Recent Trends in Dynamical Systems

In honour of Jürgen Scheurle

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Branches of periodic orbits in reversible systems

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Dedicated to
Jürgen Scheurle
on the occasion
of his 60th birthday

Based on joint work with

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Reversible Systems

$$\dot{x} = F(x)$$

$x \in \mathbb{R}^n$

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth

Reversibility:

→ a (closed) subgroup $\Gamma \subset GL(n;\mathbb{R})$

→ a nontrivial character $\chi : \Gamma \longrightarrow \{1, -1\}$

(R) $$gF(x) = \chi(g)F(gx), \quad \forall g \in \Gamma, \forall x \in \mathbb{R}^n$$

$$\Rightarrow \quad g\tilde{x}(t;x) = \tilde{x}(\chi(g)t; gx), \quad \forall g \in \Gamma, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$$

Simple case

$\Gamma = \{I, R\}, \quad \chi(R) = -1$

with $R \in \mathcal{L}(\mathbb{R}^n)$ a linear involution: $R^2 = I$

$$(R) \qquad \boxed{RF(x) = -F(Rx)} \quad \forall x \in \mathbb{R}^n$$

$$\Rightarrow \quad R\tilde{x}(t, x) = \tilde{x}(-t, Rx)$$

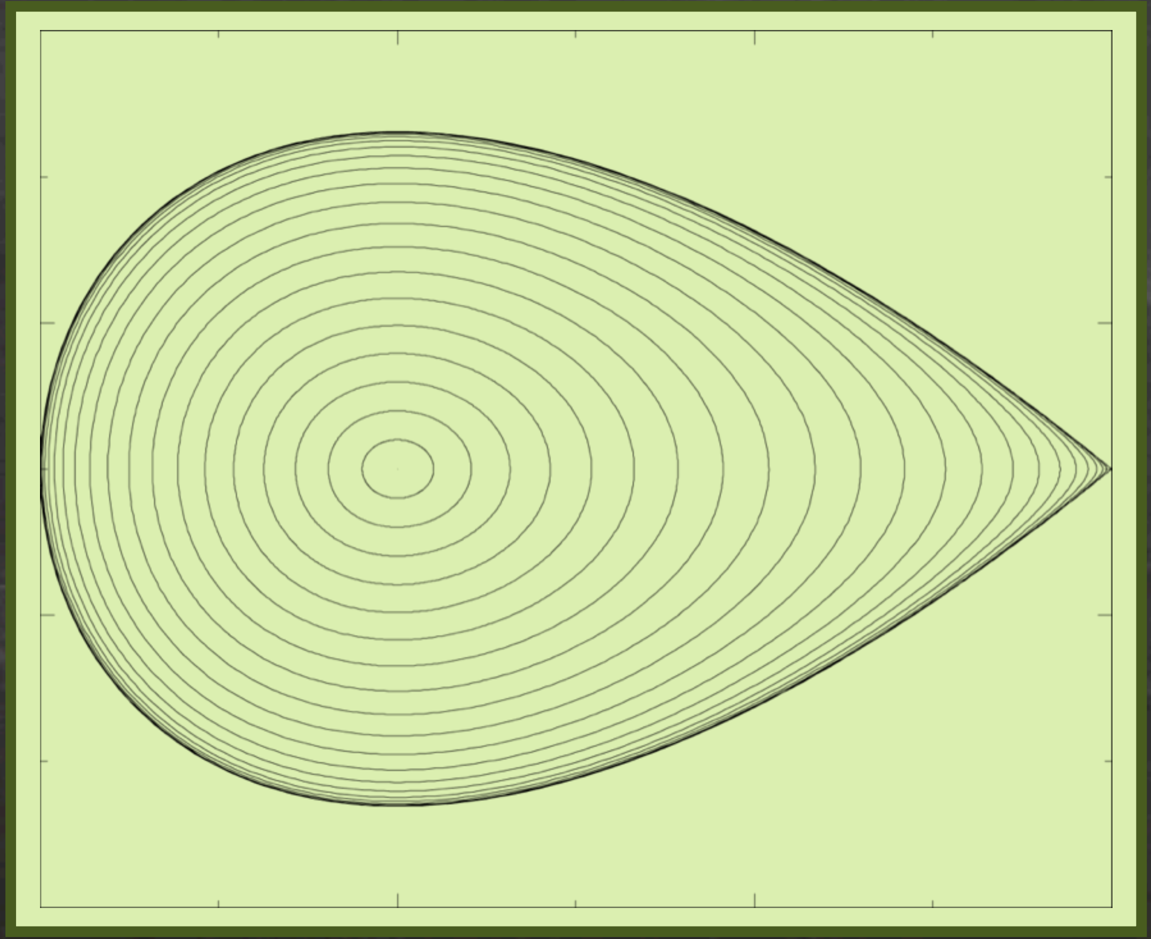
Further assumption:

$$n = 2N \quad \text{and} \quad \dim \operatorname{Fix}(R) = N$$

Example

$$\ddot{y} + f(y) = 0$$

$$\begin{cases} N = 1 \\ R(y, \dot{y}) = (y, -\dot{y}) \end{cases}$$



$$f(y) = y(1 - y)$$

Symmetric solutions

= solutions with orbit γ such that $R(\gamma) = \gamma$

Main result: an orbit γ is symmetric if and only if

$$\gamma \cap \mathrm{Fix}(R) \neq \emptyset$$

By setting $t = 0$ at one of the intersection points x_0 the solution $\tilde{x}(t, x_0)$ along such orbit satisfies

$$R\tilde{x}(t, x_0) = \tilde{x}(-t, x_0)$$

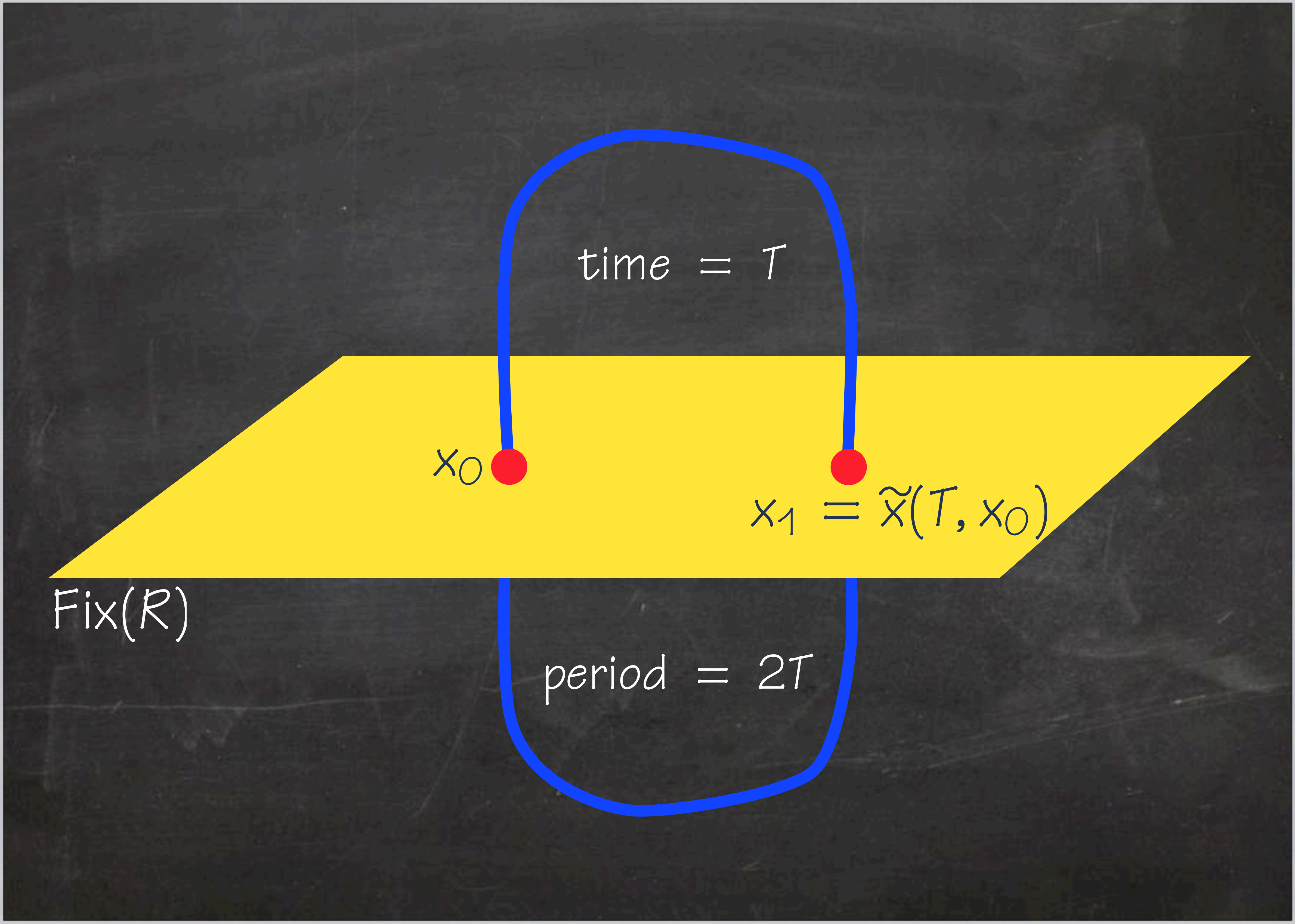
Symmetric periodic solutions

Excluding symmetric equilibria, a solution with orbit γ is symmetric and periodic if and only if

$$\gamma \cap \mathrm{Fix}(R) = \{x_0, x_1\}$$

for two distinct points $x_0 \neq x_1$.

The minimal period equals 2 times the time needed to travel from x_0 to x_1.



time = T
x_0
$x_1 = \tilde{x}(T, x_0)$
Fix(R)
period = $2T$

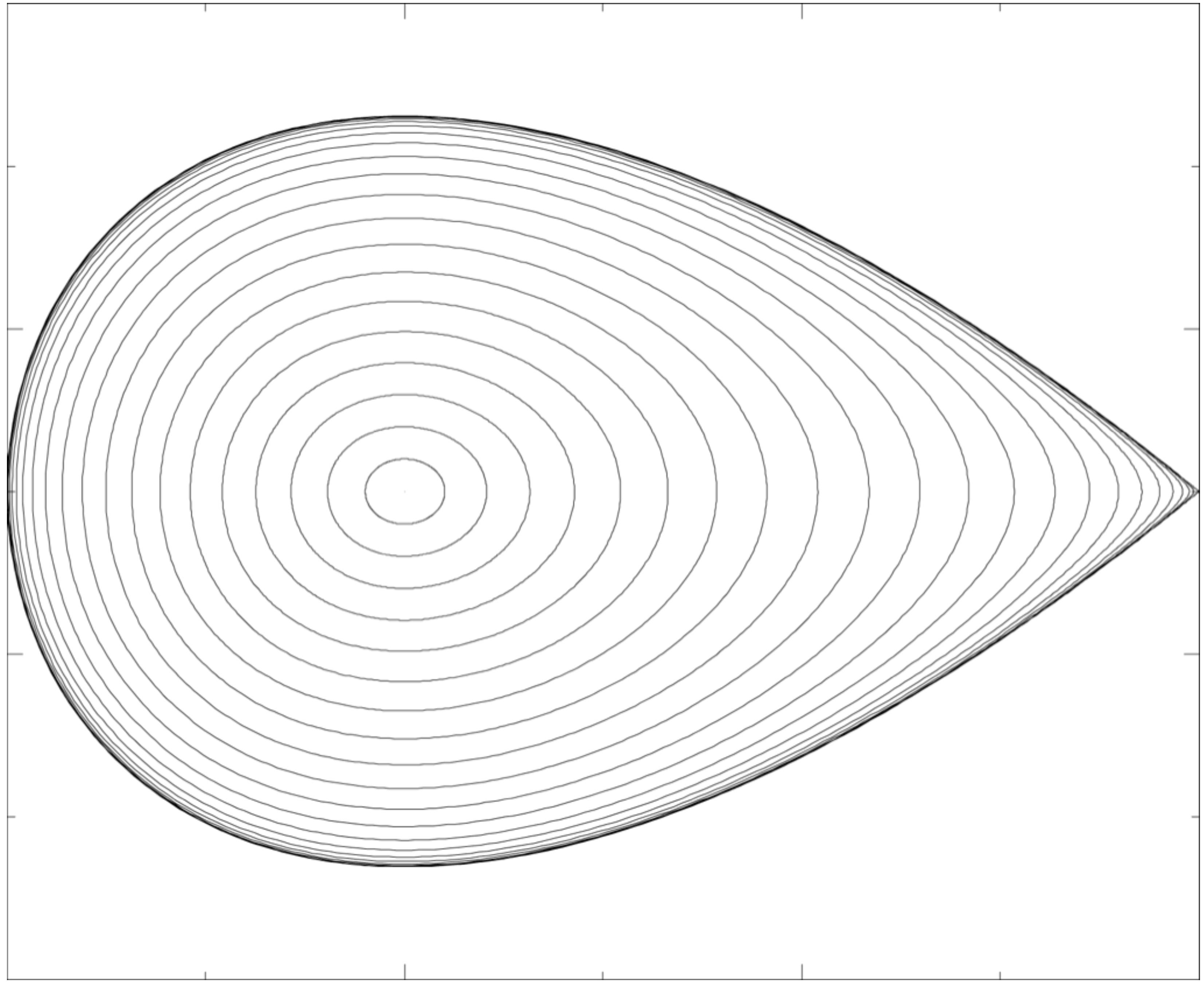
Symmetric periodic orbits are given by the intersection of

$$\mathrm{Fix}(R) \qquad (N\text{-dimensional})$$

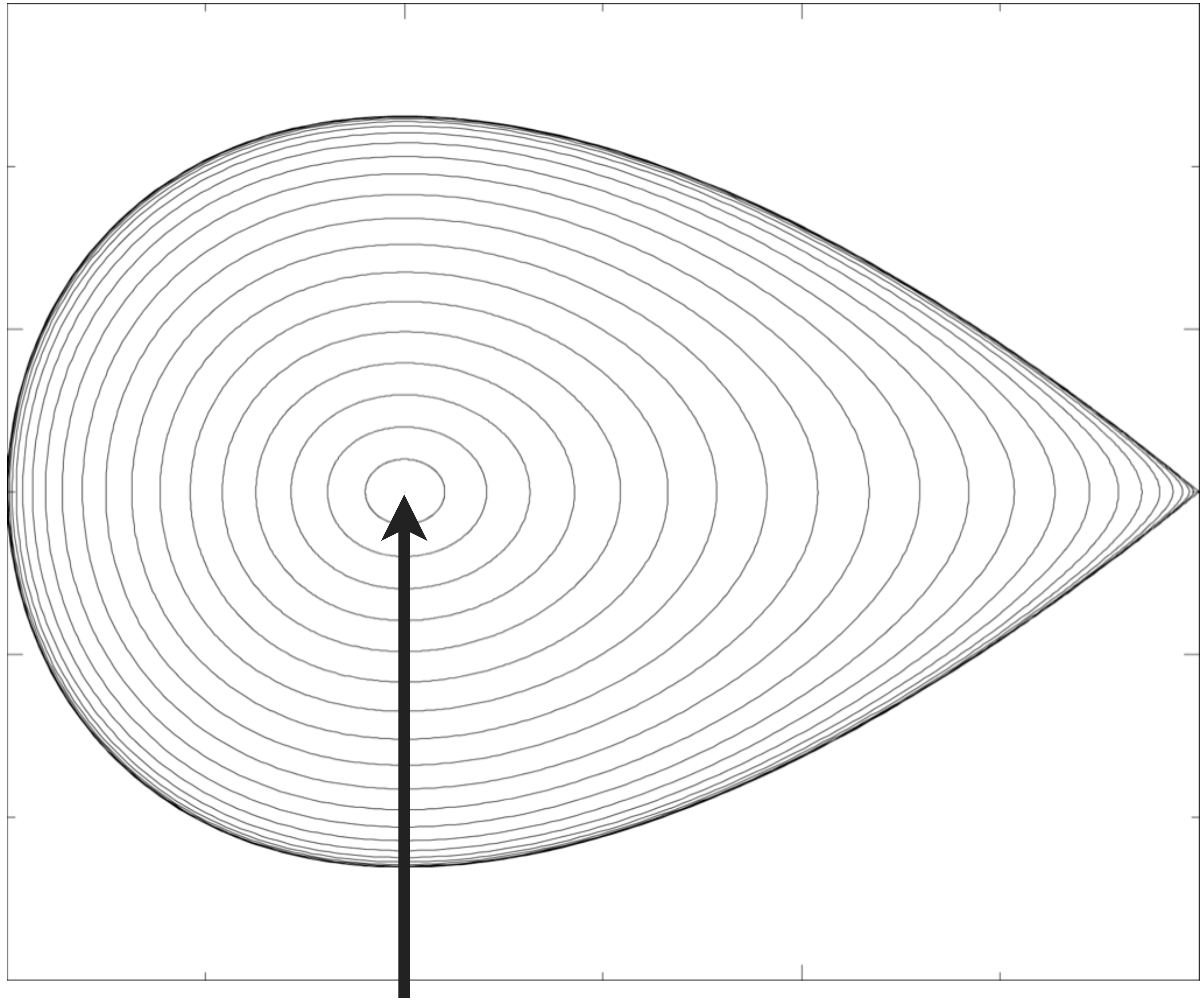
and

$$\{\tilde{x}(t;x) \mid t \in \mathbb{R},\ x \in \mathrm{Fix}(R)\} \qquad ((N+1)\text{-dimensional}).$$

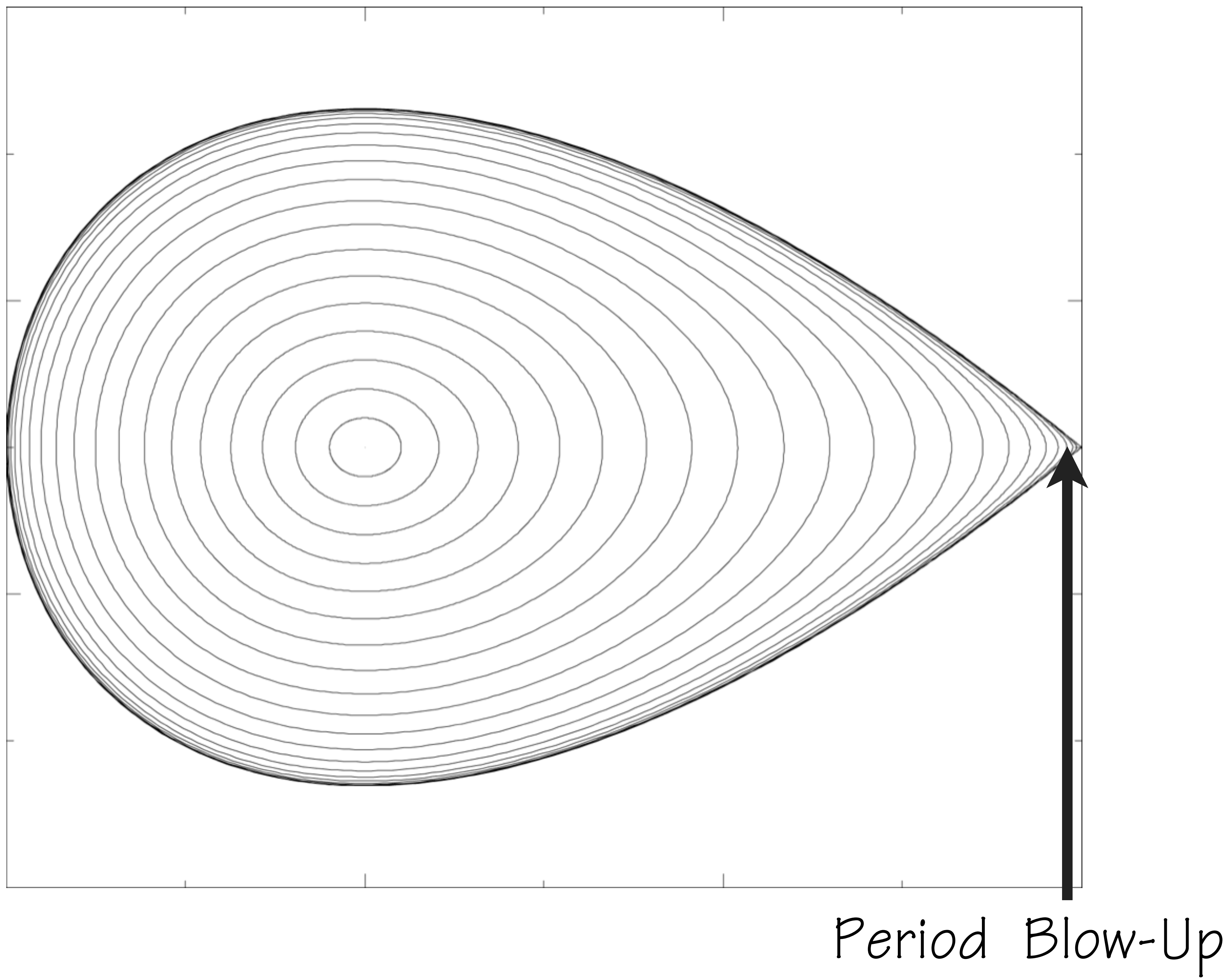
Consequence: symmetric periodic orbits appear typically in **one-parameter families**.



How do these one-parameter families of symmetric periodic orbits start, finish and (or) branch from each other?



Reversible Liapunov Center Theorem


Period Blow-Up

Many more possibilities arise in higher dimensions ($N > 1$).

$$\begin{cases} \ddot{y} + f(y) = z \\ \ddot{z} + g(z) = 0 \end{cases} \qquad g(0) = 0,\ g'(0) = 1$$

Are there, close to the symmetric periodic solutions for $z = 0$, any periodic solutions with $z \neq 0$ but small?

Adding parameters also allows to show certain transitions.

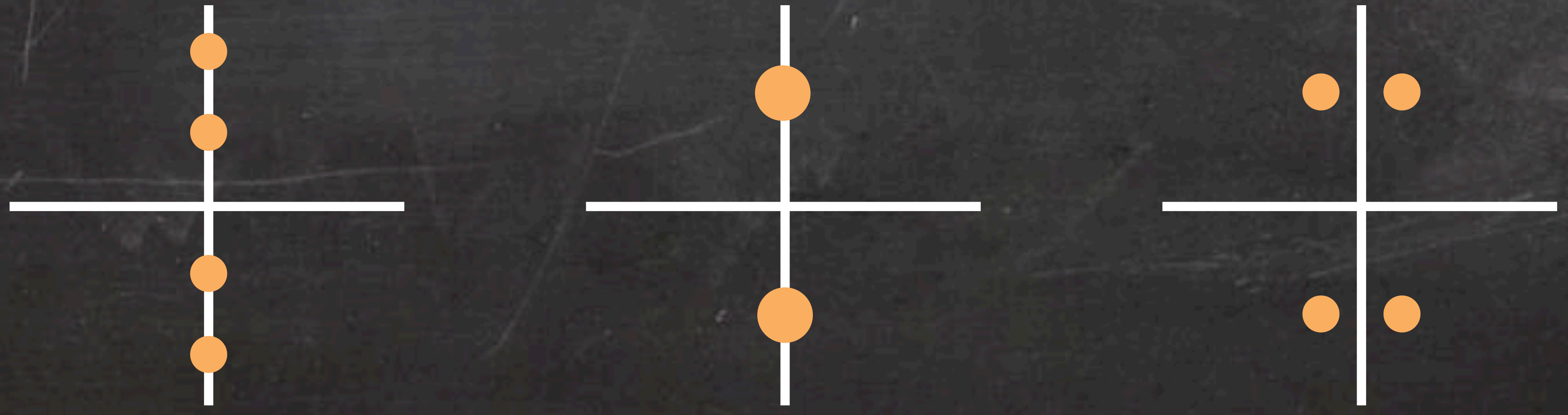
Reversible Hopf bifurcation

$$\dot{x} = F(x, \lambda)$$

$x \in \mathbb{R}^{2N}$, $\lambda \in \mathbb{R}$, $RF(x, \lambda) = -F(Rx, \lambda)$

$$F(0, \lambda) = 0$$
$$A_\lambda := D_x F(0, \lambda)$$

If $\mu \in \mathbb{C}$ is an eigenvalue of A_λ, then so is $-\mu$.



$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

$\lambda < 0$

$\lambda = 0$

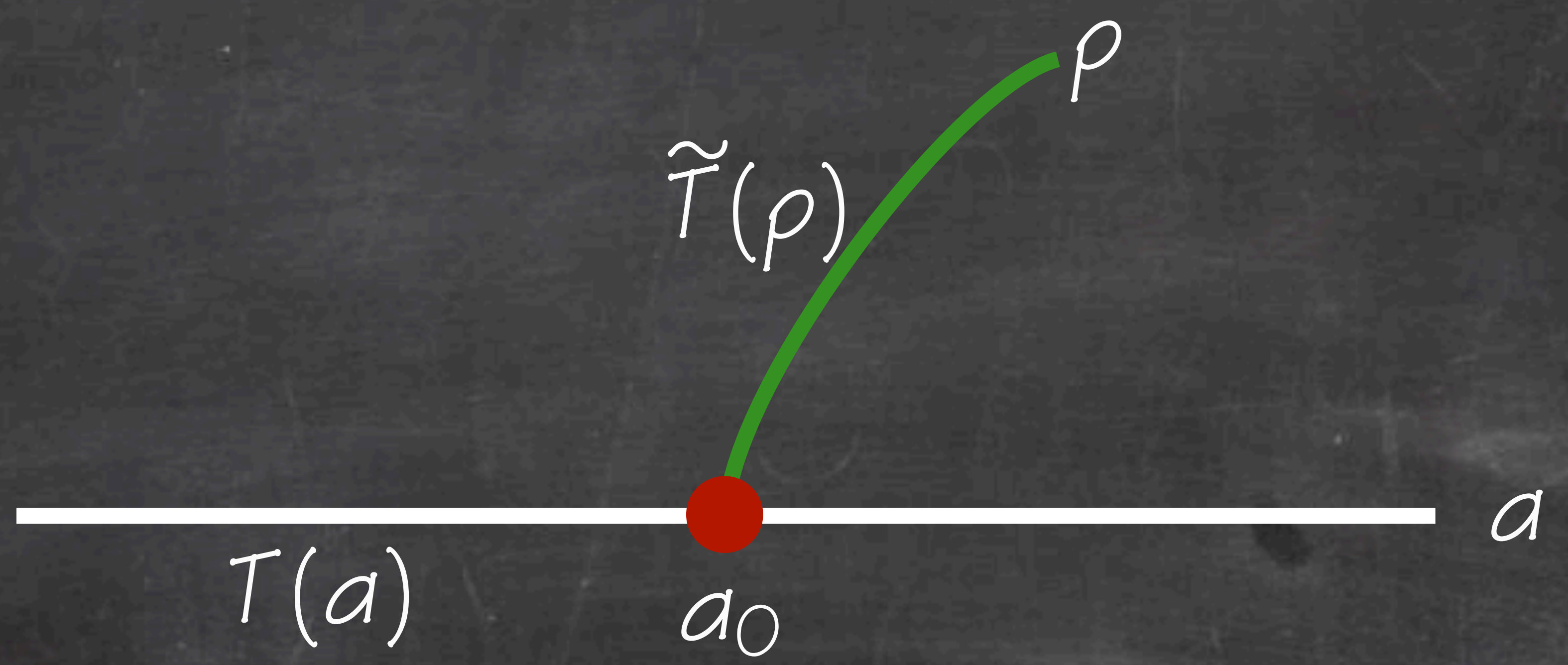
$\lambda > 0$

Branching of subharmonics

$$\begin{cases} \ddot{y} + f(y) = z \\ \ddot{z} + g(z) = 0 \end{cases} \qquad g(0) = 0,\ g'(0) = 1$$

Assume that the unperturbed system ($z = 0$) has a one-parameter family of symmetric periodic orbits — we call this the **primary family**.

When another family (with $z \neq 0$) of periodic orbits branches off this primary family, then the limiting period along the bifurcating branch must be an integer multiple of the period at the branching point along the primary branch:
$\Rightarrow$ **resonance condition**



$$\lim_{\rho \to 0} \widetilde{T}(\rho) = qT(a_0) \qquad (q \in \mathbb{N})$$

Resonance condition

In the general case a necessary condition for a symmetric periodic orbit to be at a branching point is a pair of multipliers which are roots of unity, i.e. a pair of multipliers of the form

$$\exp(\pm i\theta_0) \qquad \text{with } \theta_0 = \frac{2\pi p}{q},\ gcd(p,q) = 1$$

Approach:

Poincaré map

Poincaré map

$$P : \Sigma \longrightarrow \Sigma$$

with Σ chosen to be R-invariant; also $\dim(\Sigma) = 2N-1$.

$\Rightarrow$ $P(0) = 0$

$\Rightarrow$ $R\,P\,R = P^{-1}$

$\Rightarrow$ 1 is always an eigenvalue of $DP(0)$, with odd (algebraic) multiplicity ≥ 1

Assumption: 1 is a **simple** eigenvalue of $DP(0)$

(Res) $DP(0)$ has a pair of **simple** eigenvalues of the form

$$\exp(\pm i\theta_0) \qquad \text{with } \theta_0 = \frac{2\pi p}{q} \;\; gcd(p,q) = 1,\; q \geq 3$$

Problem:
find small q-periodic orbits of the Poincaré map P

Using Lyapunov-Schmidt method this reduces to a

3-dimensional and $\mathbb{D}_q$**-equivariant**

bifurcation problem (when $q \geq 3$)

Period-doubling $(q = 2)$

Resonance condition: -1 is an eigenvalue of $DP(0)$ with geometric multiplicity **one** and algebraic multiplicity **two**

The problem of finding period-doubled solutions reduces to a **3-dimensional** bifurcation problem with a $\mathbb{D}_2$**-symmetry** ($\mathbb{D}_2 = \mathbb{Z}_2 +$ reversibility)

$\Rightarrow$ $\mathbb{Z}_2$-symmetry: if x satisfies $P^2(x) = x$ the so does $P(x)$

$\Rightarrow$ 3-dimensional: **generalized kernel** with coordinates (a, ξ, η)

Period-doubling $(q = 2)$

The bifurcation equations reduce to

$$\xi\,\varphi(a, \xi^2) = 0 \quad \text{and} \quad \eta = 0$$

with $\varphi(0, 0) = 0$.

⇒ **primary branch** $\{(a, 0, 0) \mid a \in \mathbb{R}\}$

⇒ **period-doubled branch** $\{(a^*(\xi^2), \xi, 0) \mid \xi \in \mathbb{R}\}$

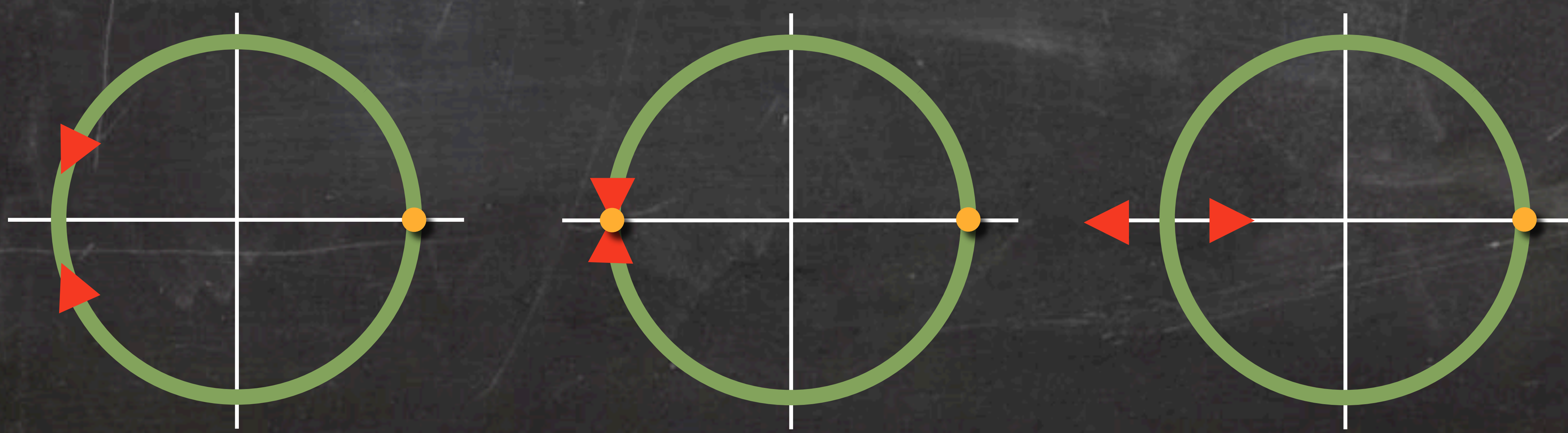
Requirement: **transversality condition**

$$\frac{\partial \varphi}{\partial a}(0, 0) \neq 0$$

Period-doubling $(q = 2)$

$(a^*(\xi^2), \xi)$ and $(a^*(\xi^2), -\xi)$ correspond to the same period-doubled orbit

Transversality condition:



$a < 0$ $a = 0$ $a > 0$

$\mathbb{D}_q$-equivariance

$q \geq 3$

$\Rightarrow$ if $x \in \Sigma$ generates a q-periodic orbit of P, then so do $P(x)$, $P^2(x), \ldots, P^q(x) = x$

$\Rightarrow$ this gives a $\mathbb{Z}_q$-equivariance, which combines with the reversibility to give a $\mathbb{D}_q$-equivariance

3-dimensional

$\Rightarrow$ critical eigenvalues of $DP(0)$: 1 and $\exp(\pm i\theta_0)$

$\Rightarrow$ coordinates: $(a, z) = (a, \rho \exp(i\theta)) \in \mathbb{R} \times \mathbb{C}$

Bifurcation equations

$$\mathcal{B}(u) = \left(h_0(u)\,\mathrm{im}(z^q), ih_1(u)z + ih_2(u)\bar{z}^{q-1}\right) = 0$$

$$u = (a, z) \in \mathbb{R} \times \mathbb{C}$$

$$h_i : \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \qquad (i = 0, 1, 2)$$

$$h_1(0) = 0$$

$$h_i(u) = h_i(S_0 u) = h_i(Ru) \qquad (i = 0, 1, 2)$$

$$S_0 u = S_0(a, z) = (a, e^{i\theta_0} z)$$

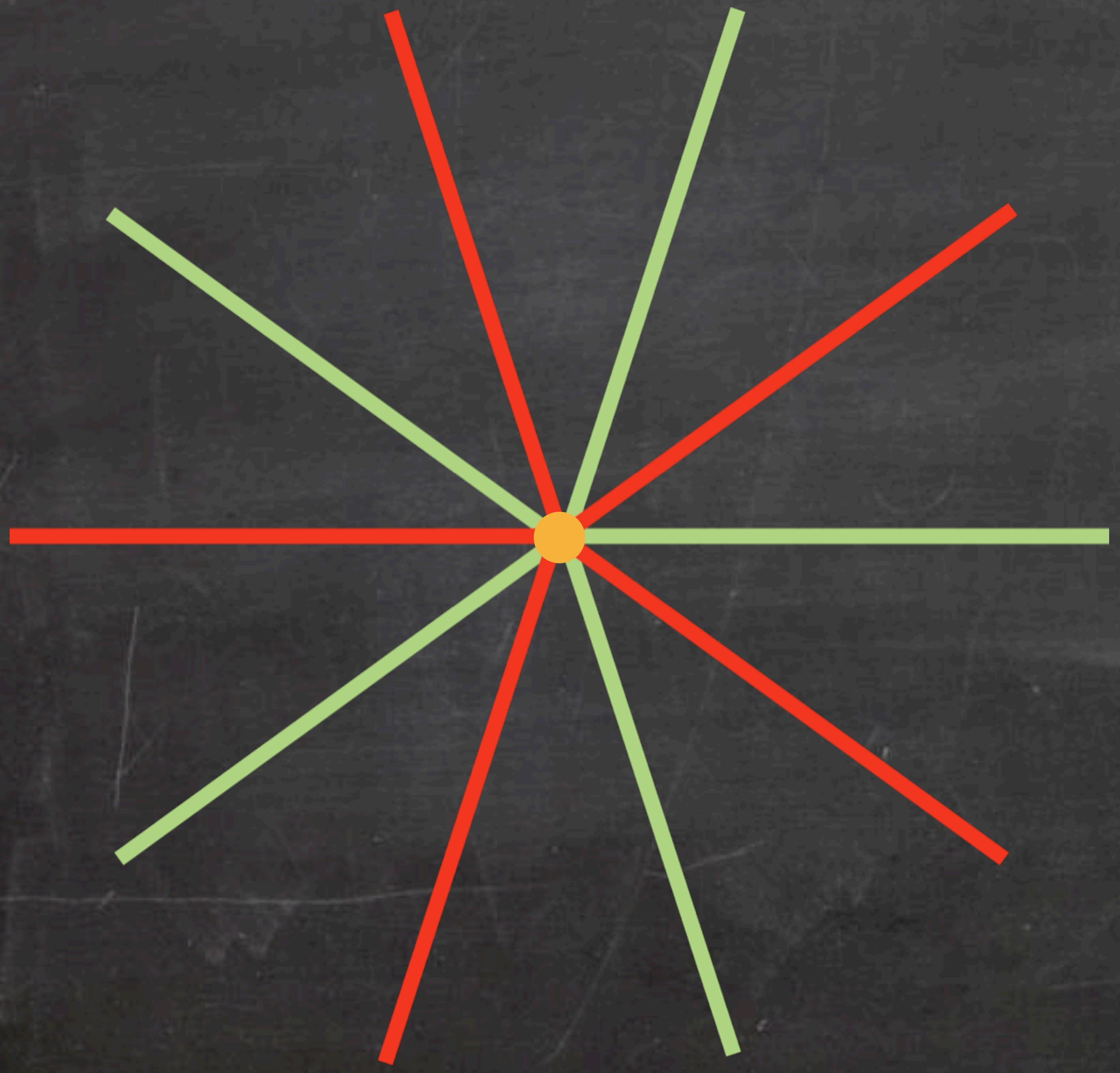
$$Ru = R(a, z) = (a, \bar{z})$$

$$\mathcal{B}(u) = \left(h_0(u)\mathrm{im}(z^q), ih_1(u)z + ih_2(u)\bar{z}^{q-1}\right) = 0$$

$\Rightarrow \ \mathcal{B}(a,0) = 0 \quad (\forall a \in \mathbb{R}) \qquad \Rightarrow$ **primary branch**

$\Rightarrow$ if either $h_0(0) \neq 0$ or $h_2(0) \neq 0$ then $\mathcal{B}(u) = 0$ implies (with $z = \rho\exp(i\theta)$)

$$\mathrm{im}(z^q) = \rho^q \sin(q\theta) = 0$$



$q = 5$

Along these "rays" the bifurcation equations reduce to a single scalar equation in the two variables a and $\rho = |z|$:

$$b_1(a, \rho) := h_1(a, \rho) + \rho^{q-2} h_2(a, \rho) = 0 \qquad (\star)$$

for $\theta = 0 \mod (2\pi/q)$, and

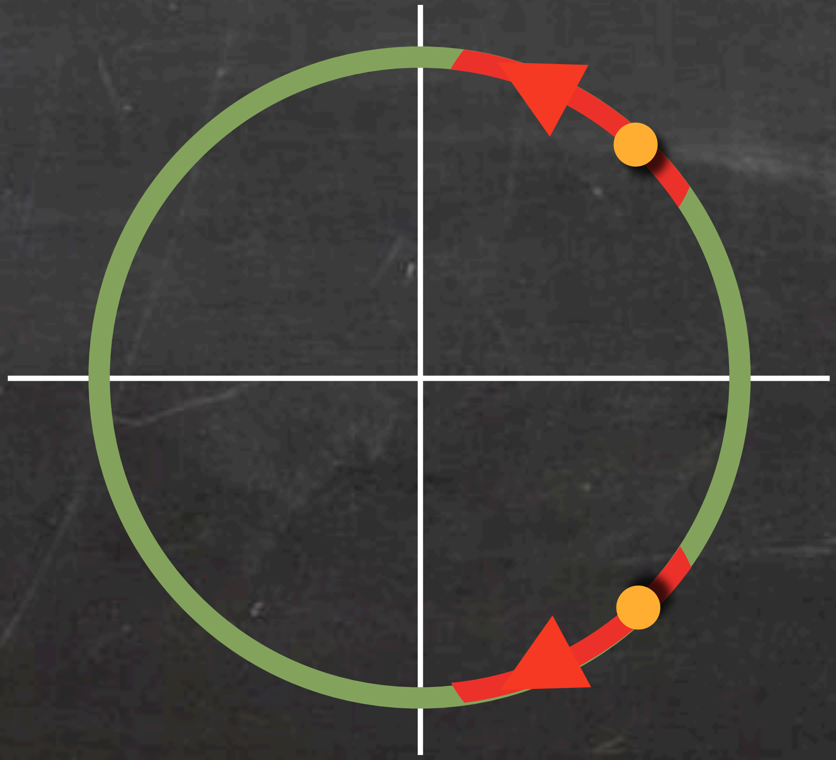
$$b_2(a, \rho) := h_1(a, \rho e^{\frac{i\pi}{q}}) - \rho^{q-2} h_2(a, \rho e^{\frac{i\pi}{q}}) = 0 \qquad (\star)$$

for $\theta = \pi/q \mod (2\pi/q)$

$\Rightarrow h_1(0,0) = 0$

$\Rightarrow$ we assume the following **transversality condition**

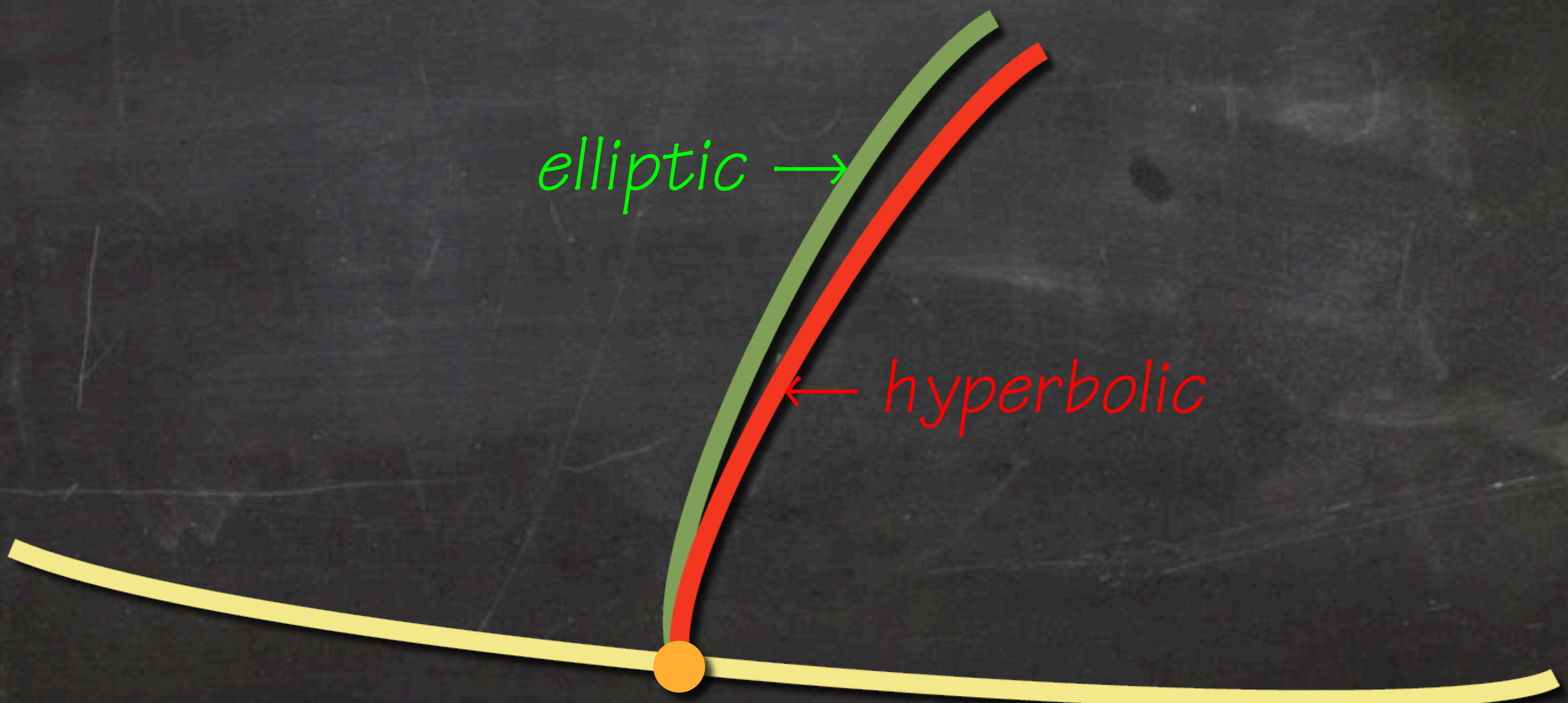
$$\frac{\partial h_1}{\partial a}(0,0) \neq 0$$



Generic subharmonic bifurcation

$(\star) \Rightarrow \quad a = a^*_+(\rho)$

$(\star) \Rightarrow \quad a = a^*_-(\rho)$



Generic subharmonic bifurcation

$$a_+^*(\rho_1) = a_-^*(\rho_2) = d \quad \text{(with } d > 0 \text{ small)} \quad \Rightarrow$$

$$|\rho_1 - \rho_2| = O\left(d^{\frac{q-2}{2}}\right) \quad \text{(Arnol'd tongue)}$$

Conditions for such generic subharmonic bifurcation:

⇒ a pair of **simple** multipliers of the form

$$\exp(\pm i\theta_0) \qquad \text{with } \theta_0 = \frac{2\pi p}{q},\ \gcd(p,q) = 1$$

⇒ **transversality condition**

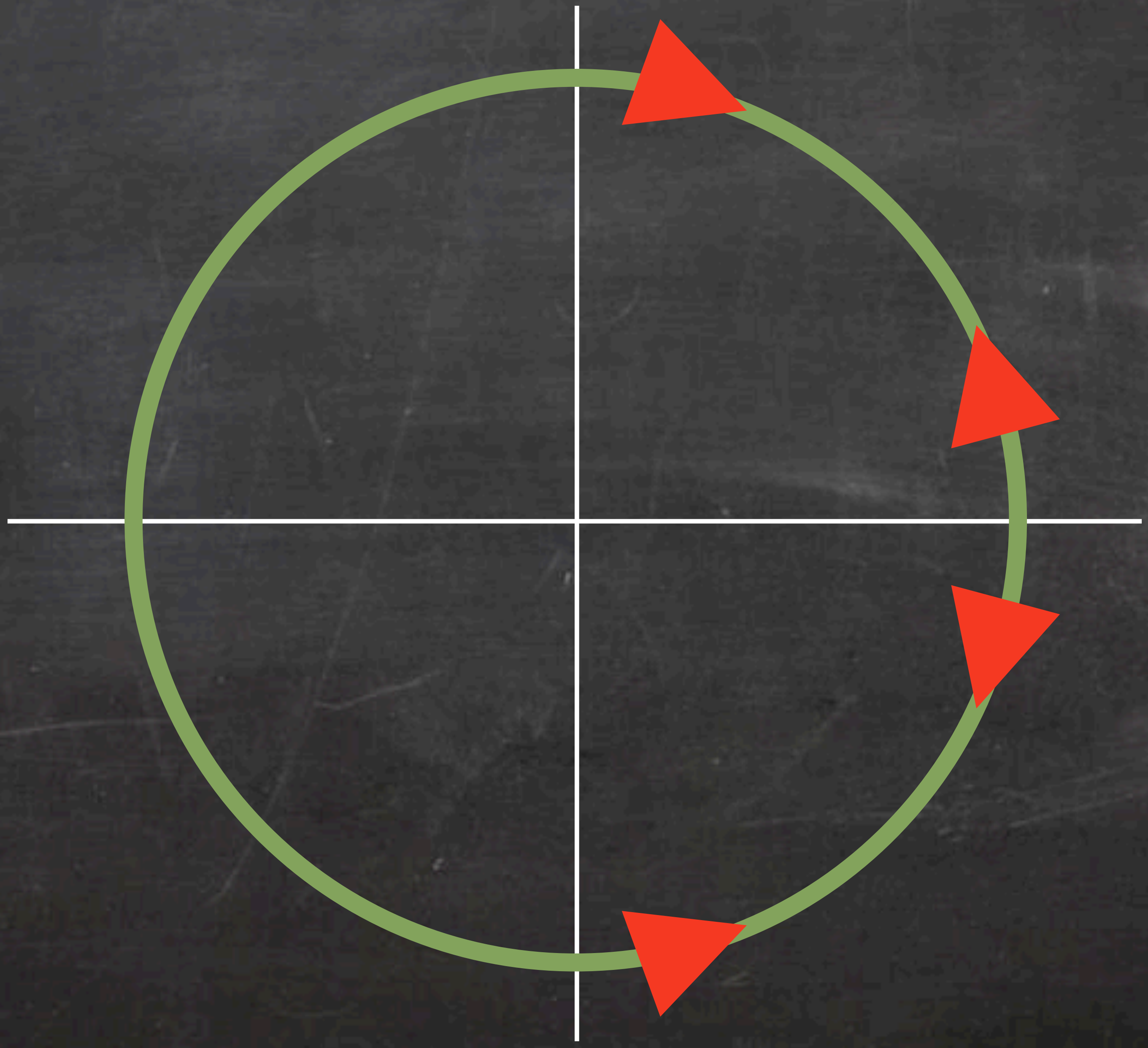
? What if any of these conditions is **not** satisfied?

Degenerate subharmonic branching

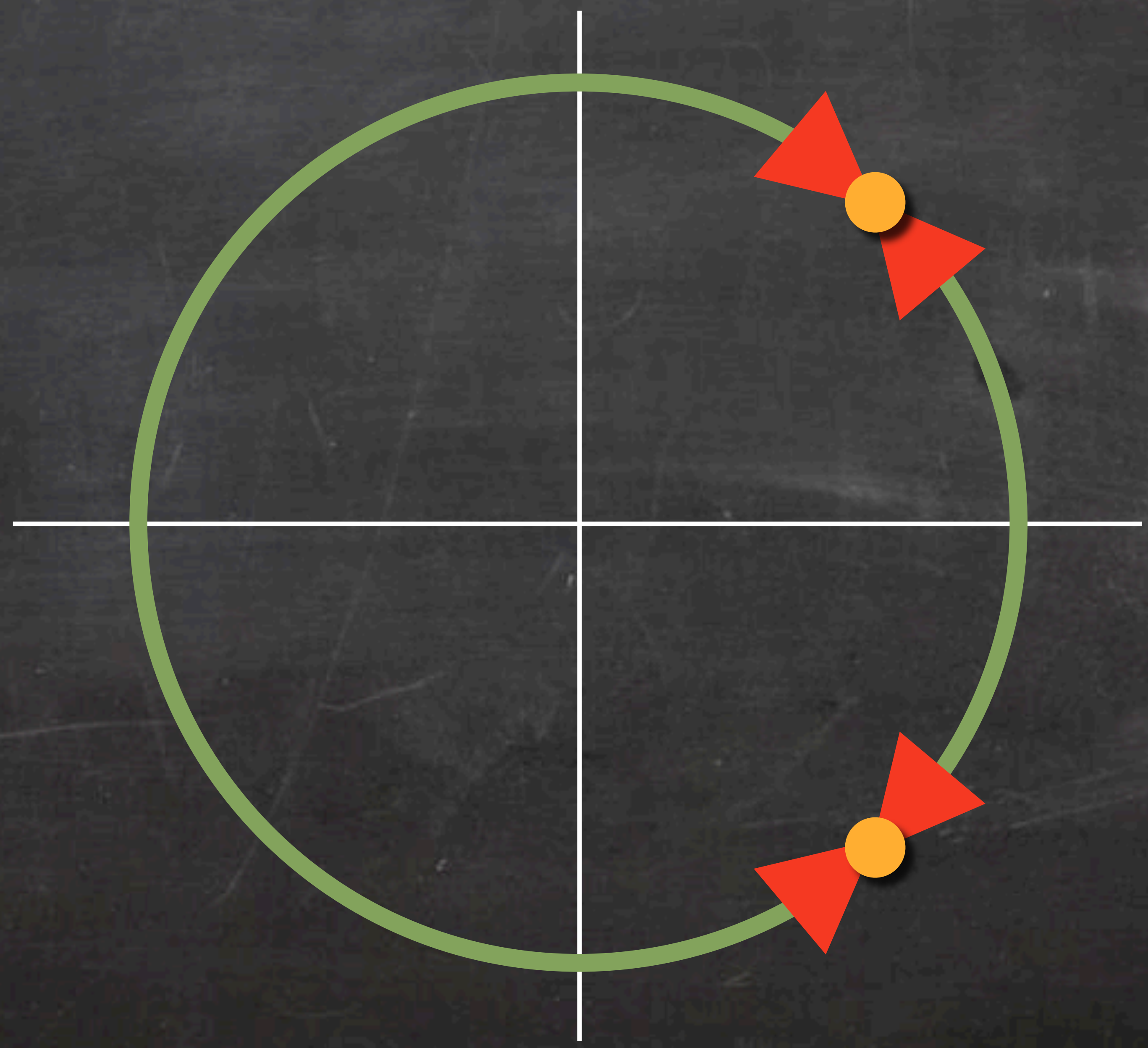
1. the critical pair of multipliers $\exp(\pm i\theta_0)$ is not simple

2. the transversality condition is not satisfied

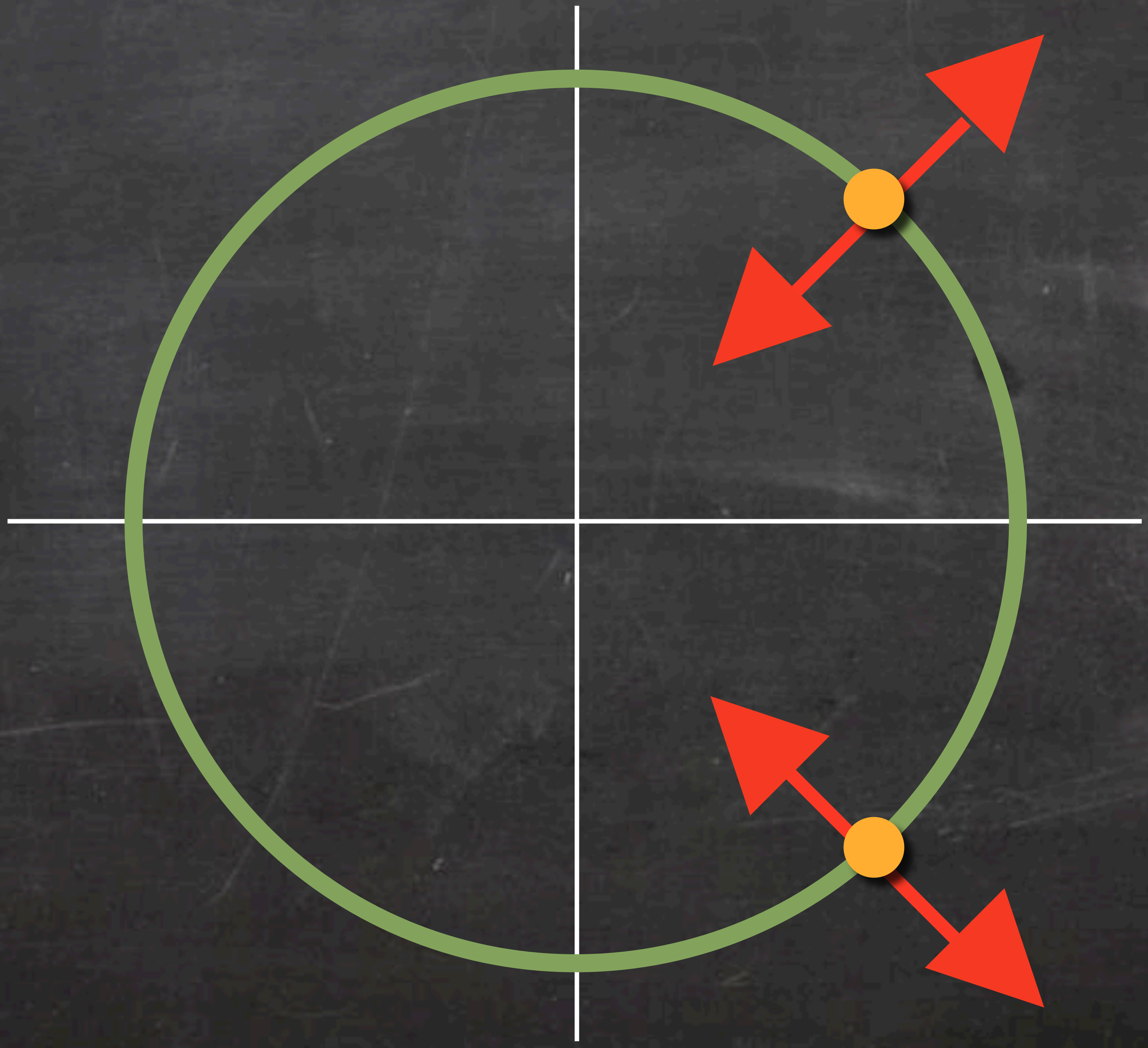
1 Nonsimple critical multipliers



1 Nonsimple critical multipliers



1 Nonsimple critical multipliers



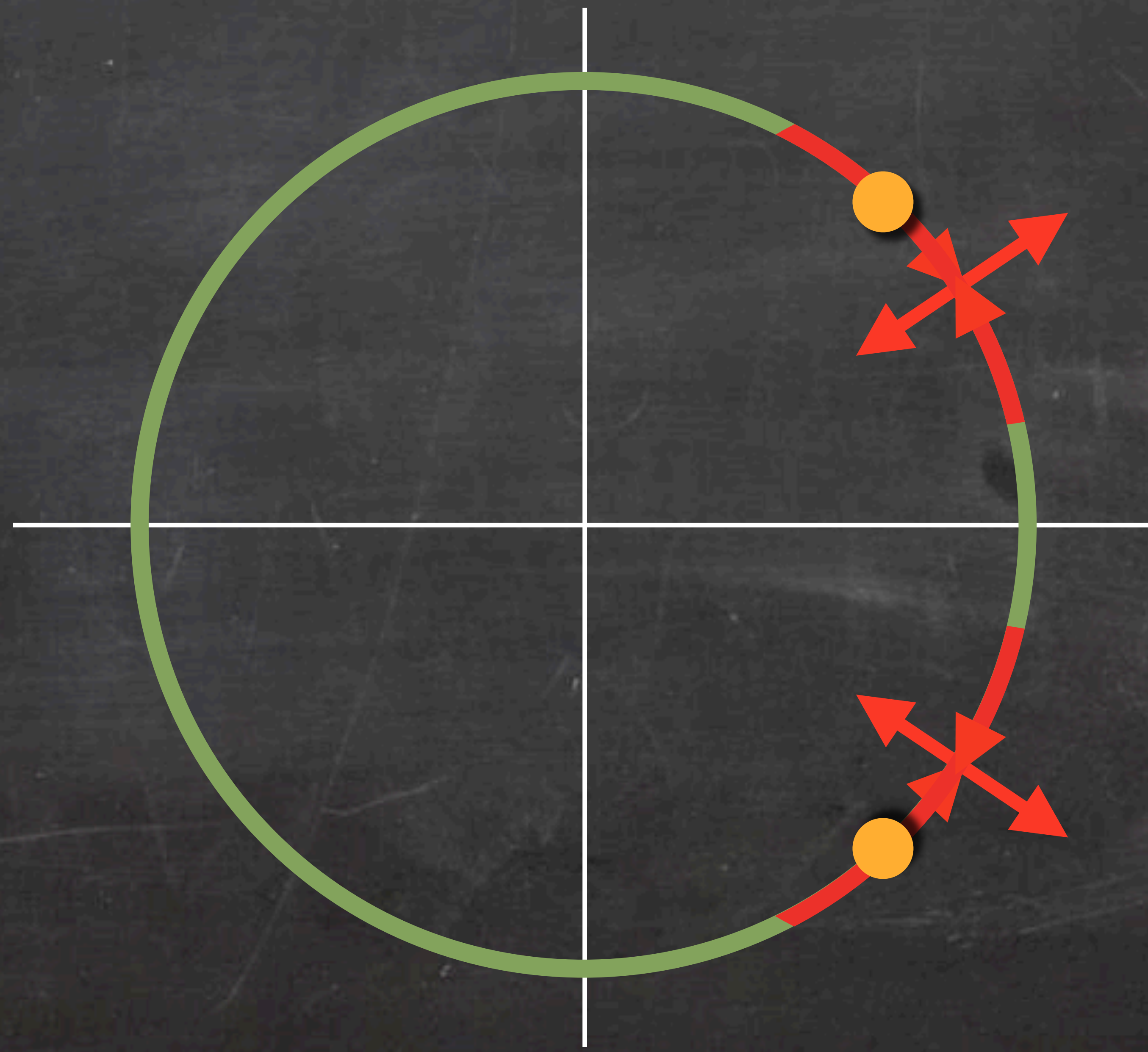
$\Rightarrow$ this requires $N \geq 3$

$\Rightarrow$ this is a **codimension one** phenomenon:

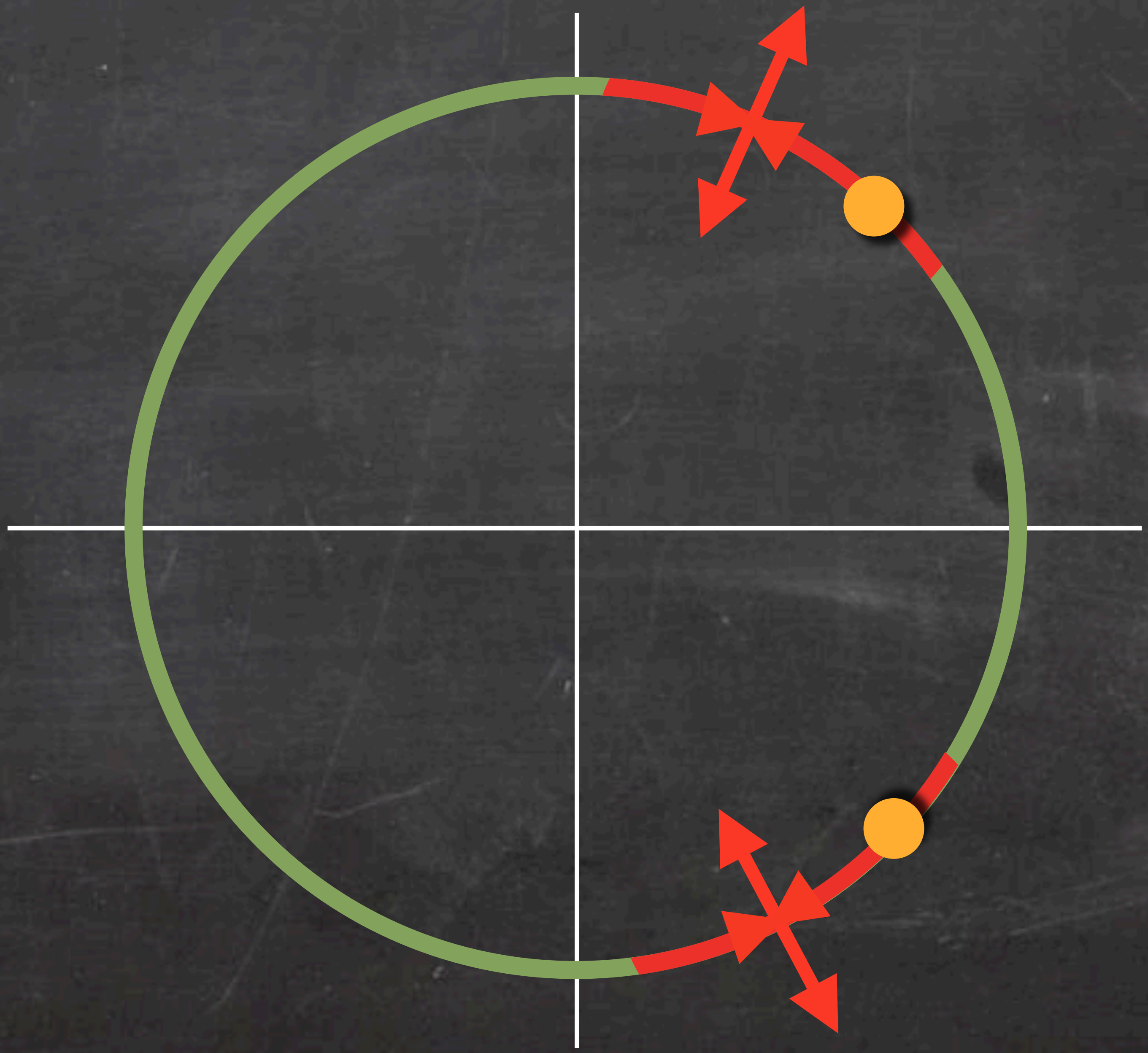
the splitting happens exactly at the q-th root of unity

$\Rightarrow$ we introduce an external parameter $\lambda \in \mathbb{R}$

$\lambda < 0$



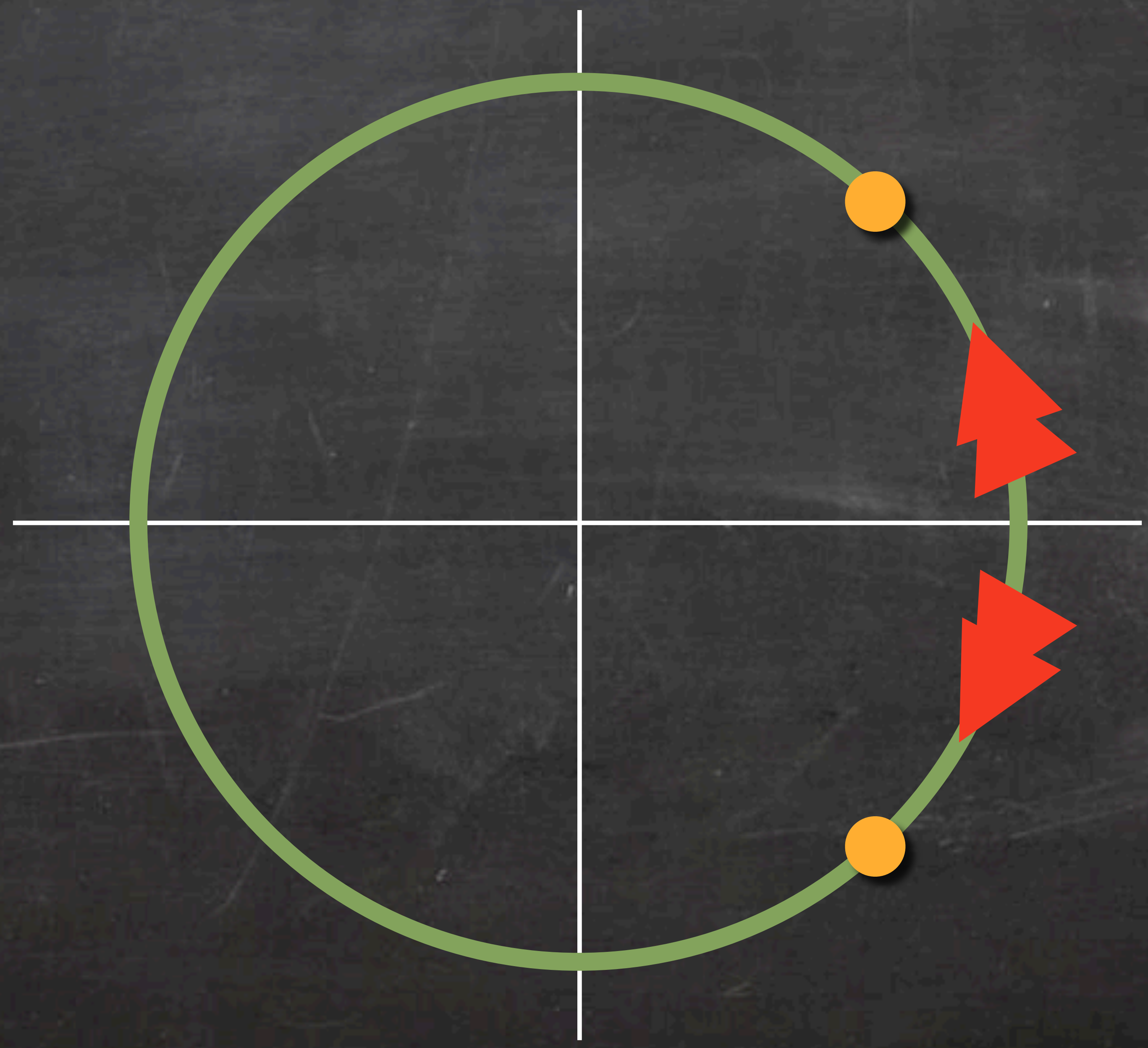
$\lambda > 0$



Both for $\lambda < 0$ and for $\lambda > 0$ we have just one generic subharmonic bifurcation along the primary branch.

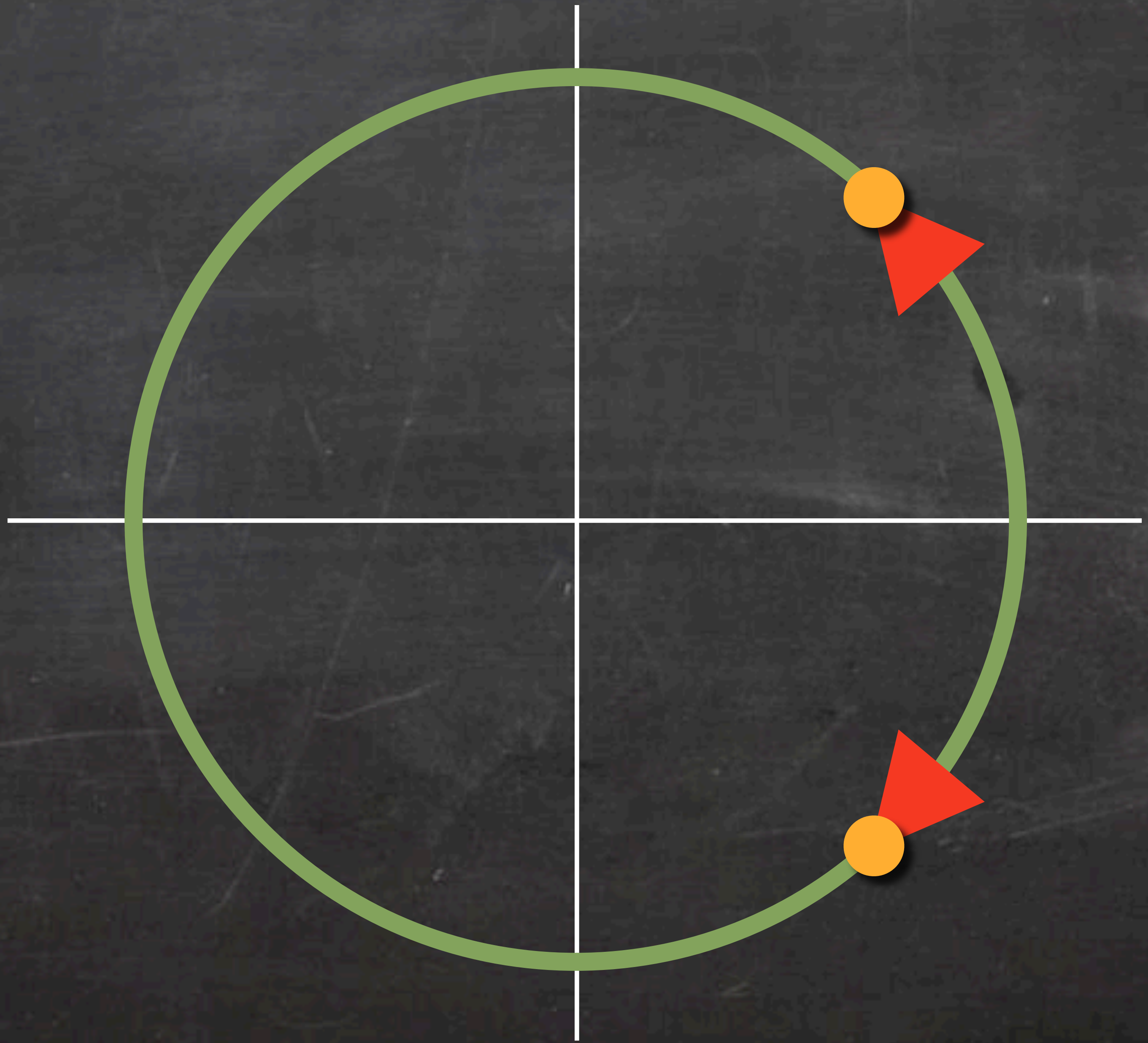
Under appropriate conditions this persists for $\lambda = 0$.

2 No transversality



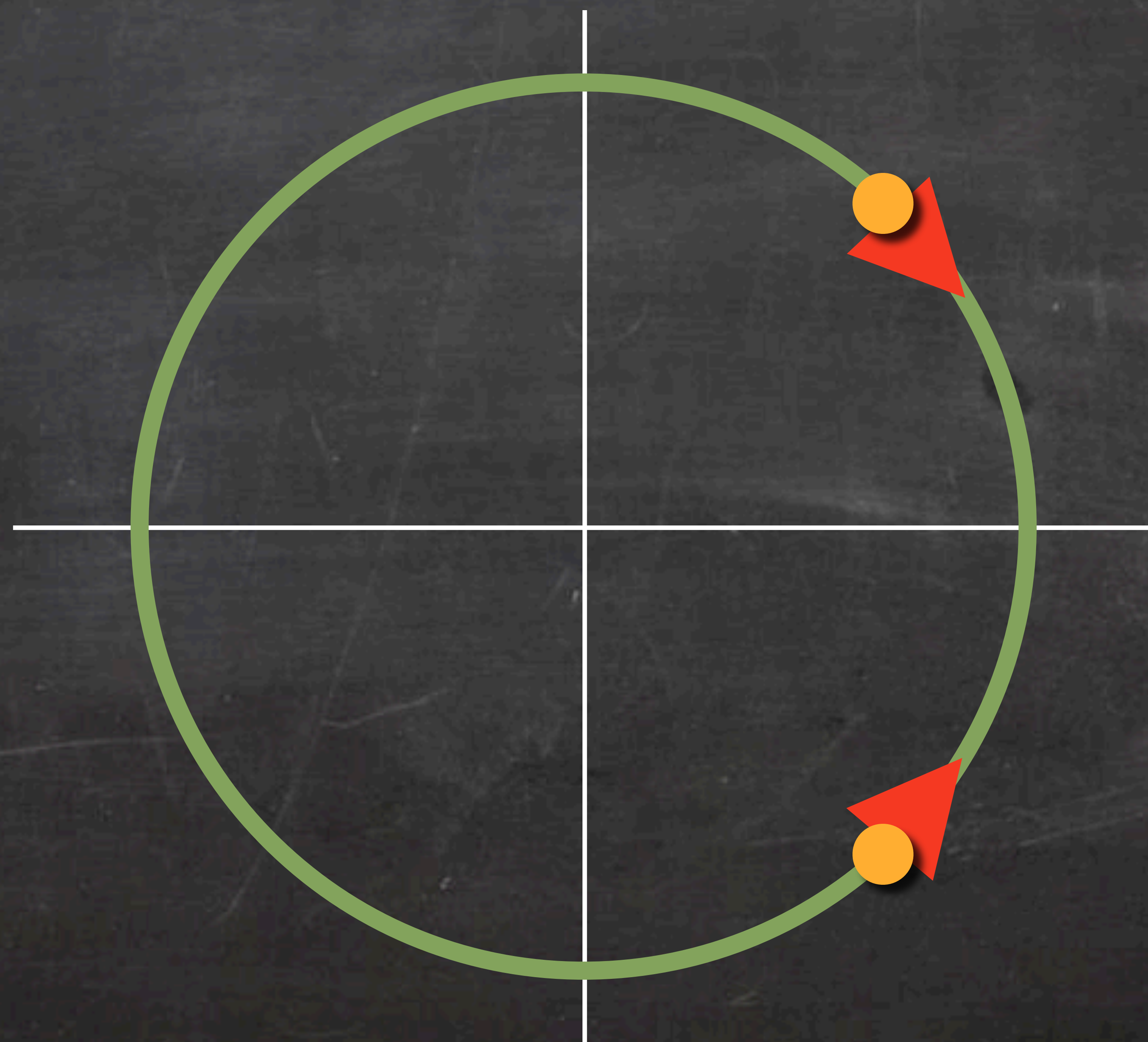
2

No transversality

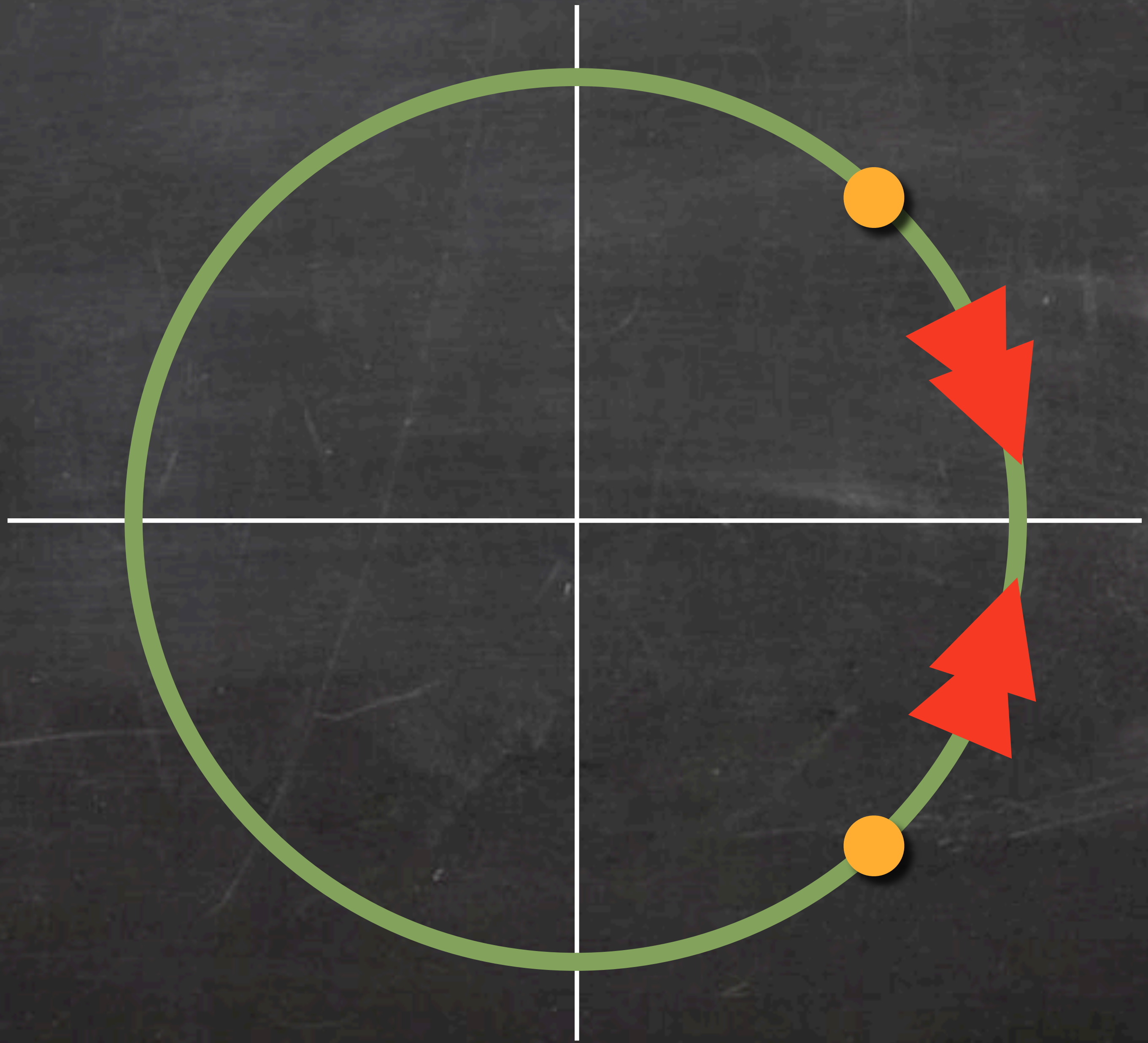


2

No transversality



2 No transversality



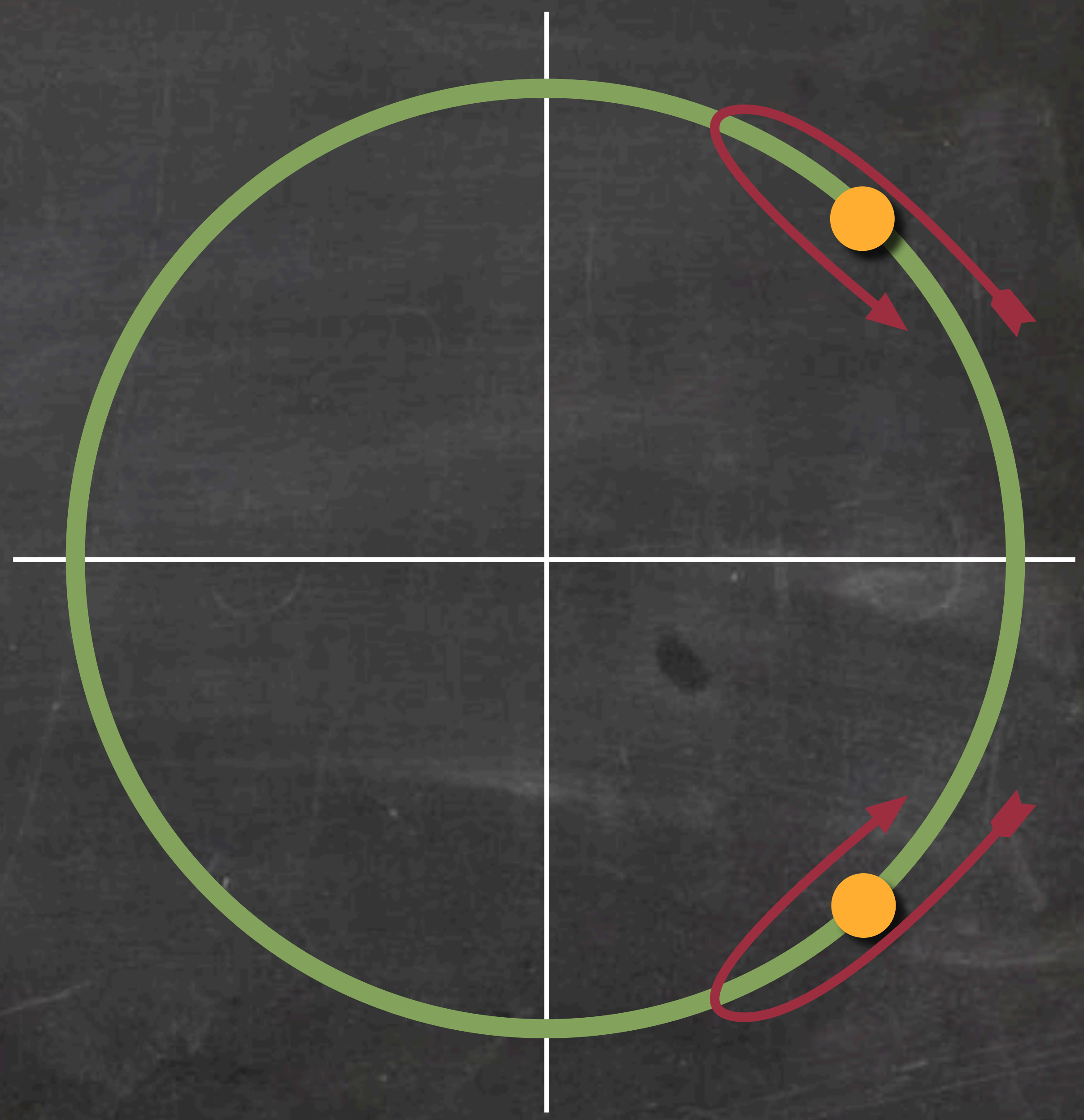
In our example

$$\begin{cases} \ddot{y} + f(y) = z \\ \ddot{z} + g(z) = 0 \end{cases}$$

this happens when along the primary family the period reaches a (local) maximum or minimum precisely at an orbit where the resonance consition is satisfied.

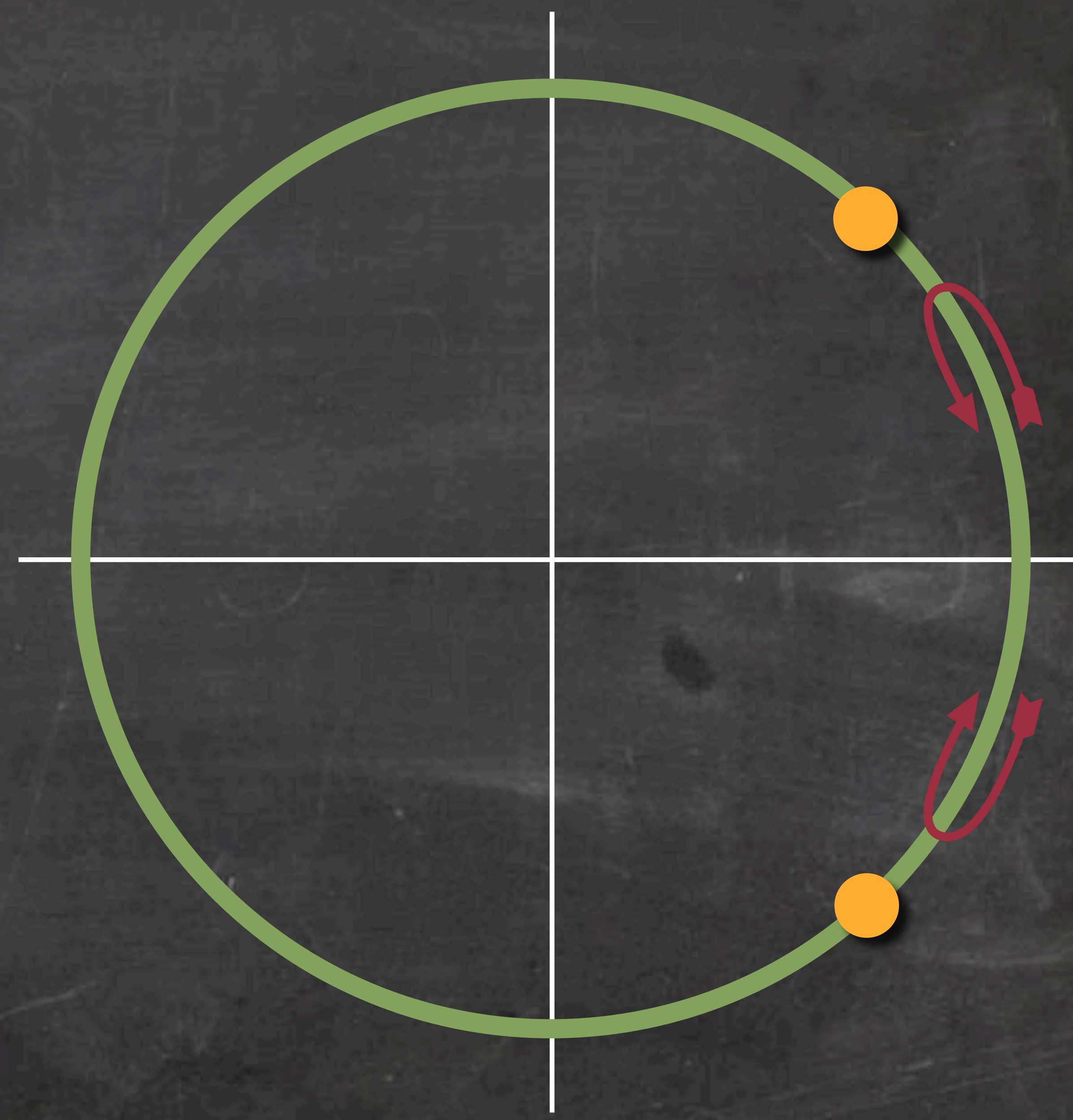
Again, this is a **codimension one** situation, so also here we add an external parameter $\lambda \in \mathbb{R}$.

$\lambda < 0$



$\Rightarrow$ two generic subharmonic bifurcations

$\lambda > 0$



$\Rightarrow$ no generic subharmonic bifurcations

Bifurcation equations

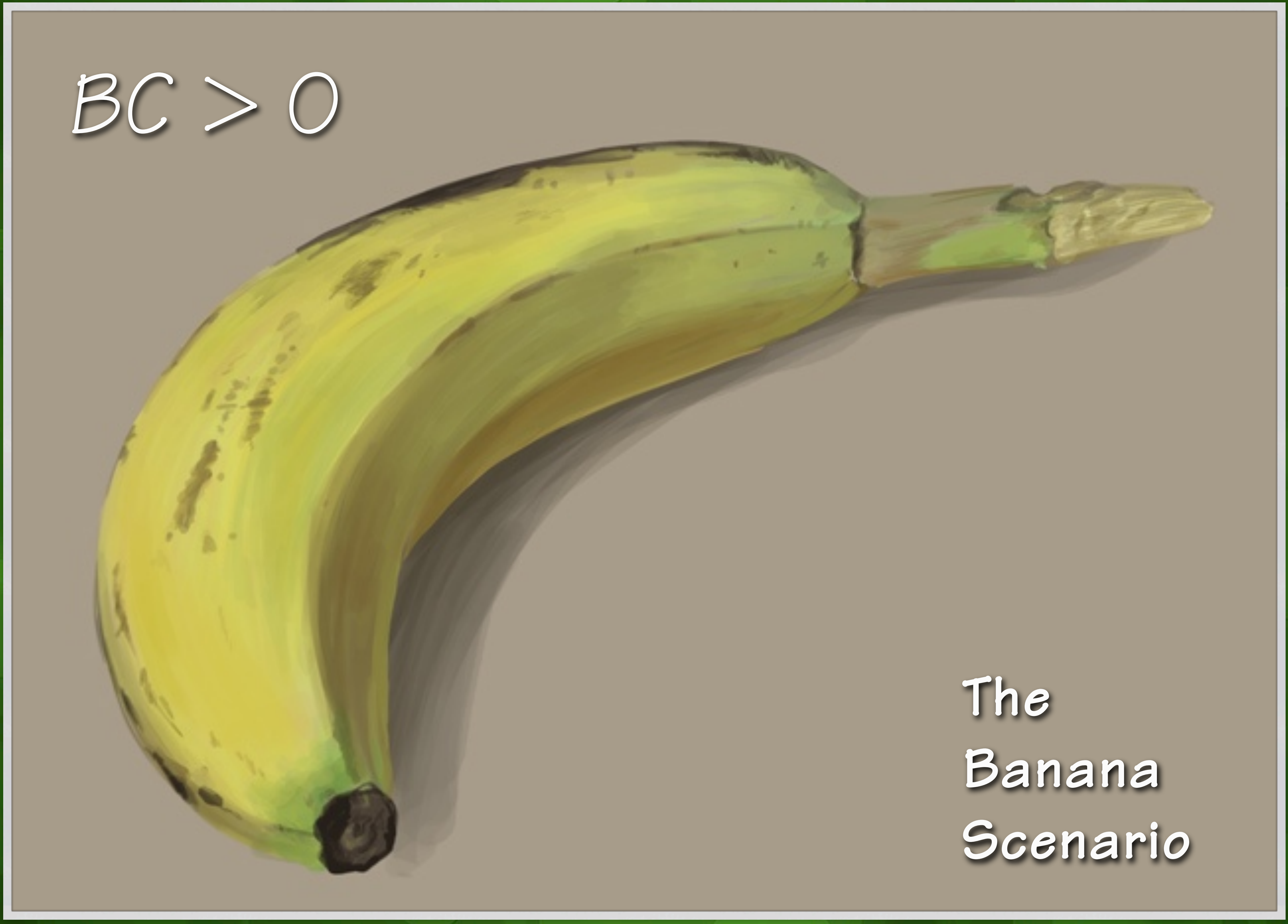
$$b_1(a,\rho,\lambda) = A\lambda + B\rho^2 + Ca^2 + \text{h.o.t.} = 0$$

and

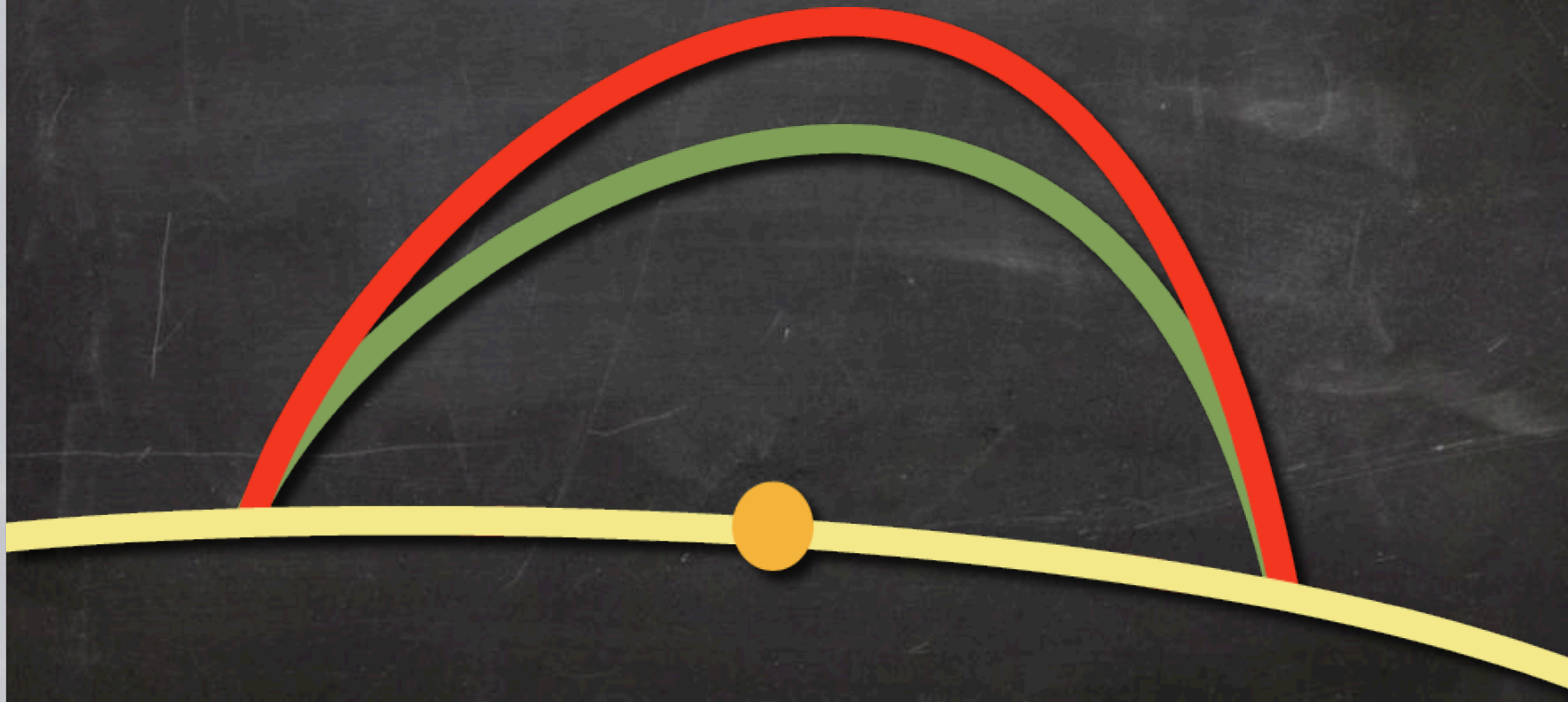
$$b_2(a,\rho,\lambda) = A\lambda + B\rho^2 + Ca^2 + \text{h.o.t.} = 0$$

The two equations differ only in the higher order terms. We assume $ABC \neq 0$.

$\Rightarrow$ two possible bifurcation scenario's.


BC > 0
The
Banana
Scenario

$AC\lambda < 0$

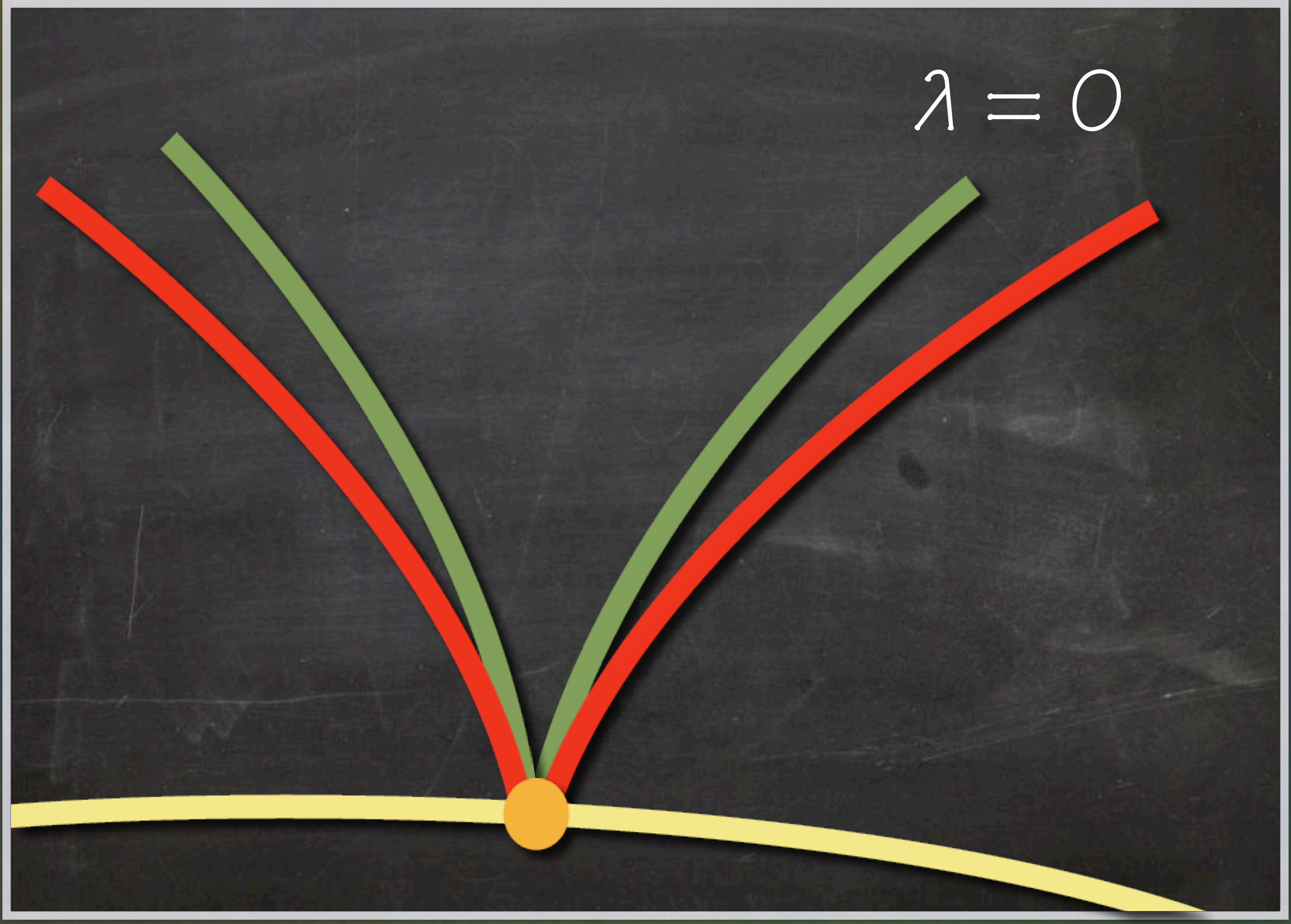


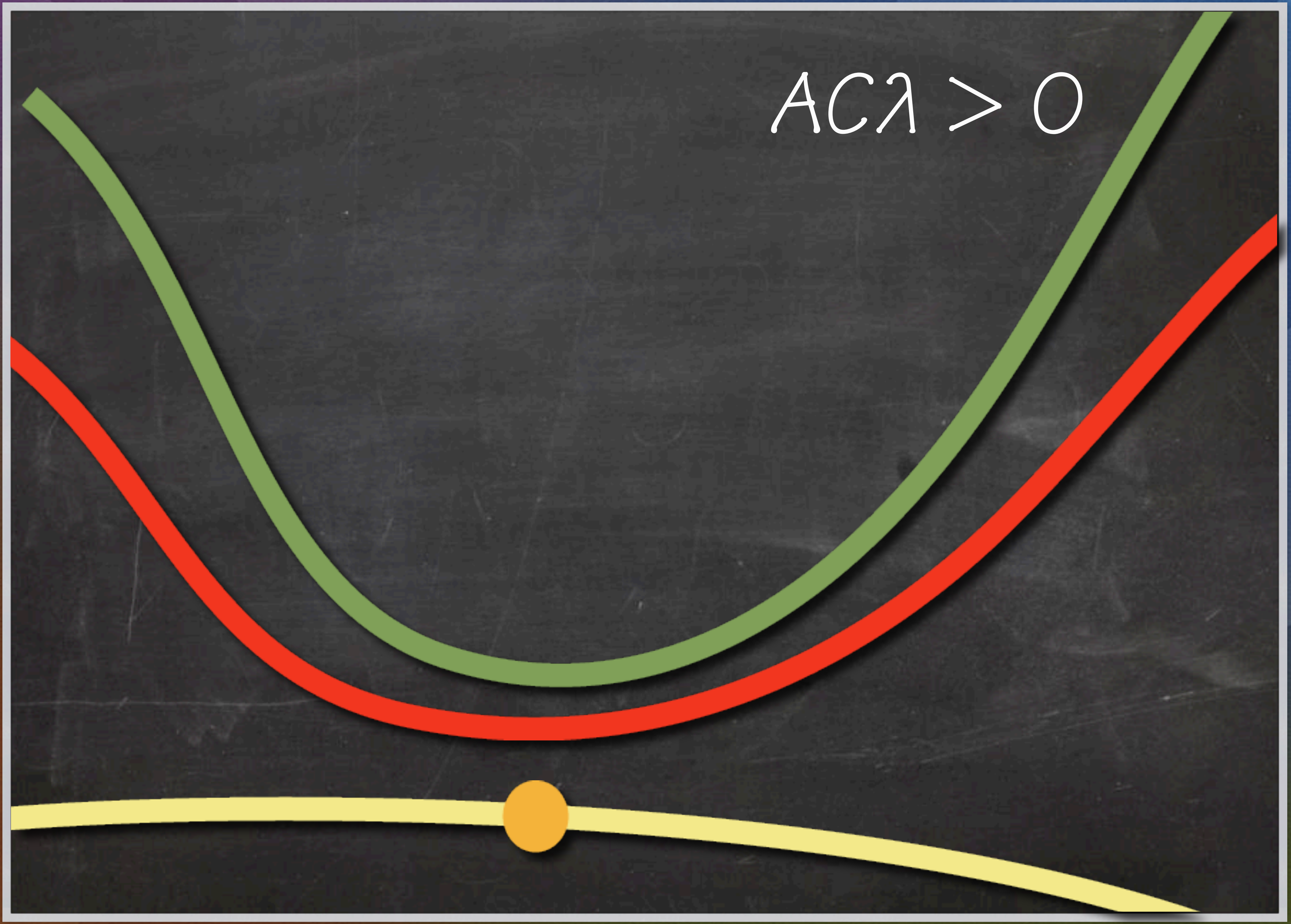
λ closer to 0

$AC\lambda \geq 0$


BC < 0
The
Banana Split
Scenario


$AC\lambda < 0$


$\lambda = 0$


ACλ > 0

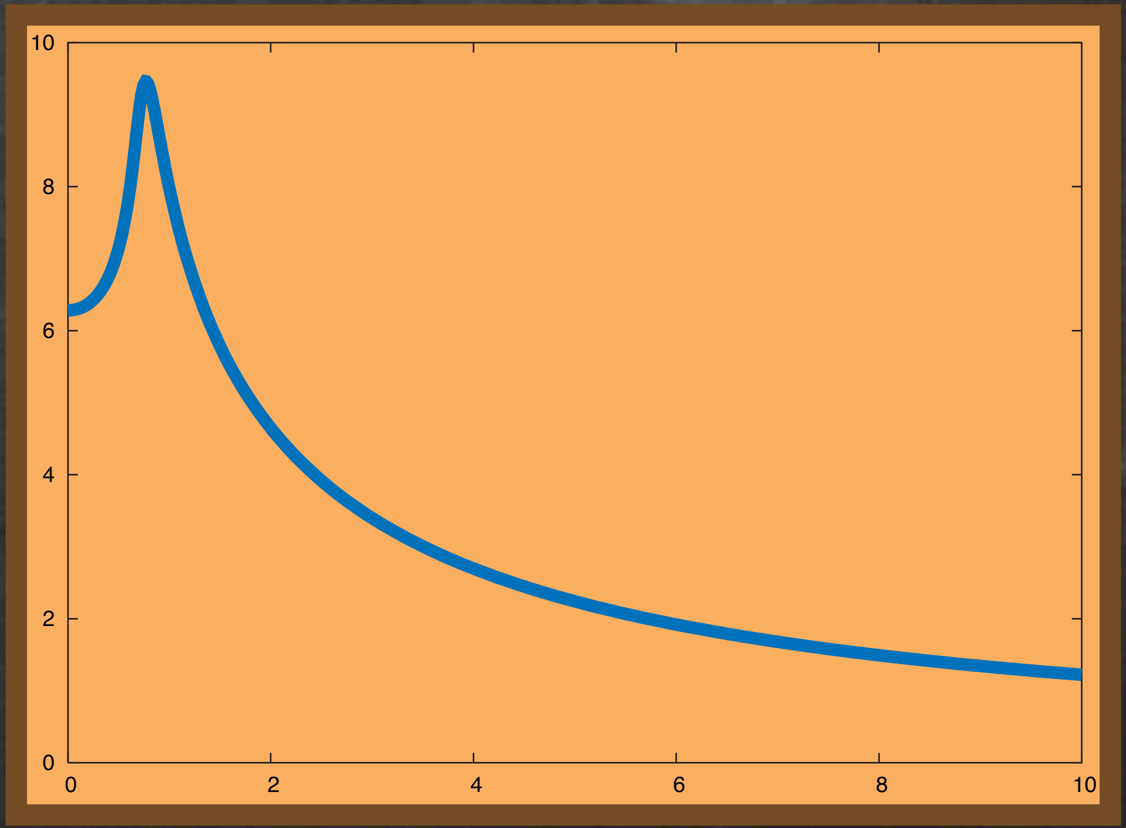
A numerical example

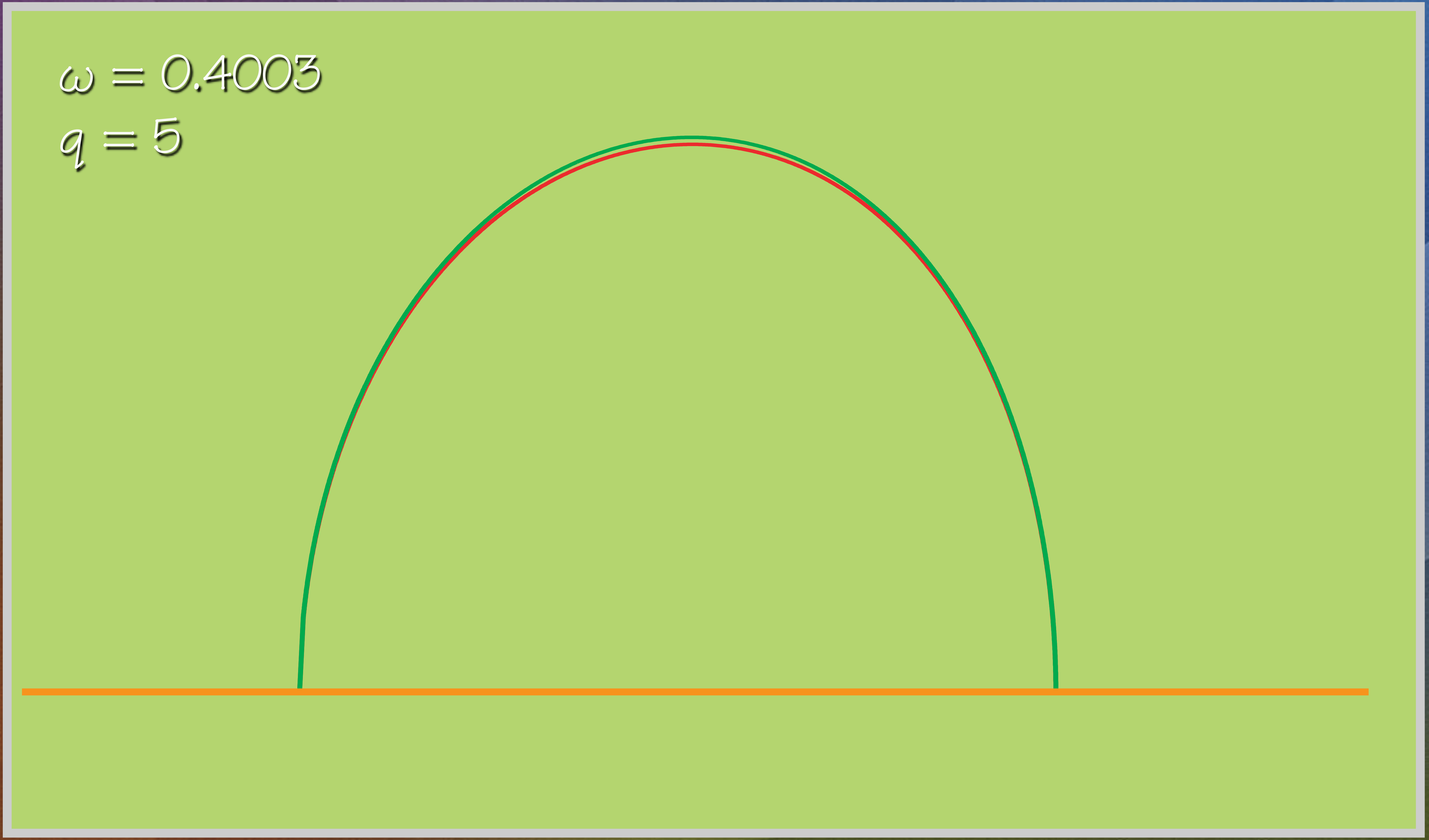
$$\begin{cases} \ddot{y} + (y + y^2 + \gamma y^3) = z \\ \ddot{z} + \omega^2 z = 0 \end{cases}$$

$\Rightarrow$ two parameters γ and ω

$\Rightarrow$ we take $q = 5$

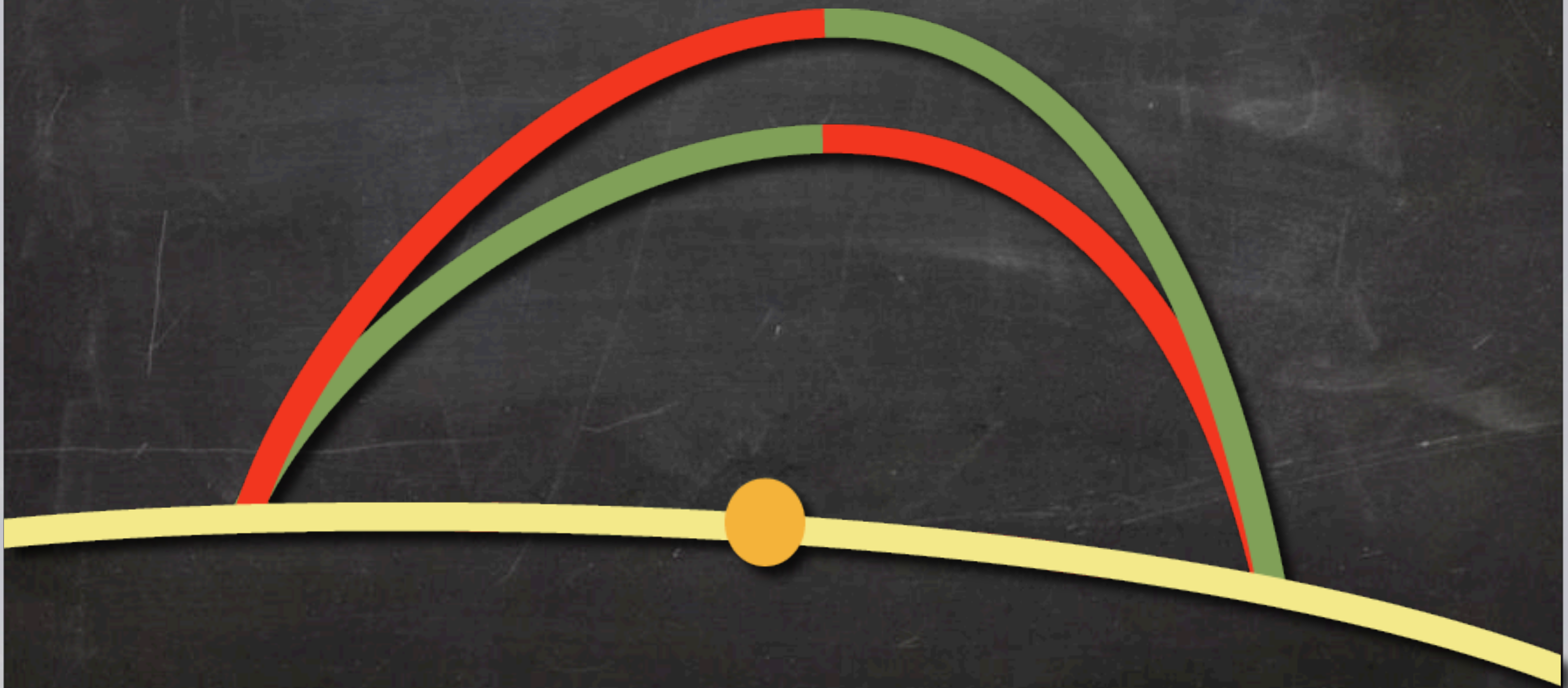
For $\gamma = 0.3$ the period map for the unperturbed y-equation ($z = 0$) shows a maximum




$\omega = 0.4003$
$q = 5$

However...

Along both arcs of the banana we found a transition from elliptic to hyperbolic or vice-versa.



Theoretically we should get

The answer

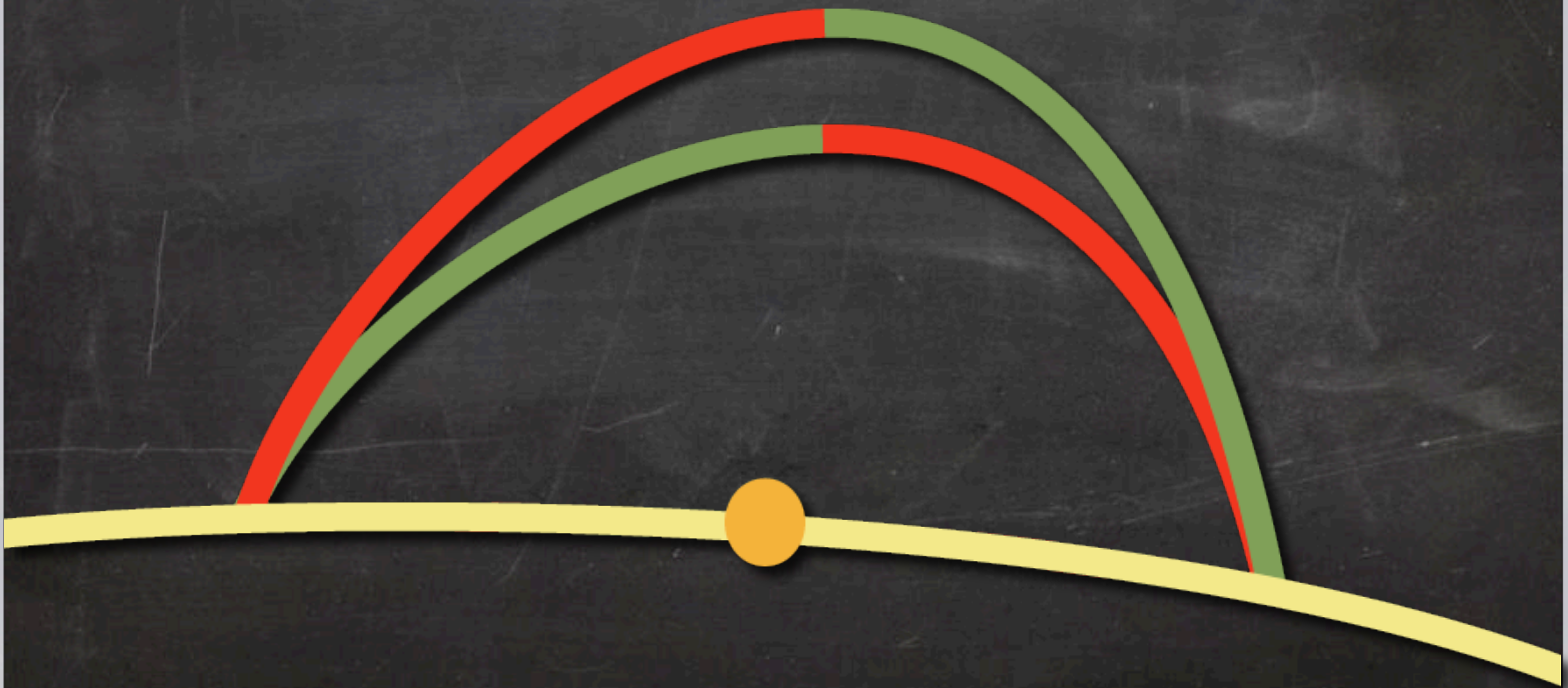
Our example is not fully generic due to the presence of a first integral — the amplitude of the forcing equation.

When along a branch of symmetric periodic orbits in a reversible system with a first integral the integral reaches a maximum or a minimum, then typically one will see at the same time a change of stability (from elliptic to hyperbolic or vice versa)

Change of stability

Non-generic banana scenario

$AC\lambda < 0$



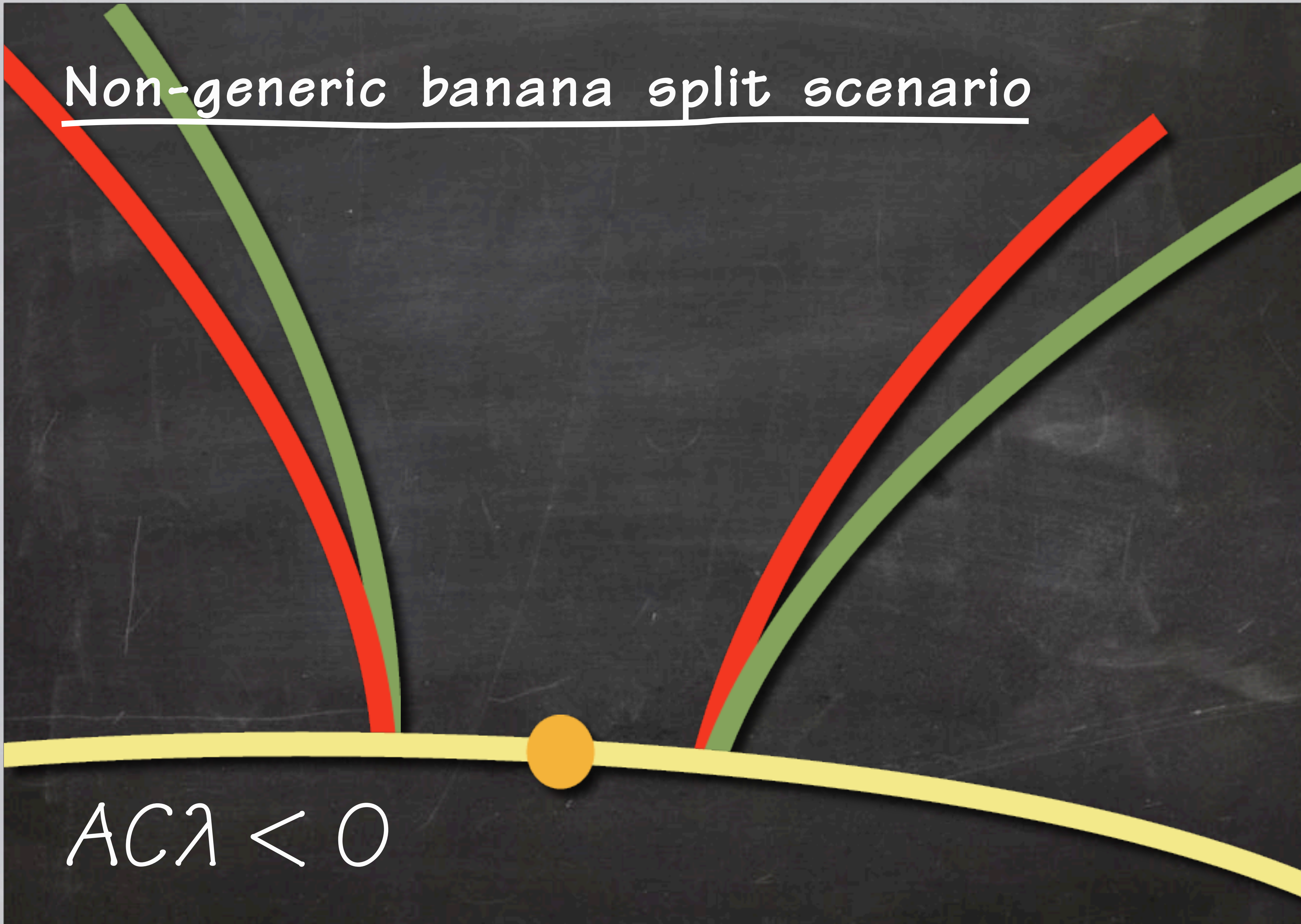
Non-generic banana scenario

λ closer to 0

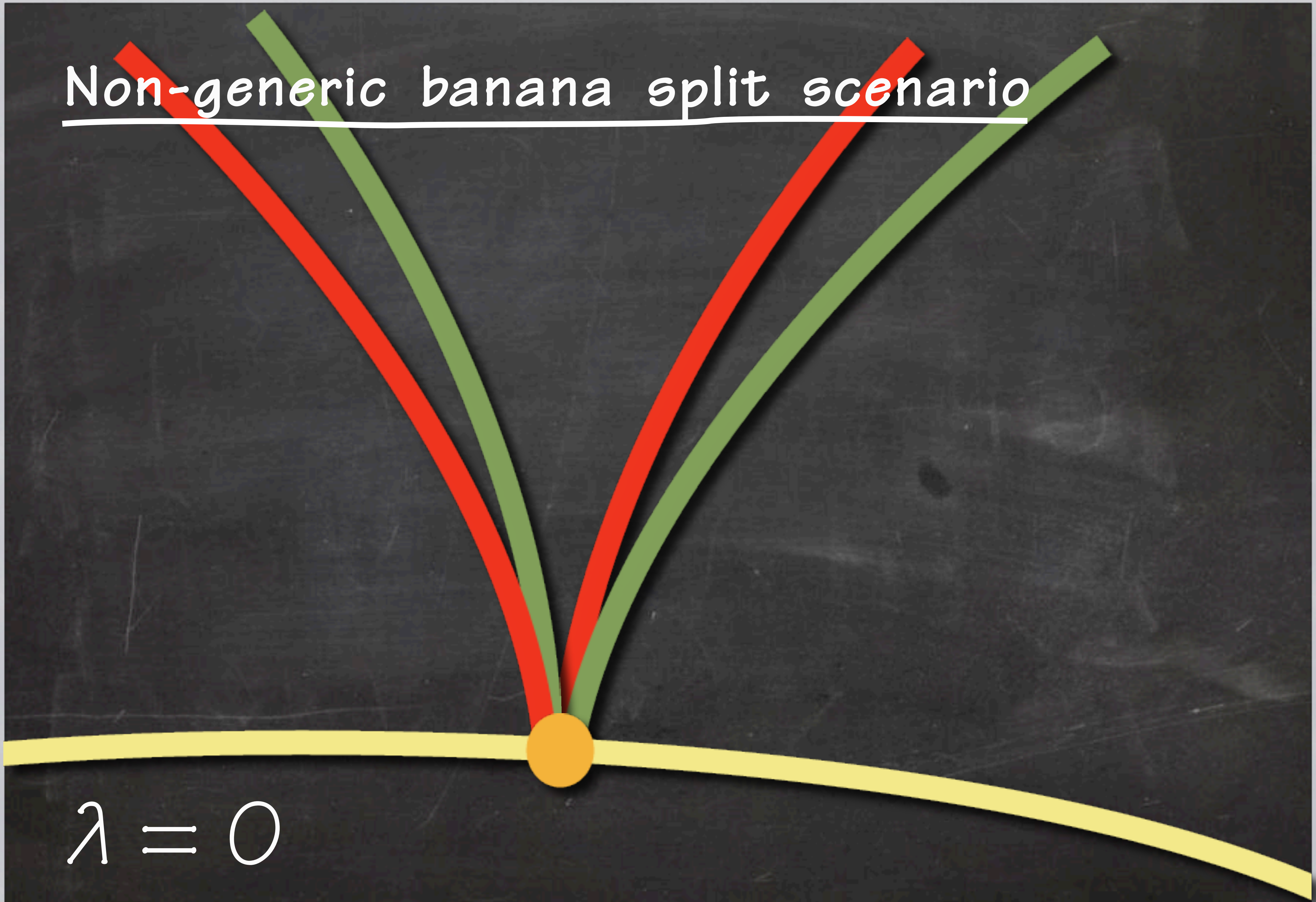
Non-generic banana scenario

$$AC\lambda \geq 0$$

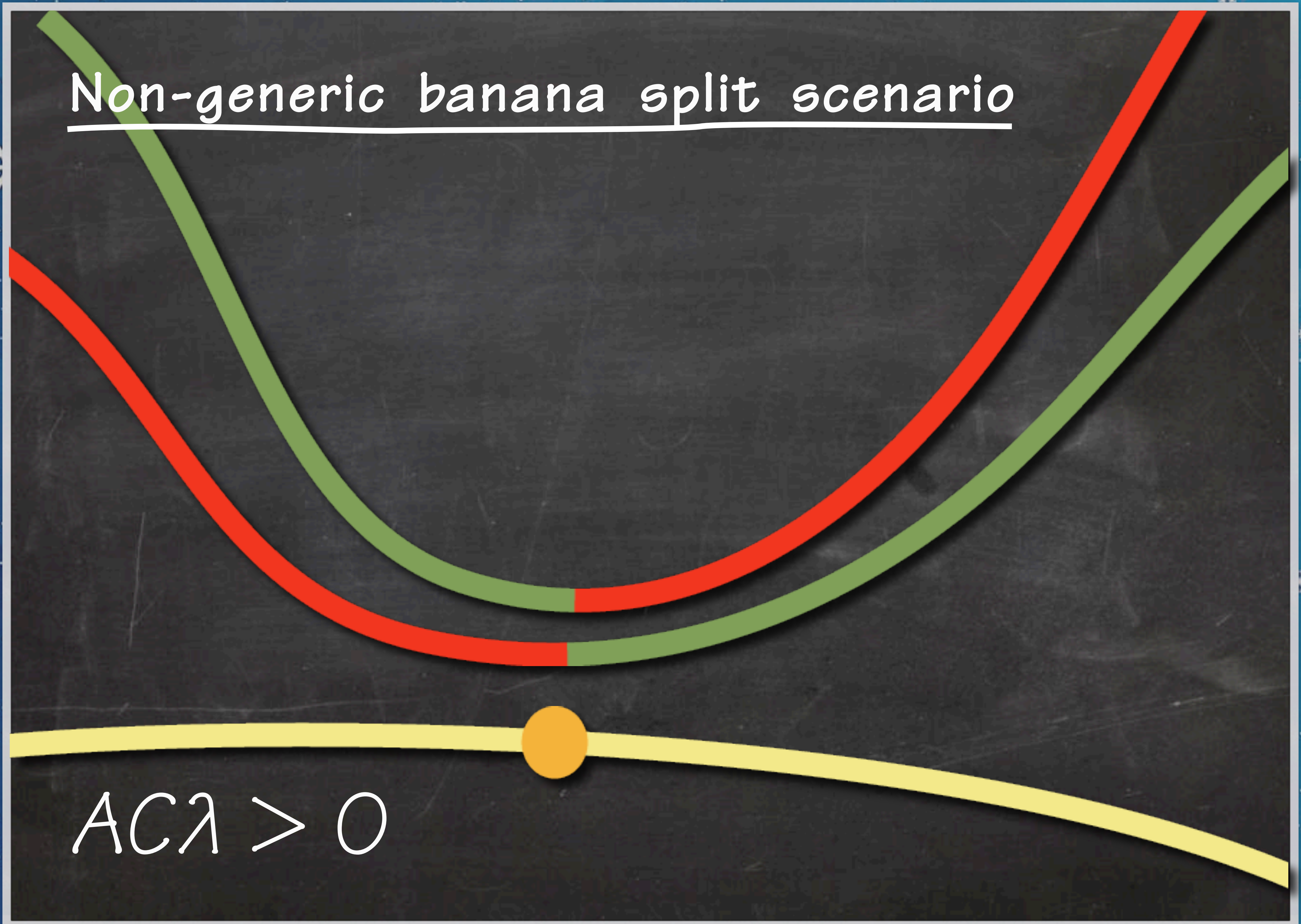
Non-generic banana split scenario



Non-generic banana split scenario



$\lambda = 0$



Non-generic banana split scenario
$AC\lambda > 0$

Transition

from non-generic to generic banana

$$\begin{cases} \ddot{y} + (y + y^2 + \gamma y^3) = z \\ \ddot{z} + (\omega^2 + \epsilon y) z = 0 \end{cases}$$

Increasing ϵ we see the following transition scenario from the non-generic situation ($\epsilon = 0$) to the generic situation — all other parameters are kept fixed.

Transition

from non-generic to generic banana

$\epsilon = 0$

Transition

from non-generic to generic banana

$\epsilon = 0.02$

Transition

from non-generic to generic banana

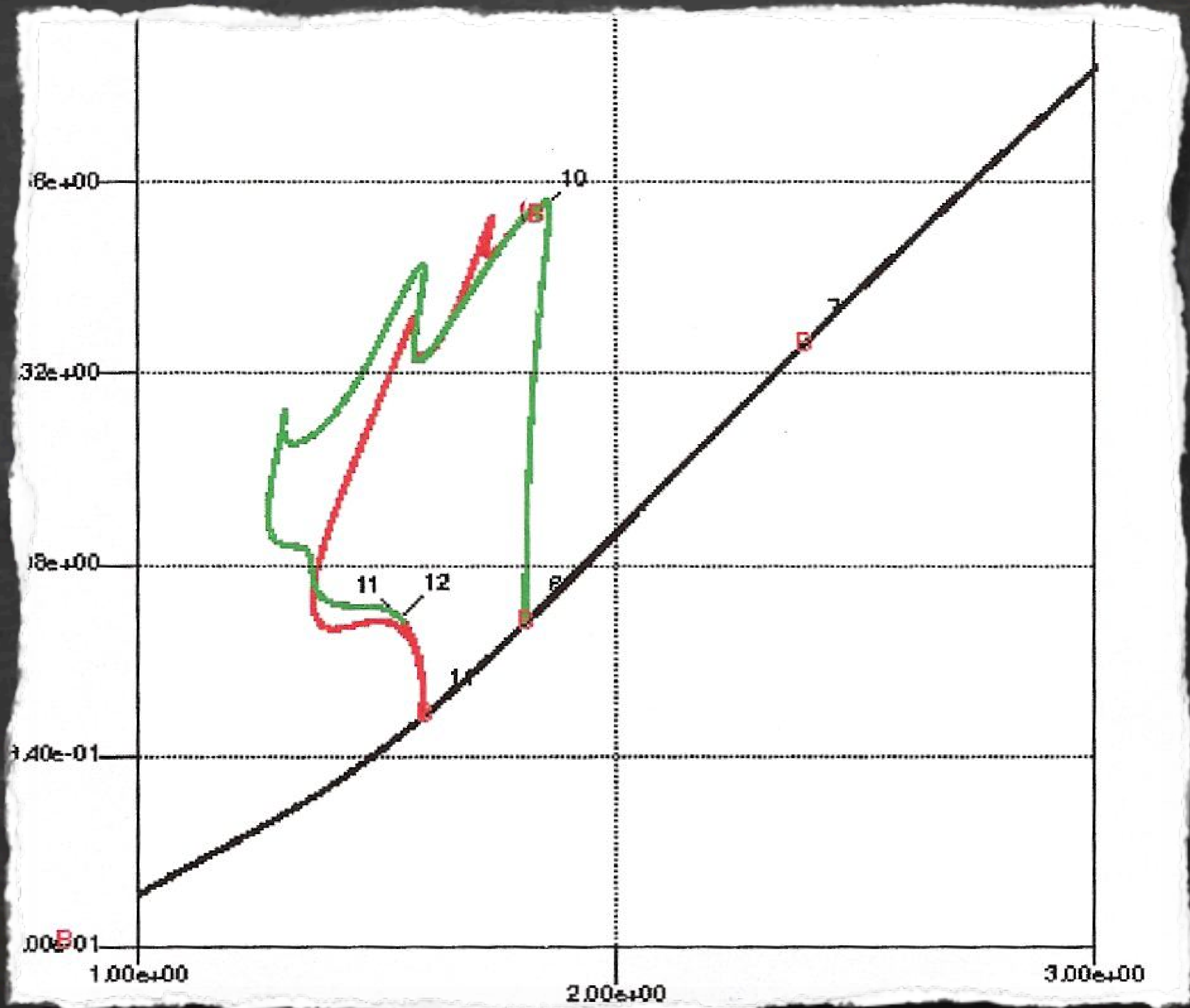
$\epsilon = 0.025$

Transition

from non-generic to generic banana

$\epsilon = 0.028$

Wild banana's!



Thank You