Symplectic Spreads

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Abstract

We construct an infinite family of symplectic spreads in spaces of odd rank and characteristic.

1 Introduction

This paper uses a technique commonly known as "net replacement" to construct new spreads in the finite symplectic polar space of odd rank and characteristic. First we give an overview of some definitions and theory of finite geometry, together with some results of the past which provide the context and background for our construction.

The definitions of a projective plane and an affine plane can be found in most standard texts on geometry such as [7]. In a projective or affine plane, a point P is a *centre* for a collineation φ if φ fixes every line incident with P. A line l is an *axis* of φ if φ fixes every point on l. It is standard knowledge that every non-identity collineation has at most one axis and at most one centre, and it has an axis if and only if it has a centre (see [7, Section 3.1.4]). A collineation which has a centre and axis which are incident with one another, is called an *elation*. A group of collineations H is called (P, l)*-transitive* if the subgroup of H consisting of those elements which have centre P and axis l, acts transitively on the non-fixed points of any line through P which is not equal to l. For two lines m and l, we say that H is (m, l)*-transitive* if H is (P, l)-transitive for all P on m. Dually, if P and Q are points, then we say that H is (P, Q)*-transitive* if H is (P, l)-transitive for every line l incident with Q. Let Γ be a projective plane and suppose that Δ is an affine plane obtained by removing the line l_∞ from Γ. Then Δ is a *translation plane* if there exists an (l_∞, l_∞)-transitive group of elations of Δ. We call l_∞ the *translation line* of Δ. The dual of a translation plane is a *shears plane*, the corresponding point being a *shears point*.

We shall identify Desarguesian projective spaces and the system of subspaces of the underlying vector space. A t*-spread* of a projective space is a collection of t-dimensional subspaces such that every point is contained in exactly one subspace. So a spread provides a partition of the points of the projective space. A *partial* t*-spread* is a collection of pairwise disjoint t-dimensional subspaces. Given a spread S, there is an associated translation plane $\pi(S)$ derived from the Andre/Bruck-Bose construction (see [6] and [2]). We call a collection $\mathcal{C}$ of $(t+1) \times (t+1)$ matrices over $\mathrm{GF}(q)$ a t*-spread set* if it satisfies the following conditions: (1) $|\mathcal{C}| = q^{t+1}$; (2) $\mathcal{C}$ contains the zero matrix; (3) if A and B are distinct matrices in $\mathcal{C}$, then $A - B$ is invertible. Every t-spread of $PG(2t+1, q)$ can be represented by a t-spread set (see [6]), as $\mathcal{S}(\mathcal{C}) = \{\{(X, XA) \in \mathrm{GF}(q^{t+1}) \oplus \mathrm{GF}(q^{t+1}) | X \in \mathrm{GF}(q^{t+1})\} : A \in \mathcal{C}\} \cup Y$ (where $Y = \{(0, y) : y \in \mathrm{GF}(q^{t+1})\}$) is a spread if and only if $\mathcal{C}$ is a spread set.

Let $\mathcal{C}$ be a spread set and let $\pi = \pi(\mathcal{S}(\mathcal{C}))$. Then π is a dual translation plane with shears point Y, if and only if $\mathcal{C}$ is closed under addition. In the finite case, the following are equivalent (see [7, Section 3.1]): (1) π has at least two translation lines; (2) every line of π is a translation line; (3) π is Desarguesian; (4) π is isomorphic to $PG(2, q^{t+1})$ for some q; (5) $\mathcal{C}$ is closed under addition and multiplication.

As defined in [7, Section 5.3], a *semifield* is a finite set $\mathcal{S}$, together with addition and multiplication such that $\mathcal{S}$ is a group with respect to addition, multiplication distributes over addition, there are no zero divisors in $\mathcal{S}$, and there exists a multiplicative identity element in $\mathcal{S}$.

R. H. Bruck in 1951 [4] introduced finite nets. A *net* is a system of points and lines satisfying: (N1) every two points lie on at most one line; (N2) every point lies on at least two distinct lines; (N3) Playfair's Axiom (for any non-incident point-line pair p, L, there exists a unique line through p which has no point in common with L;). Note that the definition of a net is the natural weakening of the axioms of an affine plane. Parallelism is an equivalence relation, and in the finite case, the number of parallel classes k is called the *degree* of the net, and the common number n of points on each line is called the *order* of the net. A net is thus equivalent to $k-2$ mutually orthogonal $n \times n$ Latin squares, as shown by Bruck in 1951. See also Bruck's 1963 paper [5] for more on nets. There is an analogue here with the Andre/Bruck-Bose construction – given a partial spread $\mathcal{S}$, one can construct a net $\nu(\mathcal{S})$, but this time the converse fails.

T. G. Ostrom [12] described a general method called net replacement for constructing new affine planes from old ones. A net is *replaceable* if there is another net with the same collinearity graph. Similarly, we say that a partial spread $\mathcal{S}$ is *replaceable* if $\nu(\mathcal{S})$ is a replaceable net. Given an affine plane containing a replaceable net, replacing the net gives another affine plane. Given a translation plane $\pi(S)$, net replacement amounts to replacing some subset U of the spread S by another partial spread V covering the same points as U. The resulting spread $S' = (S \backslash U) \cup V$ then determines a new translation plane $\pi(S')$.

The symplectic polar space $\mathrm{Sp}(2n, q)$ of rank n is the space of totally isotropic subspaces of $\mathrm{PG}(2n-1, q)$ with respect to a null polarity, and has automorphism group $\mathrm{P\Gamma Sp}(2n, q)$, for $n > 1$. Ovoids and spreads of polar spaces were introduced in full generality in Thas's seminal work [15]. An *ovoid* is a set of points meeting each maximal singular subspace in a point, and a *spread* is a set of maximal singular subspaces partitioning the points of the polar space. A spread of $\mathrm{Sp}(2n, q)$ is a spread of $\mathrm{PG}(2n-1, q)$, but not conversely. Surveys corresponding to this topic appear in [15, 16, 17, 10].

T′ep [14] in 1991, observed that the plane of Hering (1969, [9]) arises from a spread of $\mathrm{Sp}(6, 3)$. Bader, Kantor, and Lunardon (1994, [3]) give a construction for odd rank from some of Albert's twisted fields (see Albert's paper 1958/1959 [1]). Kantor [10] generalises this construction to all commutative semifields. However, in odd rank and characteristic, all previously known symplectic spreads arise from semifields, apart from Hering's spread in $\mathrm{Sp}(6, 3)$ (see [10, §6, Remark 3]).

Dempwolff [8] in 1994 classified translation planes of order 27, and so spreads of $\mathrm{PG}(5, 3)$, from which it can be deduced that every spread of $\mathrm{Sp}(6, 3)$ is regular. Hence any non-semifield spread of $\mathrm{Sp}(6, 3)$ is Hering's spread. Here we construct non-semifield spreads of symplectic spaces for odd rank and characteristic, by replacing a net in the Bader-Kantor-Lunardon-Albert spreads. For rank 3 and the field of order 3, we obtain Hering's spread, thus generalising Hering's plane of order 27, long thought to be sporadic, to an infinite family of translation planes. Here work of Suetake [13] is relevant. It appears that, the planes we obtain for rank 3 are isomorphic to those of Suetake.

Kantor [11] defined a construction of a strongly regular graph from a spread of a polar space. Given a spread Σ, the vertices of the graph $G(\Sigma)$ are the hyperplanes of the elements of the spread, and two such hyperplanes X and Y are adjacent if and only if X has a nontrivial intersection with the polar space of Y. Although non-isomorphic spreads may produce isomorphic strongly regular graphs, we do not know whether or not our family yields any new strongly regular graphs.

2 Construction

Let q be a power of an odd prime p, let t be a non-negative integer, and consider $V = \mathrm{GF}(q^{t+1})^2$ as a vector space over $\mathrm{GF}(q)$. Let $T : \mathrm{GF}(q^{t+1}) \to \mathrm{GF}(q)$ denote the *relative trace map* $x \mapsto x + x^q + \cdots + x^{q^t}$ and consider the nondegenerate $\mathrm{GF}(q)$-alternating form on V defined by

$$\langle v, w \rangle = T(v_1 w_2 - v_2 w_1)$$

for all $v = (v_1, v_2), w = (w_1, w_2) \in V$.

The regular spread in this model consists of the following subspaces of V:

$$X = \{(x, 0) : x \in \mathrm{GF}(q^{t+1})\}$$

$$Y = \{(0, y) : y \in \mathrm{GF}(q^{t+1})\}$$

$$\{(x, mx) : x \in \mathrm{GF}(q^{t+1})\}, \quad (m \in \mathrm{GF}(q^{t+1})^*).$$

Let ρ be a nontrivial automorphism of $\mathrm{GF}(q^{t+1})$ such that $-1 \notin \mathrm{GF}(q^{t+1})^{\rho-1}$ (so necessarily, t is even). Then the spread $\mathcal{B}$ of Bader-Kantor-Lunardon-Albert (BKLA) consists of the following subspaces of V:

$$Y$$

$$\{(x, mx^{\rho^{-1}} + m^{\rho}x^{\rho}) : x \in \mathrm{GF}(q^{t+1})\}, \quad (m \in \mathrm{GF}(q^{t+1})).$$

Now $\mathcal{B}$ is symplectic with respect to the aforementioned alternating form. For all $s \in \mathrm{GF}(q^{t+1})$, the map $\phi_s : V \to V$ defined by $\phi_s(v, w) = (sv, s^{-1}w)$ is an isometry of V with respect to $\langle , \rangle$. Denote the group of all ϕ_s by G. The involution τ of V, which switches coordinates $(v, w) \mapsto (w, v)$, is an isometry of V and $\tau\phi_s\tau^{-1} = \phi_{s^{-1}} = \phi_s^{-1}$ for all s. So $\mathcal{B}$ is G-invariant. Consider the G-orbit

$$\mathcal{N} = \{\phi_s(W) : s \in \mathrm{GF}(q^{t+1})^*\}$$

where $W = \{(x, x^{\rho} + x^{\rho^{-1}}) : x \in \mathrm{GF}(q^{t+1})^*\}$. Note that $\mathcal{N}$ and $\tau(\mathcal{N})$ are G-invariant. We have the following important lemma which allows us to create a new symplectic spread from the BKLA spread.

Lemma 2.1 *The collection $\mathcal{N}$ is a replaceable partial spread of the BKLA canonical spread $\mathcal{B}$ with replacement $\mathcal{N}' = \tau(\mathcal{N})$.*

Proof. We need to show that $\mathcal{N}'$ covers the same points as $\mathcal{N}$. Let $D = \{x(x^{\rho^{-1}} + x^{\rho}) : x \in \mathrm{GF}(q^{t+1})^*\}$ and let $U = \{(x, y) \in V : xy \in D\}$. Note that $\cup\mathcal{N} \subseteq U$ and $\cup\mathcal{N}' \subseteq U$. Now $|D|$ is at most $(q^{t+1} - 1)/2$ (as $x \mapsto x(x^{\rho^{-1}} + x^{\rho})$ is at least a two-to-one map) and $|\cup\mathcal{N}| = (q^{t+1} - 1)^2/2 = |\cup\mathcal{N}'|$. By definition of U, and since $|D| \leq (q^{t+1} - 1)/2$, we have that $|U| \leq (q^{t+1} - 1)^2/2$. Hence $\mathcal{N} = U = \mathcal{N}'$. ∎

Theorem 2.2 *The collection $\mathcal{S} = (\mathcal{B} \cup \mathcal{N}')\backslash\mathcal{N}$ is a symplectic spread with respect to the form $\langle , \rangle$ and $\pi(\mathcal{S})$ is not the dual of a translation plane. Hence $\mathcal{S}$ is new for $t > 2$, and $\pi(\mathcal{S})$ is Hering's plane for $(t, q) = (2, 3)$.*

Proof. Note that $\mathcal{N}$ is a replaceable partial spread for the BKLA spread, with $\nu(\mathcal{N})$ contained in $\tau(\mathcal{B})$, and so it follows from our discussion in the introduction that $\mathcal{S}$ is a spread. Moreover, it is symplectic as τ is an isometry.

Let $\mathcal{M} = \mathcal{B}\backslash(\{X, Y\} \cup \mathcal{N})$. First observe that $\mathcal{S} = (\tau(\mathcal{B})\backslash\tau(\mathcal{M})) \cup \mathcal{M}$, so that $\mathcal{S}$ may be constructed from $\tau(\mathcal{B})$ by net replacement. Thus there are two spreads, giving rise to shears planes, with substantial intersection with $\mathcal{S}$ and the intersection includes the shears point in both cases. This is sufficient to show that neither X nor Y is a shears point of $\pi(\mathcal{S})$ as follows:

> Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are spreads containing X and Y such that Y is a shears point of both $\pi(\mathcal{S}_1)$ and $\pi(\mathcal{S}_2)$. Then $|\mathcal{S}_1 \cap \mathcal{S}_2| = p^e + 1$ for some non-negative integer e, since the corresponding spread sets are closed under addition and so form subspaces over $\mathrm{GF}(p)$.

On the other hand, $|\mathcal{S} \cap \mathcal{B}| = |\mathcal{S} \cap \tau(\mathcal{B})| = \frac{q^{t+1}+3}{2}$, showing that neither X nor Y is a shears point of $\pi(\mathcal{S})$. Hence $\pi(\mathcal{S})$ is not a shears plane, for if $Z \in \mathcal{S}\backslash\{X, Y\}$ is a shears point of $\pi(\mathcal{S})$, then so is $\phi_s(Z)$ for all s, so $\pi(\mathcal{S})$ is Desarguesian, which contradicts X not being a shears point of $\pi(\mathcal{S})$. Thus if $t > 2$, then $\mathcal{S}$ is new, while if $(t, q) = (2, 3)$ it follows from Dempwolff's classification of the translation planes of order 27 that $\pi(\mathcal{S})$ is Hering's plane. ∎

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