FINITE PERMUTATION GROUPS WITH A TRANSITIVE MINIMAL NORMAL SUBGROUP.

JOHN BAMBERG AND CHERYL E. PRAEGER

Abstract. A finite permutation group is said to be innately transitive if it contains a transitive minimal normal subgroup. In this paper, we give a characterisation and structure theorem for the finite innately transitive groups, as well as describing those innately transitive groups which preserve a product decomposition. The class of innately transitive groups contains all primitive and quasiprimitive groups.

Contents

1. Introduction

The result known as the O'Nan-Scott Theorem (see [12, 16]) describes the structure of finite primitive permutation groups up to permutational isomorphism. Its applications have had significant consequences for problems within permutation group theory and in combinatorics (for a survey, see [13]). In 1992, the second author proved a similar theorem for a larger class of finite permutation groups called *quasiprimitive* groups (see [14]). A permutation group is quasiprimitive if all of its minimal normal subgroups are transitive. In particular, every primitive group is

---

2000 *Mathematics Subject Classification.* Primary 20B05, 20B15.
We would like to thank Tim Penttila, Csaba Schneider, and Laci Kovács for various enlightening discussions, and the first author would like to acknowledge the financial assistance of the Australian Postgraduate Award, UWA Winthrop Scholarship, and the Jean Rogerson Postgraduate Award.

quasiprimitive but the converse is not true. This theorem for quasiprimitive groups has been used to study finite 2-arc transitive graphs [14], line-transitive linear spaces [6], and to study Weiss's Conjecture for locally primitive graphs [7]. One of the aims of this paper is to prove an extension of these results to an even larger class of finite permutation groups, which we introduce.

A finite permutation group G is *innately transitive* if it has a transitive minimal normal subgroup. We call this subgroup a *plinth* of G. Every innately transitive group has at most two transitive minimal normal subgroups (see Lemma 5.1), and if there are two, then they are regular and isomorphic, and there is a permutational isomorphism of G that interchanges them. So up to permutational isomorphism, any of the at most two minimal normal subgroups of G can be taken to be *the* plinth of G. Clearly every finite quasiprimitive permutation group is innately transitive. Innately transitive groups occur naturally as overgroups of quasiprimitive groups and they also provide a natural setting for studying embeddings of subgroups of wreath products in their product action (see [3]). It was proved recently by Michael Giudici [9] that the Polycirculant Conjecture is true for innately transitive groups (see also [5]).

L. G. Kovács [11] introduced a "blow-up" construction for so-called primitive groups which preserve a product decomposition of the underlying set. Each primitive group of product action type can be constructed as a "blow-up" of smaller primitive groups. The construction was generalised in [2] to a blow-up construction for quasiprimitive permutation groups that produces such groups of product action type that preserve a Cartesian decomposition of the underlying set. However not all quasiprimitive groups of Product Action type can be constructed by this method. We resolve this problem in the context of innately transitive groups (see Proposition 11.1).

Define an *innate triple* to be an ordered triple (K, φ, L) consisting of a characteristically simple group K, a subgroup L of $\mathrm{Aut}(K)$, and a certain epimorphism φ with domain the subgroup of K corresponding to $L \cap \mathrm{Inn}(K)$ if K is nonabelian, or domain K if K is abelian (see Definition 6.1). In Construction 6.6, we give a method of constructing innately transitive groups that takes as input an innate triple. The first major result of this paper, Theorem 1.1, proves that up to permutational isomorphism, each finite innately transitive group can be produced by Construction 6.6. Thus Theorem 1.1 is an abstract and constructive characterisation of finite innately transitive permutation groups. The proof is given in Section 7.

Theorem 1.1. *Every finite innately transitive permutation group is permutationally isomorphic to an innately transitive group given by Construction 6.6, and every permutation group given by this construction is innately transitive.*

The second major result of this paper, Theorem 1.2, is similar to the structure theorems for finite primitive and quasiprimitive groups (see [12, 16] and [14]). It shows that the innately transitive groups may be partitioned into a number of disjoint types. These types are defined in Section 3, and are presented in a different style and case subdivision from that in [12], [16], or [14].

Theorem 1.2. *Every finite innately transitive permutation group is permutationally isomorphic to a group of one of the five types described in Section 3. Moreover, every group described in Section 3 is innately transitive.*

We give an outline of the proof of this theorem in Section 8 and complete its proof in Section 10 after examining the different types of non-quasiprimitive innately transitive groups in Sections 9 and 10.

Peter Cameron in his book [4, p. 103], calls a primitive permutation group G *basic* if it is not contained in a wreath product acting by product action. He then states a version of the O'Nan-Scott Theorem in terms of basic groups and a wreath product construction. Non-basic primitive permutation groups can be described as subgroups of wreath products $G_0 \operatorname{wr} S_k$ in product action where G_0 is a basic primitive permutation group. We need a different definition of basic for quasiprimitive groups. The non-basic finite quasiprimitive groups G are those which act (faithfully) on a G-invariant point partition as a subgroup of a wreath product in product action.

One can extend this notion to innately transitive groups. However the quotient action of G on the point partition need not be faithful in this case. In Construction 9.1, we show how to define this possibly unfaithful quotient of a non-basic innately transitive group. The quotient is a subgroup of a wreath product of basic innately transitive groups in product action (see Proposition 9.2). Given one of these quotients, Construction 9.6 gives a procedure for constructing nearly all non-basic innately transitive groups corresponding to it (see Corollary 9.8). The non-basic innately transitive groups not given by Construction 9.6 are either of *Compound Diagonal* type or *Affine Plinth* type, which we do not discuss in detail since they are quasiprimitive and so their structure is well-known (see [14]). The innate triples corresponding to the quotients given by Construction 9.6 are those triples $(K, \widetilde{\varphi}, \widetilde{L})$ such that K is nonabelian, nonsimple, and $\operatorname{Ker} \widetilde{\varphi}$ is not a subdirect subgroup of K. Given a supplement K_0 of $\operatorname{Ker} \widetilde{\varphi}$ in $\operatorname{Dom} \widetilde{\varphi}$, a supplement in $\widetilde{L}$ of the subgroup of $\operatorname{Inn}(K)$ corresponding to $\operatorname{Ker} \widetilde{\varphi}$, and a normal subgroup of K_0 contained in $\operatorname{Ker} \widetilde{\varphi}$, we can construct another innate triple via Construction 9.6. From this output, we can construct an innately transitive group via Construction 6.6.

The theorem below is similar to Cameron's structure theorem in that we have separated the basic and non-basic innately transitive groups. Essentially it shows that innately transitive groups satisfying (1), (2), or (3) of Theorem 1.3 play the role of the basic examples with other examples arising from a product type construction.

Theorem 1.3. *Let G be a finite innately transitive permutation group on a set Ω with plinth $K = T^k$ (where T is a simple group). Then one of the following is true:*

(1) *K is abelian,*
(2) *K is simple,*
(3) *G is quasiprimitive of Simple Diagonal type,*
(4) *G is quasiprimitive of Compound Diagonal type, or*
(5) *G can be obtained by Construction 9.6 from a subgroup of a wreath product $G_0 \operatorname{wr} S_k$ where G_0 is innately transitive as in (2).*

In Section 2, we give some standard definitions needed to understand the descriptions of the innately transitive types given in Section 3. In Section 4 we give some basic results and background theory. In Section 5 we explore the structure of innately transitive groups, and in Section 6 we give a general method for constructing innately transitive groups namely Construction 6.6. Theorem 1.1 is proved in Section 7 and in Section 8 we give a framework for proving Theorem 1.2. We then characterise a particular class of innately transitive groups in Section 9. We also prove Theorem 1.3 in this section. In Section 10, we give details of the structure of innately transitive groups with a regular plinth and complete the proof of Theorem 1.2. Finally Section 11 contains some observations and commentary on our results.

2. Preliminaries

First we give some preliminary definitions. A permutation group G on Ω is *semiregular* if the only element fixing a point in Ω is the identity element of G. In this situation, if $\alpha \in \Omega$, then the map defined by $g \mapsto \alpha^g$ for all $g \in G$, is injective. We say that G is *regular* on Ω if it is both transitive and semiregular on Ω. Let G be a group acting transitively on a finite set Ω. A nonempty subset Δ of Ω is called a *block* for G if for each $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \varnothing$. We call $\{\Delta^g : g \in G\}$ a *block system* for G. We say that G is *primitive* if the only blocks for G are the singleton subsets or the whole of Ω. Given a block system Σ, the group G acts naturally on Σ. We denote the induced permutation group by G^Σ. It is a standard result from permutation group theory that the orbits of a normal subgroup N of G form a block system Σ for G. In this case, G^Σ is isomorphic to the factor group $G/\hat{N}$ where $\hat{N}$ is the kernel of the action of G on Σ, and of course $\hat{N}$ contains N.

Let G be a group acting on a set Ω and H be a group acting on a set Δ. Then G on Ω is *permutationally isomorphic* to H on Δ if there is an isomorphism $\theta : G \to H$ and a bijection $\beta : \Omega \to \Delta$ such that for all $g \in G$ and $\omega \in \Omega$, we have $(\omega^g)\beta = (\omega)\beta^{(g)\theta}$. The pair (θ, β) is called a *permutational isomorphism*. If G and H are both transitive, then they are permutationally isomorphic if and only if there is an isomorphism $\theta : G \to H$ such that θ maps a point stabiliser of G onto some point stabiliser of H. In particular, if G and H are both subgroups of $\mathrm{Sym}(\Omega)$ for some set Ω, then G and H are permutationally isomorphic if and only if they are conjugate in $\mathrm{Sym}(\Omega)$.

Let $g \in G$, and let ρ_g, λ_g, and ι_g be the permutations of G defined by $(x)\rho_g = xg$, $(x)\lambda_g = g^{-1}x$, and $(x)\iota_g = g^{-1}xg = (x)\lambda_g \circ \rho_g$ for $x \in G$. The *right regular representation* of G is the subgroup of $\mathrm{Sym}(G)$ defined by $G_R := \{\rho_g : g \in G\}$. Similarly, the *left regular representation* of G is the subgroup of $\mathrm{Sym}(G)$ defined by $G_L := \{\lambda_h : h \in G\}$. The map ι_g is called an *inner automorphism* of G, and the set of all ι_g, denoted $\mathrm{Inn}(G)$, is called the *inner automorphism group* of G. For every group V and subgroup U, define $\mathrm{Inn}_U(V) = \{\iota_u \in \mathrm{Inn}(V) : u \in U\}$. In particular, note that $\mathrm{Inn}(V) = \mathrm{Inn}_V(V)$.

The *Holomorph* of a group X, denoted $\mathrm{Hol}(X)$, is the subgroup of $\mathrm{Sym}(X)$ generated by X_R and the automorphism group of X, which we denote $\mathrm{Aut}(X)$. It follows that $\mathrm{Hol}(X)$ is the semidirect product of X_R and $\mathrm{Aut}(X)$, with respect to the natural action of $\mathrm{Aut}(X)$ on X_R, and we can readily deduce that $\mathrm{Hol}(X) = N_{\mathrm{Sym}(X)}(X_R)$, $X_L = \mathrm{C}_{\mathrm{Sym}(X)}(X_R)$, and $X_R = \mathrm{C}_{\mathrm{Sym}(X)}(X_L)$. Furthermore, if X has a trivial centre, then $\langle X_L, X_R\rangle = X_L \times X_R = X_R \rtimes \mathrm{Inn}(X)$. If $G \leq \mathrm{Sym}(\Omega)$ and G has a regular normal subgroup X, then G on Ω is permutationally isomorphic to a subgroup $\hat{G}$ of $\mathrm{Hol}(X)$ on X via an isomorphism $\theta : G \to \hat{G}$ such that $(X)\theta = X_R$ and bijection $\beta : \alpha^x \mapsto x$ (for some fixed $\alpha \in \Omega$).

A group is said to be *almost simple* if it has a non-abelian simple unique minimal normal subgroup. If a group G has a normal subgroup K such that $\mathrm{C}_G(K) = 1$ (for example, if G is almost simple with minimal normal subgroup K), then G can be embedded in $\mathrm{Aut}(K)$ such that its image contains $\mathrm{Inn}(K)$.

The *core* of a subgroup H of G is the intersection of all G-conjugates of H. This is the largest normal subgroup of G contained in H and we write it as $\mathrm{Core}_G(H)$. We say that H is *corefree* in G if $\mathrm{Core}_G(H) = 1$. Let $\prod_{i\in I} H_i$ be a direct product of groups. We will denote by $\pi_j : \prod_{i\in I} H_i \to H_j$ the natural projection map for all $j \in I$. A group G is a *subdirect product* of $\prod_{i\in I} H_i$ if there is an embedding $\phi : G \to \prod_{i\in I} H_i$ such that $\phi \circ \pi_j : G \to H_j$ is an epimorphism for each $j \in I$. In the case that G is a subgroup of $\prod_{i\in I} H_i$ and ϕ is the inclusion map, we say that G is a *subdirect subgroup* of $\prod_{i\in I} H_i$. If G is a subgroup of $\prod_{i\in I} H_i$, we say that G is a *diagonal* subgroup of $\prod_{i\in I} H_i$ if the restriction of π_j to G is injective for each $j \in I$. We say that G is a *full diagonal* subgroup of $\prod_{i\in I} H_i$ if G is both a subdirect and diagonal subgroup of $\prod_{i\in I} H_i$. In this case, the direct factors H_i are isomorphic to a common group H. If $I = \{1, \dots, n\}$, then each full diagonal subgroup G of $\prod_{i\in I} H_i$ is of the form $\{((h)\gamma_1, (h)\gamma_2, \dots, (h)\gamma_n) : h \in H\}$ where for each $i \in I$, γ_i

is an isomorphism from H onto H_i. If for each $i \in I$, $H_i = H$ and γ_i is the trivial isomorphism, then we call G the *straight diagonal* subgroup of H^n.

The *graph* of a homomorphism $\varphi : B \to A$ is the subgroup of $A \times B$ defined by $\mathrm{Graph}(\varphi) := \{((b)\varphi, b) : b \in B\}$. It follows that if φ is an isomorphism, then $\mathrm{Graph}(\varphi)$ is a full diagonal subgroup of $A \times B$.

Let A be a group and let B be a subgroup of S_n. Then the *wreath product* $A \,\mathrm{wr}\, B$ of A and B is the semidirect product $A^n \rtimes B$ where the action of B on A^n is defined by $(a_1, \ldots, a_n)^{b^{-1}} = (a_{1^b}, \ldots, a_{n^b})$ for all $(a_1, \ldots, a_n) \in A^n$ and $b \in B$. The subgroup A^n of $A \,\mathrm{wr}\, B$ is known as the *base group* and the subgroup B is known as the *top group*.

Suppose now that $A \leq \mathrm{Sym}(\Gamma)$. Then there is a well-defined action of $A \,\mathrm{wr}\, B$ on Γ^n known as the *product action*, where the base group acts coordinate-wise and the top group permutes the coordinates. Specifically, for all $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, $(a_1, \ldots, a_n) \in A^n$, and $b \in B$,

$$(\gamma_1, \ldots, \gamma_n)^{(a_1,\ldots,a_n)} = (\gamma_1^{a_1}, \ldots, \gamma_n^{a_n}), \text{ and } (\gamma_1, \ldots, \gamma_n)^{b^{-1}} = (\gamma_{1^b}, \ldots, \gamma_{n^b}).$$

For $1 \leq i \leq n$, let $p_i : \Gamma^n \to \Gamma$ be the natural projection map $(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_i$. Then $\Gamma_i = \{(\gamma)p_i^{-1} : \gamma \in \Gamma\}$ is a partition of Γ^n, and it is well-known (see for example [4, pp. 103]) that $\mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_n$ in product action, is the full stabiliser in $\mathrm{Sym}(\Gamma^n)$ of the set $\{\Gamma_i : 1 \leq i \leq n\}$.

3. Description of Types

In the following, G is a permutation group on a finite set Ω with K a normal subgroup of G and $\alpha \in \Omega$. We will suppose $K = T^k$ where T is a finite simple group and k a positive integer. Theorem 1.2 states that every finite innately transitive group belongs to the following list of types, and moreover, the categories in the list are disjoint and exhaustive, and every group G that belongs to the list is innately transitive with plinth K.

Abelian Plinth Type. Here Ω is a k-dimensional vector space over a field of prime order p, that is, $\Omega = \mathbb{Z}_p^k$ and $G = K \rtimes G_0$ where K is the group of all translations and G_0 is an irreducible subgroup of $GL(k,p)$ (the group of automorphisms of the additive group of Ω). So G is contained in the holomorph of K, that is, $G \leq K \rtimes GL(k,p) = AGL(k,p)$. The subgroup K is regular on Ω and elementary abelian. In this case, G is primitive.

Simple Plinth Type. In this case, K is a non-abelian simple group, $G = KG_\alpha$, and $G_\alpha \ngeq K$. There are three subtypes:

(1) *Holomorph of a Simple Group Type*

Here $\Omega = K$ and $\mathrm{Inn}(K) \leq G_\alpha \leq \mathrm{Aut}(K)$. So G is contained in the holomorph of K, that is, $K \rtimes \mathrm{Inn}(K) \leq G \leq K \rtimes \mathrm{Aut}(K) = \mathrm{Hol}(K)$. In this case, G is primitive.

(2) *Almost Simple Type*

In this case $\mathrm{C}_G(K) = 1$ and $K \cong \mathrm{Inn}(K) \leq G \leq \mathrm{Aut}(K)$, that is, G is almost simple.

(3) *Almost Simple Quotient Type*

Here $\mathrm{C}_G(K) \neq 1$ and $G \neq \mathrm{C}_G(K)G_\alpha$.

Regular Plinth Type. In this case, K is non-abelian, non-simple, and $\Omega = K$. There is an isomorphism φ from a subgroup K_0 of K onto $\mathrm{C}_G(K)$. Here $G = K \rtimes G_\alpha$ where $\mathrm{Graph}(\varphi) \trianglelefteq G_\alpha \leq \mathrm{Aut}(K) = \mathrm{Aut}(T) \,\mathrm{wr}\, S_k$, G_α projects onto a transitive subgroup of S_k, and G_α intersects $\mathrm{Inn}(K)$ in $\mathrm{Graph}(\varphi)$. So G is contained in the holomorph of K, that is, $K \rtimes \mathrm{Graph}(\varphi) \trianglelefteq G \leq K \rtimes \mathrm{Aut}(K) = \mathrm{Hol}(K)$. Moreover, G is conjugate in $\mathrm{Sym}(K)$ to a subgroup of $V \,\mathrm{wr}\, S_k$ in product action, where V is innately transitive with regular plinth T. In particular, we have four subtypes:

(1) *Twisted Wreath Type*

Here $\mathrm{Graph}(\varphi)$ is trivial, V is of *Almost Simple* type, and K is the unique minimal normal subgroup of G. Thus G is quasiprimitive.

(2) *Product Quotient Type*

In this case, $\mathrm{Graph}(\varphi)$ is nontrivial and not subdirect in $\mathrm{Inn}(K)$, and V is of *Almost Simple Quotient* type.

(3) *Diagonal Quotient Type*

Here $\mathrm{Graph}(\varphi)$ is a proper subdirect subgroup of $\mathrm{Inn}(K)$ and V is of *Holomorph of a Simple Group* type. Moreover $\mathrm{C}_G(K)$ is a direct product of m full diagonal subgroups of $T^{k/m}$ for some proper divisor m of k, and $G \leq \mathrm{N}_{\mathrm{Hol}(K)}(\mathrm{C}_G(K)) = K \rtimes [(A \times S_{k/m}) \,\mathrm{wr}\, S_m]$, $\mathrm{Graph}(\varphi) = B^m$, where A and B are full diagonal subgroups of $\mathrm{Aut}(T)^{k/m}$ and $\mathrm{Inn}(T)^{k/m}$ respectively, and the projection of G_α onto $S_{k/m} \,\mathrm{wr}\, S_m$ is transitive.

(4) *Holomorph of a Compound Group Type*

For this case, $\mathrm{Graph}(\varphi)$ is equal to $\mathrm{Inn}(K)$, V is of *Holomorph of a Simple Group* type, and G is primitive.

Product Type. There is a G-invariant partition Ψ of Ω and Ψ is the cartesian product of k copies of a set Ψ_0. Also there is an innately transitive permutation group A on Ψ_0 of *Almost Simple* or *Almost Simple Quotient* type with non-regular plinth T. Choose $\psi_0 \in \Psi_0$ and set $U := T_{\psi_0}$. For $\psi = (\psi_0, \ldots, \psi_0) \in \Psi$, we have $K_\psi = U^k$, and for $\alpha \in \psi$, the point stabiliser K_α is a subdirect subgroup of U^k with index the size of a cell in Ψ. Replacing G by a conjugate in $\mathrm{Sym}(\Omega)$ if necessary, we may assume that $G^\Psi \leq A \,\mathrm{wr}\, S_k \leq \mathrm{Sym}(\Psi_0) \,\mathrm{wr}\, S_k$. The point stabiliser G_α projects onto a transitive subgroup of S_k.

Diagonal Type. In this case, K is non-simple, non-regular, a stabiliser K_α is a subdirect subgroup of K, and $\mathrm{C}_G(K) = 1$. Thus G is quasiprimitive. We have $\Omega = \Delta^l$ and $K = T^k \leq G \leq$

$B \operatorname{wr} S_l \leq \operatorname{Sym}(\Delta) \operatorname{wr} S_l$, in product action, for some proper divisor l of k where B is a quasiprimitive permutation group on Δ with plinth $T^{k/l}$, and G projects onto a transitive subgroup of S_l. In particular, G is quasiprimitive and we have two subtypes:

(1) *Simple Diagonal Type*

Here $l = 1$ and K_α is a full diagonal subgroup of K. For more details see [14].

(2) *Compound Diagonal Type*

In this case, $l > 1$ and B is of *Simple Diagonal* type.

Remarks 3.1.

(i) In forming this case subdivision, the first consideration was the abstract group theoretical structure of the plinth K. In the case where K is nonsimple and nonabelian, we make a further case subdivision according to the structure of a point stabiliser of K. This approach is similar to that of the structure theorem for primitive and quasiprimitive groups.

(ii) The Regular Plinth Type is not the only case where an innately transitive group can have a regular plinth. The types preceding this case in the subdivision, namely Abelian Plinth Type and Simple Plinth Type may contain innately transitive groups with a regular plinth. In fact, every group of the former type has a regular plinth.

(iii) The Almost Simple Quotient Type, Product Quotient Type, and Diagonal Quotient Type are so named because any group of one of these types has a quotient action (on the orbits of $\mathrm{C}_G(K)$) which is quasiprimitive of either Almost Simple Type, Product Type, or Diagonal Type. See Section 11(4) for more details.

4. Fundamental Theory

First we restate some well-known or easily proven results, the first of which is the structure theorem for finite quasiprimitive permutation groups which appeared in the second author's paper [14, Theorem 1].

Theorem 4.1 (Structure Theorem for Quasiprimitive Groups.)**.**
Let G be a finite quasiprimitive permutation group and let K be a minimal normal subgroup. Then G is permutationally isomorphic to a quasiprimitive group in exactly one of the following types described in the previous section:

(1) *Abelian Plinth Type,*
(2) *Holomorph of a Simple Group Type,*
(3) *Holomorph of a Compound Group Type,*
(4) *Almost Simple Type,*

(5) *Twisted Wreath Type,*
(6) *Diagonal Type, or*
(7) *Product Type (with* $\mathrm{C}_G(K) = 1$*).*

Now we give a lemma that can be found in Wielandt's classic text [17, Proposition 4.3 and Ex.4.5] and in the contemporary text of Dixon and Mortimer [8, Ex.4.2.7]. If K is a subgroup of $\mathrm{Sym}(\Omega)$, then the *full centraliser* of K is the subgroup $\mathrm{C}_{\mathrm{Sym}(\Omega)}(K)$.

Lemma 4.2. *Let* Ω *be a finite set. Then*

(1) *A subgroup of* $\mathrm{Sym}(\Omega)$ *is semiregular if and only if its full centraliser is transitive.*
(2) *The full centraliser of a transitive subgroup of* $\mathrm{Sym}(\Omega)$ *is semiregular.*

The second statement above follows from the first. The next lemma is a simple but useful result.

Lemma 4.3. *Let* G *and* G' *be groups with normal subgroups* K *and* K' *respectively. Let* θ *be an epimorphism from* G *onto* G' *which restricts to an isomorphism of* K *onto* K'. *Then* $\mathrm{C}_{G'}(K') = (\mathrm{C}_G(K))\theta$.

Proof. Let $g' \in \mathrm{C}_{G'}(K')$ and $k' \in K'$. Since θ is an epimorphism, $g' = (g)\theta$ for some $g \in G$, and since θ restricts to an isomorphism of K onto K', we have $k' = (k)\theta$ for some $k \in K$. Since $g' \in \mathrm{C}_{G'}(K')$, we have $1 = (g')^{-1}(k')^{-1}g'k' = (g^{-1}k^{-1}gk)\theta$ and hence $g^{-1}k^{-1}gk \in \mathrm{Ker}\,\theta \cap K = 1$. Since this holds for all $k' \in K'$, and hence for all $k \in K$, we conclude that $g \in \mathrm{C}_G(K)$. Therefore $\mathrm{C}_{G'}(K') \leq (\mathrm{C}_G(K))\theta$. Now let $g \in \mathrm{C}_G(K)$ and let $k' \in K'$. Then there exists $k \in K$ such that $k' = (k)\theta$ and hence $((g)\theta)^{-1}k'(g)\theta = (g^{-1}kg)\theta = (k)\theta = k'$. So $(g)\theta \in \mathrm{C}_{G'}(K')$ and hence $\mathrm{C}_{G'}(K') = (\mathrm{C}_G(K))\theta$. $\square$

The first part of the following lemma comes from a result of Scott (see [16, Lemma, p. 328]), and the second result can be found in [10, Proposition 5.2.5(i)].

Lemma 4.4. *Let* K *be a finite direct product* $\prod_{i\in I} T_i$ *of finite non-abelian simple groups, and let* M *be a subgroup of* K.

(1) *If* M *is a subdirect subgroup of* K, *then* M *is the direct product* $\prod M_j$ *of full diagonal subgroups* M_j *of subproducts* $\prod_{i\in I_j} T_i$ *where the* I_j *form a partition of* I.
(2) *If* M *is a normal subgroup of* K, *then* $M = \prod_{j\in J} T_j$ *where* J *is a subset of* I.

Consequently, we get the following result whose proof also follows from [8, Theorem 4.3A].

Lemma 4.5. *Let* G *be a group and* N *be a non-abelian normal subgroup of* G. *Suppose that* N *is a direct product of non-abelian simple groups* $T_1, T_2, \ldots, T_k$. *Then* G *acts on* $\{T_1, T_2, \ldots, T_k\}$ *by* *conjugation. Moreover, G is transitive on $\{T_1, T_2, \ldots, T_k\}$ if and only if N is a minimal normal subgroup of G.*

Below is a simple result which was stated in Scott's proof of Lemma 4.4(1).

Lemma 4.6. *A full diagonal subgroup of a finite direct product of non-abelian simple groups is self-normalising.*

Proof. Let $K = T_1 \times \cdots \times T_k$ where the T_i are non-abelian simple groups and let J be a full diagonal subgroup of K. Then there exist isomorphisms $\gamma_2 : T_1 \to T_2, \ldots, \gamma_k : T_1 \to T_k$ such that $J = \{(t, (t)\gamma_2, \ldots, (t)\gamma_k) : t \in T_1\}$. In particular, the T_i are all isomorphic so we may take $T_1 = \cdots = T_k = T$ and then the $\gamma_i \in \mathrm{Aut}(T)$.

Let $(t_1, \ldots, t_k) \in \mathrm{N}_K(J)$ and let $(j, (j)\gamma_2, \ldots, (j)\gamma_k) \in J$. Then $(j^{t_1}, ((j)\gamma_2)^{t_2}, \ldots, ((j)\gamma_k)^{t_k}) \in J$ and hence $((j)\gamma_i)^{t_i} = (j^{t_1})\gamma_i = (j)\gamma_i^{(t_1)\gamma_i}$ for $2 \leq i \leq k$. So since γ_i is bijective and j is an arbitrary element of J, we have that $(t_1)\gamma_i t_i^{-1} \in Z(T)$ for $2 \leq i \leq k$. Now T is a non-abelian simple group, and thus has trivial centre. Therefore $t_i = (t_1)\gamma_i$ for $2 \leq i \leq k$. So $(t_1, \ldots, t_k) \in J$ and $\mathrm{N}_K(J) = J$. □

Let Γ be a finite set, let n be a positive integer, and let $W := \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_n$ acting in product action. Let $\mu : W \to S_n$ be the natural projection map. One can think of μ as a permutation representation of W. Let W_1 be the point stabiliser of 1 in the induced action of W on $\{1, \ldots, n\}$. Note that W_1 can be factorised as $\mathrm{Sym}(\Gamma) \times (\mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_{n-1})$. Let ν be the natural projection map of W_1 onto the first factor $\mathrm{Sym}(\Gamma)$. Let G be a subgroup of W such that $(G)\mu$ is transitive. The *component* of G is defined as the subgroup $(G \cap W_1)\nu$ of $\mathrm{Sym}(\Gamma)$. Note that transitivity of $(G)\mu$ implies that the component of G is independent of the choice of stabiliser $G \cap W_i$ $(1 \leq i \leq n)$ up to conjugacy in G. The following lemma is a restatement of a result of L. G. Kovács [11, (2.2)].

Lemma 4.7. *Let $W = \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_n$ and let G be a subgroup of W such that $(G)\mu$ is transitive. Then G is conjugate by an element of $\mathrm{Sym}(\Gamma)^n \cap \mathrm{Ker}\, \nu = 1 \times \mathrm{Sym}(\Gamma)^{n-1}$ to a subgroup of $V \,\mathrm{wr}\, S_n$, where V is the component of G.*

Now we consider extending a transitive action of a normal subgroup to one for the group.

Lemma 4.8. *Let G be a group, K be a normal subgroup of G, and Y a subgroup of G such that $G = KY$. Then G has a well-defined transitive action on $[K : K \cap Y]$ given by*

$$[(K \cap Y)x]^{vy} = (K \cap Y)y^{-1}xvy \qquad (*)$$

for all $x, v \in K$ and $y \in Y$. Further if $K = T^k$ for some nonabelian simple group T and integer $k \geq 1$, and $K \cap Y = U^k$ for some $U < T$, then the permutation group induced by G on $[K : K \cap Y]$ is permutationally isomorphic to a subgroup of $\mathrm{Sym}([T : U]) \,\mathrm{wr}\, S_k$ in product action.

Proof. First we show that the definition in $(*)$ is independent of the coset representative x and the expression vy for an element of G. Suppose that $v_1y_1 = v_2y_2$ and $(K \cap Y)x_1 = (K \cap Y)x_2$ for some $v_1, v_2, x_1, x_2 \in K$ and $y_1, y_2 \in Y$. Thus $y_1y_2^{-1} = v_1^{-1}v_2 \in K \cap Y$. We must show that $[(K \cap Y)x_1]^{v_1y_1} = [(K \cap Y)x_2]^{v_2y_2}$, that is, $(K \cap Y)y_1^{-1}x_1v_1y_1 = (K \cap Y)y_2^{-1}x_2v_2y_2$. Thus we need to prove that $K \cap Y$ contains

$$z := (y_1^{-1}x_1v_1y_1)(y_2^{-1}v_2^{-1}x_2^{-1}y_2) = y_1^{-1}x_1x_2^{-1}y_2 = (y_1^{-1}y_2)y_2^{-1}(x_1x_2^{-1})y_2. \qquad (**)$$

Since $K \cap Y$ is normalised by Y, we have $y_1^{-1}y_2 = y_1^{-1}(y_1y_2^{-1})^{-1}y_1 \in K \cap Y$ and $y_2^{-1}(x_1x_2^{-1})y_2 \in K \cap Y$. Thus the element z in $(**)$ lies in $K \cap Y$.

Now we have to prove that we indeed have a group action. It is clear that the identity element fixes each element of $[K : K \cap Y]$. Let $v_1, v_2, u \in K$ and let $y_1, y_2 \in Y$. Then $v_1y_1v_2y_2 = v_1(y_1v_2y_1^{-1})y_1y_2$ and hence

$$[(K \cap Y)x]^{(v_1y_1)(v_2y_2)} = (K \cap Y)(y_1y_2)^{-1}xv_1(y_1v_2y_1^{-1})(y_1y_2).$$

Also

$$\begin{aligned}([(K \cap Y)x]^{v_1y_1})^{v_2y_2} &= [(K \cap Y)y_1^{-1}xv_1y_1]^{v_2y_2} \\ &= (K \cap Y)y_2^{-1}y_1^{-1}xv_1y_1v_2y_2 \\ &= (K \cap Y)(y_1y_2)^{-1}xv_1(y_1v_2y_1^{-1})(y_1y_2)\end{aligned}$$

and therefore

$$([(K \cap Y)x]^{v_1y_1})^{v_2y_2} = [(K \cap Y)x]^{(v_1y_1)(v_2y_2)}.$$

Now suppose $K = T^k \neq 1$ for some nonabelian simple group T and integer $k \geq 1$, and suppose $K \cap Y = U^k$ for some $U < T$. Let $\Gamma = [T : U]$. Note that we can identify $[K : U^k]$ with Γ^k via the bijection $\beta : U^k(x_1, \dots, x_k) \mapsto (Ux_1, \dots, Ux_k)$. Let G act on Γ^k by

$$(Ux_1, \dots, Ux_k)^g = \left(((Ux_1, \dots, Ux_k)\beta^{-1})^g\right)\beta$$

for all $(Ux_1, \dots, Ux_k) \in \Gamma^k$ and $g \in G$. Clearly this action is well-defined and faithful. By definition, the action of G on $[K : U^k]$ is permutationally isomorphic to the action of G on Γ^k. It remains to show that the subgroup of $\mathrm{Sym}(\Gamma^k)$ induced by the action of G on Γ^k, is contained in $\mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_k$ (in product action). Note that $\mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_k$ (in product action) is the full stabiliser in $\mathrm{Sym}(\Gamma^k)$ of the set of partitions $\mathcal{P} = \{\{(x)p_i^{-1} : x \in \Gamma\} \mid 1 \leq i \leq k\}$ where $p_i : \Omega \to \Gamma$ is the natural projection map $(Ux_1, \dots, Ux_k) \mapsto Ux_i$ (see the end of Section 2). By Lemma 4.5, G acts on the simple direct factors of K by conjugation, and so it follows that G stabilises $\mathcal{P}$. Therefore G is permutationally isomorphic to a subgroup of $\mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_k$ in product action. $\square$

5. Exploring Innately Transitive Groups

In this section, we will explore some properties of innately transitive groups. As mentioned earlier, we can choose any of at most two transitive minimal normal subgroups for a plinth. The following lemma justifies this claim.

Lemma 5.1. *Every finite permutation group G on Ω has at most two distinct transitive minimal normal subgroups. Moreover, if K_1 and K_2 are distinct transitive minimal normal subgroups of G, then $\mathrm{C}_G(K_1) = K_2$, $\mathrm{C}_G(K_2) = K_1$, and there is an involution of $\mathrm{N}_{\mathrm{Sym}(\Omega)}(G)$ that interchanges K_1 and K_2, and which also centralises a point stabiliser of G.*

Proof. Suppose G has three distinct transitive minimal normal subgroups K_1, K_2, and K_3. Then each pair of the K_i intersect trivially and hence any two of them are contained in the centraliser of the third. For example $K_2, K_3 \leq \mathrm{C}_G(K_1)$. However, by Lemma 4.2, $\mathrm{C}_G(K_1)$ is semiregular and it follows that K_2, K_3, and $\mathrm{C}_G(K_1)$ are regular and hence equal, which is a contradiction. Therefore, there are at most two distinct transitive minimal normal subgroups of G.

Suppose now that K_1 and K_2 are distinct transitive minimal normal subgroups of G. As was noted above, $K_1 = \mathrm{C}_G(K_2)$ and $K_2 = \mathrm{C}_G(K_1)$, and they are both regular on Ω. Let $K = K_1$. As discussed in Section 2, $\mathrm{N}_{\mathrm{Sym}(\Omega)}(K)$ on Ω is permutationally isomorphic to $\mathrm{Hol}(K)$ on K via an isomorphism θ such that $(K)\theta = K_R$. Let $\overline{G} = (G)\theta$. Now $K_2 = \mathrm{C}_G(K)$ is mapped by θ to $\mathrm{C}_{\overline{G}}(K_R) = K_L$. Let γ be the involution $x \mapsto x^{-1}$ $(x \in K)$ in $\mathrm{Sym}(K)$. Then a simple calculation shows that γ centralises $\mathrm{Aut}(K)$, and $\lambda_y^\gamma = \rho_y$ for all $y \in K$. Thus, γ interchanges K_L and K_R. Therefore, if $\overline{G}_1$ is the point stabiliser of $\overline{G}$ in its action on K, then $\overline{G}_1 \leq \mathrm{Aut}(K)$ and hence is centralised by γ. So $\overline{G}^\gamma = (K_R\overline{G}_1)^\gamma = K_L\overline{G}_1 = \overline{G}$ and hence $\gamma \in \mathrm{N}_{\mathrm{Sym}(K)}(\overline{G})$. The element $(\gamma)\theta^{-1} \in \mathrm{N}_{\mathrm{Sym}(\Omega)}(G)$ has the required properties. □

By the following lemma, if G is innately transitive with a regular (and nonabelian) plinth K, then the setwise stabiliser in K of the $\mathrm{C}_G(K)$-orbit σ containing α can be identified with $\{\rho_x : \lambda_x \in \mathrm{C}_G(K_R)\}$.

Lemma 5.2. *Let $K = T^k$ where T is a nonabelian simple group and k is a positive integer. Suppose $G \leq \mathrm{Sym}(K)$ is innately transitive with plinth K_R, and let σ be the orbit of 1 under $\mathrm{C}_G(K_R)$. Then $(K_R)_\sigma = \{\rho_x : \lambda_x \in \mathrm{C}_G(K_R)\} \cong \mathrm{C}_G(K_R)$.*

Proof. The $\mathrm{C}_G(K_R)$-orbit of the identity is equal to $\sigma = \{y^{-1} \in K : \lambda_y \in \mathrm{C}_G(K_R)\}$. Thus, $\rho_x \in (K_R)_\sigma$ if and only if, for every $y^{-1} \in K$ with $\lambda_y \in \mathrm{C}_G(K_R)$, we have $(y^{-1})\rho_x = y^{-1}x \in \sigma$, that is, $\lambda_{x^{-1}y} \in \mathrm{C}_G(K_R)$. Therefore $\rho_x \in (K_R)_\sigma$ if and only if $\lambda_x \in \mathrm{C}_G(K_R)$. □

The following proposition gives us necessary and sufficient conditions for an innately transitive group to be quasiprimitive.

Proposition 5.3. *Let G be a finite innately transitive permutation group on a set Ω with plinth K. Then G is quasiprimitive if and only if $\mathrm{C}_G(K)$ is transitive or $\mathrm{C}_G(K) = 1$.*

Proof. First suppose that K is abelian. Then $K \leq \mathrm{C}_G(K)$. But since K is transitive and $\mathrm{C}_G(K)$ is semiregular by Lemma 4.2, then K and $\mathrm{C}_G(K)$ are both regular and hence equal. Recall that each minimal normal subgroup of G distinct from K intersects K trivially and hence is contained in $\mathrm{C}_G(K)$. Thus, K is the unique minimal normal subgroup of G and hence G is quasiprimitive.

Suppose now that K is nonabelian. Since K is a minimal normal subgroup of G, we have $\mathrm{C}_G(K) \cap K = 1$. Thus, if $\mathrm{C}_G(K) = 1$, then K is the unique minimal normal subgroup of G and hence G is quasiprimitive.

Suppose that $\mathrm{C}_G(K)$ is transitive. By Lemma 4.2(1), K is semiregular and hence regular as K is transitive. By Lemma 4.2(2), $\mathrm{C}_{\mathrm{Sym}(\Omega)}(K)$ is semiregular, implying that $\mathrm{C}_G(K)$ is also semiregular and consequently $\mathrm{C}_G(K) = \mathrm{C}_{\mathrm{Sym}(\Omega)}(K)$ and is regular. Let $\alpha \in \Omega$. Since K is regular, there is an isomorphism from $\mathrm{Sym}(\Omega)$ onto $\mathrm{Sym}(K)$ which maps K, $\mathrm{C}_G(K)$, and $\mathrm{N}_{\mathrm{Sym}(\Omega)}(K)$ onto K_R, K_L, and $\mathrm{Hol}(K)$ respectively. Let $\overline{G}$ be the image of G in $\mathrm{Sym}(K)$.

Then $\overline{G}$ contains $K_L \times K_R = K_R \rtimes \mathrm{Inn}(K)$ and hence $\overline{G} = K_R \rtimes G_0$ where $\mathrm{Inn}(K) \leq G_0 \leq \mathrm{Aut}(K) = \mathrm{Aut}(T) \,\mathrm{wr}\, S_k$. Since K_R is a minimal normal subgroup of $\overline{G}$, by Lemma 4.5, G_0 induces a transitive subgroup of S_k, and hence G_0 permutes the simple direct factors of K_R, K_L, and $\mathrm{Inn}(K)$ transitively (and these actions are pairwise permutationally isomorphic). In particular, K_L is a minimal normal subgroup of $\overline{G}$ (by Lemma 4.5), so $\mathrm{C}_G(K)$ is a minimal normal subgroup of G.

If M is any minimal normal subgroup of G distinct from K, then $M \leq \mathrm{C}_G(K)$ and hence $M = \mathrm{C}_G(K)$. Since $\mathrm{C}_G(K)$ and K are transitive on Ω, it follows that G is quasiprimitive. Finally, suppose that $\mathrm{C}_G(K)$ is non-trivial and intransitive. Now $\mathrm{C}_G(K)$ is a normal subgroup of G, and so G is not quasiprimitive. □

In the study of innately transitive groups, there are three important corner stones. They are the "Centraliser Lemma" (4.2), "Scott's Lemma" (4.4), and the following result, which is a generalisation of [8, Theorem 4.2].

Lemma 5.4. *Let G be innately transitive on a set Ω with plinth K, let $\alpha \in \Omega$, and let $\sigma = \alpha^{\mathrm{C}_G(K)}$.*

(1) *For all $u \in K_\sigma$, there exists a unique element $c_u \in \mathrm{C}_G(K)$ such that $c_u\, u \in K_\alpha$.*
(2) *The map $\varphi : u \mapsto c_u$ is an epimorphism from K_σ onto $\mathrm{C}_G(K)$ with kernel K_α. In particular, K_α is a normal subgroup of the setwise stabiliser K_σ and K_σ/K_α is isomorphic to $\mathrm{C}_G(K)$.*

Proof. (1) Let $u \in K_\sigma$. Then $\alpha^u \in \sigma$ and hence there exists $c_u \in \mathrm{C}_G(K)$ such that $\alpha^u = \alpha^{c_u^{-1}}$, that is $\alpha^{uc_u} = \alpha$. Since $\mathrm{C}_G(K)$ is semiregular on Ω by Lemma 4.2, there is a unique such c_u.

(2) Let $\varphi : K_\sigma \to \mathrm{C}_G(K)$ be the mapping $u \mapsto c_u$. Let $k_1, k_2 \in K_\sigma$. So $((k_1)\varphi, k_1)$ and $((k_2)\varphi, k_2)$ are elements of $(\mathrm{C}_G(K) \times K)_\alpha$ and hence $((k_1)\varphi\,(k_2)\varphi, k_1k_2) \in (\mathrm{C}_G(K) \times K)_\alpha$. Since $k_1k_2 \in K_\sigma$, we have also that $((k_1k_2)\varphi, k_1k_2) \in (\mathrm{C}_G(K) \times K)_\alpha$. By the uniqueness proved in the previous paragraph, $(k_1k_2)\varphi = (k_1)\varphi\,(k_2)\varphi$ and so φ is a homomorphism. Let $c \in \mathrm{C}_G(K)$. Since K_σ is transitive on σ, there exists $x \in K_\sigma$ such that $\alpha^x = \alpha^c$. Therefore φ is surjective. By the definition of φ, the kernel of φ is equal to K_α. So K_α is a normal subgroup of K_σ, and by the First Isomorphism Theorem, $\mathrm{C}_G(K) \cong K_\sigma / K_\alpha$. □

The next result shows that the members of a significant family of innately transitive groups are quasiprimitive. Recall from Section 3 that an innately transitive group G with plinth K is of *Diagonal Type* if and only if K is non-abelian, non-simple, a point stabiliser K_α of K is a subdirect subgroup of K, and $\mathrm{C}_G(K) = 1$.

Proposition 5.5. *Let G be a finite innately transitive permutation group on Ω with non-abelian and non-simple plinth K, and let $\alpha \in \Omega$. Then K_α is a subdirect subgroup of K if and only if G is quasiprimitive of Diagonal Type.*

Proof. Let $K = T_1 \times \cdots \times T_k$ where each T_i is a non-abelian simple group and $k \geq 2$. Suppose first that K_α is a subdirect subgroup of K. By Lemma 4.4, $K_\alpha = D_1 \times \cdots \times D_l$ where $l \leq k$ and for all i, we have that D_i is a full diagonal subgroup of a subproduct $K_i = \prod_{j \in I_i} T_j$ and the I_i form a partition of $\{1, \ldots, k\}$. So clearly, $\mathrm{N}_K(K_\alpha) = \prod_{i=1}^{l} \mathrm{N}_{K_i}(D_i)$. By Lemma 4.6, $\mathrm{N}_{K_i}(D_i) = D_i$ for all i and hence $\mathrm{N}_K(K_\alpha) = K_\alpha$. Now Lemma 5.4 implies that $K_\alpha = K_\sigma$ and hence that $\mathrm{C}_G(K) = 1$. Thus G is quasiprimitive by Proposition 5.3. It then follows from Theorem 4.1 and the description of the types in Section 3 that G has Diagonal Type.

Conversely, if G is quasiprimitive of Diagonal Type, then by the description of this type in Section 3, K_α is a subdirect subgroup of K. □

For any group G and subgroup G_0, let $[G : G_0]$ denote the set of right cosets of G_0 in G and define the *right coset action* of G on G_0 by $(G_0x)g = G_0(xg)$ for all $g \in G$ and $G_0x \in [G : G_0]$. Note that this action of G on $[G : G_0]$ is transitive with kernel $\mathrm{Core}_G(G_0)$, and G_0 is the stabiliser of the trivial coset. We now give a definition for notational convenience.

Definition 5.6. Let $\mathcal{G}$ be the set of all triples (G, K, G_0) where G is a finite group, K is a minimal normal subgroup of G, the subgroup G_0 of G is corefree, and $G = KG_0$. Define a relation $\equiv_{\mathcal{G}}$ on $\mathcal{G}$ by $(G, K, G_0) \equiv_{\mathcal{G}} (G', K', G'_0)$ if and only if there is an isomorphism $\phi : G \to G'$ such that $(K)\phi = K'$ and $(G_0)\phi = G'_0$.

Lemma 5.7.
(1) The relation $\equiv_{\mathcal{G}}$ *is an equivalence relation on* $\mathcal{G}$*.*

(2) If $(G,K,G_0) \in \mathcal{G}$*, then* G *is innately transitive on* $[G:G_0]$ *in its right coset action, with plinth* K*. Moreover, for every* $g \in G$*,* $(G,K,G_0) \equiv_{\mathcal{G}} (G,K,G_0^g)$*.*

(3) If G *is innately transitive on a finite set* Ω *with plinth* K*, then for all* $\alpha,\beta \in \Omega$*,* $(G,K,G_\alpha),(G,K,G_\beta) \in \mathcal{G}$ *and* $(G,K,G_\alpha) \equiv_{\mathcal{G}} (G,K,G_\beta)$*. Moreover, if* G *has another transitive minimal normal subgroup* K' *distinct from* K*, then* $(G,K,G_\alpha) \equiv_{\mathcal{G}} (G,K',G_\alpha)$*.*

(4) Two elements (G,K,G_0) *and* (G',K',G_0') *of* $\mathcal{G}$ *are equivalent under* $\equiv_{\mathcal{G}}$ *if and only if the action of* G *on* $[G:G_0]$ *is permutationally isomorphic to the action of* G' *on* $[G':G_0']$*.*

Proof.
(1) Clearly $\equiv_{\mathcal{G}}$ is reflexive and symmetric, so it suffices to show that $\equiv_{\mathcal{G}}$ is transitive. Let $(G,K,G_0),(G',K',G_0'),(G'',K'',G_0'') \in \mathcal{G}$ and suppose that $(G,K,G_0) \equiv_{\mathcal{G}} (G',K',G_0')$ and $(G',K',G_0') \equiv_{\mathcal{G}} (G'',K'',G_0'')$. So there exist isomorphisms $\theta : G \to G'$ and $\theta' : G' \to G''$ such that $(K)\theta = K'$, $(G_0)\theta = G_0'$, $(K')\theta' = K''$, and $(G_0')\theta' = G_0''$. Now $\theta \circ \theta'$ is an isomorphism from G onto G'' and $(K)\theta \circ \theta' = K''$ and $(G_0)\theta \circ \theta' = G_0''$. Therefore $(G,K,G_0) \equiv_{\mathcal{G}} (G'',K'',G_0'')$ and $\equiv_{\mathcal{G}}$ is an equivalence relation on $\mathcal{G}$.

(2) Let $(G,K,G_0) \in \mathcal{G}$. By definition of $\mathcal{G}$, we already have that K is a minimal normal subgroup of G. In the coset action of G on $[G:G_0]$, we have that G_0 is a point stabiliser. The condition $G = KG_0$ implies that K is transitive in this action. Thus G is innately transitive on $[G:G_0]$ with plinth K.

Let $g \in G$ and let $\theta : G \to G$ be the inner automorphism induced by g. Indeed, θ is an isomorphism and $(K)\theta = K$ (as K is normal in G) and $(G_0)\theta = G_0^g$. Therefore $(G,K,G_0) \equiv_{\mathcal{G}} (G,K,G_0^g)$.

(3) Let G be an innately transitive group on a finite set Ω with plinth K. Let $\alpha,\beta \in \Omega$. Then K is a minimal normal subgroup of G and we have $G = KG_\alpha = KG_\beta$, so G_α and G_β are corefree subgroups of G (as G acts faithfully on Ω). Therefore $(G,K,G_\alpha),(G,K,G_\beta) \in \mathcal{G}$. Since G is transitive on Ω, there exists $g \in G$ such that $\alpha^g = \beta$. Let $\theta : G \to G$ be the inner automorphism induced by g. Then as in the previous paragraph $(K)\theta = K$ and $(G_\alpha)\theta = G_\alpha^g = G_\beta$. Therefore $(G,K,G_\alpha) \equiv_{\mathcal{G}} (G,K,G_\beta)$.

Suppose G has a second transitive minimal normal subgroup $K' \neq K$. Then by Lemma 5.1, there exists an involution $g \in \mathrm{N}_{\mathrm{Sym}(\Omega)}(G)$ that interchanges K and K' and centralises G_α for some $\alpha \in \Omega$. Let θ be the automorphism of G induced by conjugating by g. Now g centralises G_α

and hence $(G_\alpha)\theta = G_\alpha$. Since $(K)\theta = K'$, we have $(G, K, G_\alpha) \equiv_{\mathcal{G}} (G, K', G_\alpha)$. By the previous paragraph, it follows that this holds for all points $\alpha \in \Omega$.

(4) Let $(G, K, G_0), (G', K', G'_0) \in \mathcal{G}$. Suppose first that $(G, K, G_0) \equiv_{\mathcal{G}} (G', K', G'_0)$. So there exists an isomorphism $\theta : G \to G'$ such that $(K)\theta = K'$ and $(G_0)\theta = G'_0$. Since G_0 and G'_0 are stabilisers of the actions of G on $[G : G_0]$ and G' on $[G' : G'_0]$ respectively, these actions are then permutationally isomorphic. Conversely, suppose the action of G on $[G : G_0]$ is permutationally isomorphic to the action of G' on $[G' : G'_0]$. So there is an isomorphism $\theta : G \to G'$ such that $(G_0)\theta$ is a point stabiliser of G'. Also $(K)\theta$ is a minimal normal subgroup of G' and $G' = (G)\theta = (KG_0)\theta = (K)\theta(G_0)\theta$ so $(K)\theta$ is transitive. Thus $(G', (K)\theta, (G_0)\theta) \in \mathcal{G}$ and is equivalent to (G, K, G_0) under $\equiv_{\mathcal{G}}$. By part (3) (used twice), we have that $(G', (K)\theta, (G_0)\theta) \equiv_{\mathcal{G}} (G', K', G'_0)$. So since $\equiv_{\mathcal{G}}$ is transitive, we have $(G, K, G_0) \equiv_{\mathcal{G}} (G', (K)\theta, G'_0) \equiv_{\mathcal{G}} (G', K', G'_0)$. □

Now we present another property of innately transitive groups, the importance of which will be made clearer in Section 6. For $L \leq \mathrm{Aut}(K)$, we say that K is *L-simple* if the only L-invariant normal subgroups of K are 1 and K. This definition has been used, for example, in Aschbacher's book [1, p.23].

Proposition 5.8. *Let $(G, K, G_0) \in \mathcal{G}$, let $\Omega = [G : G_0]$, let $\alpha = G_0 \in \Omega$, and let $\sigma := \alpha^{\mathrm{C}_G(K)}$. Let L be the subgroup of $\mathrm{Aut}(K)$ induced by the conjugation action of G_0 on K. Then $L \cong G_0$, K is L-simple, and $L \cap \mathrm{Inn}(K) = \mathrm{Inn}_{K_\sigma}(K)$. In particular, if K is abelian, then K is elementary abelian and L is an irreducible subgroup of $GL(K)$.*

Proof. Let J be a proper L-invariant normal subgroup of K. Since J is L-invariant, J is normalised by $G_\alpha = G_0$. Now $G = KG_\alpha$ (since K is transitive) and hence J is normalised by G. But since K is a minimal normal subgroup of G, and J is a proper subgroup of K normalised by G, we have that J must be trivial. Therefore, there are no nontrivial proper L-invariant normal subgroups of K. Let $\gamma : G \to \mathrm{Aut}(K)$ be the natural map induced by the conjugation action of G on K. Note that $\mathrm{Ker}\,\gamma = \mathrm{C}_G(K)$, $G_\sigma = \mathrm{C}_G(K)G_\alpha$, and $L = (G_\alpha)\gamma = (G_\sigma)\gamma$. Since $G_\sigma \cap K = K_\sigma$, we have $L \cap \mathrm{Inn}(K) = (G_\sigma)\gamma \cap (K)\gamma \supseteq (G_\sigma \cap K)\gamma = (K_\sigma)\gamma = \mathrm{Inn}_{K_\sigma}(K)$. So it suffices to show that $(G_\sigma)\gamma \cap (K)\gamma \subseteq (G_\sigma \cap K)\gamma$. Let $x \in K$ and suppose that $(x)\gamma \in (G_\sigma)\gamma = (G_\alpha)\gamma$. Then there exists $g \in G_\alpha$ such that $(x)\gamma = (g)\gamma$. This implies that $xg^{-1} \in \mathrm{Ker}\,\gamma$ and hence $x \in \mathrm{C}_G(K)G_\alpha = G_\sigma$. Therefore $x \in G_\sigma \cap K$ and $(G_\sigma)\gamma \cap (K)\gamma \subseteq (G_\sigma \cap K)\gamma$.

Now suppose K is abelian. Since K is minimal normal, it is elementary abelian and is thus the additive group of a vector space with automorphism group $GL(K)$. Since K is L-simple, it follows by definition that L is irreducible. □

6. Constructing Innately Transitive Groups

In this section, we give a method for constructing innately transitive groups from certain abstract group theoretic information. It turns out that the subgroup L of $\mathrm{Aut}(K)$ defined in Proposition 5.8 and the epimorphism φ defined in Lemma 5.4 together with the plinth, are the crucial ingredients for understanding an innately transitive group. We call such a triple (K, φ, L) an innate triple and define these triples synthetically below.

Definition 6.1. A triple (K, φ, L) satisfying the three conditions below is called an *innate triple*.

(1) $K \cong T^k$ where T is a simple group (possibly abelian),

(2) φ is an epimorphism with domain a subgroup K_0 of K, with kernel corefree in K, and if K is abelian, then $K_0 = K$,

(3) L is a subgroup of $\mathrm{Aut}(K)$ such that K is L-simple, $\mathrm{Ker}\,\varphi$ is L-invariant, and $L \cap \mathrm{Inn}(K) = \mathrm{Inn}_{K_0}(K)$.

We denote by $\mathcal{D}$ the set of all innate triples.

Remarks 6.2.

(i) We note that if K is elementary abelian, then condition (2) implies that $\mathrm{Ker}\,\varphi = 1$ *and φ has domain K so that* $\mathrm{Im}\,\varphi \cong K$.

(ii) By Proposition 5.8, if $(G, K, G_0) \in \mathcal{G}$, and L is the subgroup of $\mathrm{Aut}(K)$ *induced by G_0 acting by conjugation on K, then L satisfies (3), and the map φ defined in Lemma 5.4 satisfies (2) and hence (K, φ, L) is an innate triple.*

Let $\Delta : \mathcal{G} \to \mathcal{D}$ be the function which maps each (G, K, G_0) to the corresponding (K, φ, L) where φ, L are defined with respect to G_0. We will show that Δ induces an equivalence relation on $\mathcal{D}$.

Lemma 6.3. *Let $(G, K, G_0), (\overline{G}, \overline{K}, \overline{G_0}) \in \mathcal{G}$ and let $(K, \varphi, L) = (G, K, G_0)\Delta$ and $(\overline{K}, \overline{\varphi}, \overline{L}) = (\overline{G}, \overline{K}, \overline{G_0})\Delta$. Then $(G, K, G_0) \equiv_{\mathcal{G}} (\overline{G}, \overline{K}, \overline{G_0})$ if and only if the following hold.*

(1) *There is an isomorphism θ from K onto $\overline{K}$ such that $\overline{L} = \theta^{-1} L \theta$.*

(2) *There is an isomorphism Θ from* $\mathrm{Im}\,\varphi$ *onto* $\mathrm{Im}\,\overline{\varphi}$.

(3) $\theta \circ \overline{\varphi} = \varphi \circ \Theta$ *on the domain* $\mathrm{Dom}\,\varphi$.

Proof. Let $\Omega = [G : G_0]$, $\overline{\Omega} = [\overline{G}, \overline{G_0}]$, $\alpha = G_0 \in \Omega$, and $\overline{\alpha} = \overline{G_0} \in \overline{\Omega}$. Let $\sigma = \alpha^{\mathrm{C}_G(K)}$ and $\overline{\sigma} = \overline{\alpha}^{\mathrm{C}_{\overline{G}}(\overline{K})}$. We will prove the forward direction first. Since $(G, K, G_0) \equiv_{\mathcal{G}} (\overline{G}, \overline{K}, \overline{G_0})$, there is an isomorphism $\Phi : G \to \overline{G}$ such that $(K)\Phi = \overline{K}$ and $(G_0)\Phi = \overline{G_0}$.

Let $\theta = \Phi|_K$, let $\Theta = \Phi|_{\mathrm{C}_G(K)}$, and let $\tau \in L$. Then there exists $g \in G_\sigma$ such that $(x)\tau = g^{-1}xg$ for all $x \in K$. Note that $(g)\theta \in \overline{G}_{\overline{\sigma}}$. Let $y \in \overline{K}$. Then $y = (x)\theta$ for some $x \in K$. So $(y)\theta^{-1}\tau\theta =$

$(x)\tau\theta = (g^{-1}xg)\theta = (g)\theta^{-1}y(g)\theta$ and hence $\theta^{-1}\tau\theta \in \overline{L}$. So $\theta^{-1}L\theta$ is equal to $\overline{L}$ (clearly they have the same size). By Lemma 4.3, we know that Φ maps $\mathrm{C}_G(K)$ onto $\mathrm{C}_{\overline{G}}(\overline{K})$, and therefore the first two conditions above are satisfied.

Now Φ maps $K_\sigma = (G_0\mathrm{C}_G(K)) \cap K$ onto $\overline{K}_{\overline{\sigma}} = (\overline{G_0}\mathrm{C}_{\overline{G}}(\overline{K})) \cap \overline{K}$. Let $(x)\Phi \in \overline{K}_{\overline{\sigma}}$. By the definition of $\overline{\varphi}$, $(x)\Phi \circ \overline{\varphi}$ is the unique element of $\mathrm{C}_{\overline{G}}(\overline{K})$ such that $(x)\Phi\,((x)\Phi \circ \overline{\varphi}) \in \overline{G}_{\overline{\alpha}}$, but this implies that $x[(x)\Phi \circ \overline{\varphi} \circ \Phi^{-1}] \in G_\alpha$ (see Lemma 5.4). Note that $(x)\Phi \in \overline{K}_{\overline{\sigma}}$ implies that $(x)\Phi\circ\overline{\varphi}\circ\Phi^{-1} \in (\mathrm{C}_{\overline{G}}(\overline{K}))\Phi^{-1} = \mathrm{C}_G(K)$, and so by the definition of φ, we have $(x)\Phi\circ\overline{\varphi}\circ\Phi^{-1} = (x)\varphi$. Since this holds for all $x \in K_\sigma$, the third condition is satisfied.

Now we will prove the converse. Assume conditions (1)-(3) hold. Let $x \in K_\alpha$. Then $(x)\varphi = 1$ (by definition of φ) and hence $(x)\theta \circ \overline{\varphi} = (x)\varphi \circ \Theta = 1$ by condition (3). Thus $(x)\theta \in \mathrm{Ker}\,\overline{\varphi} = \overline{K}_{\overline{\alpha}}$, so $(K_\alpha)\theta \leq \overline{K}_{\overline{\alpha}}$. Conversely, if $y \in \overline{K}_\alpha$, then $y = (x)\theta$ for some $x \in K$ since θ is an isomorphism, and $(x)\theta \in \mathrm{Ker}\,\overline{\varphi}$ by the definition of $\overline{\varphi}$. Thus by (3), $(x)\varphi \circ \Theta = (x)\theta \circ \overline{\varphi} = 1$, and since Θ is injective by (2), we have $(x)\varphi = 1$. So $x \in K_\alpha$ and $(K_\alpha)\theta = \overline{K}_\alpha$.

Note that $G = KG_\alpha$ and $\overline{G} = \overline{KG}_{\overline{\alpha}}$, and there are natural isomorphisms $\gamma : G_\alpha \to L$ and $\overline{\gamma} : \overline{G}_{\overline{\alpha}} \to \overline{L}$ defined by $(y)(g)\gamma = g^{-1}yg$ and $(\overline{y})(\overline{g})\overline{\gamma} = \overline{g}^{-1}\overline{yg}$ for all $g \in G_\alpha$, $\overline{g} \in \overline{G}_{\overline{\alpha}}$, $y \in K$, and $\overline{y} \in \overline{K}$. Let $\xi : G \to \overline{G}$ be defined by $(yg)\xi = (y)\theta(\theta^{-1}(g)\gamma\theta)\overline{\gamma}^{-1}$ for all $y \in K$ and $g \in G_\alpha$.

First we will show that ξ is well-defined. Let $y_1, y_2 \in K$, let $g_1, g_2 \in G_\alpha$, and suppose $y_1g_1 = y_2g_2$. So $z = g_1g_2^{-1} = y_1^{-1}y_2 \in K_\alpha$ and thus $((z)\theta)\overline{\gamma} = \iota_{(z)\theta} = \theta^{-1}\iota_z\theta = \theta^{-1}(z)\gamma\theta$. Consequently $((y_1^{-1}y_2)\theta)\overline{\gamma} = ((g_1g_2^{-1})\theta)\overline{\gamma} = \theta^{-1}(g_1g_2^{-1})\gamma\theta$.

Now we use the fact that θ is a homomorphism and that $\overline{\gamma}$ is invertible to obtain the equation

$$((y_1)\theta)^{-1}(y_2)\theta = (\theta^{-1}(g_1g_2^{-1})\gamma\,\theta)\overline{\gamma}^{-1}.$$

Using the fact that γ is a homomorphism and placing $\theta\theta^{-1}$ into the equation, the right hand side becomes

$$\left(\theta^{-1}(g_1)\gamma\,\theta\,\theta^{-1}((g_2)\gamma)^{-1}\theta\right)\overline{\gamma}^{-1}.$$

By (1), $\theta^{-1}((g_i)\gamma)^{\pm 1}\theta \in \theta^{-1}L\theta = \overline{L}$ for each i. So since $\overline{\gamma}^{-1}$ is a homomorphism, the right hand side becomes

$$\left(\theta^{-1}(g_1)\gamma\theta\right)\overline{\gamma}^{-1}\ \left(\theta^{-1}((g_2)\gamma)^{-1}\theta\right)\overline{\gamma}^{-1},$$

and so

$$(y_2)\theta(\theta^{-1}(g_2)\gamma\theta)\overline{\gamma}^{-1} = (y_1)\theta(\theta^{-1}(g_1)\gamma\theta)\overline{\gamma}^{-1}.$$

Therefore ξ is well-defined.

Now we will show that ξ is a bijection. Note by definition of ξ, that $\overline{K} = (K)\theta = (K)\xi$. Let $g \in G_\alpha$. Then $(g)\xi = (\theta^{-1}(g)\gamma\theta)\overline{\gamma}^{-1} \in \overline{G}_{\overline{\alpha}}$ since $\overline{L} = \theta^{-1}L\theta$. So $(G_\alpha)\xi \subseteq \overline{G}_{\overline{\alpha}}$. But for all $g \in G_\alpha$, $(g)\xi = (\theta^{-1}(g)\gamma\theta)\overline{\gamma}^{-1} \in \overline{G}_{\overline{\alpha}}$. Now for all $\overline{g} \in \overline{G}_{\overline{\alpha}}$, $(\theta(\overline{g})\overline{\gamma}^{-1}\theta^{-1})\gamma)\xi = \overline{g}$ and hence ξ maps G_α onto

$\overline{G}_{\overline{\alpha}}$. So $(G)\xi$ contains $\langle (K)\xi, (G_\alpha)\xi\rangle = \langle \overline{K}, \overline{G}_{\overline{\alpha}}\rangle = \overline{G}$ and hence ξ is surjective. Now by the above calculations, $|\overline{K}| = |K|$, $|\overline{G}_{\overline{\alpha}}| = |G_\alpha|$, and $|\overline{K}_{\overline{\alpha}}| = |K_\alpha|$. So $|(G)\xi| = |\overline{G}| = \frac{|\overline{K}||\overline{G}_{\overline{\alpha}}|}{|\overline{K}_{\overline{\alpha}}|} = \frac{|K||G_\alpha|}{|K_\alpha|} = |G|$. Therefore ξ is injective and hence bijective. A simple (but tedious) calculation shows that ξ is a homomorphism from G onto $\overline{G}$, and hence ξ is an isomorphism.

Since ξ maps G_α onto $\overline{G}_{\overline{\alpha}}$, $(G, K, G_0) \equiv_{\mathcal{G}} (\overline{G}, \overline{K}, \overline{G_0})$. □

By the lemma above, we get an induced relation on the innate triples which we formalise below.

Definition 6.4. Let $\equiv_{\mathcal{D}}$ be the relation on $\mathcal{D}$ defined by $(K, \varphi, L) \equiv_{\mathcal{D}} (\overline{K}, \overline{\varphi}, \overline{L})$ if and only if

(1) there is an isomorphism θ from K onto $\overline{K}$ such that $\overline{L} = \theta^{-1}L\theta$,

(2) there is an isomorphism Θ from $\operatorname{Im}\varphi$ onto $\operatorname{Im}\overline{\varphi}$, and

(3) $\theta \circ \overline{\varphi} = \varphi \circ \Theta$ on the domain $\operatorname{Dom}\varphi$.

Lemma 6.5. *The relation $\equiv_{\mathcal{D}}$ is an equivalence relation on $\mathcal{D}$.*

Proof. This follows immediately from Lemma 5.7(1) and Lemma 6.3. □

We now give a construction of an innately transitive group, given an innate triple.

Construction 6.6. *Let (K, φ, L) be an innate triple and let $K_0 = \operatorname{Dom}\varphi$ and $H = \operatorname{Graph}(\varphi) = \{(u)\varphi u : u \in K_0\} \leq \operatorname{Im}\varphi \times K$. Then let $X := (\operatorname{Im}\varphi \times K) \rtimes L$ where L acts on $\operatorname{Im}\varphi \times K$ by $((u)\varphi y)^\tau = (u^\tau)\varphi y^\tau$ for all $u \in K_0$, $y \in K$, and $\tau \in L$. Let X act by right coset multiplication on $\Omega := [X : HL]$.*

Note that the action of L on $\operatorname{Im}\varphi$ is well-defined since $\operatorname{Ker}\varphi$ is L-invariant by property (3) of Definition 6.1.

Proposition 6.7. *The kernel of the action of X on Ω (as described above) is $Z := \{(x)\varphi\, x\, \iota_{x^{-1}} : x \in K_0\}$, and X/Z is innately transitive and faithful on Ω (in its induced action) with plinth $KZ/Z \cong K$. Moreover Z is centralised by $\operatorname{Im}\varphi \times K$, and $\mathrm{C}_{X/Z}(KZ/Z) = \operatorname{Im}\varphi Z/Z \cong \operatorname{Im}\varphi$.*

Proof. By Definition 6.1(3), K is L-simple and hence K is a minimal normal subgroup of X. The kernel of the action of X on Ω is $\operatorname{Core}_X(HL)$, a normal subgroup of X. Thus $K \cap \operatorname{Core}_X(HL)$ is also normal in X, and $K \cap \operatorname{Core}_X(HL) \leq K \cap (HL) = K \cap H = \operatorname{Ker}\varphi$ which is a proper subgroup of K by Definition 6.1(2). Therefore $K \cap \operatorname{Core}_X(HL) = 1$. It is also true that L intersects $\operatorname{Core}_X(HL)$ trivially, as the following shows. Suppose $\tau \in L \cap \operatorname{Core}_X(HL)$ and $y \in K$. Then $y^{-1}y^{\tau^{-1}}\tau = y^{-1}\tau y \in \operatorname{Core}_X(HL)$ and hence $y^{-1}y^{\tau^{-1}} \in K \cap \operatorname{Core}_X(HL) = 1$. Therefore $y^{\tau^{-1}} = y$ for all $y \in K$ and so $\tau = \mathrm{id}$. Thus $L \cap \operatorname{Core}_X(HL) = 1$.

Let $u, x \in K_0$, let $y \in K$, and let $\tau \in L$. By Definition 6.1(3), K_0 is L-invariant and hence $x^\tau \in K_0$. Thus $((x)\varphi x \iota_{x^{-1}})^\tau = (x^\tau)\varphi x^\tau \iota_{(x^\tau)^{-1}} \in Z$, and hence Z is L-invariant. Since

$((x)\varphi x\iota_{x^{-1}})^y = (x)\varphi y^{-1}x\iota_{x^{-1}}y = (x)\varphi y^{-1}xy^x\iota_{x^{-1}} = (x)\varphi x\iota_{x^{-1}}$, Z is centralised by K. Also since $(u^{-1})\varphi\,(x)\varphi x\iota_{x^{-1}}(u)\varphi = (u^{-1}x)\varphi x(u^x)\varphi\iota_{x^{-1}} = (u^{-1}xu^x)\varphi x\iota_{x^{-1}} = (x)\varphi x\iota_{x^{-1}}$, it follows that Z is centralised by $\operatorname{Im}\varphi$. Thus Z is normal in X and centralised by $\operatorname{Im}\varphi \times K$. Since $Z \leq HL$, it follows that $Z \leq \operatorname{Core}_X(HL)$.

Now let $(u)\varphi y\tau \in \operatorname{Core}_X(HL)$ where $u \in K_0$, $y \in K$, and $\tau \in L$. Then $(u)\varphi y\tau \in HL$ and hence $(u)\varphi y \in H$, whence $y \in K_0$ and $(u)\varphi = (y)\varphi$. Also, $(y)\varphi y\iota_{y^{-1}} \in Z \leq \operatorname{Core}_X(HL)$, so $((y)\varphi y\iota_{y^{-1}})^{-1}(y)\varphi y\tau = \iota_y\tau \in L \cap \operatorname{Core}_X(HL) = 1$. So $\tau = \iota_{y^{-1}}$ and $(u)\varphi y\tau = (y)\varphi y\iota_{y^{-1}} \in Z$. Thus $\operatorname{Core}_X(HL) = Z$.

Therefore X/Z acts faithfully on Ω, and since $X = K(HL)$, the normal subgroup KZ/Z is transitive. Moreover, since K is a minimal normal subgroup of X, it follows that $KZ/Z \cong K$ is a minimal normal subgroup of X/Z, so X/Z is innately transitive with plinth KZ/Z. By Lemma 4.3, $\mathrm{C}_{X/Z}(KZ/Z) = \mathrm{C}_X(K)/Z$, so we need to determine $\mathrm{C}_X(K)$. Since $\operatorname{Im}\varphi$ centralises K, we have $\mathrm{C}_X(K) = (\operatorname{Im}\varphi)\mathrm{C}_{KL}(K)$. Let $y\tau \in \mathrm{C}_{KL}(K)$ where $y \in K$ and $\tau \in L$. Then for all $v \in K$, $(v)\tau = yvy^{-1} = (v)\iota_{y^{-1}}$, that is, $\tau = \iota_{y^{-1}} \in L \cap \operatorname{Inn}(K)$, which by Definition 6.1(3), is equal to $\operatorname{Inn}_{K_0}(K)$. Note that $y \in K_0$ in both cases where K is abelian or not, and hence $y\tau = (y^{-1})\varphi((y)\varphi y\iota_{y^{-1}}) \in (\operatorname{Im}\varphi)Z$. Therefore $\mathrm{C}_X(K) \leq (\operatorname{Im}\varphi)Z$, and we have already seen that $\operatorname{Im}\varphi$ and Z centralise K, so $\mathrm{C}_X(K) = (\operatorname{Im}\varphi)Z$ as required. □

7. Characterising Innately Transitive Groups

Let $\Gamma : \mathcal{D} \to \mathcal{G}$ be the function which maps each (K, φ, L) to the constructed element $(X/Z, KZ/Z, [X : HL])$ arising from Construction 6.6. We prove that Γ and the map $\Delta : \mathcal{G} \to \mathcal{D}$ defined before Lemma 6.3 preserve equivalence (see Theorems 7.2 and 7.3) and induce a bijection between equivalence classes on $\mathcal{D}$ and $\mathcal{G}$ (Proposition 7.4). This enables us to prove Theorem 1.1.

Lemma 7.1. *Let (K, φ, L) and $(\overline{K}, \overline{\varphi}, \overline{L})$ be innate triples. If $(K, \varphi, L) \equiv_{\mathcal{D}} (\overline{K}, \overline{\varphi}, \overline{L})$, then $(K, \varphi, L)\Gamma \equiv_{\mathcal{G}} (\overline{K}, \overline{\varphi}, \overline{L})\Gamma$.*

Proof. Suppose $(K, \varphi, L) \equiv_{\mathcal{D}} (\overline{K}, \overline{\varphi}, \overline{L})$ for two elements (K, φ, L) and $(\overline{K}, \overline{\varphi}, \overline{L})$ of $\mathcal{D}$. Let X, Z and H be the groups arising from (K, φ, L) in Construction 6.6 and similarly let $\overline{X}$, $\overline{Z}$, and $\overline{H}$ be the groups arising from $(\overline{K}, \overline{\varphi}, \overline{L})$. First we will show that there is an isomorphism from X/Z onto $\overline{X}/\overline{Z}$. We know by Definition 6.4 that there are isomorphisms $\theta : K \to \overline{K}$ and $\Theta : \operatorname{Im}\varphi \to \operatorname{Im}\overline{\varphi}$. So $\overline{X} = ((\operatorname{Im}\varphi)\Theta \times (K)\theta) \rtimes \theta^{-1}L\theta$. Let $\Phi : X \to \overline{X}$ be defined by $(cy\tau)\Phi = (c)\Theta(y)\theta(\theta^{-1}\tau\theta)$ for all $c \in \operatorname{Im}\varphi$, $y \in K$, and $\tau \in L$. It is clear that Φ is a well-defined bijection from X onto $\overline{X}$. We will show that Φ is a homomorphism. Let $c_1, c_2 \in \operatorname{Im}\varphi$, $y_1, y_2 \in K$, and $\tau_1, \tau_2 \in L$. Note that $(c_2^{\tau_1^{-1}})\Theta(y_2^{\tau_1^{-1}})\theta(\theta^{-1}\tau_1\theta) = (c_2^{\tau_1^{-1}})\Theta(\theta^{-1}\tau_1\theta)((y_2^{\tau_1^{-1}})^{\tau_1})\theta = (\theta^{-1}\tau_1\theta)(c_2)\Theta(y_2)\theta$ using the properties of Definition 6.4. Thus

$$\begin{aligned}(c_1y_1\tau_1c_2y_2\tau_2)\Phi &= (c_1y_1\tau_1c_2\tau_1^{-1}\tau_1y_2\tau_1^{-1}\tau_1\tau_2)\Phi = (c_1c_2^{\tau_1^{-1}}y_1y_2^{\tau_1^{-1}}(\tau_1\tau_2))\Phi \\ &= (c_1c_2^{\tau_1^{-1}})\Theta\,(y_1y_2^{\tau_1^{-1}})\theta\,(\theta^{-1}\tau_1\tau_2\theta) \\ &= (c_1)\Theta\,(y_1)\theta\,(c_2^{\tau_1^{-1}})\Theta\,(y_2^{\tau_1^{-1}})\theta\,(\theta^{-1}\tau_1\theta)(\theta^{-1}\tau_2\theta) \\ &= (c_1)\Theta\,(y_1)\theta\,(\theta^{-1}\tau_1\theta)(c_2)\Theta\,(y_2)\theta\,(\theta^{-1}\tau_2\theta) \\ &= (c_1y_1\tau_1)\Phi\,(c_2y_2\tau_2)\Phi.\end{aligned}$$

Therefore Φ is an isomorphism from X onto $\overline{X}$. Now we will show that Φ maps Z onto $\overline{Z}$. Let $(u)\varphi u\iota_{u^{-1}} \in Z$. Then $((u)\varphi u\iota_{u^{-1}})\Phi = ((u)\varphi)\Theta\,(u)\theta\,(\theta^{-1}\iota_{u^{-1}}\theta) = ((u)\theta)\overline{\varphi}\,(u)\theta\,\iota_{((u)\theta)^{-1}}$. Therefore Φ maps Z onto $\overline{Z}$. So there is an induced isomorphism Φ' from X/Z onto $\overline{X}/\overline{Z}$ given by $Zx \mapsto \overline{Z}(x)\Phi$ for all $x \in X$.

Now the stabiliser of the trivial coset HL in the action of X/Z is simply HL/Z. Moreover $(HL/Z)\Phi' = (HL)\Phi/\overline{Z} = \overline{HL}/\overline{Z}$ which is the stabiliser of the trivial coset in the action of $\overline{X}/\overline{Z}$. Finally $(KZ/Z)\Phi' = (KZ)\Phi/\overline{Z} = \overline{KZ}/\overline{Z}$, and hence $(K, \varphi, L)\Gamma \equiv_{\mathcal{G}} (\overline{K}, \overline{\varphi}, \overline{L})\Gamma$. □

By the comments preceding Lemma 6.3, every finite innately transitive permutation group gives rise to an innate triple under the map Δ defined just before Lemma 6.3. Furthermore, by Proposition 6.7, this innate triple gives rise to a finite innately transitive permutation group under the map Γ defined above. The next theorem proves that this group is permutationally isomorphic to the one we started with.

Theorem 7.2. *Let* $(G, K, G_0) \in \mathcal{G}$. *Then* $(G, K, G_0)\Delta \circ \Gamma \equiv_{\mathcal{G}} (G, K, G_0)$.

Proof. By Definition 5.6, G is innately transitive on $\Omega := [G : G_0]$ acting by right multiplication. Let $\alpha = G_0 \in \Omega$ and let $\gamma : G \to \mathrm{Aut}(K)$ be the natural map induced by G acting by conjugation on K, let φ be the epimorphism arising from (G, K, G_0) (as defined in Lemma 5.4), and let $L = (G_0)\gamma$ so $(K, \varphi, L) = (G, K, G_0)\Delta$. Let X be the group constructed in Construction 6.6 from (K, φ, L). Let $\theta : X \to G$ be defined by $((u)\varphi\, y(g)\gamma)\theta = (u)\varphi yg$ where $u \in \mathrm{Dom}\,\varphi$, $y \in K$, and $g \in G_0$. It is clear that θ is onto. Let $(u_1)\varphi\, y_1\,(g_1)\gamma, (u_2)\varphi\, y_2\,(g_2)\gamma \in X$, where $u_1, u_2 \in K_0$, $y_1, y_2 \in K$, and $g_1, g_2 \in G_0$. Then,

$$\begin{aligned}(u_1)\varphi\, y_1\,(g_1)\gamma\,(u_2)\varphi\, y_2(g_2)\gamma &= \left(u_1u_2^{(g_1^{-1})\gamma}\right)\varphi\, y_1y_2^{(g_1^{-1})\gamma}\,(g_1g_2)\gamma \\ &= (u_1g_1u_2g_1^{-1})\varphi\, y_1g_1y_2g_1^{-1}\,(g_1g_2)\gamma \\ &= (u_1)\varphi(g_1u_2g_1^{-1})\varphi\, y_1g_1y_2g_1^{-1}\,(g_1g_2)\gamma.\end{aligned}$$

So,

$$\begin{aligned}[(u_1)\varphi y_1(g_1)\gamma\,(u_2)\varphi y_2(g_2)\gamma]\,\theta &= (u_1)\varphi(g_1u_2g_1^{-1})\varphi\, y_1g_1y_2g_2\\ &= (u_1)\varphi\, y_1g_1(g_1^{-1}(g_1u_2g_1^{-1})\varphi g_1)\, y_2g_2\\ &= (u_1)\varphi\, y_1g_1\,(u_2)\varphi\, y_2g_2\\ &= ((u_1)\varphi\, y_1\,(g_1)\gamma)\,\theta\,((u_2)\varphi\, y_2\,(g_2)\gamma)\,\theta.\end{aligned}$$

Therefore, θ is an epimorphism. Let $H = \mathrm{Graph}(\varphi) = \{(x)\varphi\, x : x \in \mathrm{Dom}\,\varphi\}$. Now $(u)\varphi y(g)\gamma \in \mathrm{Ker}\,\theta$ if and only if $(u)\varphi y = g^{-1}$. But $g^{-1} \in G_0$, $(u)\varphi y \in \mathrm{Im}\,(\varphi) \times K$ and $(\mathrm{Im}\,(\varphi) \times K) \cap G_0 = H$ (by definition of φ), so $\mathrm{Ker}\,\theta = \{h(h^{-1})\gamma : h \in H\} = Z$. So there is an induced isomorphism $\Phi : X/Z \to G$ given by $(Zx)\Phi = (x)\theta$ for all $x \in X$. Now HL/Z is a point stabiliser of the action of X/Z on $[X : HL]$ and $(HL/Z)\Phi = (HL)\theta = HG_\alpha = G_0$, by definition of θ. Also $(KZ/Z)\Phi = K$ and so $(G, K, [G : G_0]) \equiv_{\mathcal{G}} (X/Z, KZ/Z, HL/Z)$. □

The next theorem is an analogous result for $\Gamma \circ \Delta$.

Theorem 7.3. *Let $(K, \varphi, L) \in \mathcal{D}$. Then $(K, \varphi, L)\Gamma \circ \Delta \equiv_{\mathcal{D}} (K, \varphi, L)$.*

Proof. Let (K, φ, L) be an innate triple and let $(X/Z, KZ/Z, HL/Z) = (K, \varphi, L)\Gamma$ acting on $\Omega = [X/Z : HL/Z]$ with X, H, Z as in Construction 6.6. Let $(\overline{K}, \overline{\varphi}, \overline{L}) = (K, \varphi, L)\Gamma\circ\Delta$. So $\overline{K} = KZ/Z$ and $\overline{L} = LZ/Z$. Let $\theta : K \to \overline{K}$ be the isomorphism arising from the Second Isomorphism Theorem, that is, the map $y \mapsto Zy$. Let $\tau \in L$. We will show that $\theta^{-1}\tau\theta \in \overline{L}$, thus satisfying the first condition of Definition 6.4. For all $y \in K$, we have $(Zy)\theta^{-1}\tau\theta = Z(y^\tau) = Z(\tau^{-1}y\tau) = (Z\tau)^{-1}(Zy)(Z\tau)$. Since $\tau \in HL$, we have $\theta^{-1}\tau\theta \in \overline{L}$.

Note that $\mathrm{Dom}\,\overline{\varphi} = (\mathrm{Dom}\,\varphi)Z/Z$ and by Proposition 6.7, $\mathrm{Im}\,\overline{\varphi} = \mathrm{C}_{X/Z}(KZ/Z) = (\mathrm{Im}\,\varphi)Z/Z$. Let $\Theta : \mathrm{Im}\,\varphi \to \mathrm{Im}\,\overline{\varphi}$ be the isomorphism $(u)\varphi \mapsto Z(u)\varphi$. This is the isomorphism required in (2) of Definition 6.4. For all $x \in \mathrm{Dom}\,\varphi$, $(Zx)\overline{\varphi}Zx = Z((x)\varphi x) \in ZH$. By the definition of Θ, $(x)(\varphi \circ \Theta) = Z((x)\varphi)$, and by the definition of θ and $\overline{\varphi}$, $(x)(\theta \circ \overline{\varphi}) = (Zx)\overline{\varphi} = Z((x)\varphi)$. Thus $\varphi\circ\Theta = \theta\circ\overline{\varphi}$ on $\mathrm{Dom}\,\varphi$, and so condition (3) of Definition 6.4 holds. So $(K, \varphi, L) \equiv_{\mathcal{D}} (\overline{K}, \overline{\varphi}, \overline{L})$. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let G be an innately transitive group on a set Ω with plinth K, and let $\alpha \in \Omega$. Then $(G, K, G_0) \in \mathcal{G}$ where $G_0 = G_\alpha$. By Theorem 7.2, $(G, K, G_0) \equiv_{\mathcal{G}} (G, K, G_0)\Delta \circ \Gamma$. So G on Ω is permutationally isomorphic to a permutation group constructed by Construction 6.6. By Proposition 6.7, every permutation group constructed by Construction 6.6 is innately transitive. □

The next result is a culmination of all the results we have derived so far in this section. Basically, we can transfer our study of innately transitive permutation groups to the study of innate triples.

Proposition 7.4. *The maps* Δ *and* Γ *induce mutually inverse bijections between* $\mathcal{G}/\equiv_{\mathcal{G}}$ *and* $\mathcal{D}/\equiv_{\mathcal{D}}$.

Proof. Let $f : \mathcal{D}/\equiv_{\mathcal{D}} \to \mathcal{G}/\equiv_{\mathcal{G}}$ be the map defined by $([(K,\varphi,L)]_{\mathcal{D}})f = [(K,\varphi,L)\Gamma]_{\mathcal{G}}$ for all equivalence classes $[(K,\varphi,L)]_{\mathcal{D}} \in \mathcal{D}/\equiv_{\mathcal{D}}$. By Lemma 7.1, f is well-defined. By Theorem 7.2, it follows that f is surjective. Suppose $([(K,\varphi,L)]_{\mathcal{D}})f = ([(\overline{K},\overline{\varphi},\overline{L})]_{\mathcal{D}})f$ for some $(K,\varphi,L),(\overline{K},\overline{\varphi},\overline{L}) \in \mathcal{D}$. Thus $(K,\varphi,L)\Gamma \equiv_{\mathcal{G}} (\overline{K},\overline{\varphi},\overline{L})\Gamma$, and so by Proposition 6.3 and Definition 6.4, $(K,\varphi,L)\Gamma\circ\Delta \equiv_{\mathcal{D}} (\overline{K},\overline{\varphi},\overline{L})$. Finally by Theorem 7.3, we have $(K,\varphi,L) \equiv_{\mathcal{D}} (\overline{K},\overline{\varphi},\overline{L})$. Hence f is injective. □

The innately transitive types given in Section 3 can be characterised by structural properties of permutation groups such as whether the plinth is abelian, regular, simple, or has trivial centraliser. So in characterising finite innately transitive permutation groups as innate triples, it is useful to translate the properties used to separate the types into properties of innate triples.

Lemma 7.5. *Let* (K,φ,L) *be an innate triple and let* $(G,\overline{K},G_0) = (K,\varphi,L)\Gamma$ *and* $\Omega = [G : G_0]$. *Then*

(1) $\overline{K}$ *is regular on* Ω *if and only if* $\operatorname{Ker}\varphi = 1$,
(2) $\mathrm{C}_G(\overline{K}) = 1$ *if and only if* $\operatorname{Im}\varphi = 1$, *and*
(3) $\mathrm{C}_G(\overline{K})$ *is transitive on* Ω *if and only if* $\operatorname{Dom}\varphi = K$ *and* $\operatorname{Ker}\varphi = 1$.

Proof. First recall that $G = X/Z$, $\overline{K} = KZ/Z$, $\Omega = [X : HL]$, and $G_0 = HL/Z$ with X, Z, H as given in Construction 6.6. We know already from Proposition 6.7 that $\mathrm{C}_G(\overline{K}) = \operatorname{Im}\varphi Z/Z \cong \operatorname{Im}\varphi$ and it is simple to calculate that the stabiliser in $\overline{K}$ of the point $\alpha = HL$ is equal to $\operatorname{Ker}\varphi Z/Z \cong \operatorname{Ker}\varphi$. So (1) and (2) follow. For (3), observe that since $\mathrm{C}_G(\overline{K})$ is semiregular, it is transitive if and only if $\overline{K}$ is regular (see Lemma 4.2), that is $|\mathrm{C}_G(\overline{K})| = |K|$. However by Lemma 5.4, $|\mathrm{C}_G(\overline{K})| = |\operatorname{Dom}\varphi|/|\operatorname{Ker}\varphi|$ and hence $|\mathrm{C}_G(\overline{K})| = |K|$ if and only if $\operatorname{Dom}\varphi = K$ and $\operatorname{Ker}\varphi = 1$. □

Recall that K is isomorphic to KZ/Z and so properties of the plinth such as being abelian or simple are also known from the innate triple. Define an innate triple (K,φ,L) to be *quasiprimitive* if $(K,\varphi,L)\Gamma$ is quasiprimitive. We have the following corollary to Lemma 7.5.

Corollary 7.6. *An innate triple* (K,φ,L) *is quasiprimitive if and only if either* $\operatorname{Dom}\varphi = \operatorname{Ker}\varphi$ *or* $\operatorname{Dom}\varphi = K$ *and* $\operatorname{Ker}\varphi = 1$.

Proof. Follows from Lemma 7.5 and Lemma 5.3. □

Note that we could also give an analogue of primitivity for innate triples. Recall that a transitive permutation group G is primitive, on a set of more than one element, if every point stabiliser of G is a maximal subgroup of G. Let (K, φ, L) be an innate triple. By Construction 6.6, we see that $(K, \varphi, L)\Gamma$ is primitive if and only if $\mathrm{Graph}(\varphi) \rtimes L$ is a maximal subgroup of $(\mathrm{Im}\,\varphi \times K) \rtimes L$.

8. Framework for a Proof of Theorem 1.2

In this section, we prove first that every permutation group G with a normal subgroup K satisfying one of the types described in Section 3 is innately transitive with plinth K (see Lemma 8.1). We then give a framework for completing the proof of Theorem 1.2 namely we outline our program to show that each innately transitive group satisfies the conditions of one of these types.

Lemma 8.1. *Each group G with normal subgroup K of one of the types described in Section 3 is innately transitive with plinth K.*

Proof. Let G be a group belonging to one of the types listed in Section 3, and let K be the corresponding normal subgroup of G. Clearly in the cases *Abelian Plinth*, *Simple Plinth*, and *Regular Plinth*, K is a transitive minimal normal subgroup of G. Suppose G is of *Diagonal Type*. Then K is a minimal normal subgroup of G since G acts transitively on the simple direct factors of K (see Lemma 4.5). Now for some proper divisor l of k, $T^{k/l}$ acts transitively on a set Δ and $\Omega = \Delta^l$. So by definition of product action, K acts transitively on Ω.

All that remains is to show that if G satisfies the conditions of *Product Type*, then G is innately transitive. Now since G_α projects onto a transitive subgroup of S_k, it follows that G is transitive on the simple direct factors of K. So by Lemma 4.5, K is a minimal normal subgroup of G. We want to show now that K is transitive on Ω, that is, $|K : K_\alpha| = |\Omega|$. First note that $|K_\psi : K_\alpha|$ is equal to the size of a cell of Ψ, which is in turn equal to $|\Omega|/|\Psi|$. So $|K : K_\alpha| = |K : K_\psi||K_\psi : K_\alpha| = |\Psi|(|\Omega|/|\Psi|) = |\Omega|$. Therefore, K is transitive on Ω and so G is innately transitive. □

From now on, assume G is innately transitive on a set Ω with plinth K. If K is abelian, then by Proposition 5.3, G is quasiprimitive and by Theorem 4.1, G is of *Abelian Plinth Type* as described in Section 3. So assume K is non-abelian. Recall from Proposition 5.3, that if G is not quasiprimitive, then $\mathrm{C}_G(K)$ is nontrivial and intransitive. Suppose next that K is a simple group. If G is quasiprimitive, then by Theorem 4.1, G is of *Holomorph of a Simple Group Type* or *Almost Simple Type* as described in Section 3. On the other hand, if G is not quasiprimitive, then G satisfies all the conditions of *Almost Simple Quotient* type given in Section 3. We can also characterise *Simple Plinth* type by observing the following simplification of the conditions of Definition 6.4 for (K, φ, L) to be an innate triple when K is a nonabelian simple group.

Lemma 8.2. *Let K be a finite non-abelian simple group, let φ be an epimorphism with domain $K_0 \leq K$, and let $L \leq \mathrm{Aut}(K)$. Then (K, φ, L) is an innate triple if and only if* $\mathrm{Ker}\,\varphi \neq K$ *and* $L \cap \mathrm{Inn}(K) = \mathrm{Inn}_{K_0}(K)$.

Proof. The second condition of Definition 6.1 is that φ is an epimorphism with domain a subgroup K_0 of K such that $\mathrm{Ker}\,\varphi$ is corefree in K. Since the core of $\mathrm{Ker}\,\varphi$ is a normal subgroup of K, and K is simple, $\mathrm{Ker}\,\varphi$ is corefree if and only if $\mathrm{Core}_K(\mathrm{Ker}\,\varphi) \neq K$. Finally $\mathrm{Core}_K(\mathrm{Ker}\,\varphi) \neq K$ is equivalent to $\mathrm{Ker}\,\varphi \neq K$.

The third condition of Definition 6.1 is that K is L-simple and $L \cap \mathrm{Inn}(K) = \mathrm{Inn}_{K_0}(K)$. Since K is simple, clearly K has no proper nontrivial L-invariant normal subgroups, and K is automatically L-simple. □

So now let us assume that K is not simple. Let $K = T^k$ for some non-abelian simple group T and integer $k > 1$. Since $G = KG_\alpha$ and K is a minimal normal subgroup of G, then by Lemma 4.5, G_α is transitive on the simple direct factors of K and G_α normalises K_α. For $i \in \{1, \ldots, k\}$, let $\pi_i : K \to T_i$ be the natural projection map of K onto the ith simple direct factor T_i of K. Then G_α permutes the $(K_\alpha)\pi_i$ transitively and thus they are pairwise isomorphic.

By Proposition 5.5, if K_α is a subdirect subgroup of K then G is quasiprimitive, and by Theorem 4.1, G and K satisfy the conditions of the *Diagonal Type* as described in Section 3. Thus we may assume that K_α is not a subdirect subgroup of K. We have two cases; $K_\alpha = 1$ (i.e. K is regular) or, $K_\alpha \neq 1$ and $(K_\alpha)\pi_i$ is a proper subgroup of T_i for all i. In the next section we examine these cases in more detail and in particular, we prove that if $K_\alpha \neq 1$ then (G, K, Ω) satisfy the conditions of the Product Type in Section 3. In Section 10, we analyse the sub-case where K_α is trivial and prove that G satisfies the conditions of Twisted Wreath type, Product Quotient type, Diagonal Quotient type, or Holomorph of a Compound Group type. This will then complete the proof of Theorem 1.2.

9. Innately Transitive Groups Preserving Product Decompositions

In this section, we characterise innately transitive groups (G, K, G_0) where K is nonabelian, nonsimple, and a point stabiliser K_α is not a subdirect subgroup of K (including the case when $K_\alpha = 1$). Let $\mathcal{G}_{prod}$ be the set of all such (G, K, G_0). Note that $\mathcal{G}_{prod}$ is closed under the equivalence relation $\equiv_{\mathcal{G}}$. We shall, in particular, complete the proof of Theorem 1.2 in the case where $K_\alpha \neq 1$ (see Lemma 9.4). In addition, we prove Theorem 1.3 (after Corollary 9.9).

We construct, from a given element of $\mathcal{G}_{prod}$ a new innately transitive group that preserves a product decomposition of the underlying set. We use the notation introduced at the end of Section

8. Since the $(K_\alpha)\pi_i$ $(1 \le i \le k)$ are permuted transitively by conjugation by G_α, we may identify each T_i with T in such a way that $(K_\alpha)\pi_i = U$ is independent of i.

Construction 9.1. *Let $(G, K, G_0) \in \mathcal{G}_{prod}$ where $K = T^k$ for some nonabelian simple group T and $k > 1$, let $\Omega = [G : G_0]$, $\alpha \in \Omega$, and let $U = (K_\alpha)\pi_i$ for all $i \in \{1, \dots, k\}$. Let $\Psi := [K : U^k]$ and let K^Ψ be the permutation group of Ψ induced by K by right multiplication. Let $Y = U^k G_\alpha$ and let G^Ψ be the permutation group of Ψ induced by the action given in Lemma 4.8 (noting that $K \cap G_0 = U^k$). The output of this construction is the triple (G^Ψ, K^Ψ, Ψ).*

Proposition 9.2. *Let $(G, K, G_0) \in \mathcal{G}_{prod}$, where $K = T^k$ for some nonabelian simple group T and $k > 1$, and let $(G^\Psi, K^\Psi, G_0^\Psi)$ be the triple constructed from (G, K, G_0) via Construction 9.1. Then*

(1) $(G^\Psi, K^\Psi, G_0^\Psi) \in \mathcal{G}_{prod}$,

(2) $\mathrm{C}_{G^\Psi}(K^\Psi) = \mathrm{C}_G(K)^\Psi$,

(3) *Ψ can be identified with a cartesian product of k copies of a set Ψ_0 and the component V of G^Ψ in $\mathrm{Sym}(\Psi_0)$ is innately transitive of Simple Plinth type with plinth T. Moreover G^Ψ is conjugate in $\mathrm{Sym}(\Psi)$ to a subgroup of $V \operatorname{wr} S_k \le \mathrm{Sym}(\Psi_0) \operatorname{wr} S_k$ and G projects onto a transitive subgroup of S_k.*

Proof.
(1) Let $\psi = U^k \in \Psi$ so that $G_\psi = Y$. Since Y contains G_α, it follows that Ψ may be identified with a G-invariant partition of Ω with cells of size $|Y : G_\alpha| = |U^k : K_\alpha|$. Since $G = G_\alpha K = YK$, it follows that K^Ψ is a transitive minimal normal subgroup of G^Ψ, so G^Ψ is innately transitive with plinth K^Ψ. Thus $(G^\Psi, K^\Psi, G_0^\Psi) \in \mathcal{G}$. Now K^Ψ is isomorphic to K (since K acts faithfully on Ψ) and hence K^Ψ is nonabelian and nonsimple. Also $K_\psi = K \cap Y = U^k$ and for each i, $U = (K_\psi)\pi_i = (K_\alpha)\pi_i$. Hence K_ψ is not a subdirect subgroup of K. Therefore, $(G^\Psi, K^\Psi, G_0^\Psi) \in \mathcal{G}_{prod}$.

(2) The map $g \mapsto g^\Psi$ is a well-defined epimorphism from G onto G^Ψ, and its restriction to K is an isomorphism from K onto K^Ψ (since K acts faithfully on Ψ). So by Lemma 4.3, $\mathrm{C}_G(K)^\Psi = \mathrm{C}_{G^\Psi}(K^\Psi)$.

(3) Since $K_\psi = U^k$, it follows from Lemma 4.8 that we can identify Ψ with the Cartesian product of k-copies of $\Psi_0 := [T : U]$ in such a way that $G^\Psi \le \mathrm{Sym}(\Psi_0) \operatorname{wr} S_k$ in product action. Since the action of G on the simple direct factors of K is transitive, G projects onto a transitive subgroup of S_k. Let G_1 be the stabiliser of 1 in this action and let $\nu : G_1 \to \mathrm{Sym}(\Psi_0)$ so that $V = (G_1)\nu$ is the component of G in $\mathrm{Sym}(\Psi_0)$. Note that $K \le G_1$, $(K)\nu = T$ acts transitively on Ψ_0, and T is a minimal normal subgroup of V. Thus V is innately transitive with simple plinth T and $K \subseteq G \cap \left(\mathrm{Sym}(\Psi_0)^k\right)$. Applying Lemma 4.7, we get that G^Ψ is conjugate in $\mathrm{Sym}(\Psi_0) \operatorname{wr} S_k$ to a subgroup of $V \operatorname{wr} S_k$. □

In the case that K acts regularly on Ω, we can say even more.

Corollary 9.3. *Let $(G,K,G_0)\in\mathcal{G}_{prod}$, let T, Ω, Ψ, Ψ_0, V be as in Construction 9.1 and Proposition 9.2, and suppose K acts regularly on Ω. Then $\Omega=\Psi$, and we may identify Ψ_0 with T so that $T_R\leq V\leq \mathrm{Hol}(T)$. Moreover, one of the following holds:*

1. *G is quasiprimitive of Twisted Wreath type or Holomorph of a Compound Group type, and V is of Almost Simple type or Holomorph of a Simple Group type respectively,*
2. *K_σ is a proper subdirect subgroup of K and V is of Holomorph of a Simple Group type, or*
3. *K_σ is not a subdirect subgroup of K and V is of Almost Simple Quotient type.*

Proof. Since $K_\alpha=1$ and K_α is a subdirect subgroup of K_ψ, it follows that $U=1$. Thus $K_\psi=1$ and hence $|\Psi|=|K|=|\Omega|$, So in this case, $\Omega=\Psi=\Psi_0^k$ and hence by Proposition 9.2, G is conjugate in $\mathrm{Sym}(\Omega)$ to a subgroup of $V\,\mathrm{wr}\,S_k\leq \mathrm{Sym}(\Psi_0)\,\mathrm{wr}\,S_k$. Moreover, $\Psi_0=[T:U]$ in this case may be identified with T and T is a regular normal subgroup of V, so we may identify T with T_R and hence $T_R\leq V\leq\mathrm{Hol}(T)$. Under this identification, $K_R=T_R^k\leq G\leq \mathrm{Hol}(K)$ and $\mathrm{C}_G(K)=G\cap K_L$.

Suppose that $\mathrm{C}_G(K)=1$ or $\mathrm{C}_G(K)\cong K_L$. Then by Proposition 5.3, G is quasiprimitive and by Theorem 4.1, G is of Twisted Wreath type or Holomorph of a Compound Group type respectively and (1) holds. Thus we may assume that $\mathrm{C}_G(K)\neq 1$ and $|\mathrm{C}_G(K)|\neq|K|$. Now $(\mathrm{C}_G(K)\times K_R)\nu\cong(\mathrm{C}_G(K))\nu\times T_R$. Since $\mathrm{C}_G(K)$ is G-invariant and G is transitive on the simple direct factors of K, it follows that the projections of $\mathrm{C}_G(K)$ to the simple direct factors of K_L are pairwise isomorphic. In particular, $(\mathrm{C}_G(K))\nu\neq 1$. If $(\mathrm{C}_G(K))\nu\neq T_L$, then $\mathrm{C}_G(K)$ is not a subdirect subgroup of K_L and hence K_σ is not a subdirect subgroup of K_R and (3) holds. On the other hand if $(\mathrm{C}_G(K))\nu=T_L$, then $\mathrm{C}_G(K)$ is a subdirect subgroup of K_L and K_σ is a subdirect subgroup of K_R and (2) holds. □

Let $Q:\mathcal{G}_{prod}\to\mathcal{G}_{prod}$ be the map where for each $(G,K,G_0)\in\mathcal{G}_{prod}$, the image $(G,K,G_0)Q=(G^\Psi,K^\Psi,G_0^\Psi)$ is the element of $\mathcal{G}_{prod}$ constructed from (G,K,G_0) via Construction 9.1. Corollary 9.3 elaborates on the case where K is regular. Suppose now that K is not regular and recall that K_α is not a subdirect subgroup of K.

Lemma 9.4. *Let $(G,K,G_0)\in\mathcal{G}_{prod}$. Then G is of Product Type (as described in Section 3) if and only if K is not regular. In this case, the group V in Proposition 9.2 is of Almost Simple type or Almost Simple Quotient type.*

Proof. Let $(G,K,G_0)\in\mathcal{G}_{prod}$ and let $(G^\Psi,K^\Psi,\Psi)=(G,K,G_0)Q$. As in Proposition 9.2, suppose $K=T^k$ for some nonabelian simple group T and $k\geq 2$, and let $\Omega:[G:G_0]$, $\alpha=U$, and $\Psi=\Psi_0^k$ for some set Ψ_0. Let $\psi\in\Psi$ such that $\alpha\in\Psi$. The innately transitive Product Type occurs precisely in this situation where K is not regular. For suppose that K is not regular on Ω. Then K is not regular on Ψ as K_α is a subdirect subgroup of K_ψ. So the simple direct factor T is not regular on Ψ_0 and hence V has either Almost Simple or Almost Simple Quotient Type. Also by Proposition 9.2, G projects onto a transitive subgroup of S_k. The description of Product Type in Section 3 then follows. Conversely, if G is of Product Type, then by definition K is non-regular and K_α is not subdirect in K. □

We will characterise the set $\mathcal{G}_{prod}$ in terms of its subset $(\mathcal{G}_{prod})Q$. It is more convenient to work with innate triples than innately transitive permutation groups, so we first determine the innate triples corresponding to the groups given by Construction 9.1. It follows from the definition that an innate triple (K, φ, L) corresponds to an element of $\mathcal{G}_{prod}$ if and only if K is nonabelian and nonsimple, and $\operatorname{Ker}\varphi$ is not a subdirect subgroup of K. Let $\mathcal{D}_{prod}$ denote the set of such triples.

Lemma 9.5. *Let $(G, K, G_0) \in \mathcal{G}_{prod}$, let $(G^\Psi, K^\Psi, G^\Psi) = (G, K, G_0)Q$, and let (K, φ, L) and $(K^\Psi, \widetilde{\varphi}, \widetilde{L})$ be the elements of $\mathcal{D}_{prod}$ corresponding to (G, K, G_0) and (G^Ψ, K^Ψ, G^Ψ) respectively. Then for some $\psi \in \Psi$, $\operatorname{Ker}\widetilde{\varphi} = K_\psi$, $\operatorname{Dom}\widetilde{\varphi} = (\operatorname{Dom}\varphi)K_\psi$, and $\widetilde{L} = L\operatorname{Inn}_{K_\psi}(K)$.*

Proof. First we consider $M = G_{(\Psi)}$. Now M is a normal subgroup of G and $M \cap K = 1$, so $M \leq \mathrm{C}_G(K)$. Recall that $\mathrm{C}_G(K)^\Psi = \mathrm{C}_{G^\Psi}(K^\Psi)$ by Proposition 9.2(2). Let N be the normal subgroup of G containing M such that $N/M = \mathrm{C}_{G^\Psi}(K^\Psi)$. Then $1 = (N/M) \cap K^\Psi = (N \cap KM)/M$ so $N \cap KM = M$. Since N contains M, it follows that $N \cap K = 1$. Thus $N \leq \mathrm{C}_G(K)$. On the other hand, $|\mathrm{C}_G(K)/M| = |\mathrm{C}_{G^\Psi}(K^\Psi)|$, so $N = \mathrm{C}_G(K)$. As in the proof of Construction 9.1, we identify Ψ with a partition of Ω in such a way that α lies in the cell ψ. Then the $\mathrm{C}_G(K)^\Psi$-orbit $\hat{\sigma}$ containing ψ is the set of all cells of Ψ containing at least one point of $\sigma := \alpha^{\mathrm{C}_G(K)}$. It follows that $K_{\hat{\sigma}}$ contains $K_\sigma K_\psi$.

Moreover, since $\mathrm{C}_G(K)^\Psi$ is regular on $\hat{\sigma}$ with kernel M, σ is the union of the $|\hat{\sigma}| = |\mathrm{C}_G(K)/M|$ cells of Ψ contained in σ, each of size $|M|$. Now $|K_{\hat{\sigma}} : K_\psi| = |\hat{\sigma}| = |\mathrm{C}_G(K)/M|$ and $|K_\sigma K_\psi : K_\psi| = |K_\sigma : K_\sigma \cap K_\psi| = |\hat{\sigma}|$ as $K_\sigma \cap K_\psi$ is the stabiliser in K_σ of ψ, one of the $|\hat{\sigma}|$ cells in σ. Thus $K_{\hat{\sigma}} = K_\sigma K_\psi$, that is, $\operatorname{Dom}\widetilde{\varphi} = K_\sigma K_\psi = (\operatorname{Dom}\varphi)K_\psi$. Hence by Lemma 5.4, $\operatorname{Ker}\widetilde{\varphi} = K_\psi$.

Finally, $\widetilde{L}$ is the subgroup of $\operatorname{Aut}(K)$ induced by G_ψ acting by conjugation on K. Since $G_\psi = G_\alpha K_\psi$, it follows that $\widetilde{L} = L\operatorname{Inn}_{K_\psi}(K)$. □

Let $(\mathcal{D}_{prod})Q$ denote the subset of $\mathcal{D}_{prod}$ of elements corresponding to $(\mathcal{G}_{prod})Q$. Given an element (G^Ψ, K^Ψ, G^Ψ) of $(\mathcal{G}_{prod})Q$, we now give a second construction that produces a subset of elements of $\mathcal{G}_{prod}$. The output of this construction is a subset of innate triples in $\mathcal{D}_{prod}$ which corresponds to a family of innately transitive groups via Construction 6.6.

Construction 9.6. *Given $(K, \widetilde{\varphi}, \widetilde{L}) \in (\mathcal{D}_{prod})Q$, let*

(1) *K_0 be a supplement of* $\operatorname{Ker}\widetilde{\varphi}$ *in* $\operatorname{Dom}\widetilde{\varphi}$*,*

(2) *L be a supplement of* $\operatorname{Inn}_{\operatorname{Ker}\widetilde{\varphi}}(K)$ *in $\widetilde{L}$ such that* $L\cap\operatorname{Inn}(K)=\operatorname{Inn}_{K_0}(K)$*,*

(3) *M be an L-invariant normal subgroup of K_0 contained in* $\operatorname{Ker}\widetilde{\varphi}$*,*

(4) *$\varphi : K_0 \to K_0/M$ be the natural quotient map.*

Then $(K,\varphi,L)\in\mathcal{D}_{prod}$.

Proof. By (4), $\operatorname{Ker}\varphi$ is equal to M, and is L-invariant by (3). Since $\operatorname{Ker}\widetilde{\varphi}$ is corefree in K and $M \le \operatorname{Ker}\widetilde{\varphi}$ by (3), it follows that M is corefree in K. By definition of $\mathcal{D}$, we have that $\widetilde{L}$ is transitive on the simple direct factors of K, so by (2), L is also transitive on the simple direct factors of K. Hence the only L-invariant normal subgroups of K are 1 and K. Therefore K is L-simple. Now K is nonabelian and nonsimple and since $\operatorname{Dom}\widetilde{\varphi}$ is not a subdirect subgroup of K, it follows from (1) that $\operatorname{Dom}\varphi = K_0$ is not subdirect in K. Thus $(K,\varphi,L)\in\mathcal{D}_{prod}$. □

Remarks 9.7. *Let $(K,\widetilde{\varphi},\widetilde{L})\in(\mathcal{D}_{prod})Q$.*

(i) There is always at least one output of Construction 9.6. One can set K_0, L, and M in (1)-(3) of Construction 9.6 to be $\operatorname{Dom}\widetilde{\varphi}$, $\widetilde{L}$, *and* $\operatorname{Ker}\widetilde{\varphi}$ *respectively. We will see in Corollary 9.9 that the element of $\mathcal{D}_{prod}$ obtained with these parameters is equivalent to $(K,\widetilde{\varphi},\widetilde{L})$ under $\equiv_{\mathcal{D}}$.*

(ii) By definition of $(\mathcal{D}_{prod})Q$, $\operatorname{Ker}\widetilde{\varphi}=U^k$ *for some $U<T$. Recall, that if $(G,K,G_0)\in\mathcal{G}_{prod}$ (where $K=T^k$) and $(G^\Psi,K^\Psi,G^\Psi)=(G,K,G_0)Q$, then for $\psi\in\Psi$, there exists $\alpha\in\Omega$ such that $K_\psi=\prod_{i=1}^k(K_\alpha)\pi_i$. By Lemma 5.4,* $\operatorname{Ker}\widetilde{\varphi}=K_\psi$ *for some ψ, and since the $(K_\alpha)\pi_i$ are pairwise isomorphic,* $\operatorname{Ker}\widetilde{\varphi}$ *can be identified with U^k for some proper subgroup U of T.*

(iii) For some triples in $(\mathcal{D}_{prod})Q$, it is possible to obtain an output different from the example in (i). Suppose $\operatorname{Ker}\widetilde{\varphi}=U^k$ *for some proper subgroup U of T. If U is abelian, it is possible to have* $\operatorname{Dom}\widetilde{\varphi}=\operatorname{Ker}\widetilde{\varphi}=U^k\neq 1$*. So in Construction 9.6, we could chose $K_0=1$, which means that* $\operatorname{Ker}\varphi=1$ *is not subdirect in* $\operatorname{Ker}\widetilde{\varphi}$*.*

As explained in the remarks above, it is not always true that $\operatorname{Ker}\varphi$ is a subdirect subgroup of $\operatorname{Ker}\widetilde{\varphi}$. However, when $\operatorname{Ker}\varphi$ is a subdirect subgroup of $\operatorname{Ker}\widetilde{\varphi}$, it turns out that the corresponding output of Construction 9.6 is equivalent to $(K,\widetilde{\varphi},\widetilde{L})$.

Proposition 9.8. *Let $(K,\varphi,L)\in\mathcal{D}_{prod}$ be obtained from $(K,\widetilde{\varphi},\widetilde{L})\in(\mathcal{D}_{prod})Q$ by Construction 9.6. Then $(K,\varphi,L)\Gamma\in\mathcal{G}_{prod}$. Moreover $(K,\varphi,L)\Gamma\circ Q$ corresponds to an innate triple equivalent to $(K,\widetilde{\varphi},\widetilde{L})$ if and only if* $\operatorname{Ker}\varphi$ *is a subdirect subgroup of* $\operatorname{Ker}\widetilde{\varphi}$*.*

Proof. Let $(K,\widetilde{\varphi},\widetilde{L})\in(\mathcal{D}_{prod})Q$ and let (K,φ,L) be an element of $\mathcal{D}_{prod}$ obtained by Construction 9.6. Recall from Construction 6.6, that $(K,\varphi,L)\Gamma=(\overline{G},\overline{K},\overline{G_0})$ where $\overline{G}=X/Z$, $\overline{K}=KZ/Z$, and $\overline{G_0} = \mathrm{Graph}(\varphi)L/Z$ acting on $\overline{\Omega} = [X : \mathrm{Graph}(\varphi)L]$, where $X = (\mathrm{Im}\,\varphi \times K) \rtimes L$ and $Z = \{(x)\varphi x\iota_{x^{-1}} : x \in \mathrm{Dom}\,\varphi\}$. Since $(K, \varphi, L) \in \mathcal{D}_{prod}$, K is nonabelian and nonsimple, and $\mathrm{Ker}\,\varphi$ is not a subdirect subgroup of K. Therefore $\overline{K}$ is nonabelian and nonsimple. The point stabiliser of the trivial coset $\mathrm{Graph}(\varphi)L$ in $\overline{K}$, is equal to $(\mathrm{Graph}(\varphi)L/Z) \cap \overline{K} = (\mathrm{Ker}\,\varphi)Z/Z$, and is not a subdirect subgroup of $\overline{K}$. Therefore $(K, \varphi, L)\Gamma \in \mathcal{G}_{prod}$.

Suppose $\mathrm{Ker}\,\varphi$ is a subdirect subgroup of $\mathrm{Ker}\,\widetilde{\varphi}$. Then $\mathrm{Ker}\,\widetilde{\varphi} = \prod_{i=1}^{k}(\mathrm{Ker}\,\varphi)\pi_i$ where π_i is the natural projection map of K onto its ith simple direct factor. Now we apply Construction 9.1 to obtain $(K, \varphi, L)\Gamma \circ Q$. Let $(\overline{K}, \overline{\varphi}, \overline{L})$ be the element of $\mathcal{D}_{prod}$ corresponding to $(K, \varphi, L)\Gamma \circ Q$. We verify th three conditions of Definition 6.4 to show that $(\overline{K}, \overline{\varphi}, \overline{L}) \equiv_{\mathcal{D}} (K, \varphi, L)$. As in the proof of Lemma 9.5, the domain of $\widetilde{\varphi}$ is $(\mathrm{Dom}\,\varphi)(\mathrm{Ker}\,\widetilde{\varphi})$ and so $\widetilde{\varphi}$ maps $\mathrm{Dom}\,\varphi$ onto $\mathrm{Im}\,\widetilde{\varphi}$.

Let $\theta : K \to \overline{K}$ be the isomorphism which maps each $y \in K$ to $Zy \in \overline{K}$. The domain of $\overline{\varphi}$ is $(\mathrm{Dom}\,\varphi)\theta(\mathrm{Ker}\,\overline{\varphi})$, and hence $\overline{\varphi}$ maps $(\mathrm{Dom}\,\varphi)\theta$ onto $\mathrm{Im}\,\overline{\varphi}$. Now the natural projection map $\overline{\pi}_i : \overline{K} \to (T_i)\theta$, is equal to $\theta^{-1} \circ \pi_i \circ \theta$. Note that $(\mathrm{Ker}\,\widetilde{\varphi})\theta = \left(\prod_{i=1}^{k}(\mathrm{Ker}\,\varphi)\pi_i\right)\theta = \prod_{i=1}^{k}((\mathrm{Ker}\,\varphi)\pi_i)\theta = \prod_{i=1}^{k}((\mathrm{Ker}\,\varphi)\theta)\overline{\pi}_i = \mathrm{Ker}\,\overline{\varphi}$.

Let $\Theta : \mathrm{Im}\,\widetilde{\varphi} \to \mathrm{Im}\,\overline{\varphi}$ be the map defined by $((x)\widetilde{\varphi})\Theta = ((x)\theta)\overline{\varphi}$ for all $x \in \mathrm{Dom}\,\varphi$. We check first that Θ is well-defined. Let $(x_1)\widetilde{\varphi} = (x_2)\widetilde{\varphi}$. Then $x_1x_2^{-1} \in \mathrm{Ker}\,\widetilde{\varphi}$. So $(x_1x_2^{-1})\theta \in \mathrm{Ker}\,\overline{\varphi}$ and hence $((x_1)\theta)\overline{\varphi} = ((x_2)\theta)\overline{\varphi}$. Therefore, Θ is well-defined. By reversing this argument we see that Θ is injective. So Θ is a bijection, since it is clearly surjective. To see that Θ is a homomorphism, let $(x_1)\widetilde{\varphi}, (x_2)\widetilde{\varphi} \in \mathrm{Im}\,\widetilde{\varphi}$ where $x_1, x_2 \in \mathrm{Dom}\,\varphi$. Then $((x_1)\widetilde{\varphi}(x_2)\widetilde{\varphi})\Theta = ((x_1x_2)\widetilde{\varphi})\Theta = ((x_1x_2)\theta)\overline{\varphi} = ((x_1)\theta)\overline{\varphi}\,((x_2)\theta)\overline{\varphi} = ((x_1)\widetilde{\varphi})\Theta\,((x_2)\widetilde{\varphi})\Theta$. Therefore, Θ is a homomorphism.

Now we show that $\theta \circ \overline{\varphi} = \widetilde{\varphi} \circ \Theta$. Let $x \in \mathrm{Dom}\,\varphi$ and $y \in \mathrm{Ker}\,\widetilde{\varphi}$. Then $(xy)\widetilde{\varphi} \circ \Theta = ((x)\widetilde{\varphi})\Theta = ((x)\theta)\varphi = (xy)\theta \circ \varphi$. Therefore $\theta \circ \overline{\varphi} = \widetilde{\varphi} \circ \Theta$. Finally, we show that $\overline{L} = \theta^{-1}\widetilde{L}\theta$. We have, $\theta^{-1}\widetilde{L}\theta = (\theta^{-1}L\theta)[\theta^{-1}(\mathrm{Inn}_{\mathrm{Ker}\,\widetilde{\varphi}}(K))\theta] = (\theta^{-1}L\theta)\mathrm{Inn}_{(\mathrm{Ker}\,\widetilde{\varphi})\theta}((K)\theta) = (\theta^{-1}L\theta)\mathrm{Inn}_{\mathrm{Ker}\,\overline{\varphi}}(\overline{K}) = \overline{L}$. Therefore, $(K, \widetilde{\varphi}, \widetilde{L}) \equiv_{\mathcal{D}} (\overline{K}, \overline{\varphi}, \overline{L})$.

Conversely, suppose $(K, \varphi, L)\Gamma \circ Q$ corresponds to a triple $(\overline{K}, \overline{\varphi}, \overline{L})$ of $\mathcal{D}$ equivalent to $(K, \widetilde{\varphi}, \widetilde{L})$. Then there exist isomorphisms $\theta : K \to \overline{K}$ and $\Theta : \mathrm{Im}\,\widetilde{\varphi} \to \mathrm{Im}\,\overline{\varphi}$ such that $\theta \circ \overline{\varphi} = \widetilde{\varphi} \circ \Theta$. So $\mathrm{Ker}\,\widetilde{\varphi} = (\mathrm{Ker}\,\overline{\varphi})\theta^{-1}$. Now by Construction 9.1 (and by the same argument used previously in this proof), $\mathrm{Ker}\,\overline{\varphi} = \prod_{i=1}^{k}((\mathrm{Ker}\,\varphi)\theta)\overline{\pi}_i$. So $\mathrm{Ker}\,\widetilde{\varphi} = \left(\prod_{i=1}^{k}((\mathrm{Ker}\,\varphi)\theta)\overline{\pi}_i\right)\theta^{-1} = \prod_{i=1}^{k}(\mathrm{Ker}\,\varphi)\pi_i$ and therefore $\mathrm{Ker}\,\varphi$ is a subdirect subgroup of $\mathrm{Ker}\,\widetilde{\varphi}$. □

Let us review what we have found so far. It is our main objective in this section to characterise the elements of $\mathcal{G}_{prod}$ by their quotient constructions $(\mathcal{G}_{prod})Q$. So far we have shown that we can complete the circle – that is, given an element of $(\mathcal{G}_{prod})Q$ we construct (possibly several) innate triples which yield elements of $\mathcal{G}_{prod}$, and the image under Q of at least one of these is equivalent to the element of $(\mathcal{G}_{prod})Q$ we started with. It remains to show that all elements of $\mathcal{G}_{prod}$ arise by this process using Construction 9.6.

Corollary 9.9. *Let $(K, \varphi, L) \in \mathcal{D}_{prod}$ and let $(K^\Psi, \widetilde{\varphi}, \widetilde{L})$ be the element of $(\mathcal{D}_{prod})Q$ corresponding to the output of Construction 9.1 applied to $(K, \varphi, L)\Gamma$ so that $\widetilde{L} = L\mathrm{Inn}_{K_1}(K)$ by Lemma 9.5, where $K_1 = \mathrm{Ker}\,\widetilde{\varphi}$. If in Construction 9.6, the subgroups K_0, L, and M chosen in parts (1), (2), and (3) are $\mathrm{Dom}\,\widetilde{\varphi}$, L, and $\mathrm{Ker}\,\varphi$ respectively, then the element of $\mathcal{D}$ constructed from $(K^\Psi, \widetilde{\varphi}, \widetilde{L})$ is equivalent to (K, φ, L) under $\equiv_{\mathcal{D}}$.*

Proof. It is clear that $M = \mathrm{Ker}\,\varphi$, $K_0 = \mathrm{Dom}\,\varphi$, and L satisfy the conditions of Construction 9.6. Let θ be the identity isomorphism of K onto itself (so clearly $L = \theta^{-1}L\theta$), and let $\hat{\varphi}$ be the natural quotient map of K_0 onto $C := K_0/M$. Let $\Theta : \mathrm{C}_G(K) \to C$ be defined by $((x)\varphi)\Theta = Mx$. First we show that Θ is well-defined. Let $x_1, x_2 \in K_0$ and suppose $(x_1)\varphi = (x_2)\varphi$. Then $x_1x_2^{-1} \in \mathrm{Ker}\,\varphi = M$ and hence $Mx_1 = Mx_2$, and Θ is well-defined. It is clear that Θ is an isomorphism. Now for all $x \in \mathrm{Dom}\,\varphi$, $(x)\varphi \circ \Theta = Mx = (x)\hat{\varphi} = (x)\theta \circ \hat{\varphi}$ and hence $\theta \circ \hat{\varphi} = \varphi \circ \Theta$ on $\mathrm{Dom}\,\varphi$. Therefore $(K, \hat{\varphi}, L) \equiv_{\mathcal{D}} (K, \varphi, L)$. □

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Let G be an innately transitive group on a set Ω with plinth K. We may assume that K is nonabelian and nonsimple, as otherwise K satisfies (1) or (2) of Theorem 1.3. Let $\alpha \in \Omega$. We have two cases; K_α is or is not a subdirect subgroup of K. By Proposition 5.5 and the description of Diagonal type in Section 3, if K_α is a subdirect subgroup of K, then G satisfies (3) or (4) of Theorem 1.3. So suppose K_α is not a subdirect subgroup of K. Then $(G, K, G_\alpha) \in \mathcal{G}_{prod}$. Let $(G^\Psi, K^\Psi, G_0^\Psi) = (G, K, G_0)Q$ as in Construction 9.1.

By Proposition 9.2, Ψ can be identified with k copies of a set Ψ_0 and G^Ψ is conjugate to a subgroup of $V \,\mathrm{wr}\, S_k$ in product action, where V is the component of G^Ψ in $\mathrm{Sym}(\Psi_0)$. It follows from Corollary 9.9 that G is permutationally isomorphic to an innately transitive group produced by Construction 9.6 using the innate triple corresponding to $(G^\Psi, K^\Psi, G_0^\Psi)$ as input.

If K is regular, then by Corollary 9.3 it follows that V is innately transitive with a simple plinth (in all cases). Also if K is not regular, then by Lemma 9.4, V is again innately transitive with a simple plinth. Therefore G satisfies part (5) of Theorem 1.3. □

10. Innately Transitive Groups with a Regular Plinth

In this section, we analyse innately transitive groups whose plinths act regularly. Let G be an innately transitive group on a set Ω, let $\alpha \in \Omega$, let K be the plinth of G, and suppose K is nonabelian, nonsimple, and acts regularly on Ω, that is, $K_\alpha = 1$. As discussed in Section 2, we may can identify Ω with K, identify K with K_R, identify α with the identity element of K, identify $\mathrm{C}_{\mathrm{Sym}(\Omega)}(K)$ with $\mathrm{C}_{\mathrm{Sym}(K)}(K_R) = K_L$, and identify $\mathrm{N}_{\mathrm{Sym}(\Omega)}(K)$ with $\mathrm{Hol}(K)$. Let $K = T^k$ where T is a nonabelian simple group and $k \geq 2$. There are four sub-cases according to the structure of $\mathrm{C}_G(K)$.

(1) $\mathrm{C}_G(K_R) = 1$: Here G is quasiprimitive by Proposition 5.3, and by Theorem 4.1, G is of Twisted Wreath type.
(2) $\mathrm{C}_G(K_R) \neq 1$ and $\mathrm{C}_G(K_R)$ is not subdirect in K_L: We shall show in Proposition 10.3(ii) that G is of Product Quotient type.
(3) $1 < \mathrm{C}_G(K_R) < K_L$ and $\mathrm{C}_G(K_R)$ is subdirect in K_L: It will turn out (see Proposition 10.3(iii)) that G is of Diagonal Quotient type.
(4) $\mathrm{C}_G(K_R) = K_L$: Here G is quasiprimitive by Proposition 5.3, and by Theorem 4.1, G is of Holomorph of a Compound Group Type.

Let (K_R, φ, L) be the innate triple associated to (G, K_R, G_α). By Lemma 5.2, φ is an isomorphism from $(K_R)_\sigma$ onto $\mathrm{C}_G(K_R)$ and $(\rho_x)\varphi = \lambda_x$ for all $\rho_x \in (K_R)_\sigma$. The graph of φ is useful for analysing this case. Recall that $\mathrm{Graph}(\varphi)$ is the subgroup of $\mathrm{C}_G(K_R) \times (K_R)_\sigma)$ defined by $\mathrm{Graph}(\varphi) = \{(u)\varphi\, u : u \in (K_R)_\sigma)\}$. It has the following properties.

Lemma 10.1. *Let $G \leq \mathrm{Sym}(K)$ be innately transitive with non-abelian plinth K_R, let σ be the orbit of the identity element α of K under $\mathrm{C}_G(K_R)$, and let φ be the isomorphism from $(K_R)_\sigma$ onto $\mathrm{C}_G(K_R)$ given in Lemma 5.4. Then:*

(1) $\mathrm{Graph}(\varphi)$ *is a full diagonal subgroup of* $\mathrm{C}_G(K_R) \times (K_R)_\sigma$*,*
(2) $\mathrm{Graph}(\varphi) = (\mathrm{C}_G(K_R) \times K_R)_\alpha$*, and*
(3) $\mathrm{Graph}(\varphi) = \mathrm{Core}_{G_\alpha}((\mathrm{Inn}(T_R)\phi^{-1})$ *where T is a simple direct factor of K and $\phi : \mathrm{N}_{G_\alpha}(T_R) \to \mathrm{Aut}(T_R)$ is the map induced by the conjugation action of G_α on T_R.*

Proof. Part (1) Follows from the definition of $\mathrm{Graph}(\varphi)$.

(2) Since $K_L \times K_R = K_R \rtimes \mathrm{Inn}(K)$, K_R is regular on K, and $\mathrm{Inn}(K)$ fixes α, it follows that $(K_L \times K_R)_\alpha = \mathrm{Inn}(K)$. Now $\mathrm{Graph}(\varphi)$ is a subgroup of $\mathrm{C}_G(K_R) \times K_R$, so $\mathrm{Graph}(\varphi) \leq (\mathrm{C}_G(K_R) \times K_R)_\alpha$. On the other hand $\mathrm{C}_G(K_R) \times K_R$ is transitive on K and hence

$$|(\mathrm{C}_G(K_R) \times K_R)_\alpha| = |\mathrm{C}_G(K_R) \times K_R|/|K_R| = |\mathrm{C}_G(K_R)| = |\mathrm{Graph}(\varphi)|.$$

Therefore $\mathrm{Graph}(\varphi) = (\mathrm{C}_G(K_R) \times K_R)_\alpha$.

(3) Let $M = (\mathrm{Inn}(T_R))\phi^{-1}$. First we will show that $\mathrm{Graph}(\varphi) \leq \mathrm{Core}_{G_\alpha}(M)$. Clearly, $\mathrm{Graph}(\varphi)$ lies in the domain $\mathrm{N}_{G_\alpha}(T_R)$ of ϕ and $(\mathrm{Graph}(\varphi))\phi \leq \mathrm{Inn}(T_R)$ since $\mathrm{Graph}(\varphi) \leq \mathrm{Inn}(K)$. So

$\mathrm{Graph}(\varphi) \leq M$ and since $\mathrm{Graph}(\varphi) \trianglelefteq G_\alpha$, it follows that $\mathrm{Graph}(\varphi) \leq g^{-1}Mg$ for all $g \in G_\alpha$. Therefore $\mathrm{Graph}(\varphi) \leq \mathrm{Core}_{G_\alpha}(M)$.

Now we prove the reverse inclusion. For all i, let T_i be the i-th simple direct factor of K, $\phi_i : \mathrm{N}_{G_\alpha}((T_i)_R) \to \mathrm{Aut}((T_i)_R)$ be the induced conjugation map, and $M_i = (\mathrm{Inn}((T_i)_R))\phi_i^{-1}$. Since G_α is transitive on the simple direct factors of K, $\mathrm{Core}_{G_\alpha}(M) = \bigcap_{i=1}^k M_i$. Let $x \in \mathrm{Core}_{G_\alpha}(M)$. Then $(x)\phi_i \in \mathrm{Inn}((T_i)_R)$ for all i. So for all i, there exists $\rho_{t_i} \in (T_i)_R$ such that for all $\rho_{y_i} \in (T_i)_R$, $x^{-1}\rho_{y_i}x = \rho_{t_i}^{-1}y_i\rho_{t_i}$. Hence for all $\rho_{y_1}\cdots\rho_{y_k} \in (T_1)_R \times \cdots \times (T_k)_R = K_R$, we have $x^{-1}(\rho_{y_1}\cdots\rho_{y_k})x = \rho_{(t_1\cdots t_k)^{-1}(y_1\cdots y_k)(t_1\cdots t_k)}$, and therefore x induces an inner automorphism of K. The subgroup inducing inner automorphisms is $K_L \times K_R = K_R \rtimes \mathrm{Inn}(K)$ (see Section 2), and thus $x \in (K_L \times K_R) \cap G_\alpha = \mathrm{Graph}(\varphi)$ (by (2)). Therefore $\mathrm{Core}_{G_\alpha}(M) \leq \mathrm{Graph}(\varphi)$ as required. □

Remarks 10.2. *We can translate the properties of* $\mathrm{C}_G(K_R)$ *in the case subdivision in Lemma 10.1 to equivalent properties for* $\mathrm{Graph}(\varphi)$*:*

(1) $\mathrm{C}_G(K_R) = 1$ *if and only if* $\mathrm{Graph}(\varphi) = 1$*,*
(2) $\mathrm{C}_G(K_R)$ *is subdirect in* K_L *if and only if* $\mathrm{Graph}(\varphi)$ *is subdirect in* $\mathrm{Inn}(K)$*, and*
(3) $\mathrm{C}_G(K_R) = K_L$ *if and only if* $\mathrm{Graph}(\varphi) = \mathrm{Inn}(K)$*.*

We now give a structure theorem for *Regular Plinth* type. Recall by Corollary 9.3, that we have $G \leq V \,\mathrm{wr}\, S_k$ in product action on T^k where V, the component of G, is innately transitive on T with a regular plinth.

Proposition 10.3. *Let* $K = T^k$ *where* $k \geq 2$ *and* T *is a nonabelian simple group, let* $G \leq \mathrm{Hol}(K) \cong \mathrm{Hol}(T) \,\mathrm{wr}\, S_k$*, and let* V *be the component of* G *in* $\mathrm{Hol}(T)$*.*

(i) *If* $\mathrm{Graph}(\varphi) = 1$*, then* V *is of Almost Simple type,* K *is the unique minimal normal subgroup of* G*, and* G *is quasiprimitive of Twisted Wreath type.*
(ii) *If* $\mathrm{Graph}(\varphi)$ *is nontrivial and not subdirect in* $\mathrm{Inn}(K)$*, then* V *is of Almost Simple Quotient type and* G *is of Product Quotient type.*
(iii) *If* $\mathrm{Graph}(\varphi)$ *is nontrivial and subdirect in* $\mathrm{Inn}(K)$*, then* V *is of Holomorph of a Simple Group Type, and* $\mathrm{C}_G(K_R)$ *is a direct product of* m *full diagonal subgroups of* T^k *where* m *is a proper divisor of* k*. Up to permutational isomorphism, we have that* $G \leq \mathrm{N}_{\mathrm{Hol}(K)}(\mathrm{C}_G(K)) = K_R \rtimes [(A \times S_{k/m}) \,\mathrm{wr}\, S_m]$ *and* $\mathrm{Graph}(\varphi) = B^m$*, where* A *and* B *are full diagonal subgroups of* $\mathrm{Aut}(T)^{k/m}$ *and* $\mathrm{Inn}(T)^{k/m}$ *respectively. Thus* G *is of Diagonal Quotient type.*
(iv) *If* $\mathrm{Graph}(\varphi) = \mathrm{Inn}(K)$*, then* V *is of Holomorph of a Simple Group type, and* G *is primitive of Holomorph of a Compound Group type.*

Proof. Parts (i) and (iv) follow from Corollary 9.3. For part (ii), it follows from Corollary 9.3 that V is of Almost Simple Quotient type and hence G is of Product Quotient type as described in Section 3. It remains to prove part (iii). Let σ be the orbit of the identity element of K under $\mathrm{C}_G(K_R)$. By Remarks 10.2, $\mathrm{C}_G(K_R)$ is a subdirect subgroup of K_L. Equivalently (see the proof of Corollary 9.3), the setwise stabiliser $(K_R)_\sigma$ is a subdirect subgroup of K_R. So by Corollary 9.3, V is of Holomorph of a Simple Group type.

By Lemma 4.4, there exists a positive integer m, such that $\mathrm{C}_G(K_R)$ is a direct product of m full diagonal subgroups of subproducts of T^k. Since G_α acts transitively on the simple direct factors of K, it also acts transitively on the simple direct factors of $\mathrm{C}_G(K_R)$, and hence m is a divisor of k. Now each full diagonal subgroup of a subproduct of T^k is of the form $\{(t^{\gamma_1},\ldots,t^{\gamma_{k/m}}) : t \in T\}$, where each γ_i is an automorphism of T. So the image of G under some element γ of $\mathrm{Aut}(K)$ yields a permutation group $\hat{G} = (G)\gamma$ on $[\hat{G} : (G_0)\gamma]$ that is permutationally isomorphic to G on Ω such that $\mathrm{C}_{\hat{G}}((K_R)\gamma)$ is a direct product of straight diagonal groups of the form $\{(t,\ldots,t) : t \in T\}$. We will identify G with $\hat{G}$.

Let $\Sigma = \{\underline{1},\ldots,\underline{m}\}$ where each $\underline{i} = \{(i-1)r+1,\ldots,ir\}$ and $r = k/m$. Note that Σ forms a partition of $\{1,\ldots,k\}$. It is a simple fact (see [8, Exercise 2.6.2]), that $(S_k)_\Sigma \cong S_r \,\mathrm{wr}\, S_m$. We relabel the simple direct factors of K such that $(K_R)_\sigma = D_{\underline{1}} \times \cdots \times D_{\underline{m}}$ where each $D_{\underline{i}}$ is a straight diagonal subgroup of T^r.

Note that $\mathrm{N}_{\mathrm{Hol}(K)}(\mathrm{C}_G(K_R)) = \mathrm{N}_{\mathrm{Sym}(K)}(\mathrm{C}_G(K_R)) \cap (K_R \rtimes \mathrm{Aut}(K)) = K_R \rtimes \mathrm{N}_{\mathrm{Aut}(K)}(\mathrm{C}_G(K_R))$. Now $\tau \in \mathrm{N}_{\mathrm{Aut}(K)}(\mathrm{C}_G(K_R))$ if and only if $\lambda_{(x)\tau} = \tau^{-1}\lambda_x\tau \in \mathrm{C}_G(K_R)$ for all λ_x in $\mathrm{C}_G(K_R)$, or equivalently, $\rho_{(x)\tau} \in (K_R)_\sigma$ where $\rho_x \in (K_R)_\sigma$. So $\mathrm{N}_{\mathrm{Aut}(K)}(\mathrm{C}_G(K_R)) = \mathrm{Aut}(K)_{(K_R)_\sigma}$ in the natural action of $\mathrm{Aut}(K)$ on K_R. Let $(a_1,\ldots,a_k)\pi \in \mathrm{Aut}(K) = \mathrm{Aut}(T) \,\mathrm{wr}\, S_k$. So $(a_1,\ldots,a_k)\pi$ fixes $(K_R)_\sigma$ setwise if and only if $\pi \in (S_k)_\Sigma$ and $a_{(i-1)r+1} = \cdots = a_{ir}$ for each $i \in \{1,\ldots,k\}$. Therefore $\mathrm{Aut}(K)_{(K_R)_\sigma} = A_0 \times (S_k)_\Sigma$ where $A_0 = A_1 \times \cdots \times A_k$ and each A_i is a straight diagonal subgroup of $\mathrm{Aut}(T)^r$. Hence $\mathrm{N}_{\mathrm{Hol}(K)}(\mathrm{C}_G(K_R)) = K_R \rtimes [(A \times S_{k/m}) \,\mathrm{wr}\, S_m]$ where A is a full diagonal subgroup of $\mathrm{Aut}(T)^r$. By Lemma 10.1, $\mathrm{Graph}(\varphi) = \mathrm{Inn}_{(K_R)_\sigma}(K)$ and so by a similar argument, $\mathrm{Graph}(\varphi) = B^m$ where B is a full diagonal subgroup of $\mathrm{Inn}(T)^r$. Thus G is of Diagonal Quotient type as described in Section 3. □

Finally in this section we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By the discussion in Section 8, we may assume that G is innately transitive with nonabelian plinth $K = T^k$ where $k \geq 2$, and K_α is not a subdirect subgroup of K. Thus $(G, K, G_\alpha) \in \mathcal{G}_{prod}$. By Lemma 9.4, we may assume further that K is regular. In this last case the various types of innately transitive groups are classified in Proposition 10.3. □

11. Concluding Remarks

We conclude this paper with some remarks.

(1) Construction 6.6 gives a general construction method for all finite innately transitive groups up to permutational isomorphism. We designed this construction so that each innately transitive group G determines a unique set of "input data" for this construction. In the case where G has an abelian plinth K, we chose the subgroup K_0 to be K and φ to be an automorphism of K. In this case, Construction 6.6 produced a group X with an unfaithful innately transitive action having kernel $Z \cong K_0$ such that the induced permutation group X/Z is permutationally isomorphic to G. We could equally well have chosen $K_0 = 1$, with φ the trivial map, for our input data and then Construction 6.6 would have produced a different group X with a faithful action permutationally isomorphic to G. We decided on the former choice since for it we have $\mathrm{C}_{X/Z}(KZ/Z) = KZ/Z$ so that Proposition 6.7 holds for all cases.

(2) We would like to draw attention to the key role played by $\mathrm{Graph}(\varphi)$ in the structure of innately transitive groups with a regular plinth, and in particular, the result of Proposition 10.1(3). For the description of *Twisted Wreath type* of primitive groups given in [12], $G = KP$ where $K = T_1 \times \cdots \times T_k = T^k$ and $P = G_\alpha$. The condition for these primitive groups that P acts faithfully as a group of permutations of $\{T_1, \ldots, T_k\}$, is equivalent to the condition $\mathrm{Core}_{G_\alpha}((\mathrm{Inn}(T_R))\phi^{-1}) = 1$. In the Twisted Wreath type for quasiprimitive groups in [14], the requirement that $\mathrm{Core}_{G_\alpha}((\mathrm{Inn}(T_R))\phi^{-1}) = 1$ was explicitly stated in this form. In the more general setting of innately transitive groups we can replace the condition "$\mathrm{Core}_{G_\alpha}((\mathrm{Inn}(T_R))\phi^{-1}) = 1$" with the natural and equivalent requirement that $\mathrm{Graph}(\varphi)$ be trivial. This form of the condition is helpful in differentiating Twisted Wreath type from other types of innately transitive groups which involve $\mathrm{Graph}(\varphi)$ in their descriptions.

(3) As mentioned in the introduction to this paper, there was until now, no general construction method that produced all quasiprimitive groups of Product Action type. By specialising Construction 9.6 to innate triples $(K, \widetilde{\varphi}, \widetilde{L})$ with $\mathrm{Im}\,\widetilde{\varphi} = 1$ and choosing $M = K_0$ (see part (3) of Construction 9.6) so that the constructed groups are quasiprimitive, we obtain a general construction for these groups.

Proposition 11.1. *Let G be a quasiprimitive group on a finite set Ω with plinth K, let $\alpha \in \Omega$, and let (K, φ, L) be the innate triple associated to (G, K, G_α). Then G is of Product Type if and only if there exists $(\tilde{K}, \widetilde{\varphi}, \widetilde{L}) \in (\mathcal{D}_{prod})Q$ such that (K, φ, L) is obtained by Construction 9.6 applied to $(\tilde{K}, \widetilde{\varphi}, \widetilde{L})$, with the parameters M and K_0 in part (3) of Construction 9.6, equal and nontrivial.*

Proof. Let G, K, Ω, and α be as given in the statement. Suppose first that G is quasiprimitive of Product Type. Then by Theorem 4.1 and the description of Product Type in Section 3, K is nonabelian and nonsimple, $\mathrm{Ker}\,\varphi = K_\alpha$ is nontrivial and not a subdirect subgroup of K, and $\mathrm{Im}\,\varphi = \mathrm{C}_G(K) = 1$. So $(K,\varphi,L) \in \mathcal{D}_{prod}$ and by Corollary 9.9, there exists $(\tilde{K},\widetilde{\varphi},\widetilde{L})$ such that (K,φ,L) is equivalent to an innate triple obtained by Construction 9.6 applied to $(\tilde{K},\widetilde{\varphi},\widetilde{L})$. Now the parameters M and K_0 in part (3) of Construction 9.6 are precisely the kernel and domain of φ. Since G is quasiprimitive, by Proposition 5.3, either $M = K_0$, or $M = 1$ and $K_0 = K$. The latter is impossible as $\mathrm{Im}\,\varphi = 1$. Therefore M and K_0 are equal, and since G is of Product Type, M is nontrivial.

Conversely suppose there exists $(\tilde{K},\widetilde{\varphi},\widetilde{L}) \in (\mathcal{D}_{prod})Q$ such that (K,φ,L) is obtained by Construction 9.6 applied to $(\tilde{K},\widetilde{\varphi},\widetilde{L})$, with the parameters M and K_0 in part (3) of Construction 9.6, equal and nontrivial. By Corollary 7.6, (K,φ,L) is quasiprimitive and hence G is quasiprimitive on Ω. Since $(K,\varphi,L) \in \mathcal{D}_{prod}$ and $\mathrm{Ker}\,\varphi = M \neq 1$, by definition we must have $(G,K,G_\alpha) \in \mathcal{G}_{prod}$ with $K_\alpha \neq 1$. Therefore G is of Product Type. □

(4) In many investigations involving a transitive permutation group G, the *normal quotient* actions of G, play an important role (see for example [15]). These are the transitive actions of G on the orbit sets of intransitive normal subgroups. In the case where G is innately transitive and imprimitive with nonabelian plinth K, the centraliser $\mathrm{C}_G(K)$ is a maximal intransitive normal subgroup of G. If Σ denotes the set of $\mathrm{C}_G(K)$-orbits then the corresponding normal quotient $G^\Sigma \cong G/\mathrm{C}_G(K)$ has a unique minimal normal subgroup, namely $K^\Sigma \cong K$, and hence G^Σ is quasiprimitive. If $\mathrm{C}_G(K) = 1$ then G is quasiprimitive on Ω and of course is permutationally isomorphic to G^Σ on Σ. Table 1 lists the types of innately transitive groups G that are not quasiprimitive (so $1 < |\mathrm{C}_G(K)| < |K|$) together with the types of the corresponding quasiprimitive normal quotient.

Type of G	Type of G^Σ
Almost Simple Quotient	Almost Simple
Diagonal Quotient	Diagonal
Product Quotient	Product Action
Product Action	Product Action

TABLE 1. The quasiprimitive type of the group G^Σ corresponding to the innately transitive type of G.

References

[1] M. Aschbacher. *Finite group theory.* Cambridge University Press, Cambridge, second edition, 2000.

[2] Robert W. Baddeley and Cheryl E. Praeger. On primitive overgroups of quasiprimitive permutation groups. *to appear in J. Algebra*.

[3] Robert W. Baddeley, Cheryl E. Praeger, and Csaba Schneider. Recognizing cartesian decompositions for transitive permutation groups. *preprint*.

[4] Peter J. Cameron. *Permutation groups*. Cambridge University Press, Cambridge, 1999.

[5] Peter J. Cameron, Michael Giudici, Gareth A. Jones, William M. Kantor, Mikhail H. Klin, Dragan Marušič, and Lewis A. Nowitz. Transitive permutation groups without semiregular subgroups. *J. London Math. Soc. (2)*, 66(2):325–333, 2002.

[6] Alan R. Camina and Cheryl E. Praeger. Line-transitive, point quasiprimitive automorphism groups of finite linear spaces are affine or almost simple. *Aequationes Math.*, 61(3):221–232, 2001.

[7] Marston D. Conder, Cai Heng Li, and Cheryl E. Praeger. On the Weiss conjecture for finite locally primitive graphs. *Proc. Edinburgh Math. Soc. (2)*, 43(1):129–138, 2000.

[8] John D. Dixon and Brian Mortimer. *Permutation groups*. Springer-Verlag, New York, 1996.

[9] M. Giudici. Quasiprimitive permutation groups with no fixed point free elements of prime order. *to appear J. London Math. Soc.*

[10] Peter Kleidman and Martin Liebeck. *The subgroup structure of the finite classical groups*. Cambridge University Press, Cambridge, 1990.

[11] L. G. Kovács. Primitive subgroups of wreath products in product action. *Proc. London Math. Soc. (3)*, 58(2):306–322, 1989.

[12] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl. On the O'Nan-Scott theorem for finite primitive permutation groups. *J. Austral. Math. Soc. Ser. A*, 44(3):389–396, 1988.

[13] Cheryl E. Praeger. Finite primitive permutation groups: a survey. In *Groups—Canberra 1989*, pages 63–84. Springer, Berlin, 1990.

[14] Cheryl E. Praeger. An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs. *J. London Math. Soc. (2)*, 47(2):227–239, 1993.

[15] Cheryl E. Praeger, Cai Heng Li, and Alice C. Niemeyer. Finite transitive permutation groups and finite vertex-transitive graphs. In *Graph symmetry (Montreal, PQ, 1996)*, pages 277–318. Kluwer Acad. Publ., Dordrecht, 1997.

[16] Leonard L. Scott. Representations in characteristic p. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, pages 319–331. Amer. Math. Soc., Providence, R.I., 1980.

[17] Helmut Wielandt. *Finite permutation groups*. Academic Press, New York, 1964.

School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009

E-mail address: `john.bam@maths.uwa.edu.au, praeger@maths.uwa.edu.au`