# The Crystallographic Restriction, Permutations, and Goldbach's Conjecture

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**1. INTRODUCTION.** The object of this paper is to make an observation connecting Goldbach's conjecture, the crystallographic restriction, and the orders of the elements of the symmetric group. First recall that for an element $g$ of a group $G$ the *order* $\mathrm{Ord}(g)$ of $g$ is defined to be the smallest natural number such that $g^{\mathrm{Ord}(g)} = \mathrm{id}$ if such a number exists, and $\mathrm{Ord}(g) = \infty$ otherwise. In dimension $n$, the *crystallographic restriction* (CR) is the set $\mathrm{Ord}_n$ of finite orders realized by $n \times n$ integer matrices:

$$\mathrm{Ord}_n = \{m \in \mathbb{N} \mid \exists A \in GL(n, \mathbb{Z}) \text{ with } \mathrm{Ord}(A) = m\}.$$

Its name comes from the fact that it coincides with the set of possible orders of symmetries of lattices in dimension $n$, the connection being that for a given lattice there is an obvious choice of basis for which the symmetries are represented by integer matrices. In dimension two, one has the classic CR: $\mathrm{Ord}_2 = \{1, 2, 3, 4, 6\}$, which has been known since the crystallographic work of René-Just Haüy in 1822. (For an introduction to the mathematics of crystals, see [**31**] and [**26**]. For some of the history of the early crystallographic works, see [**23**] and the historical comments at the ends of chapters in [**3**] and [**6**].) It was also known to nineteenth century crystallographers that the same restriction applies in dimension three. For the CRs in higher dimensions, many authors refer to the founding work of Hermann [**7**]. In fact, the CR was already correctly described by Vaidyanathaswamy [**29**], [**30**] in 1928, and later rediscovered independently by many authors [**28**], [**11**], [**18**], [**8**], [**5**]. The CR has also been incorrectly presented in a number of places; e.g., in Schwarzenberger's book [**24**], and subsequently in such places as [**9**]. The mistake was corrected in [**17**]. Coincidentally, Schwarzenberger discusses the errors in the works of the early crystallographers in his entertaining article [**25**].

To describe the CR, define a function $\psi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ as follows. For an odd prime $p$ and $r = 1, 2, \dots$, set $\psi(p^r) = \phi(p^r)$, where $\phi$ is the Euler totient function: $\phi(p^r) = p^r - p^{r-1}$. Set $\psi(2^r) = \phi(2^r)$ for $r > 1$, $\psi(2) = 0$, and $\psi(1) = 0$. For each $i$ in $\mathbb{N}$, let $p_i$ denote the $i$th prime, and for $m$ in $\mathbb{N}$ with prime factorization $m = \prod_i p_i^{r_i}$ set

$$\psi(m) = \sum_i \psi(p_i^{r_i}),$$

which should be compared to the standard formula $\phi(m) = \prod_i \phi(p_i^{r_i})$. Then the CR in dimension $n$ is given by (see [**29**], [**30**]):

**Theorem 1.** $\mathrm{Ord}_n = \{m \in \mathbb{N} \mid \psi(m) \le n\}$.

For a proof of this theorem, and for more information about finite subgroups of $GL(n, \mathbb{Z})$, see the excellent introductory account in [**12**]. In particular, for the asymptotic behaviour of $\max(\mathrm{Ord}_n)$, see [**12**], [**5**], [**13**], and [**16**].

Notice that $\psi(m)$ is even for all $m$, so $\mathrm{Ord}_{2k+1} = \mathrm{Ord}_{2k}$ for all $k \geq 1$. Hence it suffices to consider $\mathrm{Ord}_n$ for even $n$. For $n$ even, Theorem 1 gives $\mathrm{Ord}_n \setminus \mathrm{Ord}_{n-1} = \psi^{-1}\{n\}$, but there is no known formula for $\psi^{-1}\{n\}$, as one might well appreciate by considering the graph of $\psi$ shown in Figure 1.

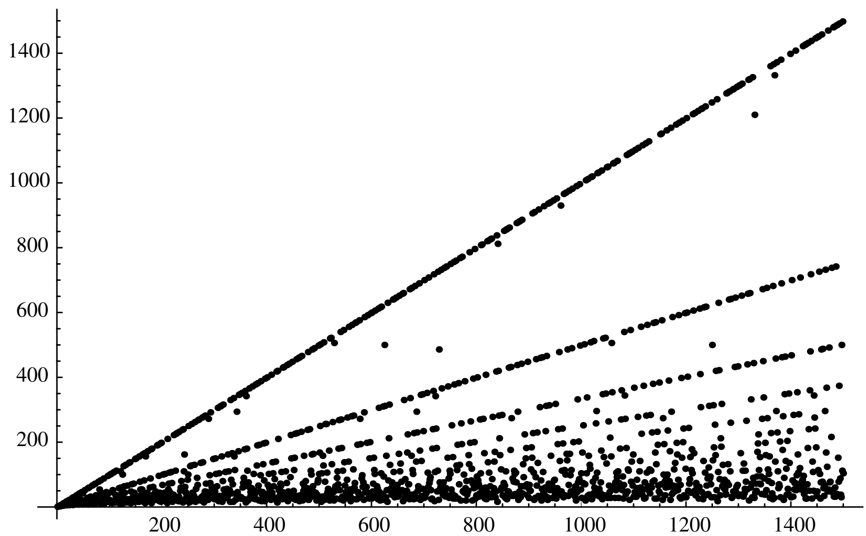


**Figure 1.** Values of $\psi(n)$ for $n \leq 1500$. The prominent lines are numbers of the form $kp$ with $p$ prime and $k$ small, the isolated points between the top two lines are prime powers (with the points hanging just below the top line being squares of primes), the points between the 2nd and 3rd lines are twice prime powers, etc.

**2. COMPUTING THE CR.** Let $\mathrm{Ord}_n^+$ and $\mathrm{Ord}_n^-$ denote the subsets of $\mathrm{Ord}_n$ of even and odd elements, respectively. One has the following formula, due to Hiller [**8**, prop. 2.2]:

$$\mathrm{Ord}_n = \bigcup_{0 \leq i \leq L(2,n)} 2^i \, \mathrm{Ord}^-_{n-\psi(2^i)}, \tag{1}$$

where $L(2, n)$ denotes the largest integer such that $\psi(2^{L(2,n)}) \leq n$; that is,

$$L(2, n) = \begin{cases} \lfloor \log_2 n \rfloor + 1 & \text{if } n > 0, \\ 1 & \text{if } n = 0, \end{cases}$$

in which $\lfloor x \rfloor$ signifies the integer part of $x$. The proof of (1) involves little more than the observation that every element of $\mathrm{Ord}_n$ can be written in the form $2^i x$ for some $i \geq 0$ and odd integer $x$. Formula (1) has the practical advantage that it reduces the problem to the computation of the odd elements of $\mathrm{Ord}_n$; Hiller computed $\mathrm{Ord}_n \setminus \mathrm{Ord}_{n-1}$ for $n \leq 22$ [**8**]. One can extend Hiller's idea by considering amongst the elements of $\mathrm{Ord}_n^-$ those that are not divisible by 3, and amongst them, those that are not divisible by 5, etc. In the limit, one obviously obtains

$$\begin{aligned} \mathrm{Ord}_n = \{2^{r_1} 3^{r_2} \cdots p_l^{r_l} \mid\ & 0 \leq r_1 \leq L(2, n), 0 \leq r_2 \leq L(3, n - \psi(2^{r_1})), \\ & \ldots, 0 \leq r_l \leq L(p_l, n - \psi(2^{r_1} 3^{r_2} \cdots p_{l-1}^{r_{l-1}}))\}, \end{aligned}$$

where $p_l$ is the largest prime with $p_l \leq n + 1$ and $L(p, n)$ denotes the largest integer such that $\psi(p^{L(p,n)}) \leq n$. Explicitly, for any odd prime $p$ we have

$$L(p,n) = \begin{cases} \lfloor \log_p \left(\frac{n}{p-1}\right) \rfloor + 1 & \text{if } p \leq n+1, \\ 0 & \text{otherwise.} \end{cases}$$

This simple and direct method provides a rapid means of computing $\mathrm{Ord}_n$ (Table 1 shows the values for $n \leq 24$). The method also gives a way of computing the size of $\mathrm{Ord}_n$; namely,

$$|\,\mathrm{Ord}_n\,| = \sum_{\substack{0 \leq r_1 \leq L(2,n) \\ 0 \leq r_2 \leq L(3, n-\psi(2^{r_1})) \\ \vdots \\ 0 \leq r_l \leq L(p_l, n-\psi(2^{r_1}3^{r_2}\ldots p_{l-1}^{r_{l-1}}))}} 1.$$

TABLE 1. The crystallographic restriction in dimension $n \leq 24$.

| $n$ | $\psi^{-1}\{n\} = \mathrm{Ord}_n \setminus \mathrm{Ord}_{n-1}$ |
|---|---|
| 2 | 3,4,6 |
| 4 | 5,8,10,12 |
| 6 | 7,9,14,15,18,20,24,30 |
| 8 | 16,21,28,36,40,42,60 |
| 10 | 11,22,35,45,48,56,70,72,84,90,120 |
| 12 | 13,26,33,44,63,66,80,105,126,140,168,180,210 |
| 14 | 39,52,55,78,88,110,112,132,144,240,252,280,360,420 |
| 16 | 17,32,34,65,77,99,104,130,154,156,165,198,220,264,315,330,336,504,630,840 |
| 18 | 19,27,38,51,54,68,91,96,102,117,176,182,195,231,234,260,308,312,390,396, 440,462,560,660,720,1260 |
| 20 | 25,50,57,76,85,108,114,136,160,170,204,208,273,364,385,468,495,520,528 546,616,770,780,792,924,990,1008,1320,1680 |
| 22 | 23,46,75,95,100,119,135,143,150,152,153,190,216,224,228,238,255,270,286 288,306,340,408,455,480,510,585,624,693,728,880,910,936,1092,1155,1170 1386,1540,1560,1848,1980 |
| 24 | 69,92,133,138,171,189,200,266,272,285,300,342,357,378,380,429,456,476 540,570,572,612,672,680,714,819,858,1020,1040,1232,1365,1584,1638,1820 |

Computationally, it is more efficient to use the following algorithm. Set $T(n,0) = 1$ for all $n$ in $\mathbb{N} \cup \{0\}$, and define

$$T(n,k) = \sum_{0 \leq r \leq L(p_k,n)} T(n - \psi(p_k^r), k-1)$$

for all positive integers $n$ and $k$. Then for $n \geq 2$, $T(n,k) \to |\,\mathrm{Ord}_n\,|$ as $k \to \infty$, and it achieves the limit as soon as $\psi(p_k) > n$. Figure 2 shows a plot of

$$\frac{\log \log |\,\mathrm{Ord}_n\,|}{\log n}$$

for $n \leq 40{,}000$; that graph suggests that $\log |\,\mathrm{Ord}_n\,| \sim n^c$ for some constant $c$ satisfying $0.45 < c < 0.5$.

**3. PERMUTATIONS AND THE CR.** There is an obvious connection between the symmetric group $S_n$ and the general linear group $GL(n, \mathbb{Z})$: each permutation $\sigma$ in $S_n$

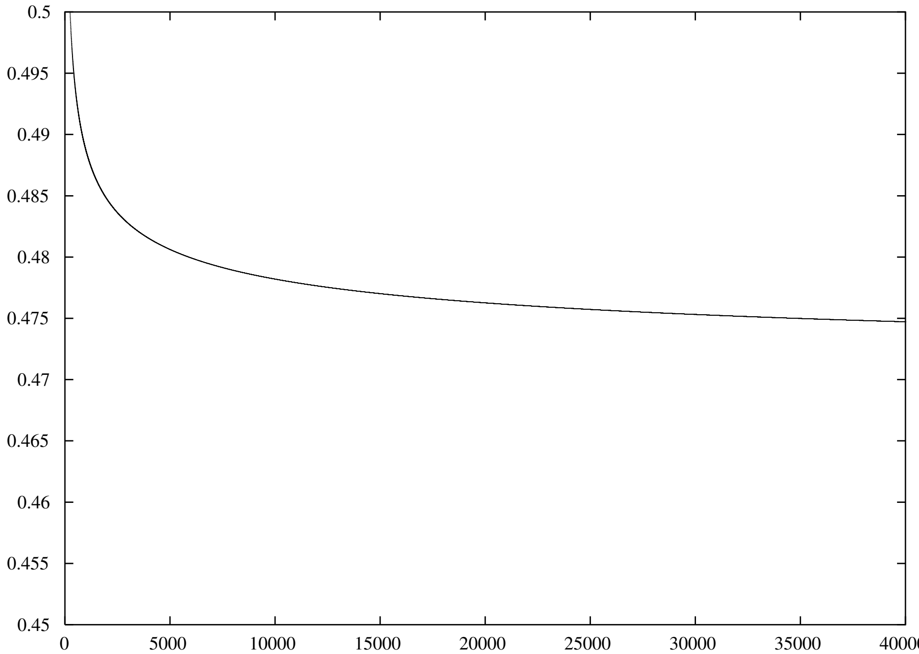


**Figure 2.** $(\log\log|\operatorname{Ord}_n|)/\log n$ for $n \leq 40{,}000$.

gives rise to a linear transformation, which is determined by the action of $\sigma$ on the standard basis elements $e_1, \ldots, e_n$ of $\mathbb{R}^n$. This gives a group homomorphism $S_n \to GL(n, \mathbb{Z})$ (see [**12**, Exercise 1.1]) whose image is called the *Weyl subgroup*. However, this representation of $S_n$ is not irreducible, for the vector $e_1 + \cdots + e_n$ is invariant. Instead, the *standard irreducible representation* of $S_n$ is the group homomorphism $S_n \to GL(n-1, \mathbb{Z})$ defined as follows. Consider the hyperspace $V$ of $\mathbb{R}^n$ perpendicular to $e_1 + \cdots + e_n$; that is, $V$ consists of those vectors for which the sum of the coordinates is zero. Clearly, $V$ is invariant under the indicated action of $S_n$, so we obtain an injective group homomorphism $\rho : S_n \to \operatorname{End}(V)$, where $\operatorname{End}(V)$ is the group of linear transformations of $V$. The vector space $V$ has basis $\{e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\}$, whence $\rho$ may be regarded as taking its values in $GL(n-1, \mathbb{Z})$. For example, for $n = 3$ one finds that, relative to the specified basis for $V$, $\rho(S_3)$ consists of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

which have orders 1, 2, 2, 2, 3, and 3, respectively.

Two important properties of the standard representation $\rho : S_n \to GL(n-1, \mathbb{Z})$ are that it is faithful (i.e., $\rho$ is injective) and that there are no faithful representations of smaller degree (i.e., there is no injective homomorphism $S_n \to GL(k, \mathbb{Z})$ for any $k$ smaller than $n-1$) [**10**]. In other words, $S_n$ is a subgroup of $GL(n-1, \mathbb{Z})$, but it isn't a subgroup of $GL(n-2, \mathbb{Z})$.

The possible orders of the elements of $S_n$ can be computed in a manner similar to the way they were for the crystallographic restriction. Consider the function $S : \mathbb{N} \to \mathbb{N}$

defined as follows: $S(1) = 1$ and for $m > 1$ with prime factorization $m = \prod_i p_i^{r_i}$, $S(m) = \sum_i p_i^{r_i}$. Analogous to Theorem 1, one has [**15**]:

**Theorem 2.** *$S_n$ has an element of order $m$ if and only if $S(m) \leq n$.*

Analogous to the equation $\psi^{-1}\{n\} = \text{Ord}_n \setminus \text{Ord}_{n-1}$, Theorem 2 shows that $S^{-1}\{n\}$ is the set of orders that are realized in $S_n$ but are not realized in $S_{n-1}$. The sets $S^{-1}\{n\}$ can be computed using the procedure described for $\psi^{-1}\{n\}$ in the previous section (Table 2 shows the values of $S^{-1}\{n\}$ for $n \leq 24$). As Tables 1 and 2 indicate, despite the connections between $S_n$ and $GL(n, \mathbb{Z})$ there is little obvious relation between $S^{-1}\{n\}$ and $\psi^{-1}\{n\}$. For example, although there is no injective homomorphism from $S_4$ to $GL(2, \mathbb{Z})$, all the orders realized in $S_4$ are also realized in $GL(2, \mathbb{Z})$. There is a simple reason for this, as the following proposition proves.

TABLE 2. $S^{-1}\{n\}$ for $n \leq 24$.

| $n$ | $S^{-1}\{n\}$ |
|---|---|
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5,6 |
| 6 | |
| 7 | 7,10,12 |
| 8 | 8,15 |
| 9 | 9,14,20 |
| 10 | 21,30 |
| 11 | 11,18,24,28 |
| 12 | 35,42,60 |
| 13 | 13,22,36,40 |
| 14 | 33,45,70,84 |
| 15 | 26,44,56,105 |
| 16 | 16,39,55,63,66,90,120,140 |
| 17 | 17,52,72,210 |
| 18 | 65,77,78,110,126,132,168,180 |
| 19 | 19,34,48,88,165,420 |
| 20 | 51,91,99,130,154,156,220,252,280 |
| 21 | 38,68,80,104,195,231,315,330 |
| 22 | 57,85,102,117,182,198,260,264,308,360 |
| 23 | 23,76,112,273,385,390,462,630,660,840 |
| 24 | 95,114,119,143,170,204,234,240,312,364,396,440,504 |

**Proposition 1.** *Let $n > 2$ be even. If $S_n$ contains an element of order $m$, then $GL(n-2, \mathbb{Z})$ contains an element of the same order.*

*Proof.* Suppose that $S_n$ contains an element of order $m = \prod_i p_i^{r_i}$. By Theorem 2, $S(m) \leq n$. Thus

$$\psi(m) \leq \sum_i \phi(p_i^{r_i}) = S(m) - \sum_i p_i^{r_i - 1} \leq n - \sum_i p_i^{r_i - 1}.$$

Using Theorem 1, it remains to see that $\sum_i p_i^{r_i - 1} \geq 2$. This clearly holds if the prime decomposition of $m$ involves more than one prime. Finally, $m = p^r$ and $p^{r-1} < 2$ is impossible, since otherwise $r = 1$ and $m = p$, contradicting the hypothesis that $m$ is even. ■

A further connection between $\mathrm{Ord}_n$ and the orders of $S_n$ is established in:

**Proposition 2.** *If $m = p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are distinct odd primes, then there exists an $n \times n$ integer matrix of order $m$ if and only if $S_{n+k}$ has an element of order $m$.*

*Proof.* In view of Theorems 1 and 2, it suffices to notice that $\psi(m) \leq n$ if and only if $(p_1 - 1) + \cdots + (p_k - 1) \leq n$, which translates to $S(m) \leq n + k$. ■

Notice that the case $k = 2$ of Proposition 2 gives a converse to Proposition 1 for the particular case when $m$ is a product of two primes. Looking at Tables 1 and 2, yet another feature becomes apparent:

**Proposition 3.** *The following statements hold:*

1. *$S^{-1}\{n\}$ is nonempty for all $n > 6$;*
2. *$\psi^{-1}\{n\}$ is nonempty for all even $n \geq 2$.*

Before providing the proof, we observe that part (2) of the proposition says that the function $\psi$ maps $\mathbb{N}$ onto the set of even nonnegative integers. This contrasts nicely with the case of the $\phi$-function, which is not onto the even numbers (nothing maps to 14 under $\phi$, for example); instead Carmichael's conjecture is that for every even $x$ the set $\phi^{-1}\{x\}$ is either empty or has at least two elements (see [**22**], [**4**]).

*Proof.* Part (1) follows immediately from Richert's theorem, which states that every integer greater than six can be written as the sum of distinct primes [**19**]. To establish part (2), we prove a similar result: for every even number $n \geq 2$ there are distinct odd primes $p_1, \ldots, p_k$ such that $\psi(p_1 \cdots p_k) = n$. The proof is by induction on $n$. First note that $\psi(3) = 2$. Suppose that $n = 2x \geq 4$. By Bertrand's postulate (see [**2**]), there exists a prime $p$ with $x + 1 < p \leq 2x + 1 = n + 1$. If $p = n + 1$, then $\psi(p) = n$ and we are done. Otherwise, let $n' = n - p + 1$. Then $n'$ is even and less than $n$, so by the inductive hypothesis there are distinct primes $p_1, \ldots, p_k$ such that $\psi(p_1 \cdots p_k) = n'$. Note that for each of the primes $p_i$,

$$p_i = \phi(p_i) + 1 \leq \psi(p_1 \cdots p_k) + 1 = n' + 1.$$

Thus, since $x + 1 < p$, we have

$$p_i \leq n - p + 2 < n - (x + 1) + 2 = x + 1 < p.$$

In particular, $p_i \neq p$ for each $i$, and consequently

$$\psi(p_1 \cdots p_k p) = \psi(p_1 \cdots p_k) + \phi(p) = n' + p - 1 = n,$$

which completes the induction. ■

**4. THE CONNECTION WITH GOLDBACH.** Recall that Goldbach's conjecture asserts that every even natural number $x$ greater than 4 can be written as the sum of two odd primes. A common variation on this is the:

**Strong Goldbach Conjecture.** *Every even natural number $x$ greater than six can be written as the sum of two distinct odd primes.*

Schinzel proved that Goldbach's conjecture implies that every odd integer larger than 17 is the sum of three distinct primes [**21**], while Sierpiński proved that the strong Goldbach conjecture is equivalent to the condition that every integer greater than 17 is the sum of three distinct primes [**27**]. Goldbach's conjecture has been verified up to $4 \cdot 10^{14}$ [**20**], and these calculations also support the strong Goldbach conjecture [**20**, Table 1].

We can now state the connection between the strong Goldbach conjecture, the crystallographic restriction, and the orders of the elements of the symmetric group:

**Theorem 3.** *The following statements are equivalent:*

1. *the strong Goldbach conjecture is true;*
2. *for each even $n \geq 6$ there is an $n \times n$ integer matrix of order $pq$ for distinct odd primes $p$ and $q$, and there is no smaller integer matrix of this order;*
3. *for each even $n > 6$ there is an element of $S_n$ of order $pq$ for distinct odd primes $p$ and $q$, and there is no element of $S_{n-1}$ of this order.*

*Proof.* To prove that (1) and (2) are equivalent, it suffices to note that for $n \geq 6$ one has $n + 2 = p + q$ for distinct odd primes $p$ and $q$ if and only if $n = (p - 1) + (q - 1) = \psi(pq)$. The equivalence of (1) and (3) follows immediately from Theorem 2. ■

Finally, let us make two comments to put Theorem 3 in context. First, as the reader may easily verify, the equivalence between (1) and (3) can be proved directly, using the fact that every permutation is a product of disjoint cycles. Not surprisingly, this latter fact forms the basis of the proof of Theorem 2. Second, recall Erdős's conjecture that for every even $x$ there exist natural numbers $a$ and $b$ such that $\phi(a) + \phi(b) = x$. That (1) implies (2) in Theorem 3 is just a reformulation of the obvious and well-known fact that the strong Goldbach conjecture implies Erdős's conjecture (see, for example, [**1**]). Similarly, statement (2) of Proposition 3 can be rephrased in the following "Erdős-type" fashion: For every even number $n \geq 2$, there exist distinct odd primes $p_1, \ldots, p_k$ such that $\phi(p_1) + \cdots + \phi(p_k) = n$.

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