NON-FREE POINTS FOR GROUPS GENERATED BY A PAIR OF 2×2 MATRICES.

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Abstract. A point λ in the complex plane is said to be *free* if the group generated by the matrices $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\\lambda&1\end{smallmatrix}\right)$ is free. In this paper we give an infinite family of polynomials whose roots are the non-free points. The main idea in this paper is to employ a symmetry relation.

Introduction

A point λ in the complex plane is said to be *free* if the group generated by the matrices $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\\lambda&1\end{smallmatrix}\right)$ is free. Many papers have been written on this topic (see 'References' and references therein). It is a classical result that λ is free if $|\lambda| \geq 2$ ([19, 3]). In fact, λ is free in each of the following cases:

(1) $\lambda \in \mathbb{C}$ lies outside the unit discs centered at -1, 0 and 1 ([5]).
(2) $\lambda \in \mathbb{C}$ lies outside the open discs of radii $\frac{1}{2}$ centered at $i/2$ and $-i/2$, and outside the unit open discs centred at -1 and 1 ([16]).
(3) $\lambda \in \mathbb{C}$ lies outside the convex hull containing the unit circle at the origin and the points ± 2 ([16]).
(4) $\lambda \in \mathbb{C}$ satisfies $|\lambda - 1| > \frac{1}{2}$ and $1 \leq |Re(\lambda)| < \frac{5}{4}$ ([8]).
(5) λ lies outside the unit circle and $|Im\,\lambda| \geq \frac{1}{2}$ ([9]).

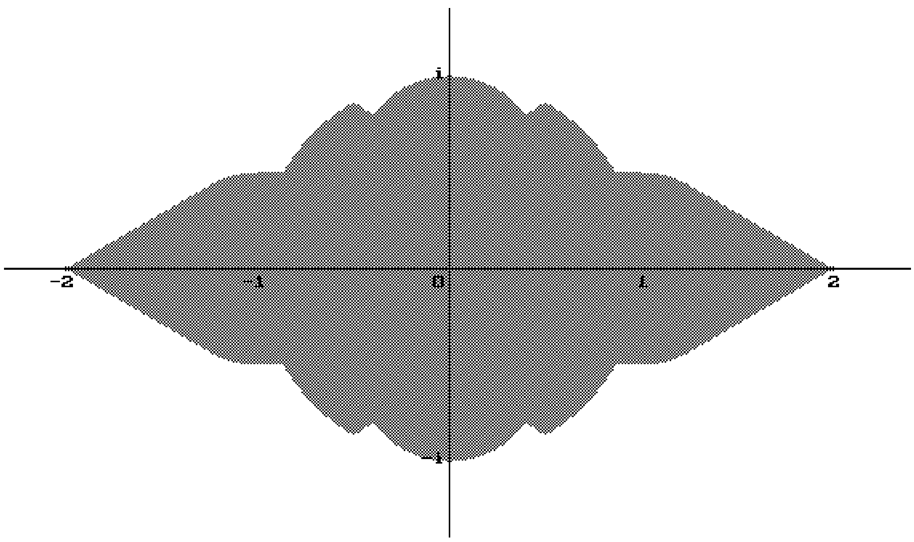

Figure 1. Known free points in the complex plane (unshaded)

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Figure 1 summarizes these results. It is well known that if z is free then $\bar{z}$ and $-z$ are free. It is also known that if ρ is a point in the "Riley Slice of Schottky Space" then $\frac{1}{2}\rho$ is free ([13]). A computer generated picture of the 'Riley Slice' is given in [13]. It is somewhat similar to figure 1 except larger with a fractal boundary. Inside the 'eye', it is also known that free points are abundant. For example, the transcendental points are all free ([7]), algebraic free points are dense in the complex plane ([5]), and points with a free algebraic conjugate are free ([16]). Also, if λ is not free, then neither is λ/n for all $n \in \mathbb{Z}\backslash\{0\}$ ([16]).

In the study of non-free points, there have also been numerous results. Many of these results give domains for which non-free points are densely distributed; for example, Rimhak Ree showed (see [18]) that the segments $(-2, 2)$ and $(-i, i)$ on the real and imaginary axes respectively, reside in open sets in which non-free points are densely distributed. Evans proved ([6]) Newman's conjecture ([17]) (which was also independently proved in [15]) that if μ is a root of unity, then $\frac{1}{2}\mu$ is a non-free point. According to [15], the closure of the set of non-free points is connected. The introduction of Alan Beardon's paper ([2]) has a brief summary of the literature on real non-free points. It seems that the set of all known explicit non-free points is very small. In particular, there are few known rational non-free points greater than 1. In Ignatov's papers [10, 11], it is stated that the rationals $\frac{1}{2}(1/n)^2, \frac{1}{2}(2/n)^2, \frac{1}{2}(3/n)^2, \ldots, \frac{1}{2}(8/n)^2$ are non-free for all non-zero integers n, and that numbers of the form $\frac{(m+n)^2}{2m^2n^2}$ are non-free points.

Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$. Generally, when studying non-free points, one considers the word $W = A^{h_1}B^{h_2}A^{h_3}\ldots A^{h_{2k-1}}B^{h_{2k}}$. From a computational point of view, the difficulty in obtaining non-free points, is that the relation $W = I$ is a system of four simultaneous polynomials in λ of degree k. Moreover if $W = I$, then necessarily $k \geq 3$ ([4]). In this present note, we consider words of the form $W = A^{h_1}B^{h_2}\ldots B^{h_{2n}}A^{h_{2n+1}}$ with symmetry relation $CWC^{-1} = W^{-1}$, where $C = \left(\begin{smallmatrix} 0 & 2 \\ \lambda & 0 \end{smallmatrix}\right)$ (in dynamics, this is commonly known as "time-reversal" symmetry, see [14, 1]). The matrix C has the property that it simultaneously conjugates A to B and B to A. So the relation $CWC^{-1} = W^{-1}$ is indeed a relation between the matrices A and B. This is surprisingly well adapted to computations.

Results

Below, we now define the polynomials $B_n(\lambda)$ central to the statement of the main theorem.

Definition 1. *Let $h_1, \ldots, h_{2n+1} \in \mathbb{Z}\backslash\{0\}$. Define the following recurrence relation for all $n \in \mathbb{N}$:*

$$\begin{aligned} a_n &= a_{n-1} + \lambda h_{2n} b_{n-1} \\ b_n &= 2h_{2n+1}a_{n-1} + (1 + 2\lambda h_{2n}h_{2n+1})b_{n-1} \\ c_n &= c_{n-1} + \lambda h_{2n} d_{n-1} \\ d_n &= 2h_{2n+1}c_{n-1} + (1 + 2\lambda h_{2n}h_{2n+1})d_{n-1} \end{aligned}$$

where $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2h_1 \\ 0 & 1 \end{smallmatrix}\right)$. Then B_n is the nth degree polynomial defined by:

$$B_n(\lambda) = \tfrac{1}{2}b_n + \tfrac{1}{\lambda}c_n \qquad \text{for all } \lambda \in \mathbb{C}.$$

Here are the first two polynomials, B_1 and B_2:

- $B_1(\lambda) = 2h_1h_2h_3\lambda + h_1 + h_2 + h_3$
- $B_2(\lambda) = 4h_1h_2h_3h_4h_5\lambda^2 + 2(h_1h_2h_3 + h_2h_3h_4 + h_3h_4h_5 + h_4h_5h_1 + h_5h_1h_2)\lambda + h_1 + h_2 + h_3 + h_4 + h_5$

The polynomials $B_n(\lambda)$ are essentially the continuant polynomials of Euler that arise naturally in the context of matrix products such as the above (see for example [12, Section 6.4]). It turns out that the polynomials $B_n(\lambda)$ have a high level of symmetry.

Definition 2. *Let $P_c(k)$ denote the set of all strings with $k - 2c$ variables obtained by taking out c distinct pairs of adjacent variables from the string $\prod_{i=1}^{k} h_i$.*

Here "adjacent" is understood in the cyclic sense. For example, $P_1(5) = \{h_1h_2h_3, h_2h_3h_4, h_3h_4h_5, h_4h_5h_1, h_5h_1h_2\}$. It is easy to prove the following result by induction. We will not require it in what follows.

Proposition 1. *For all $n \in \mathbb{N}$ and $h_1, \ldots, h_{2n+1} \in \mathbb{Z}\backslash\{0\}$,*

$$B_n(\lambda) = \sum_{r=0}^{n} \left[(2\lambda)^r \sum P_{n-r}(2n+1)\right]$$

The main result of this paper is

Theorem 1. *If $\lambda \neq 0$ is a root of B_n for some $n \in \mathbb{N}$ and $h_1, \ldots, h_{2n+1} \in \mathbb{Z}\backslash\{0\}$, then λ is non-free. Conversely, every non-free point is a root of some B_n.*

Proof. Suppose $B_n(\lambda) = 0$. Let $W_n = A^{h_1}B^{h_2} \ldots B^{h_{2n}}A^{h_{2n+1}}$. The recurrence relation has been chosen such that $W_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. One finds by direct calculation that

$$CW_nC^{-1} = \begin{pmatrix} d_n & \frac{2}{\lambda}c_n \\ \frac{\lambda}{2}b_n & a_n \end{pmatrix} \text{ and } W_n^{-1} = \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix}.$$

By hypothesis it follows that $CW_nC^{-1} = W_n^{-1}$. Conversely, if λ is non-free, then $W = I$ for some word $W = A^{h_1}B^{h_2}A^{h_3} \ldots A^{h_{2n+1}}$. So $CWC^{-1} = W^{-1}$ and it follows that λ is a root of the corresponding B_n. □

The above result is similar to that of Lyndon and Ullman ([16, Proposition 1]) and Ignatov ([9, Theorem 4]) except that it is better adapted to computations.

The following corollaries to Theorem 1 use $B_1(\lambda)$.

Corollary 1. *If $\lambda = \frac{1}{2} + \frac{1}{h}$ or $\lambda = \frac{1}{2} - \frac{1}{h}$ for some $h \in \mathbb{Z}\backslash\{0\}$, then λ is non-free.*

Proof. Use $B_1(\lambda)$ by assigning $h_1 = h$, $h_2 = \pm 1$, and $h_3 = \pm 1$, and note that λ is non-free if $-\lambda$ is non-free. □

To the author's knowledge, the above corollary was not previously known.

Corollary 2. *The points $\lambda = \frac{1}{2h}$ for all $h \in \mathbb{Z}\backslash\{0\}$ are limit points of the set of non-free points.*

Proof. Using $B_1(\lambda)$ with $h_3 = 1$, we have that for all $h_1, h_2 \in \mathbb{Z}\backslash\{0\}$, $-\lambda = \frac{1}{2h_2} + \frac{1}{2h_1} + \frac{1}{2h_1h_2}$ is a non-free point and hence $\frac{1}{2h_2} + \frac{1}{2h_1} + \frac{1}{2h_1h_2}$ is also a non-free point. The result follows by letting h_1 tend to infinity. □

From a result of Ignatov (see [10, 11]), one can deduce that numbers of the form $\frac{1}{2n^2}$ are accumulation points of non-free points, and from a result of Beardon ([2]), that $\lambda = \frac{1}{2N}$, where $\sqrt{N}$ is irrational, is also an accumulation point of non-free points.

Corollary 3. *$\frac{1}{2}$, 1, and $\frac{3}{2}$ are non-free.*

Proof. One chooses (h_1, h_2, h_3) to be $(1, 2, 3)$, $(1, 2, 1)$, and $(1, 1, 1)$ respectively. $\square$

It is well known that 1 and $\frac{1}{2}$ are non-free (see [16, Proposition 2], $\mu = \sqrt{2}$ and $\mu = 1$), but it is well known (see [2, Section 2]) that for λ equal to $\frac{1}{2}$, 1, or $\frac{3}{2}$, that the element AB^{-1} has order 6, 4, or 3 respectively.

The following corollaries to Theorem 1 use $B_2(\lambda)$.

Corollary 4. *$\frac{1}{2}e^{\pi i/3}$, $\frac{1+\sqrt{13}}{4}$, $\frac{5+\sqrt{5}}{4}$, $\frac{i}{2}$, and $\frac{1}{\sqrt{2}}$ are non-free.*

Proof. Choose $(h_1, h_2, h_3, h_4, h_5)$ to be $(-1, -1, -1, 1, 1)$, $(-1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1)$, $(1, 2, 1, -1, -1)$, and $(1, -1, 1, 1, 2)$ respectively. $\square$

The fact that $\lambda = e^{\pi i/3}/2$ and $\lambda = i/2$ are non-free points follows from the main result in Evans (see [6], both points correspond to μ being a root of unity). The point $1/\sqrt{2}$ was also previously known to be non-free (see [16, Proposition 2], $\mu = \sqrt[4]{2}$). As far as we are aware, $\frac{1+\sqrt{13}}{4}$ and $\frac{5+\sqrt{5}}{4}$ were not previously known to be non-free points.

In addition, $B_2(\lambda)$ enables one to recover the results of Beardon [2, section 6].

Corollary 5. *$\frac{9}{50}$, $\frac{8}{25}$, $\frac{25}{72}$, $\frac{9}{98}$, $\frac{8}{81}$, $\frac{25}{162}$, and $\frac{25}{98}$ are non-free.*

Proof. Choose $(h_1, h_2, h_3, h_4, h_5)$ to be $(-1, 5, 5, 4, 5)$, $(1, -1, -3, -47, 50)$, $(1, -3, 1, -8, 9)$, $(-2, -1, 7, 7, 7)$, $(-2, 5, -3, -9, 9)$, $(-4, -2, -3, -9, 18)$, and $(-2, -1, -49, 3, 49)$ respectively. $\square$

Similarly, one can recover Lyndon and Ullman's simple result (see [16, Proposition 6]) using various B_n.

Corollary 6. *$\frac{9}{8}$, $\frac{8}{9}$, $\frac{25}{32}$, and $\frac{25}{18}$ are non-free.*

Proof. Choose $(h_1, h_2, \ldots, h_5) = (1, -1, 1, -2, 10)$ for B_2, $(h_1, h_2, \ldots, h_7) = (-1, 3, -1, 1, -2, -3, 3)$ for B_3, $(h_1, h_2, \ldots, h_5) = (-1, 1, -2, -6, 8)$ for B_2, and $(h_1, h_2, \ldots, h_{11}) = (-1, 1, -1, 1, -3, -1, 1, -3, 3, -1, 4)$ for B_5 respectively. $\square$

Similarly, one can obtain points on the arc considered by Shkurat-skii [20] in the same way. For example using B_4 with $(h_1, h_2, \ldots, h_9) = (1, -1, 1, 1, -1, 1, 1, -1, 1)$.

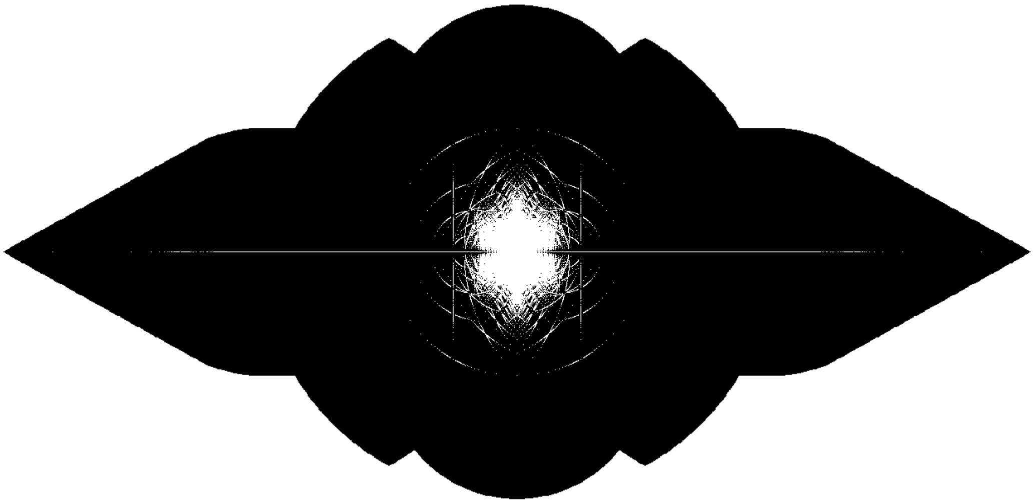

FIGURE 2. Non-free points: roots of $B_2(\lambda)$.

Figure 2 is a diagram of some of the roots of $B_2(\lambda)$ with $h_1, h_2, \ldots, h_5 \in [-60, 60] \cap \mathbb{Z}\backslash\{0\}$. These diagrams were generated by a computer program written by the author in 'C++'. Figure 3 is a diagram of some of the roots of $B_3(\lambda)$ with $h_1, h_2, \ldots, h_7 \in [-10, 10] \cap \mathbb{Z}\backslash\{0\}$. Figure 4 is a diagram of some of the roots of $B_4(\lambda)$ with $h_1, h_2, \ldots, h_9 \in [-5, 5] \cap \mathbb{Z}\backslash\{0\}$.

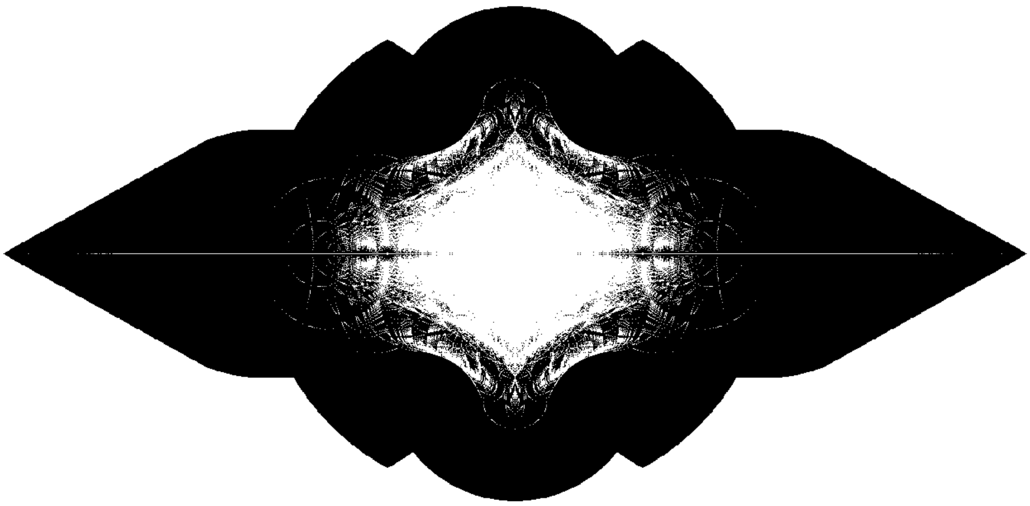

FIGURE 3. Non-free points: roots of $B_3(\lambda)$.

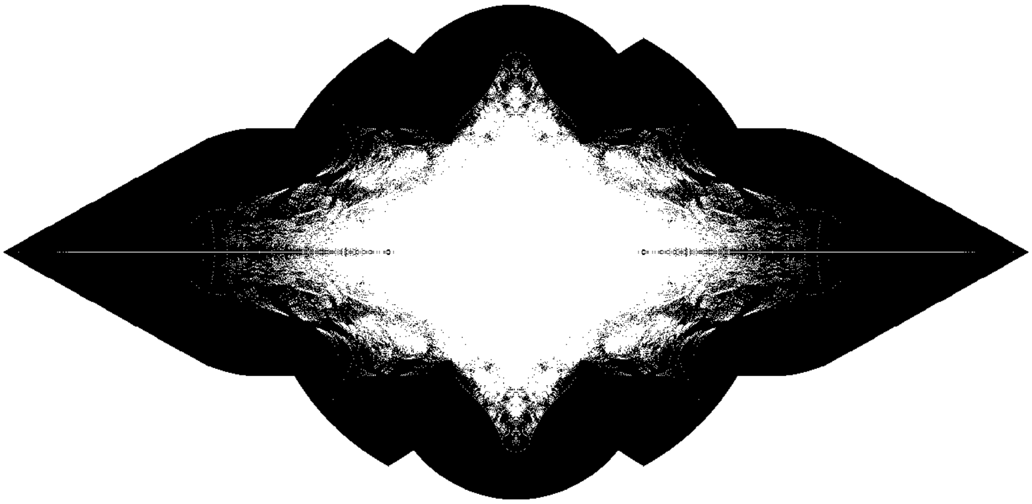

FIGURE 4. Non-free points: roots of $B_4(\lambda)$.

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