(y), x_0 + y) \in x_0 + \Sigma \quad \text{and} \quad \tau(0) = T_0.$$

Then

$$R \circ P \circ R = P^{-1}.$$

Also

$$P(0) = 0 \quad \text{and} \quad \dim(\ker(A_s - I_\Sigma)) = 3,$$

where A_s is the semisimple part of $A := DP(0)$.

We look then for fixed points of P near x_0.

The simple answer:
That is what Sebius did...

The more complicated answer:
For each $k \geq 2$ there are a finite number of conditions on the eigenvalues of $DP(x_0)$ which ensure that all q-periodic points of P in a sufficiently small neighborhood of x_0 and with $q \leq k$ are fixed points of P.

So we are interested in the small fixed points $y \in \Sigma$ of the Poincaré-map P.

To describe these fixed points we can use a Lyapunov-Schmidt reduction; this reduction involves the 3-dimensional and R-invariant subspace

$$U := \ker(A_s - I_\Sigma).$$

Observe that

$$\dim\left(U \cap \mathrm{Fix}(R)\right) = 2$$

and

$$\dim\left(U \cap \mathrm{Fix}(-R)\right) = 1$$

The Lyapunov-Schmidt reduction tells us that there is a smooth 1-1-relation between the small fixed points of P and the small fixed points of a smooth mapping $P_0 : U \to U$ which has the following properties:

(1) $P_0(0) = 0$ and $DP_0(0) = DP(0)\big|_U$;

(2) $R_0 \circ P_0 \circ R_0 = P_0^{-1}$, where $R_0 = R\big|_U$.

Moreover, the small fixed points of P_0 coincide with the solutions of the equation

$$B(u) := P_0(u) - P_0^{-1}(u) = 0.$$

Observe that

$$B(R_0 u) = -R_0 B(u), \quad \forall u \in U.$$

Moreover, $B(0) = 0$ and $DB(0) = 2A_0$, where A_0 is a nilpotent linear operator on U defined by

$$DP_0(0) = e^{A_0}$$

and satisfying

$$R_0 A_0 = -A_0 R_0.$$

Using normal forms we can approximate $P_0(u)$ and $B(u)$ up to any order; these normal forms give also information on the stability of the fixed points.

Taking into account that

$$\dim(\mathrm{Fix}(R_0) = 2 \quad \text{and} \quad \dim(\mathrm{Fix}(-R_0) = 1,$$

and restricting to cases which have codimension 1 on the linear level, we find the following cases.

Using normal forms we can approximate $P_0(u)$ and $B(u)$ u… …rmal forms give also inf… …fixed points.

> The singularity which we want to study should appear along one-parameter families of symmetric periodic orbits

dim(… …$-R_0) = 1$,

and restricting to cases which have codimension 1 on the linear level, we find the following cases.

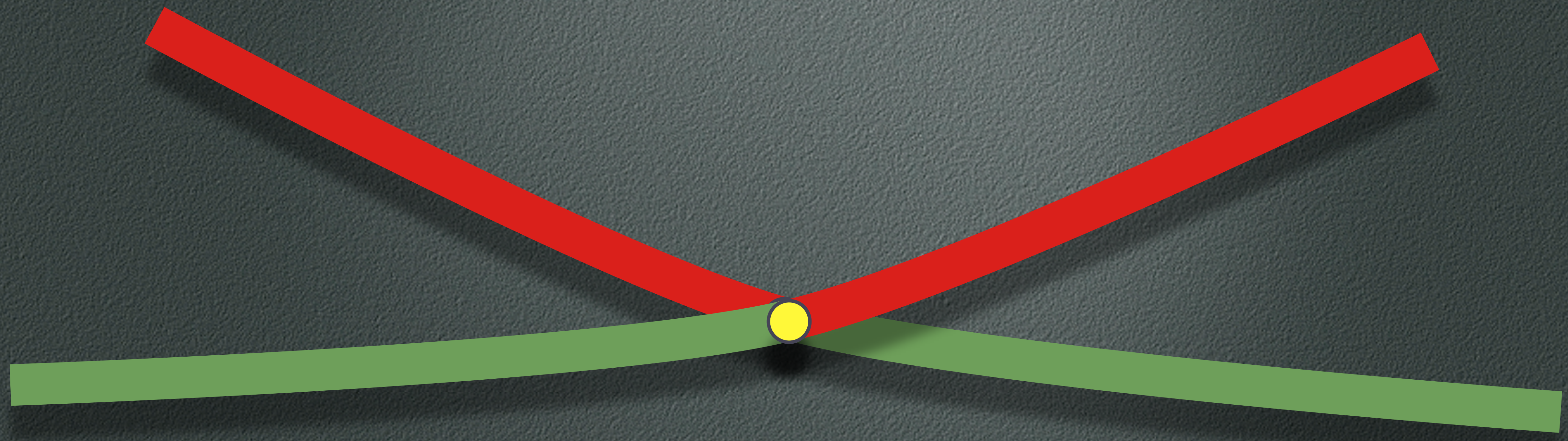
The codimension condition implies that $A_0 \neq 0$.

❶ $A_0^2 = 0$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

❶ $A_0^2 = 0$



Two crossing branches of symmetric periodic orbits; there is an exchange of stability at the crossing.

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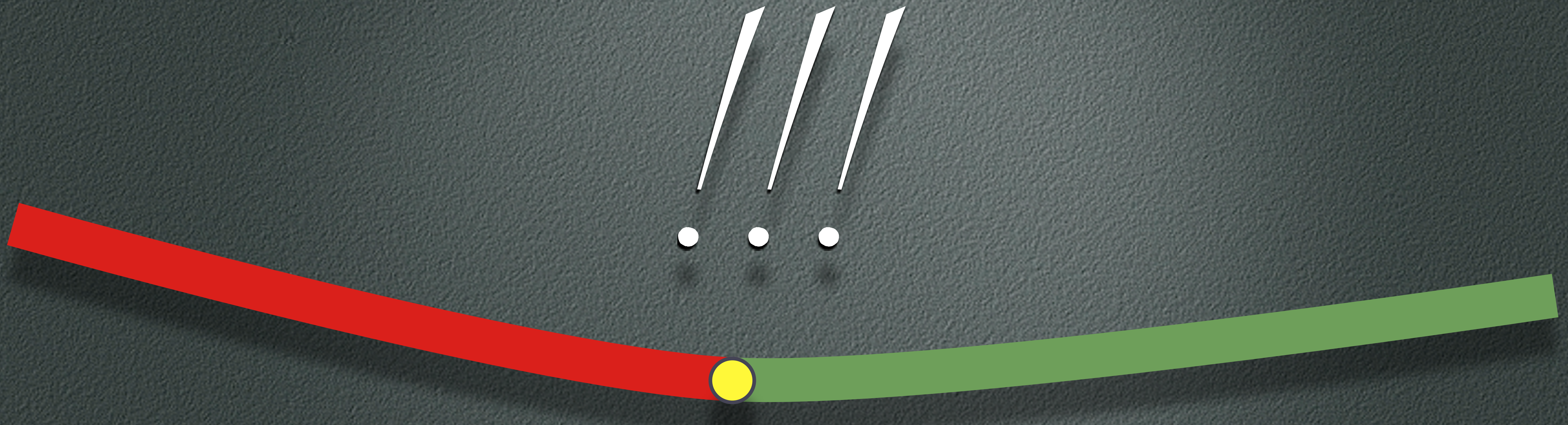

A single branch of symmetric periodic orbits. There is a change of stability, but no bifurcation.

③ $A_0^2 \neq 0$

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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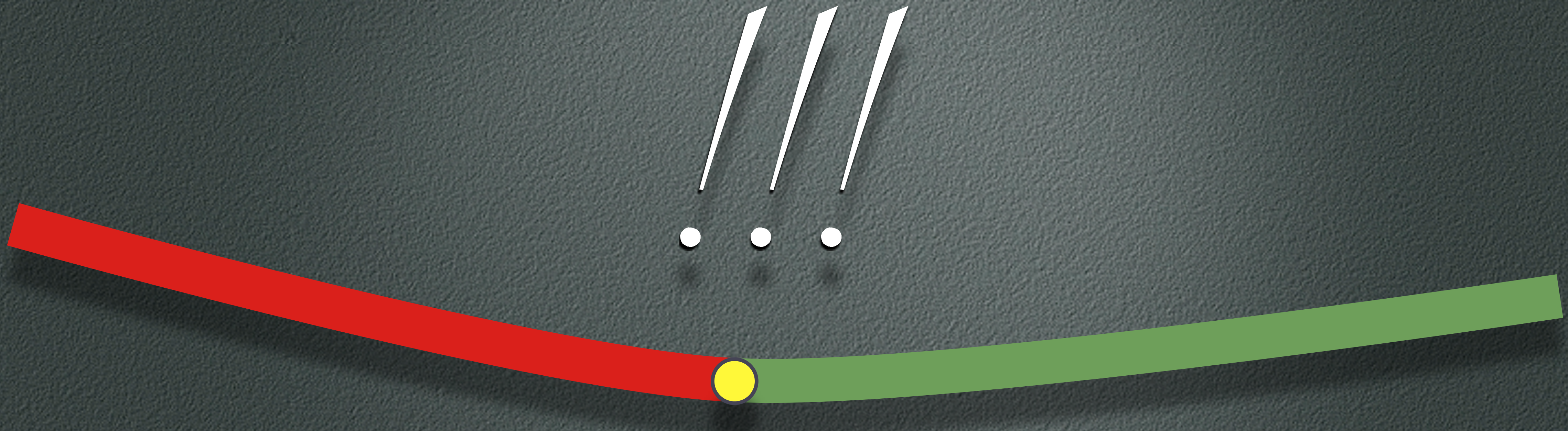


A single branch of symmetric periodic orbits. There is a change of stability, but no bifurcation.

When we turn to the case where our reversible system has an additional $S^1 \times \mathbb{Z}_2$-symmetry and when we suppose that γ_0 is a symmetric relative equilibrium which is also symmetric under the $\mathbb{Z}_2$-symmetry, then the Poincaré-map has a special structure and some further possibilities arise.

First we have to clarify the set-up for this particular case.

We consider again a smooth system

$$\dot{x} = X(x), \quad (\text{with } x \in \mathbb{R}^{2N}),$$

and assume that there exist orthogonal linear operators J_0, S and R such that the following holds:

(1) $J_0^2 = -\text{Id}$, $S^2 = \text{Id}$ and $R^2 = \text{Id}$;

(2) $J_0 S = S J_0$, $J_0 R = -R J_0$ and $SR = RS$;

(3) $\dim(\text{Fix}(R)) = N$ and $\dim(\text{Fix}(RS)) = N$;

(4) $X(e^{J_0\theta}x) = e^{J_0\theta}X(x)$ for all $\theta \in S^1 = \mathbb{Z}/2\pi\mathbb{Z}$;

(5) $X(Sx) = SX(x)$;

(6) $X(Rx) = -RX(x)$.

Observe: there are lots of reversors:

$$R, \; RS, \; Re^{J_0\theta}, \; RSe^{J_0\theta}.$$

Now let $x_0 \in \mathbb{R}^{2N}$ be such that

$$Rx_0 = Sx_0 = x_0$$

and

$$X(x_0) = \Omega_0 J_0 x_0, \quad (\Omega_0 \neq 0).$$

Then x_0 generates a relative equilibrium

$$\tilde{x}(t, x_0) = e^{\Omega_0 J_0 t} x_0,$$

with an orbit γ_0 which is invariant under both the reversors R and RS, and with minimal period $T_0 = 2\pi/\Omega_0$ (by replacing J_0 by $-J_0$ when necessary we may w.l.o.g. assume that $\Omega_0 > 0$).

To study bifurcations near γ_0 we construct a Poincaré-map, as follows.

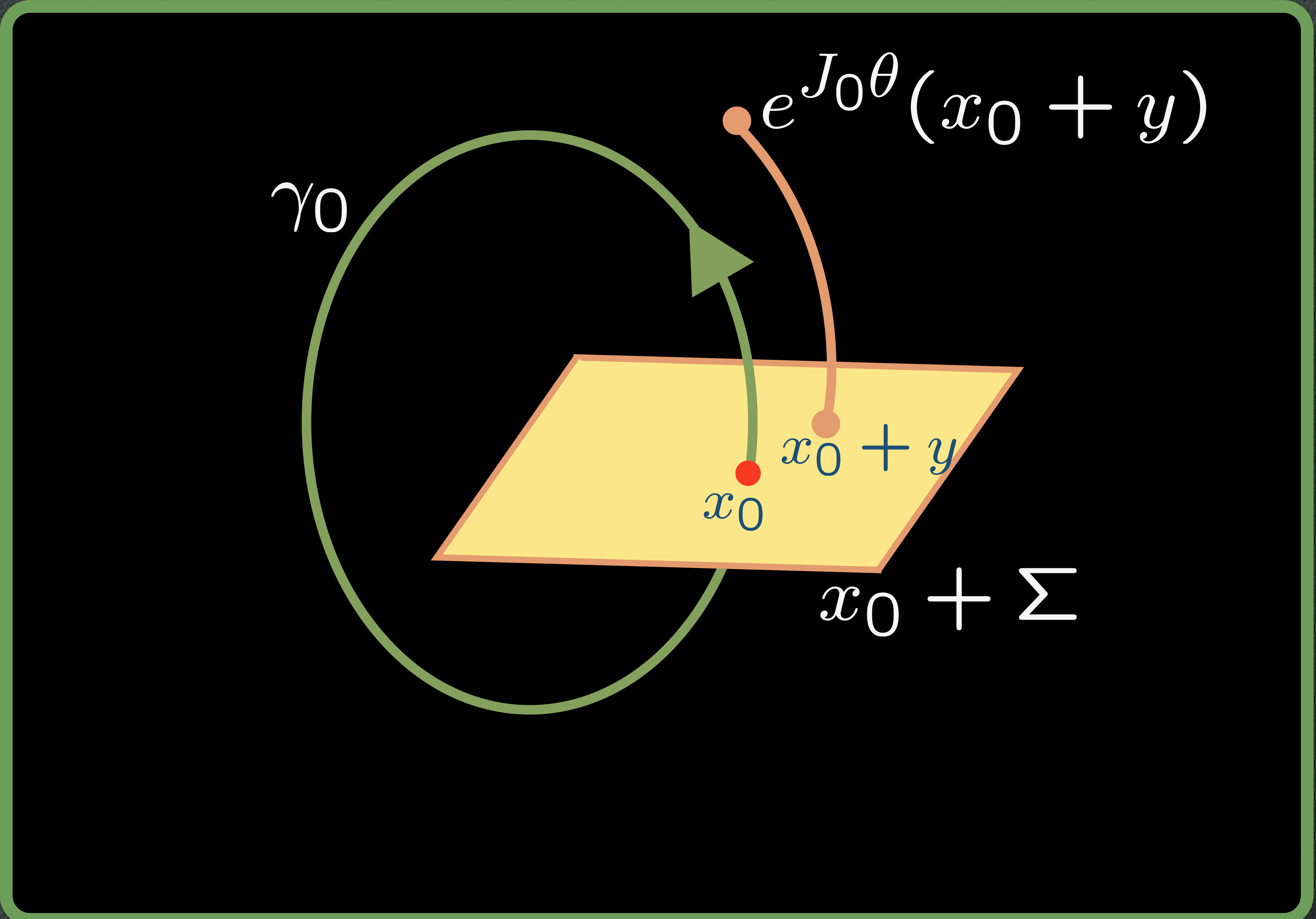
We set $\Sigma = (J_0 x_0)^\perp$ and observe that (by the orthogonality of J_0, R and S) the transversal section $x_0 + \Sigma$ is invariant under R, S and RS.

We have

$$\dim\Big(\mathrm{Fix}(R) \cap \Sigma\Big) = \dim\Big(\mathrm{Fix}(RS) \cap \Sigma\Big) = N.$$

All points in a sufficiently small tubular neighborhood of γ_0 have a unique representation of the form

$$e^{J_0\theta}\Big(x_0 + y\Big), \quad \text{with } (\theta, y) \in S^1 \times \Sigma.$$



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of the
form

$$e^{J_0\theta}\big(x_0 + y\big), \quad \text{with } (\theta, y) \in S^1 \times \Sigma.$$

We have

$$\mathbb{R}^{2N} = \mathbb{R}(J_0 x_0) \oplus \Sigma,$$

and therefore, by continuity and for $y \in \Sigma$ sufficiently small,

$$\mathbb{R}^{2N} = \mathbb{R}\Big(J_0(x_0 + y)\Big) \oplus \Sigma.$$

Therefore, for $y \in \Sigma$ sufficiently small, we have

$$X(x_0 + y) = \Omega(y) J_0(x_0 + y) + Y(y),$$

with $\Omega : \Sigma \to \mathbb{R}$ and $Y : \Sigma \to \Sigma$ such that:

(1) $\Omega(0) = \Omega_0$ and $Y(0) = 0$;
(2) $\Omega(Ry) = \Omega(Sy) = \Omega(y)$;
(3) $Y(Ry) = -RY(y)$ and $Y(Sy) = SY(y)$.

As a consequence, it is not hard to prove that

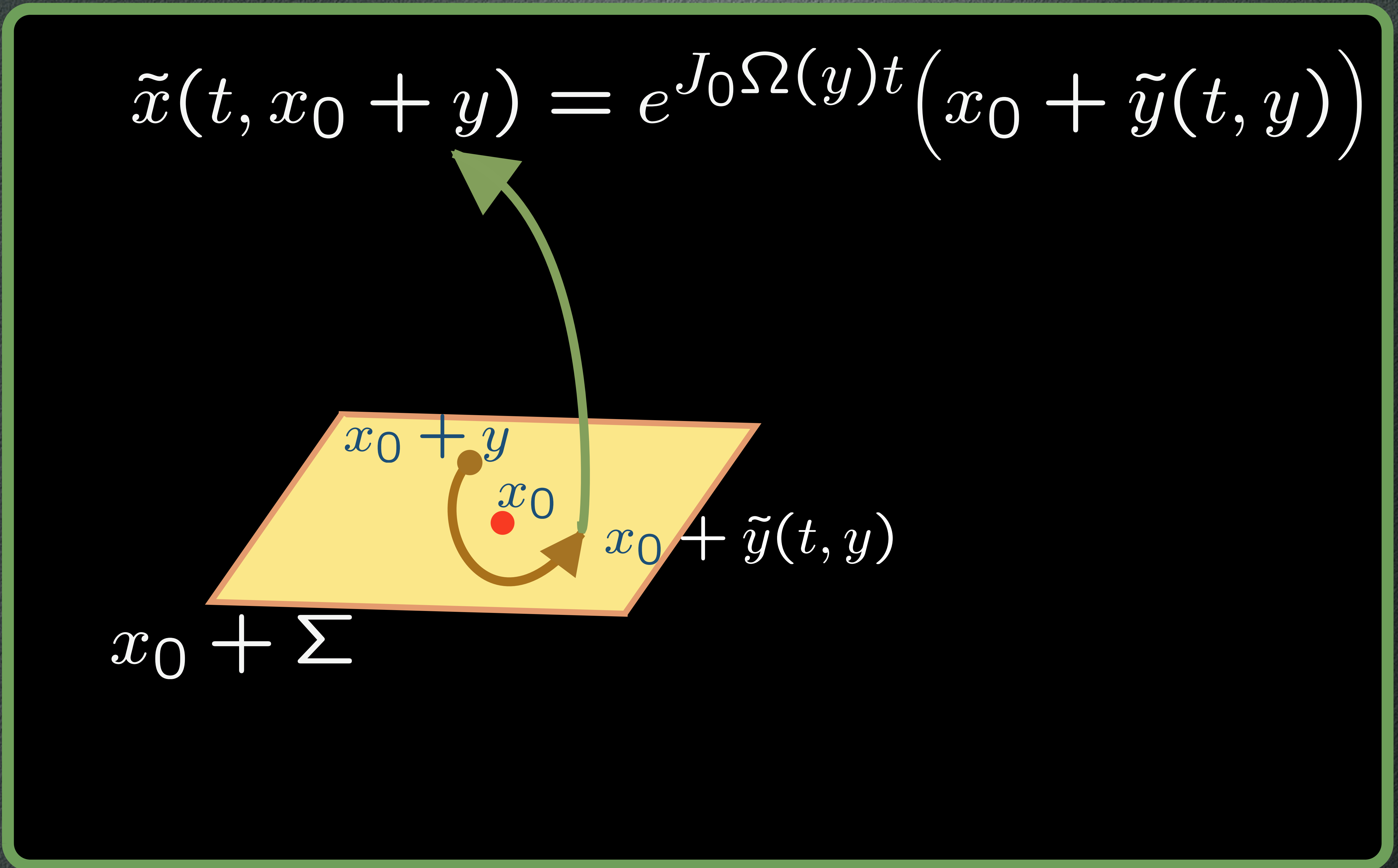
$$\tilde{x}(t, x_0 + y) = e^{J_0 \Omega(y) t}\Big(x_0 + \tilde{y}(t, y)\Big),$$

where $\tilde{y}(t, y)$ is the flow of the reversible, $\mathbb{Z}_2$-equivariant and $(2N-1)$-dimensional system

$$\dot{y} = Y(y).$$

As a consequence, it is not hard to prove that

$$\tilde{x}(t, x_0 + y) = e^{J_0 \Omega(y) t}\Big(x_0 + \tilde{y}(t, y)\Big),$$



It follows that the Poincaré-map $P : \Sigma \to \Sigma$ is given by

$$P(y) = \tilde{y}(2\pi\Omega(y)^{-1}, y).$$

Also $P(0) = 0$ and

$$DP(0) = \exp\left(2\pi\Omega_0^{-1}A\right), \quad \text{with } A = DY(0).$$

We assume (as before):

$DP(0)$ has the eigenvalue 1
with algebraic multiplicity $m_a = 3$.

We look then for small fixed points of P.

Due to the relation

$$P(y) = \tilde{y}(2\pi\Omega(y)^{-1}, y)$$

a fixed point $y \in \Sigma$ of P is

- either an equilibrium of the system

$$\dot{y} = Y(y);$$

- or a point belonging to a periodic orbit of the same system with minimal period of the form

$$\frac{2\pi}{m\Omega(y)}, \qquad (m \geq 1).$$

So we have to study (small) equilibria and periodic orbits (with minimal period near $2\pi(m\Omega_0)^{-1}$) of the reversible and $\mathbb{Z}_2$-equivariant system

$$\dot{y} = Y(y),$$

under the further condition that

$$DP(0) = \exp\left(2\pi\Omega_0^{-1}A\right) \qquad (A = DY(0))$$

has the eigenvalue 1 with algebraic multiplicity 3.

Again we use a Lyapunov-Schmidt reduction.

We set

$$U = \ker\Big((DP(0) - \mathrm{Id})^3\Big).$$

This subspace of Σ is 3-dimensional and invariant under A, R and S.

Setting

$$A_0 = A\Big|_U, \ \ R_0 = R\Big|_U \text{ and } S_0 = S\Big|_U,$$

we have the following properties:

(1) $\Big(\exp(2\pi\Omega_0^{-1}A_0) - \mathrm{Id}_U\Big)^3 = 0$, $R_0^2 = \mathrm{Id}_U$ and $S_0^2 = \mathrm{Id}_U$;

(2) $A_0R_0 = -R_0A_0$, $A_0S_0 = S_0A_0$ and $R_0S_0 = S_0R_0$;

(3) $\dim(\mathrm{Fix}(R_0)) = \dim(\mathrm{Fix}(R_0S_0)) = 2$.

Appropriate versions of Lyapunov-Schmidt tell us that it is sufficient to study equilibria and periodic orbits of a reduced system

$$\dot{u} = Z(u),$$

where the reduced vectorfield $Z : U \to U$ has the following properties:

(1) $Z(0) = 0$ and $DZ(0) = A_0$;
(2) Z commutes with the semi-simple part of A_0;
(3) $Z(R_0 u) = -R_0 Z(u)$ and $Z(S_0 u) = S_0 Z(u)$.

With these ingredients we can start a detailed study, exploring the different possibilities for A_0, R_0 and S_0.

Assuming a number of generic conditions on higher order coefficients in the Taylor expansion of $Z(z)$, imposing again our "codimension 1" condition, and possibly interchanging R and RS the problem boils down to 5 cases; 3 of these correspond to the 3 cases which we found in the general case, but there are also 2 new situations which depend on the additional symmetries.

❶ $A_0^2 = 0, \quad R_0 S_0 = R_0$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = R_0 S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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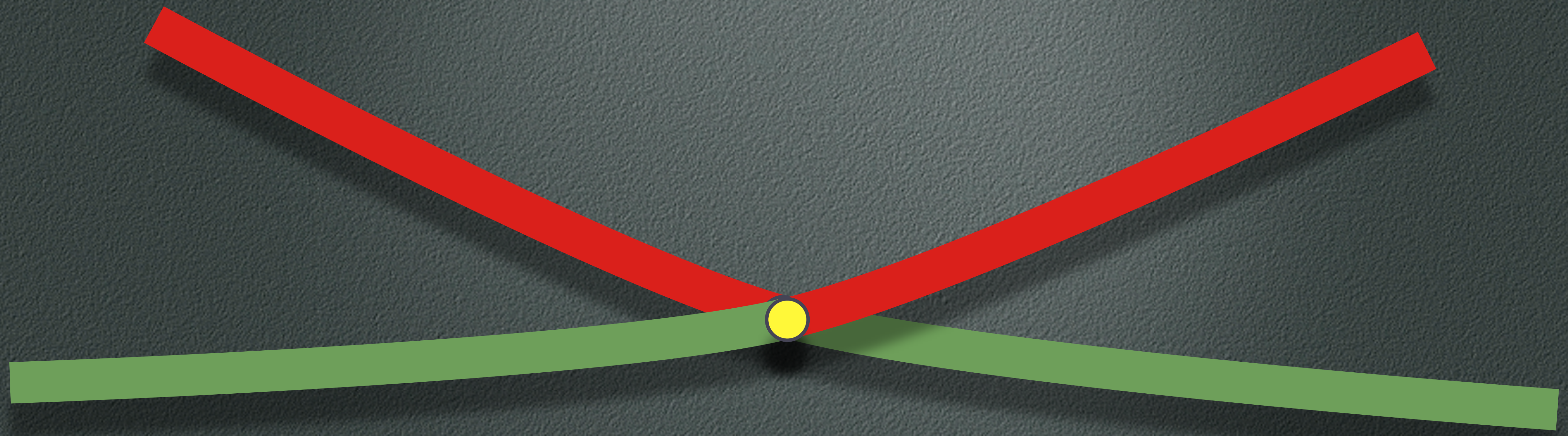

Two crossing branches of relative equilibria symmetric w.r.t. both R and RS; exchange of stability at the crossing.

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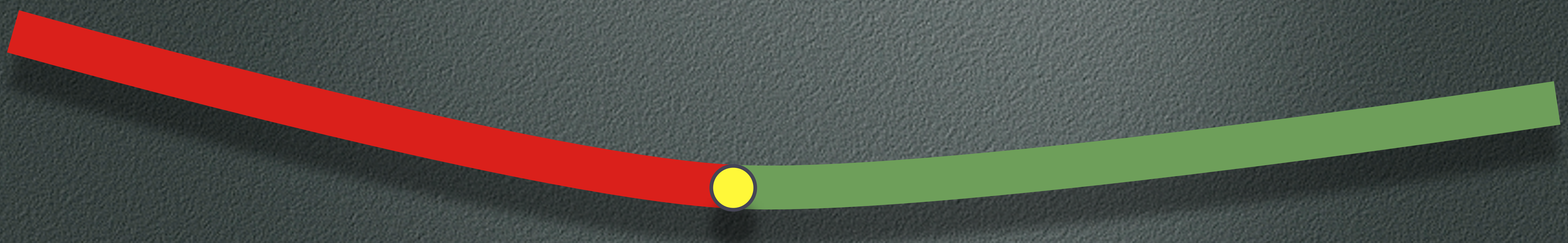


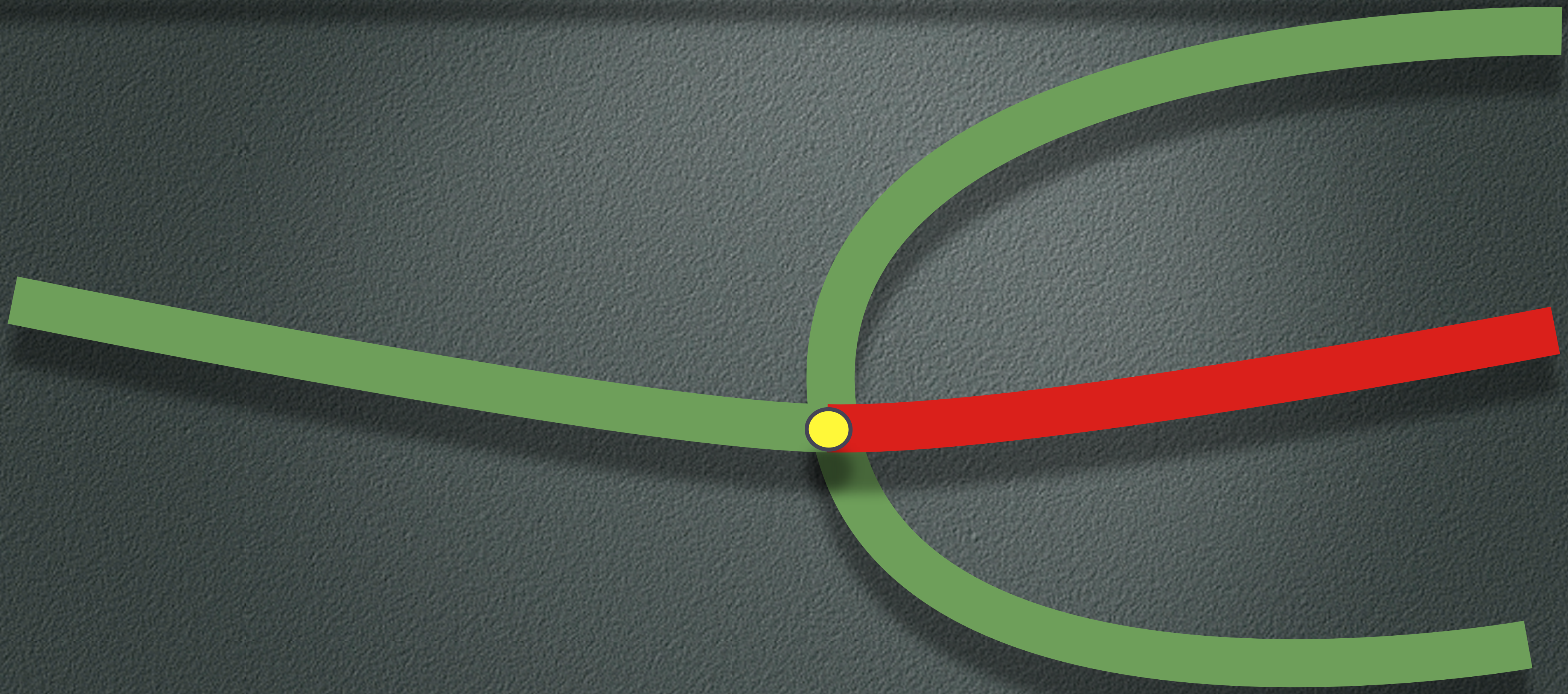
A single branch of relative equilibria, symmetric w.r.t. both R and RS. A change of stability, and no bifurcation.

③ $A_0^2 = 0, \quad R_0 S_0 \neq R_0$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad R_0 S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

❸ $A_0^2 = 0, \quad R_0 S_0 \neq R_0$



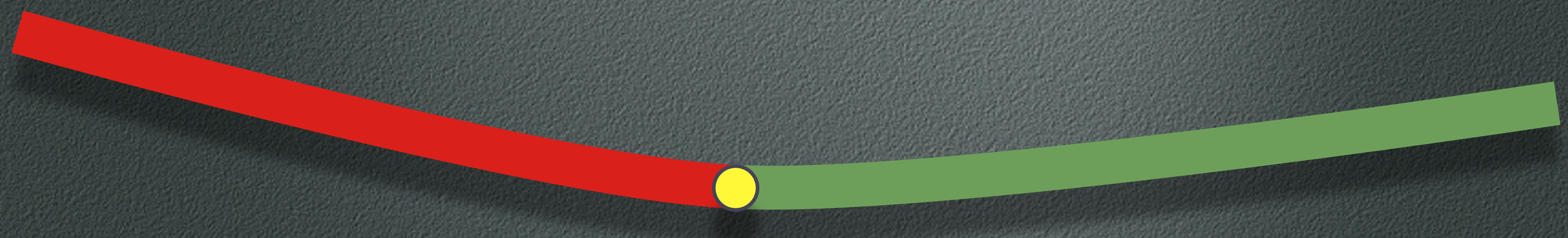
A pitchfork bifurcation of relative equilibria, with the usual exchange of stability and symmetry-breaking. Central branch is symmetric w.r.t. both R and RS, the bifurcating branch is only symmetric w.r.t. R.

④ $A_0^2 \neq 0, \quad R_0 S_0 = R_0$

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = R_0 S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

④ $A_0^2 \neq 0, \quad R_0 S_0 = R_0$



A single branch of relative equilibria, symmetric w.r.t. both R and RS. A change of stability, and no bifurcation.

❺ $\dim \mathrm{Ker}(A_0) = 1,\ R_0 = R_0 S_0$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m\Omega_0 \\ 0 & -m\Omega_0 & 0 \end{pmatrix}$$

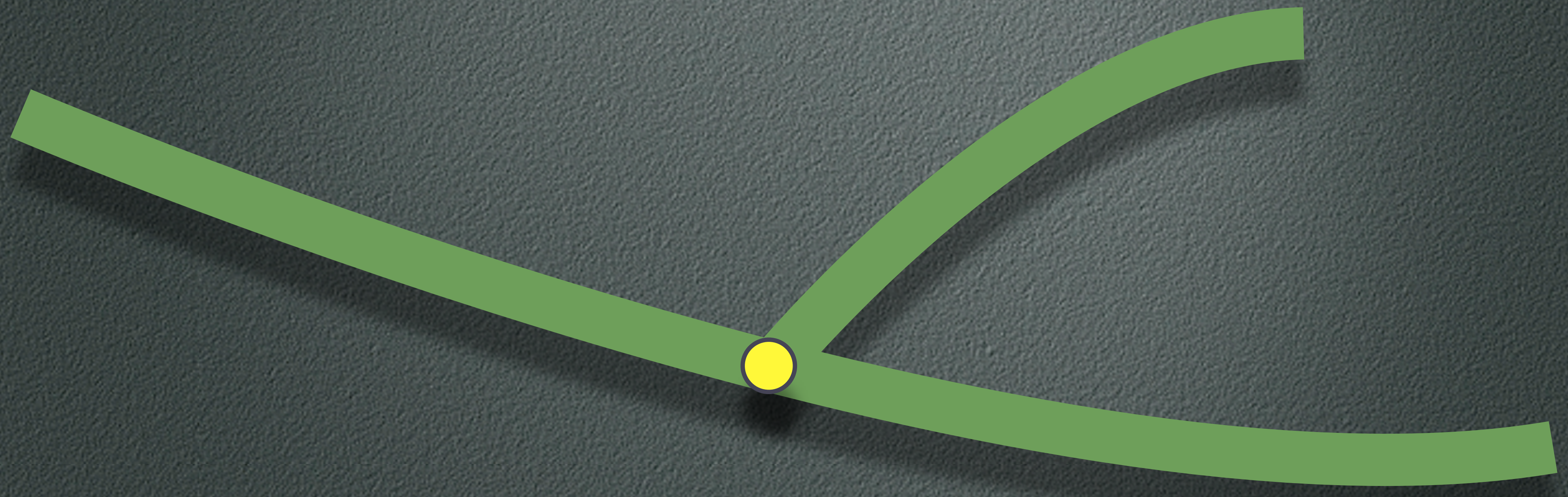
$$R_0 = R_0 S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

❺ $\dim \mathrm{Ker}(A_0) = 1,\ R_0 \neq R_0 S_0$

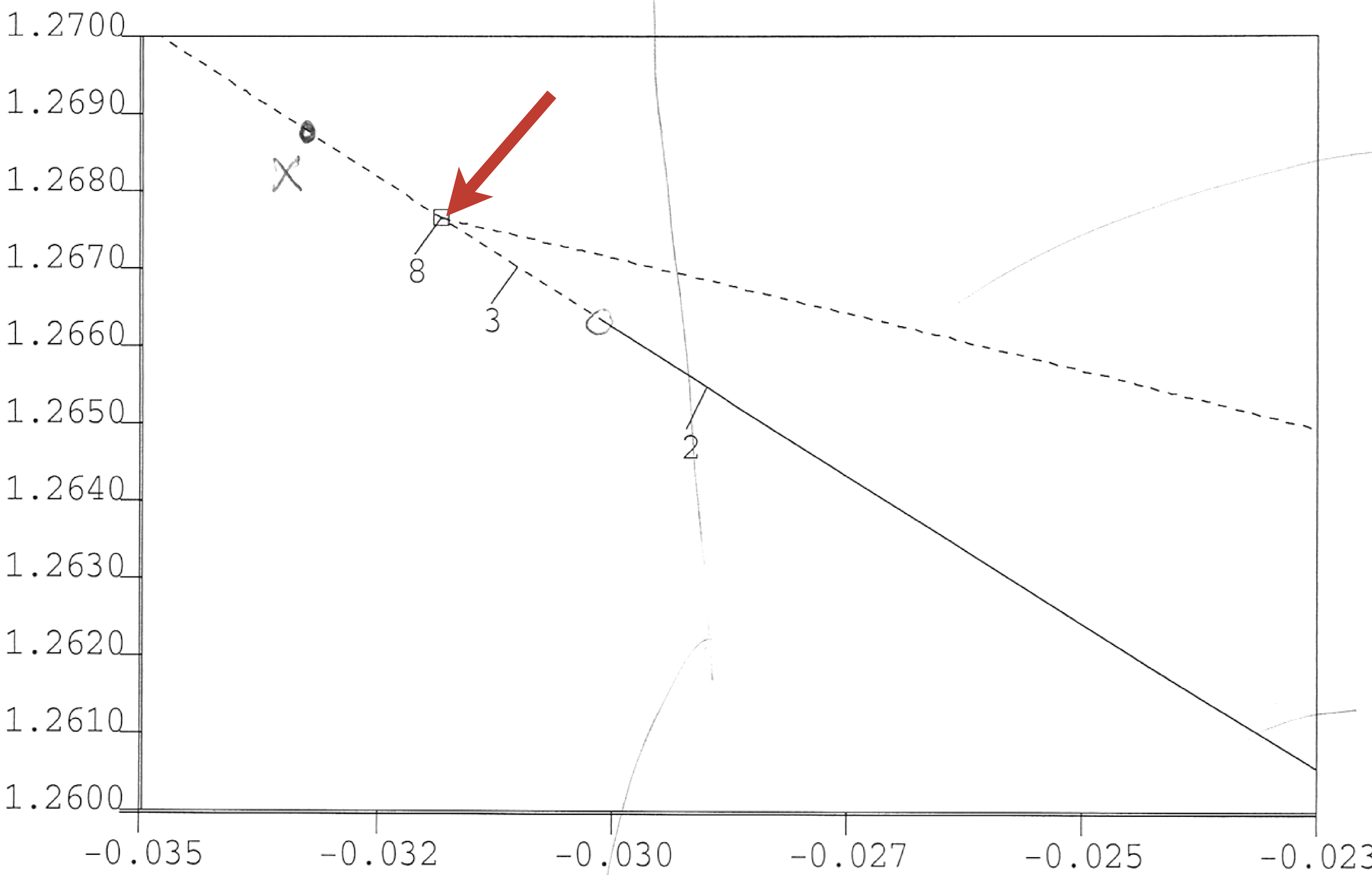
$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m\Omega_0 \\ 0 & -m\Omega_0 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad R_0 S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

❺ $\dim \operatorname{Ker}(A_0) = 1$



A branch of relative equilibria, symmetric w.r.t. both R and RS, and a bifurcating (half-)branch of 2-tori filled with periodic orbits; the symmetry of these depend on the case. No change of stability.



1.2700
1.2690
1.2680
1.2670
1.2660
1.2650
1.2640
1.2630
1.2620
1.2610
1.2600
-0.035
-0.032
-0.030
-0.027
-0.025
-0.023
8
3
2

Branching without...

...Change of Stability!

Muchas Gracias!

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