Tight sets and m-ovoids of finite polar spaces

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Abstract

An *intriguing set* of points of a generalised quadrangle was introduced in [2] as a unification of the pre-existing notions of *tight set* and *m-ovoid*. It was shown in [2] that every intriguing set of points in a finite generalised quadrangle is a tight set or an m-ovoid (for some m). Moreover, it was shown that an m-ovoid and an i-tight set of a common generalised quadrangle intersect in mi points. These results yielded new proofs of old results, and in this paper, we study the natural analogue of intriguing sets in finite polar spaces of higher rank. In particular, we use the techniques developed in this paper to give an alternative proof of a result of Thas [36] that there are no ovoids of $\mathsf{H}(2r, q^2)$, $\mathsf{Q}^-(2r+1, q)$, and $\mathsf{W}(2r-1, q)$ for $r > 2$. We also strengthen a result of Drudge on the non-existence of tight sets in $\mathsf{W}(2r-1, q)$, $\mathsf{H}(2r+1, q^2)$, and $\mathsf{Q}^+(2r+1, q)$, and we give a new proof of a result of due to De Winter, Luyckx, and Thas [9,28] that an m-system of $\mathsf{W}(4m+3, q)$ or $\mathsf{Q}^-(4m+3, q)$ is a pseudo-ovoid of the ambient projective space.

Key words: m-ovoid, tight set, m-system, egg
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1 Introduction

For generalised quadrangles, the point sets known as *tight sets* and *m-ovoids* have been well studied (see [2] and the references within) and there are known connections with two-character sets and strongly regular graphs. Both of these types of point sets share the property that they exhibit two intersection numbers with respect to perps of points (one for points in the set and the other for points not in the set), and such sets were coined *intriguing* in [2]. It was shown in [2] that every intriguing set of points in a finite generalised quadrangle is a tight set or an m-ovoid (for some m). Moreover, it was shown that an m-ovoid and an i-tight set of a common generalised quadrangle intersect in mi

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points. These results yielded new proofs of old results, and in this paper, we study the natural analogue of intriguing sets in finite polar spaces of higher rank.

The concept of an m-ovoid was extended to finite polar spaces in a paper of Shult and Thas [35] where it was defined to be a set of points meeting each maximal in m points. Similarly, the notion of a *tight set* of points was extended to finite polar spaces in the Ph. D thesis of K. Drudge [15]. A set of points $\mathcal{T}$ of a finite polar space $\mathcal{P}$ is tight if the average number of points of $\mathcal{T}$ collinear with a given point of $\mathcal{P}$ equals the maximum possible value. We will give a more precise definition in Section 3. In this paper, the theory of intriguing sets is extended to finite polar spaces of rank at least 3 where the aforementioned results for generalised quadrangles have a natural analogue.

The central results of this paper provide tools for clean proofs of interesting results such as those connecting m-systems and eggs. The earliest occurrence of an instance of this appears to be in Segre's 1965 monograph [34] on the hermitian surface, where in Chapter X on "Regular systems" (i.e., m-ovoids of the dual generalised quadrangle) the intersection with a particular type of 2-tight set is used to analyse the structure of a hemisystem. In Section 6, results of Govaerts and Storme on minihypers [19,20] find application to classification of small tight sets in hermitian spaces of odd projective dimension, symplectic spaces and hyperbolic spaces, thus improving a result of Drudge [15, Corollary 9.1]. Also in this section, we prove the non-existence of m-ovoids with small m in hermitian spaces of even projective dimension, symplectic spaces and elliptic spaces, using a method similar to that of Hamilton and Mathon for m-systems (see [21]).

Shult and Thas [35] noted that the set of points covered by the elements of an m-system of a polar space over $\mathsf{GF}(q)$ is a $(q^{m+1}-1)/(q-1)$-ovoid. Together with the constructions of m-systems there and in [28] and [22], m-systems provide a rich source of m'-ovoids. De Clerck, Delanote, Hamilton and Mathon [8] found an interesting 3-ovoid of $\mathsf{W}(5,3)$ giving rise to a partial geometry with parameters $(8,20,2)$. It arises from what they call a *perp-system* of lines of projective space, and this 3-ovoid is the set of points lying on the 21 pairwise opposite non-singular lines of the perp-system they describe. The 3-ovoid has stabiliser S_5 inside $\mathrm{PGSp}_6(3)$.

Notation and conventions

In this paper, we will mostly be concerned with finite polar spaces that are not generalised quadrangles. Hence such a polar space arises from a finite vector space equipped with a non-degenerate reflexive sesquilinear form. A maximal totally isotropic/totally singular subspace of a polar space will simply be referred to as a *maximal*. By Witt's Theorem, the maximals have the same algebraic dimension, which is called the *rank* of the polar space. Note that the rank will be in terms of algebraic dimension (i.e., the vector space dimension), and is also the number of variety types (points, lines, planes, etc). Throughout this paper, θ_r will be used exclusively to refer to the quantity in Table 1, which is the size of an ovoid or spread of the polar space. It will be convenient to use θ_r to state generic results about polar spaces. For example, a finite polar space of rank r defined over $\mathsf{GF}(q)$ has $\theta_r(q^r-1)/(q-1)$ points (note that the number of points on a maximal is

$(q^r-1)/(q-1))$.

Polar Space	$\mathsf{W}(2r-1,q)$	$\mathsf{H}(2r-1,q^2)$	$\mathsf{H}(2r,q^2)$	$\mathsf{Q}^+(2r-1,q)$	$\mathsf{Q}(2r,q)$	$\mathsf{Q}^-(2r+1,q)$
θ_r	q^r+1	$q^{2r-1}+1$	$q^{2r+1}+1$	$q^{r-1}+1$	q^r+1	$q^{r+1}+1$

Table 1
The values of θ_r, the size of an ovoid and/or spread, of the classical polar spaces.

We will often use the fact that for $r \geqslant 2$ we have $\theta_r = \beta\theta_{r-1} - \beta + 1$ where β is the order of the underlying field (q or q^2). We will say that two subspaces S and T are opposite (with respect to a polarity $\perp$) if $S^\perp$ and T are disjoint. On subspaces of the same dimension, the relation of being opposite is symmetric. For more on the properties of finite polar spaces, we refer the reader to [23].

2 m-ovoids of Polar Spaces

The concept of an m-ovoid of a generalised quadrangle was introduced by Thas in [37], and was extended naturally to finite polar spaces of higher rank in the work of Shult and Thas (see [35]). A set of points $\mathcal{P}$ of a finite polar space $\mathcal{S}$ is an m-ovoid if each maximal of $\mathcal{S}$ meets $\mathcal{P}$ in m points. There are some straightforward methods for constructing m-ovoids. Let $\mathcal{S}$ be a finite classical polar space of rank r over the field $\mathsf{GF}(q)$, and let A and B be an m-ovoid and an n-ovoid of $\mathcal{S}$ respectively. If $A \subseteq B$, then $B \backslash A$ is an $(n-m)$-ovoid. Dually, if A and B are disjoint, then $B \cup A$ is an $(n+m)$-ovoid. Note that the whole point set is a $\frac{q^r-1}{q-1}$-ovoid, and hence the complement of an m-ovoid is a $\left(\frac{q^r-1}{q-1} - m\right)$-ovoid. We say that an m-ovoid is *reducible* if it contains a smaller m'-ovoid for some positive integer m' less than m. (An m-ovoid is *irreducible* if it is not reducible.)

In the generalised quadrangle situation, we have that a non-degenerate hyperplane section sometimes produces an m-ovoid. Specifically, we have that a non-degenerate hyperplane section of $\mathsf{H}(3,q^2)$ or of $\mathsf{Q}^+(3,q)$ yields an ovoid, and in $\mathsf{Q}(4,q)$, we obtain an ovoid if the hyperplane section is an elliptic quadric (n.b., a non-degenerate hyperplane section of $\mathsf{Q}(4,q)$ is either an elliptic or hyperbolic quadric). In higher rank, we have similar such examples of m-ovoids of $\mathsf{H}(2r-1,q^2)$, $\mathsf{Q}(2r,q)$, and $\mathsf{Q}^+(2r-1,q)$, which is explored in Section 5.

The concept of an m-system was introduced in a paper by Shult and Thas [35]. A *partial m-system* of a finite classical polar space $\mathcal{S}$ is a set of totally isotropic m-subspaces of $\mathcal{S}$ which are pairwise opposite. In [35], it was shown that a partial m-system has at most θ_r elements. This number is the common size of an ovoid and a spread (if they exist), and if equality in the bound occurs, then we obtain an *m-system*. Let π be an m-system of the finite classical polar space $\mathcal{S}$ and let $\tilde{\pi}$ be the set of points covered by elements of π. By [35, 6.1], any maximal meets $\tilde{\pi}$ in $\frac{q^{m+1}-1}{q-1}$ points; a corollary of which is that $\tilde{\pi}$ is a $\frac{q^{m+1}-1}{q-1}$-ovoid of $\mathcal{S}$.

3 Tight sets of Polar Spaces

An i-tight set of a finite generalised quadrangle was introduced by Payne (see [31]) and was generalised to polar spaces of higher rank by Drudge [15]. The latter proved that if $\mathcal{T}$ is a set of points of a finite polar space of rank $r \geqslant 2$ over a field of order q, then the average number of points of $\mathcal{T}$ collinear with a given point is bounded above by

$$i\frac{q^{r-1}-1}{q-1}+q^{r-1}-1$$

where i is determined by the size of $\mathcal{T}$. If equality occurs, then we say that $\mathcal{T}$ is *i-tight* (see [15, Theorem 8.1]). A set of points is *tight* if it is i-tight for some i. We prefer an equivalent alternative definition highlighting the two intersection number behaviour of tight sets. A set of points $\mathcal{T}$ is i-tight if

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^{r-1}-1}{q-1}+q^{r-1} & \text{if } P \in \mathcal{T} \\ i\frac{q^{r-1}-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

As for m-ovoids, we have some straightforward ways for constructing tight sets of a finite polar space $\mathcal{S}$. Let A and B be i-tight and j-tight sets of points of $\mathcal{S}$ respectively. If $A \subseteq B$, then $B \backslash A$ is a $(j-i)$-tight set; if A and B are disjoint, then $B \cup A$ is an $(i+j)$-tight set. A maximal is an example of a 1-tight set, and hence a partial spread is also a tight set. We say that an i-tight set is *reducible* if it contains a smaller i'-tight set for some positive integer i' less than i. (A tight set is *irreducible* if it is not reducible.)

It was observed in [15, Theorem 8.1] that an i-tight set of a finite polar space meets every ovoid in i points. We prove a generalisation of this result to m-ovoids (see Corollary 5), that is, that an i-tight set meets every m-ovoid in mi points. Drudge remarks that the Klein correspondence maps Cameron-Liebler line classes (see [6]) to tight sets. He also proves that a 1-tight set is the set of points on a maximal [15, Theorem 9.1], and proves that for q at least 4, any i-tight set with $i \leqslant \sqrt{q}$ in a hyperbolic (orthogonal) space is the set of points covered by a partial spread of i maximals [15, Corollary 9.1]; hence, deducing that none such exist if additionally $i > 2$ and the rank is odd [15, Corollary 9.2]. We show in Section 6 that there are no irreducible i-tight sets in $\mathsf{W}(2r-1,q)$, $\mathsf{H}(2r-1,q^2)$, or $\mathsf{Q}^+(2r-1,q)$ for $2 < i < \epsilon_q$, where $q+\epsilon_q$ is the size of the smallest non-trivial blocking set in $\mathsf{PG}(2,q)$.

In the generalised quadrangle situation, we have that a non-degenerate hyperplane section sometimes produces a tight set. In Section 5, we show that an $\mathsf{H}(2r-1,q^2)$ embedded in $\mathsf{H}(2r,q^2)$ is tight, a $\mathsf{Q}(2r,q)$ embedded in $\mathsf{Q}^-(2r+1,q)$ is tight, and a $\mathsf{Q}^+(2r-1,q)$ embedded in $\mathsf{Q}(2r,q)$ is tight.

4 Intriguing Sets

First we begin with some basic algebraic graph theory. A regular graph Γ, with v vertices and valency k, is *strongly regular* with parameters (v, k, λ, μ) if (i) any two adjacent vertices are both adjacent to λ common vertices; (ii) any two non-adjacent vertices are both adjacent to μ common vertices. If A is the adjacency matrix of the strongly regular graph Γ, then A has three eigenvalues and satisfies the equation $A^2 = kI + \lambda A + \mu(J - I - A)$ where I is the identity matrix and J is the all-ones matrix. The all-ones vector η, which can be thought of as the characteristic function of the vertex set of Γ, is an eigenvector of A with eigenvalue k. The remaining two eigenvalues e_1 and e_2 satisfy the quadratic equation $x^2 = k + \lambda x + \mu(-1 - x)$. Hence $\mu - k = e_1 e_2$ and $\lambda - \mu = e_1 + e_2$.

Let $\mathcal{S}$ be a polar space of rank r over $\mathsf{GF}(q)$. The point graph of $\mathcal{S}$ is strongly regular with parameters:

$$
\begin{aligned}
v &= \theta_r \frac{q^r - 1}{q - 1} \\
k &= q \cdot (\#\text{ points of the quotient geometry of a point}) = q\frac{q^{r-1} - 1}{q - 1}\theta_{r-1} \\
\lambda &= q^{r-1} - 1 + q\theta_{r-1}\frac{q^{r-2} - 1}{q - 1} \\
\mu &= \#\text{ points of the quotient geometry of a point} = \theta_{r-1}\frac{q^{r-1} - 1}{q - 1}.
\end{aligned}
$$

(see [26,27] or [15, Lemma 8.3]) and hence has three eigenvalues, one of which is the valency. (Note that the point graph is taken to have no loops, and hence the diagonal of the adjacency matrix only has zeros as our graphs are simple. In other words two points are adjacent if they are collinear *and distinct*.) The other two eigenvalues are

$$q^{r-1} - 1 \text{ and } -\theta_{r-1}.$$

Note that these eigenvalues are easily distinguishable: one is positive whilst the other is negative. As an example, consider a classical generalised quadrangle $\mathcal{S}$, which is a rank 2 polar space. Then the point graph of $\mathcal{S}$ is a strongly regular graph with parameters

$$(\theta_2(q + 1), q\theta_1, q - 1, \theta_1).$$

The eigenvalues of this graph are $q - 1$ and $-\theta_1$.

We say that a set of points $\mathcal{I}$ in a polar space $\mathcal{S}$ is *intriguing* if

$$|P^\perp \cap \mathcal{I}| = \begin{cases} h_1 & \text{when } P \in \mathcal{I} \\ h_2 & \text{when } P \notin \mathcal{I} \end{cases}$$

for some constants h_1 and h_2 (where P ranges over the points of $\mathcal{S}$). We will refer to h_1 and h_2 as the *intersection numbers* of $\mathcal{I}$. Note that every maximal of $\mathcal{S}$ is an intriguing set with parameters

$$h_1 = q^{r-1} + \frac{q^{r-1}-1}{q-1} \text{ and } h_2 = \frac{q^{r-1}-1}{q-1}.$$

In general, we have the following observation:

Lemma 1 *Let $\mathcal{S}$ be a finite polar space of rank r at least 2 defined over $\mathsf{GF}(q)$.*

(a) An i-tight set of $\mathcal{S}$ is an intriguing set with intersection numbers

$$\begin{aligned} h_1 &= i\frac{q^{r-1}-1}{q-1} + q^{r-1} \\ h_2 &= i\frac{q^{r-1}-1}{q-1}. \end{aligned}$$

(b) An m-ovoid of $\mathcal{S}$ is an intriguing set with intersection numbers

$$\begin{aligned} h_1 &= m\theta_{r-1} - \theta_{r-1} + 1 \\ h_2 &= m\theta_{r-1}. \end{aligned}$$

PROOF. Part (a) follows from the definition of an i-tight set. Let $\mathcal{O}$ be an m-ovoid of $\mathcal{S}$ and let $P \in \mathcal{S}$. First suppose that $P \in \mathcal{O}$ and let $x = |P^\perp \cap \mathcal{O}|$. We will count the number of pairs (M, Q) where Q is a point of $\mathcal{O}$, and M is a maximal containing P and Q. There are n maximals, say, through P, and each one contains m points of $\mathcal{O}$. There are $x-1$ points of $\mathcal{O}$ other than P, collinear with P, and for each one there are n', say, maximals containing P and Q. Since there are also n maximals through P, we have $nm = (x-1)n' + n$ and hence $x = n(m-1)/n' + 1 = \theta_{r-1}(m-1) + 1$.

Now suppose that $P \notin \mathcal{O}$. Again, we will count the number of pairs (M, Q) where Q is a point of $\mathcal{O}$, and M is a maximal containing P and Q. However, this time we have the equation $nm = xn'$ where n is the number of maximals through P, the quantity n' is again number of maximals through both P and Q, and x is the size of $P^\perp \cap \mathcal{O}$. Hence, in this case, $x = nm/n' = m\theta_{r-1}$. □

The entire point set is an example of an intriguing set, and so we will call smaller intriguing sets *proper*. Let A be the adjacency matrix of the point graph of $\mathcal{S}$ (remember, A only has zeros on its diagonal). Now the characteristic function of the point set is the constant function η, which is an eigenvector of A with eigenvalue k; the valency of the point graph. The following result states that if $\mathcal{I}$ is a proper intriguing set, then $A\mathcal{X}_\mathcal{I} - k\mathcal{X}_\mathcal{I}$ is an eigenvector of A.

Lemma 2 *Let $\mathcal{I}$ be a proper intriguing set of a finite polar space $\mathcal{S}$ (with rank at least 2) with intersection numbers h_1 and h_2, let k be the number of points in $\mathcal{S}$ collinear (but not equal) to a given point of $\mathcal{S}$, and let A be the adjacency matrix of the point graph of $\mathcal{S}$. Then*

$$(h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} + h_2\eta$$

is an eigenvector of A with eigenvalue $h_1 - h_2 - 1$.

PROOF. By definition of an intriguing set, we have $A\mathcal{X}_\mathcal{I} = (h_1 - h_2 - 1)\mathcal{X}_\mathcal{I} + h_2\eta$, and so

$$\begin{aligned} A\left((h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} + h_2\eta\right) &= (h_1 - h_2 - 1 - k)\left((h_1 - h_2 - 1)\mathcal{X}_\mathcal{I} + h_2\eta\right) + h_2 k\eta \\ &= (h_1 - h_2 - 1)\left((h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} + h_2\eta\right). \end{aligned}$$

Therefore $(h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} + h_2\eta$ is an eigenvector with eigenvalue $h_1 - h_2 - 1$. □

Lemma 3 *Let $\mathcal{I}$ be a proper intriguing set of a finite polar space $\mathcal{S}$ with intersection numbers h_1 and h_2, and let $\mathcal{P}$ be the set of points of $\mathcal{S}$. Then the number of points in $\mathcal{I}$ is*

$$|\mathcal{I}| = \frac{h_2}{k + 1 - h_1 + h_2}|\mathcal{P}|$$

where k is the valency of the point graph of $\mathcal{S}$.

PROOF. Let A be the adjacency matrix of $\mathcal{S}$. Since $\mathcal{I}$ is proper and A is a real symmetric matrix, the eigenvector $(h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} + h_2\eta$ is orthogonal to the constant function η with eigenvalue k. So $((h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} + h_2\eta) \cdot \eta = 0$ and hence:

$$-(h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{I} \cdot \eta = h_2\eta \cdot \eta$$

from which the conclusion follows. □

So a corollary of the above lemma is that an m-ovoid has size $m\theta_r$ (i.e., m times the size of an ovoid) and an i-tight set has size $i(q^r - 1)/(q - 1)$ (i.e., i times the size of a maximal).

Theorem 4 *Let $\mathcal{S}$ be a finite polar space, let $\mathcal{O}$ and $\mathcal{T}$ be intriguing sets of $\mathcal{S}$ with different associated eigenvalues, and let $\mathcal{P}$ be the set of points of $\mathcal{S}$. Then $\mathcal{O}$ and $\mathcal{T}$ intersect in $|\mathcal{O}||\mathcal{T}|/|\mathcal{P}|$ points.*

PROOF. Let h_1 and h_2 be the intersection numbers of $\mathcal{T}$ and let m_1 and m_2 be the intersection numbers of $\mathcal{O}$. We let A be the adjacency matrix of the point graph of $\mathcal{S}$ and k be the valency of this graph. We know from Lemma 2 that the eigenvector $(m_1 - m_2 - k - 1)\mathcal{X}_\mathcal{O} + m_2\eta$ of A has eigenvalue $m_1 - m_2 - 1$ and that the eigenvector $(h_1 - h_2 - k - 1)\mathcal{X}_\mathcal{T} + h_2\eta$ has eigenvalue $h_1 - h_2 - 1$ (and so $m_1 - m_2 \neq h_1 - h_2$). Since A is symmetric and the two eigenvectors come from different eigenspaces, they are orthogonal, and so

$$\Big((m_1 - m_2 - 1 - k)\mathcal{X}_\mathcal{O} + m_2\eta\Big) \cdot \Big((h_1 - h_2 - 1 - k)\mathcal{X}_\mathcal{T} + h_2\eta\Big) = 0.$$

We can expand and rearrange this to get $\mathcal{X}_\mathcal{O} \cdot \mathcal{X}_\mathcal{T}$; the size of the intersection of $\mathcal{O}$ and $\mathcal{T}$.

$$
\begin{aligned}
|\mathcal{O} \cap \mathcal{T}| &= \mathcal{X}_{\mathcal{O}} \cdot \mathcal{X}_{\mathcal{T}} \\
&= \frac{-h_2(m_1 - m_2 - 1 - k)|\mathcal{O}| - m_2(h_1 - h_2 - 1 - k)|\mathcal{T}| - h_2 m_2 |\mathcal{P}|}{(h_1 - h_2 - 1 - k)(m_1 - m_2 - 1 - k)}
\end{aligned}
$$

By Lemma 3, we have

$$
|\mathcal{O} \cap \mathcal{T}| = \frac{h_2 m_2 + m_2 h_2 - h_2 m_2}{(h_1 - h_2 - 1 - k)(m_1 - m_2 - 1 - k)} |\mathcal{P}| = \frac{h_2 m_2 |\mathcal{P}|}{(h_1 - h_2 - 1 - k)(m_1 - m_2 - 1 - k)}.
$$

This equation then simplifies to $|\mathcal{O} \cap \mathcal{T}| = |\mathcal{O}||\mathcal{T}|/|\mathcal{P}|$. □

We have the following consequence, which is a generalisation of [2, Theorem 4.3].

Corollary 5 *Let $\mathcal{S}$ be a finite polar space, and let $\mathcal{O}$ and $\mathcal{T}$ be an m-ovoid and an i-tight set respectively. Then $\mathcal{O}$ and $\mathcal{T}$ intersect in mi points.*

The following is a generalisation of [2, Theorem 4.1].

Theorem 6 *Let $\mathcal{I}$ be a proper intriguing set of a finite polar space $\mathcal{S}$ and suppose that its associated eigenvalue e is not equal to k.*

(a) If e is negative, then $\mathcal{I}$ is an m-ovoid (for some m).
(b) If e is positive, then $\mathcal{I}$ is a tight set.

Hence a proper intriguing set is a tight set or an m-ovoid (for some m).

PROOF. Suppose that $\mathcal{I}$ has intersection numbers h_1 and h_2, and let k be the valency of the point graph of $\mathcal{S}$. By [2, Theorem 4.1], we can suppose that $r > 2$.

(a) Suppose that e is negative, that is, $h_1 - h_2 - 1 = -\theta_{r-1}$. Recall that a maximal M is a 1-tight set (and so has positive associated eigenvalue) and therefore by Theorem 4, $|M \cap \mathcal{I}| = |M||\mathcal{I}|/|\mathcal{P}|$. Now $k + 1 - h_1 + h_2 = \theta_{r-1}\frac{q^r-1}{q-1}$ and hence by Lemma 3, $|M \cap \mathcal{I}| = h_2/\theta_{r-1}$. Therefore $\mathcal{I}$ is an m-ovoid where $m = h_2/\theta_{r-1}$.

(b) Now suppose that e is positive, that is, $h_1 - h_2 - 1 = q^{r-1} - 1$. It suffices to show that h_2 is divisible by $(q^{r-1} - 1)/(q - 1)$. We have that $k + 1 - h_1 + h_2 = \frac{q^{r-1}-1}{q-1}\theta_r = \frac{q^{r-1}-1}{q^r-1}|\mathcal{P}|$ and hence by Lemma 3

$$
|\mathcal{I}|(q^{r-1} - 1) = h_2(q^r - 1).
$$

Now $q^{r-1} - 1$ is coprime to $q^r - 1$ (as $r > 2$), and so it follows that h_2 is divisible by $(q^{r-1} - 1)/(q - 1)$. □

5 Some Constructions of Intriguing Sets

In Sections 2 and 3, we saw some natural constructions of intriguing sets of finite polar spaces. Here we give some less straightforward constructions.

5.1 Non-degenerate hyperplane sections

Recall from Sections 2 and 3 that there are natural constructions of intriguing sets of generalised quadrangles arising by taking non-degenerate hyperplane sections. Here we extend these constructions to higher rank polar spaces.

Lemma 7 *Let $\mathcal{S}$ be a finite classical polar space in the table below, and let H be a non-degenerate hyperplane. Then $\mathcal{S} \cap H$ is an intriguing set as follows:*

$\mathcal{S}$	*Conditions*	*Type*
$\mathsf{H}(2r, q^2)$	–	$(1+q^{2r-1})$-*tight set*
$\mathsf{H}(2r-1, q^2)$	–	$\frac{q^{2r-2}-1}{q^2-1}$-*ovoid*
$\mathsf{Q}^-(2r+1, q)$	–	$(1+q^r)$-*tight set*
$\mathsf{Q}(2r, q)$	$\mathcal{S} \cap H = \mathsf{Q}^+(2r-1, q)$, q *odd*	$(1+q^{r-1})$-*tight set*
$\mathsf{Q}(2r, q)$	$\mathcal{S} \cap H = \mathsf{Q}^-(2r-1, q)$, q *odd*	$\frac{q^{r-1}-1}{q-1}$-*ovoid*
$\mathsf{Q}^+(2r-1, q)$	–	$\frac{q^{r-1}-1}{q-1}$-*ovoid.*

PROOF. First note that H can be identified with $Q^\perp$ where Q is a point of the ambient projective space not in $\mathcal{S}$. Let P be a point of $\mathcal{S}$, and suppose first that P is in $Q^\perp$. Then P belongs to the non-degenerate polar space $\mathcal{S} \cap H$ and it is clear that $|P^\perp \cap \mathcal{S} \cap H|$ is a constant (not depending on the location of P on H). Now suppose that P is not in $Q^\perp$. Then $\mathcal{S} \cap P^\perp \cap Q^\perp$ is isomorphic to the quotient geometry $\mathcal{S}/P$ (by natural projection) and hence $|P^\perp \cap (\mathcal{S} \cap H)|$ is a constant (not depending on the location of P in $\mathcal{S} \backslash H$). Hence $\mathcal{S} \cap H$ is an intriguing set of $\mathcal{S}$. So by Theorem 6, $\mathcal{S} \cap H$ is a tight set or an m-ovoid for some m. For $\mathcal{S} \cap H$ a tight set, we can easily calculate the intersection numbers. However, if $\mathcal{S} \cap H$ is an m-ovoid, note that in each case the rank of the polar space $\mathcal{S} \cap H$ is one less than that of $\mathcal{S}$. Let M be a maximal of $\mathcal{S}$. So either H contains M or H intersects M in a subspace L of codimension 1 in M. Since the rank of $\mathcal{S} \cap H$ is one less than that of $\mathcal{S}$, containment is not possible so each maximal intersects $\mathcal{S} \cap H$ in $|L|$ points. Hence, $\mathcal{S} \cap H$ is an $|L|$-ovoid of $\mathcal{S}$. □

In some cases, when we have a polar space embedded in another, an intriguing set of the smaller polar space naturally gives rise to an intriguing set of the larger polar space. Let $\mathcal{S}$ be one of the polar spaces $\mathsf{H}(2r-1, q^2)$ or $\mathsf{Q}^+(2r-1, q)$ (where $r \geqslant 2$) and let P be a point of the ambient projective space not in $\mathcal{S}$. Then

(a) an m-ovoid of $P^\perp \cap \mathcal{S}$ is an m-ovoid of $\mathcal{S}$;
(b) if an m-ovoid of $\mathcal{S}$ is contained in $P^\perp$, it is an m-ovoid of $P^\perp \cap \mathcal{S}$.

For (a), note that if M is a maximal of $\mathcal{S}$, then $P^\perp \cap M$ is a maximal of $P^\perp \cap \mathcal{S}$. For (b), note that every maximal M' of $P^\perp \cap \mathcal{S}$ is a subspace of codimension 1 in at least one maximal M of $\mathcal{S}$.

Now suppose that $\mathcal{S}$ is one of the polar spaces $\mathsf{H}(2r, q^2)$ or $\mathsf{Q}^-(2r+1, q)$ (where $r \geqslant 2$) and let P be a point of the ambient projective space not in $\mathcal{S}$. Since two points in $P^\perp \cap \mathcal{S}$ are collinear if and only if they are collinear in $\mathcal{S}$, it follows that if an i-tight set of $\mathcal{S}$ is contained in $P^\perp$, it is an i-tight set of $P^\perp \cap \mathcal{S}$.

5.2 Subfield embeddings

Let $\mathsf{W}_q = \mathsf{W}(2r-1, q)$ and consider the embedding of W_q in $\mathsf{H}(2r-1, q^2)$ (see [25, pp. 144] for details). The socle of the stabiliser of W_q in $\mathrm{P\Gamma U}(2r, q^2)$, namely $\mathrm{PSp}(2r, q)$, has two orbits on the points of $\mathsf{H}(2r-1, q^2)$; the points of W_q and its complement. Hence it follows that the set of points of W_q is an intriguing set in $\mathsf{H}(2r-1, q^2)$. Moreover, since every point in W_q is collinear with $(q^{2r-1}-1)/(q-1)$ points of W_q, we have that W_q is a $(q+1)$-tight set of $\mathsf{H}(2r-1, q^2)$. Other subfield embeddings also give rise to tight sets (although, the stabiliser of the subgeometry does not have only two orbits on points in these cases).

Theorem 8 *Let $r \geqslant 2$ and let $\mathcal{S}$ be one of the polar spaces $\mathsf{W}(2r-1, q^2)$ or $\mathsf{Q}(2r, q^2)$. Let $\mathcal{S}_q$ be the subgeometry of $\mathcal{S}$ obtained by taking those points and lines which are fixed by an automorphism of the extension of $\mathsf{GF}(q^2)$ over $\mathsf{GF}(q)$; i.e., $\mathsf{W}(2r-1, q)$ or $\mathsf{Q}(2r, q)$ respectively. The stabiliser of $\mathcal{S}_q$ in the collineation group of $\mathcal{S}$ has three orbits on points of $\mathcal{S}$, and each orbit is a tight set of $\mathcal{S}$.*

PROOF. Throughout, we will use the fact that $\theta_r = (q^2)^r + 1$ for the polar space $\mathcal{S}$. The three orbits of the stabiliser of $\mathcal{S}_q$ are (c.f. [25, §4.5]):

(a) the points which are not collinear with their conjugate;
(b) the points which are collinear with, but not equal to, their conjugate;
(c) the points which are equal to their conjugate (i.e., $\mathcal{S}_q$).

First notice that the lines of $\mathcal{S}$ meet $\mathcal{S}_q$ in 0, 1, or $q+1$ points. Thus we will refer to these lines (temporarily) as *external*, *tangent*, and *real* respectively. Let $X = (q^{2(r-1)}-1)/(q^2-1)$ (i.e., the size of a maximal in a quotient geometry $P^\perp/P$, where P is a point of $\mathcal{S}$) and let $\mathcal{T}$ be the second of the orbits above; those points which are collinear with, but not equal to their conjugate. Note that if $P \in \mathcal{T}$, then there is a unique real line on P and there are $(q+1)(X-1)$ tangent lines on P. Thus $|P^\perp \cap \mathcal{T}| = (q+1) + (q+1)(X-1) = (q+1)X$. Now if P is not in $\mathcal{S}_q$ or $\mathcal{T}$, then a line through P is either a tangent or external line (since P is not collinear with its conjugate). Since there are $(q+1)X$ tangent lines on P, it follows that $|P^\perp \cap \mathcal{T}| = (q+1)X$. Clearly, if $P \in \mathcal{S}_q$ we have that $|P^\perp \cap \mathcal{T}|$ is $(q+1)X + (q^2)^{r-1}$ and hence $\mathcal{S}_q$ is $(q+1)$-tight.

It remains to show that $\mathcal{T}$ is a tight set since by taking the complement of $\mathcal{T} \cup \mathcal{S}_q$ will yield that the first of our orbits above is a tight set. Let $P \in \mathcal{S}_q$. Then a line through P is either a tangent line or real line.

Suppose ℓ is a tangent line on P, and suppose that R is a point on ℓ which is in $\mathcal{T}$. Note that $\bar{R}$ is collinear with both R and P (as $P = \bar{P}$), and thus all points of ℓ. Suppose that Q is a point on ℓ and that $\bar{Q}$ is not on ℓ. Then $\bar{R}$ is collinear with Q and hence $\bar{Q}$ is collinear with R. Since $\bar{Q}$ is collinear with at least two points of ℓ, namely R and P, we have that $\bar{Q}$ is collinear with all points of ℓ. Therefore Q is collinear with $\bar{Q}$ and it follows that $Q \in \mathcal{T}$. Thus tangent lines through P either meet $\mathcal{T}$ in q^2 points or no points at all. Moreover, there are $(q^{2r-2} - q)X$ tangent lines through P, but nX say, which meet $\mathcal{T}$ in a point.

Suppose ℓ is a real line on P. Then there are $q^2 - q$ points of ℓ not in $\mathcal{S}_q$. Note that $\ell = \bar{\ell}$ as ℓ is incident with at least two points of $\mathcal{S}_q$. Hence if $Q \in \ell$, then $\bar{Q} \in \ell$ and so $Q \in \mathcal{T}$. Therefore, the set of lines on P contains $(q^{2(r-1)} - 1)/(q-1)$ real lines each meeting $\mathcal{T}$ in $q^2 - q$ points. So

$$|P^{\perp} \cap \mathcal{T}| = q^2 nX + q(q^{2(r-1)} - 1).$$

Now suppose that $P \notin \mathcal{S}_q \cup \mathcal{T}$. Then a line through P is either a tangent line or external line. Recall that tangent lines through P meet $\mathcal{T}$ in 0 or q^2 points. The latter cannot occur as tangent lines through P occupy two points not in $\mathcal{T}$. So suppose ℓ is an external line on P.

There are two cases:

Case $|\ell^{\perp} \cap \bar{\ell}| = 1$: Here we have a unique point Q on ℓ which is collinear to all the points of $\bar{\ell}$. Hence Q is the sole element of ℓ in $\mathcal{T}$ and hence $|\ell \cap \mathcal{T}| = 1$. Now we will show that there are nX possibilities for ℓ. Let R be a real point not collinear to P and let ℓ_R be a line on R. Since P is not collinear with all of the points of ℓ_R, there is a unique line on P concurrent with ℓ_R. Thus there is a bijection between the set of lines ℓ_R on R with $\bar{\ell_R} \subseteq \ell_R^{\perp}$ and the set of lines ℓ on P with $|\bar{\ell} \cap \ell^{\perp}| = 1$. The former was calculated above as having size $(q+1)X + nX$, from which it follows that there are nX external lines ℓ on P with $|\ell^{\perp} \cap \bar{\ell}| = 1$.

Case $|\ell^{\perp} \cap \bar{\ell}| = 0$: We will show that ℓ meets $\mathcal{T}$ in $q+1$ points. Since $\bar{\ell}$ does not meet $\ell^{\perp}$, there are precisely $q^2 + 1$ lines of $\mathcal{S}$ which are concurrent with both ℓ and $\bar{\ell}$. Moreover, of these lines, $q+1$ of them are real lines. Hence there are $q+1$ points of ℓ which are collinear with their conjugate. There are bX possibilities for ℓ where the total number of lines on any point of $\mathcal{S}$ is $(q+1+b+n)X$.

Therefore $|P^{\perp} \cap \mathcal{T}| = (q+1)bX + nX$. The table below summarises the above argument:

Case	Type of line ℓ	$\lvert\ell \cap \mathcal{T}\rvert$	Number of such lines
$P \in \mathcal{S}_q$	Tangent $\bar{\ell} \subset \ell^\perp$	q^2	nX
	Tangent $\lvert\bar{\ell} \cap \ell^\perp\rvert = 1$	0	bX
	Real $\bar{\ell} = \ell$	$q^2 - q$	$(q+1)X$
			$\lvert P^\perp \cap \mathcal{T}\rvert = (q^2 n + q(q^2-1))\, X$
$P \notin \mathcal{S}_q \cup \mathcal{T}$	Tangent $\lvert\bar{\ell} \cap \ell^\perp\rvert = 1$	0	$(q+1)X$
	External $\lvert\bar{\ell} \cap \ell^\perp\rvert = 0$	$q+1$	bX
	External $\lvert\bar{\ell} \cap \ell^\perp\rvert = 1$	1	nX
			$\lvert P^\perp \cap \mathcal{T}\rvert = ((q+1)b + n)\, X$

We also know that $b+n = q(q^{2r-3}-1)$. So in order to show that $|P^\perp \cap \mathcal{T}|$ is constant for $P \in \mathcal{S}_q$ and $P \in (\mathcal{S}_q \cup \mathcal{T})'$, it suffices to show that $n = q(q^{2r-4}-1)$. Let P be an element of $\mathcal{S}_q$ and consider the set $\mathcal{N}$ of tangent lines ℓ on P which satisfy $\bar{\ell} \subset \ell^\perp$. Note that nX is the total number of such lines ℓ through P. The quotient geometry $P^\perp/P$ is important in the enumeration of $\mathcal{N}$. Recall that the number of lines of $\mathcal{S}$ incident with P is equal to the number of points in the quotient geometry $P^\perp/P$; namely $\theta_{r-1}X$. The number of real lines on P is then the number of real points of $P^\perp/P$ (namely $(q+1)X$) and hence the number of tangents on P is simply $(\theta_{r-1} - q - 1)X$. Now in $\mathcal{S}$, the number of nonreal points which are collinear with their conjugate (the set $\mathcal{T}$ above) is $q(q^{2r-2}-1)(q^{2r}-1)/(q^2-1)$. Note that $\mathcal{N}$ corresponds to the nonreal points of $P^\perp/P$ which are collinear to their conjugate, and therefore

$$\mathcal{N} = q(q^{2(r-1)-2}-1)(q^{2(r-1)}-1)/(q^2-1) = q(q^{2r-4}-1)X.$$

Hence, we have that $n = q(q^{2r-4}-1)$ as required. So for all P not in $\mathcal{T}$ we have that $|P^\perp \cap \mathcal{T}| = iX$ where $i = q(q^{2(r-1)}-1)$, and hence $\mathcal{T}$ is i-tight. $\square$

Remark:

For the parabolic quadric $\mathsf{Q}(2r, q^2)$, we could have used a different argument in the proof of Theorem 8 which makes use of Corollary 5. Let $\mathcal{T}$ be the set of nonreal points of $\mathsf{Q}(2r, q^2)$ which lie on a real line. Now a non-degenerate hyperplane section of $\mathsf{Q}(2r, q^2)$ is an elliptic or hyperbolic quadric. Recall from Lemma 7 that a $\mathsf{Q}^-(2r-1, q^2)$ embedding is an m-ovoid of $\mathsf{Q}(2r, q^2)$ where $m = (q^{2(r-1)}-1)/(q^2-1)$. Now $\mathsf{Q}^-(2r-1, q^2) \cap \mathcal{T}$ is simply the set of nonreal points of $\mathsf{Q}^-(2r-1, q^2)$ which are on a real line of $\mathsf{Q}^-(2r-1, q^2)$, and thus, $\mathsf{Q}^-(2r-1, q^2) \cap \mathcal{T}$ consists of

$$q(q^{2r-2}-1)\frac{q^{2r-2}-1}{q^2-1}$$

points. Since $\mathrm{P}\Omega_{2r+1}(q)$ has three orbits on the points of $\mathsf{Q}(2r, q^2)$, the three eigenspaces corresponding to the adjacency matrix of the point graph of $\mathsf{Q}(2r, q^2)$ are irreducible. The eigenspace E corresponding to the eigenvalue $-\theta_{r-1}$ is the orthogonal complement of the sum of the other two eigenspaces. Similarly, the eigenspace corresponding to the eigenvalue $q^{2(r-1)}-1$ is the orthogonal complement of the sum of the other two eigenspaces.

Recall from Section 4 (and in particular, Lemma 2) that a set $\mathcal{O}$ of points is an m-ovoid if and only if

$$\mathcal{X}_{\mathcal{O}} - \frac{m}{(q^{2r}-1)/(q^2-1)}\eta$$

is an eigenvector of the adjacency matrix with eigenvalue $-\theta_{r-1}$, and that a set $\mathcal{I}$ of points is i-tight if and only if

$$\mathcal{X}_{\mathcal{I}} - \frac{i}{\theta_r}\eta$$

is an eigenvector of the adjacency matrix with eigenvalue $q^{2(r-1)}-1$. To show that $\mathcal{T}$ is i-tight, it is sufficient to show that $\mathcal{X}_{\mathcal{T}} - \frac{i}{\theta_r}\eta$ is orthogonal to E. Since E is irreducible and spanned by the elliptic quadrics of rank $r-1$, we have that $\mathcal{X}_{\mathcal{T}} - \frac{i}{\theta_r}\eta$ is orthogonal to $\mathcal{X}_{\mathcal{O}} - \frac{m}{(q^{2r}-1)/(q^2-1)}\eta$ over all elliptic quadrics $\mathcal{O} = \mathsf{Q}^-(2r-1,q^2)$. So by Corollary 5, $\mathcal{T}$ meets every $\mathsf{Q}^-(2r-1,q^2)$ in mi points. Therefore $\mathcal{T}$ is i-tight.

5.3 Field reduction and derivation

In a forthcoming paper of the second author (see [24]), there are two explicit methods for obtaining new intriguing sets from old ones. The first of these is by field reduction where one begins with an intriguing set of a finite polar space over a field $\mathsf{GF}(q^b)$ of algebraic dimension d, from which field reduction yields a new non-degenerate polar space over a field $\mathsf{GF}(q)$ of algebraic dimension db. Under this mapping, and suitable conditions on the input, one obtains an intriguing set of the polar space of larger dimension. Here is a summary of these polar space mappings and the induced intriguing sets (given an intriguing set of the initial polar space).

Mapping $\mathcal{S} \to \mathcal{S}'$	Intriguing set of $\mathcal{S}$	Intriguing set of $\mathcal{S}'$
$\mathsf{H}(2r,q^{2e}) \to \mathsf{H}(e(2r+1)-1,q^2)$, e odd	m-ovoid	$m\frac{q^{2e-1}-1}{q^2-1}$-ovoid
$\mathsf{H}(2r,q^{2e}) \to \mathsf{W}(2e(r+1)-1,q)$	m-ovoid	$m\frac{q^{2e}-1}{q-1}$-ovoid
$\mathsf{H}(2r,q^{2e}) \to \mathsf{Q}^-(2e(r+1)-1,q)$	m-ovoid	$m\frac{q^{2e}-1}{q-1}$-ovoid
$\mathsf{H}(2r-1,q^{2e}) \to \mathsf{H}(2er-1,q^2)$, e odd	i-tight	i-tight
$\mathsf{H}(2r,q^{2e}) \to \mathsf{W}(4er-1,q)$	i-tight	i-tight
$\mathsf{H}(2r,q^{2e}) \to \mathsf{Q}^+(4er-1,q)$	i-tight	i-tight
$\mathsf{W}(2r-1,q^e) \to \mathsf{W}(2er-1,q)$	m-ovoid	$m\frac{q^e-1}{q-1}$-ovoid
$\mathsf{W}(2r-1,q^e) \to \mathsf{W}(2er-1,q)$	i-tight	i-tight
$\mathsf{Q}^+(2r-1,q^e) \to \mathsf{Q}^+(2er-1,q)$	i-tight	i-tight
$\mathsf{Q}^-(2r+1,q^e) \to \mathsf{Q}^-(e(2r+2)-1,q)$	m-ovoid	$m\frac{q^e-1}{q-1}$-ovoid

The second author has also generalised the so-called "derivation" method of Payne and Thas, which was used to obtain new ovoids of generalised quadrangles. In [24], it is shown that derivation in higher rank hermitian spaces and hyperbolic quadrics yields new m-ovoids from old ones.

Theorem 9 ([24, Theorem 2.1]) *Let $\mathcal{O}$ be an m-ovoid of a finite polar space $\mathcal{S}$ of rank r in $\mathsf{PG}(2r-1,q)$ and let ℓ be an r-subspace of $\mathsf{PG}(2r-1,q)$ such that $\ell^\perp \cap \mathcal{O} \subseteq \ell$. Then*

$\mathcal{O}' := (\mathcal{O}\backslash\ell) \cup (\ell^\perp \cap \mathcal{S})$ *is an m-ovoid of $\mathcal{S}$.*

By [24, Lemma 2.2], when $\mathcal{S}$ is one of $\mathsf{H}(2r-1,q^2)$ or $\mathsf{Q}^+(2r-1,q)$, the existence of ℓ in the above theorem is assured.

5.4 m-systems and eggs

Recall that an m-system of a finite polar space $\mathcal{S}$ is a maximal set of m totally isotropic subspaces of $\mathcal{S}$ which are pairwise opposite. In this section, we show that in symplectic spaces and elliptic quadrics, there is a strong connection to eggs. We first point out that the results in this section are already known, but for which a direct proof has not been given. It is shown in [28] that if $\mathcal{S}$ is a symplectic space, an elliptic quadric, or a hermitian space (even dimension), then all m-systems of $\mathcal{S}$ are SPG-reguli of the ambient projective space (see [28, §2.1] for the definition of an *SPG-regulus*). From this observation, and [9, §2], it follows that every m-system of $\mathsf{W}(4m+3,q)$ and $\mathsf{Q}^-(4m+3,q)$ is a pseudo-ovoid (n.b. a more general result was proved in [9]).

An *egg* of the projective space $\mathsf{PG}(2n+m-1,q)$ is a set $\mathcal{E}$ of q^m+1 subspaces of dimension $(n-1)$ such that every three are independent (i.e., span a $(3n-1)$-dimensional subspace), and such that each element of $\mathcal{E}$ is contained in a common complement to the other elements of $\mathcal{E}$ (i.e., each element of $\mathcal{E}$ is contained in a $(n+m-1)$-dimensional subspace having no point in common with any other element of $\mathcal{E}$). The theory of eggs is equivalent to the theory of translation generalised quadrangles (see [32, Chapter 8]). If q is even, then $m=n$ or $m=2n$, and for q odd, the only known examples of eggs have $m=n$ or $m=2n$. The latter are known as *pseudo-ovoids* (notice that an ovoid is an egg with $m=2n=2$). All known eggs are *elementary*, that is, they arise by field reduction of an ovoid of $\mathsf{PG}(3,q)$ or an oval of $\mathsf{PG}(2,q)$.

Theorem 10 *Let $\mathcal{S}$ be one of the polar spaces $\mathsf{W}(4m+3,q)$ or $\mathsf{Q}^-(4m+3,q)$. Then an m-system of $\mathcal{S}$ is a pseudo-ovoid of $\mathsf{PG}(4m+3,q)$.*

PROOF. Let E_1 and E_2 be two elements of an m-system $\mathcal{M}$, and let Σ be the elements of $\mathcal{S}$ in the span of these two elements. If $\mathcal{S}=\mathsf{W}(4m+3,q)$, then Σ is isomorphic to $\mathsf{W}(2m+1,q)$ as E_1 and E_2 are opposite. In the case that $\mathcal{S}=\mathsf{Q}^-(4m+3,q)$, we have that Σ is isomorphic to $\mathsf{Q}^+(2m+1,q)$. In the following, we will denote by Σ a subspace of $\mathcal{S}$ described below:

$\mathcal{S}$	Σ
$\mathsf{W}(4m+3,q)$	$\mathsf{W}(2m+1,q)$
$\mathsf{Q}^-(4m+3,q)$	$\mathsf{Q}^+(2m+1,q)$

We will show that $\Sigma\cup\Sigma^\perp$ is 2-tight. First we look at the case where $\mathcal{S}$ is $\mathsf{W}(4m+3,q)$. Let P be a point of $\mathcal{S}$, and suppose firstly that it resides in the exterior of $\Sigma\cup\Sigma^\perp$. Then $P^\perp$ intersects Σ and $\Sigma^\perp$ each in subspaces of dimension $2m$. If P lies in Σ, then $P^\perp$ intersects Σ in a $2m$-dimensional subspace and $\Sigma^\perp$ in a $(2m+1)$-dimensional subspace. A similar argument applies for the case that P is in $\Sigma^\perp$, and hence there are two constants for the value $|P^\perp \cap (\Sigma \cup \Sigma^\perp)|$ depending on P being in $\Sigma \cup \Sigma^\perp$ or not. By the values we derived for these intersections, it follows that $\Sigma \cup \Sigma^\perp$ is 2-tight.

Now suppose that $\mathcal{S}$ is $\mathsf{Q}^-(4m+3, q)$. First note that $\Sigma^\perp$ is isomorphic to $\mathsf{Q}^-(2m+1, q)$ and hence $|\Sigma \cup \Sigma^\perp| = 2(q^{2m+1} - 1)/(q-1)$. Let P be a point of $\mathcal{S}$, and suppose firstly that it resides in the exterior of $\Sigma \cup \Sigma^\perp$. Then $P^\perp$ intersects Σ and $\Sigma^\perp$ each in parabolic quadrics of dimension $2m$. So $|P^\perp \cap (\Sigma \cup \Sigma^\perp)| = 2(q^{2m} - 1)/(q-1)$. If P lies in Σ, then $P^\perp$ intersects Σ in a cone with vertex P subtended by a hyperbolic quadric of dimension $2m-1$. So $|P^\perp \cap \Sigma| = q(q^m - 1)(q^{m-1} + 1)/(q-1) + 1$. Since P is not in $\Sigma^\perp$, we have that $P^\perp$ meets meets $\Sigma^\perp$ in a parabolic quadric of dimension $2m$. Therefore

$$\begin{aligned}|P^\perp \cap (\Sigma \cup \Sigma^\perp)| &= |P^\perp \cap \Sigma| + |P^\perp \cap \Sigma^\perp| \\ &= q(q^m - 1)(q^{m-1} + 1)/(q-1) + 1 + (q^{2m} - 1)/(q-1) \\ &= q^m + 2(q^{2m} - 1)/(q-1).\end{aligned}$$

Again, a similar argument applies for the case that P is in $\Sigma^\perp$, and hence there are two constants for the value $|P^\perp \cap (\Sigma \cup \Sigma^\perp)|$ depending on P being in $\Sigma \cup \Sigma^\perp$ or not. Therefore $\Sigma \cup \Sigma^\perp$ is 2-tight. So $\Sigma \cup \Sigma^\perp$ meets the set of underlying points of $\mathcal{M}$ in

$$2\frac{q^{m+1} - 1}{q-1}$$

points, which is the total size of $E_1 \cup E_2$. Hence if E_3 is an element of $\mathcal{M}$ distinct from E_1 and E_2, then we have that E_3 is disjoint from Σ, and so $\langle E_1, E_2, E_3 \rangle$ is as large as possible; a $(3m+2)$-subspace. If we define $T(E) = E^\perp$ for every element E of $\mathcal{M}$, we see that the $T(E)$ play the role of tangent spaces in the axioms for an egg. $\square$

There are no known m-systems of $\mathsf{W}(4m+3, q)$, q odd. By a result of Hamilton and Mathon [21], there are no m-systems of $\mathsf{W}(2n+1, q)$ for $n > 2m+1$ (we have $n = 2m+1$ here). Luyckx and Thas (see [29] and [30]) showed that there is a unique 1-system of $\mathsf{Q}^-(7, q)$, namely that arising by field reduction of the elliptic quadric $\mathsf{Q}^-(3, q^2)$. Thus the unique 1-system of $\mathsf{Q}^-(7, q)$ is an elementary pseudo-ovoid.

5.5 The Split Cayley hexagon

The analogue of an intriguing set for the Split Cayley hexagon $\mathcal{H}(q)$ gives rise to an intriguing set of $\mathsf{Q}(6, q)$ in its natural embedding and has been studied by De Wispelaere in her Ph. D. thesis [10]. We say that a set of points $\mathcal{I}$ of a generalised hexagon is *intriguing*, if there are two constants h_1 and h_2, such that $|P^\perp \cap \mathcal{I}|$ equals h_1 for points $P \in \mathcal{I}$, and h_2 for points P in the exterior of $\mathcal{I}$. Let A be the adjacency matrix of the point graph of $\mathcal{H}(q)$ and let η be the constant function with value 1. The following facts can either be established easily or found in [3, §6.5]:

(i) the eigenvalues of A are $q(q+1)$, $2q-1$, -1, and $-q-1$;
(ii) η is an eigenvector of A with eigenvalue $k = q(q+1)$;

(iii) $(h_1 - h_2 - k - 1)\mathcal{X}_{\mathcal{I}} + h_2\eta$ is an eigenvector of A with eigenvalue $h_1 - h_2 - 1$.

So the possible values for $h_1 - h_2$ of a proper intriguing set of $\mathcal{H}(q)$ are $2q$, 0, and $-q$. We will call an intriguing set *positive*, *zero*, or *negative*, if the difference $h_1 - h_2$ is positive, zero, or negative respectively.

A *distance-2-ovoid* of $\mathcal{H}(q)$ is a set of points such that each line of $\mathcal{H}(q)$ meets it in precisely one point. For more on distance-2-ovoids, see [10]. A distance-2-ovoid is an example of a negative intriguing set of $\mathcal{H}(q)$ with intersection numbers $(1, q+1)$, and each such intriguing set of $\mathcal{H}(q)$ is a distance-2-ovoid. Moreover, if $\mathcal{I}$ is a set of points of $\mathcal{H}(q)$ such that each line of $\mathcal{H}(q)$ meets $\mathcal{I}$ in m points, then $\mathcal{I}$ is a negative intriguing set of $\mathcal{H}(q)$ with intersection numbers $(m(q+1) - q, m(q+1))$. An *ovoid* of $\mathcal{H}(q)$ is a set of q^3+1 pairwise opposite points of $\mathcal{H}(q)$. Dually, a *spread* of $\mathcal{H}(q)$ is a set of q^3+1 pairwise opposite lines of $\mathcal{H}(q)$. The point-sets underlying ovoids and spreads of $\mathcal{H}(q)$ are examples of zero intriguing sets of $\mathcal{H}(q)$ with intersection numbers $(1,1)$ and $(q+1, q+1)$ respectively. The converse is also true. Finally, the points of a subhexagon of order $(q,1)$ in $\mathcal{H}(q)$ is a positive intriguing set of $\mathcal{H}(q)$ with intersection numbers $(2q+1, 1)$ (and vice versa).

To date, there are only four known distance-2-ovoids of $\mathcal{H}(q)$ and they have been classified for $q \in \{2,3,4\}$. (See [11–13] for the cases $q = 2$ and $q = 3$. The case $q = 4$ was achieved recently by Sven Reichard [33]). Thas [36] showed that an ovoid of $\mathcal{H}(q)$ exists if and only if $\mathsf{Q}(6,q)$ has an ovoid. Moreover, $\mathsf{Q}(6,q)$ has an ovoid for q an odd power of 3 (see [38]), and these are the only field orders for which ovoids are known to exist.

Now the stabiliser of $\mathcal{H}(q)$ in $\mathrm{GO}_7(q)$ (i.e., the Ree group $G_2(q)$) has rank 3 on the points of $\mathsf{Q}(6,q)$. Hence, $G_2(q)$ admits four irreducible submodules, compared to the three irreducible submodules of $\mathrm{GO}_7(q)$. The dimensions of these modules are precisely the multiplicities of the eigenvalues corresponding to the associated point graph in question (i.e., that for $\mathcal{H}(q)$ or $\mathsf{Q}(6,q)$). It turns out that two of the eigenspaces of $\mathcal{H}(q)$ combine to yield one of the eigenspaces of $\mathsf{Q}(6,q)$ and thus the negative and positive intriguing sets of $\mathcal{H}(q)$ are in fact tight sets of $\mathsf{Q}(6,q)$; the zero intriguing sets of $\mathcal{H}(q)$ with parameters (m,m) are m-ovoids of $\mathsf{Q}(6,q)$.

6 Sets of Type (a,b) and Non-existence Results

It was shown in the seminal work of Shult and Thas [35], that an m-system sometimes gives rise to a strongly regular graph. To be more specific, the points underlying the elements of an m-system of $\mathsf{H}(2r,q^2)$, $\mathsf{Q}^-(2r+1,q)$, or $\mathsf{W}(2r-1,q)$, have two intersection numbers with respect to hyperplanes. Such a set gives rise naturally to a strongly regular graph (see [5] for a description of Delsarte's construction). We have analogous results for m-ovoids and i-tight sets of finite polar spaces.

Theorem 11 *Let $\mathcal{S}$ be one of the polar spaces $\mathsf{H}(2r,q^2)$, $\mathsf{Q}^-(2r+1,q)$, or $\mathsf{W}(2r-1,q)$ and let $\mathcal{O}$ be a proper m-ovoid that spans the ambient projective space. Then the set of points covered by $\mathcal{O}$ have two intersection numbers with respect to hyperplanes and so defines a* *strongly regular graph with parameters (v, k, λ, μ) as follows:*

$\mathcal{S}$	$\mathsf{H}(2r, q^2)$	$\mathsf{Q}^-(2r+1, q)$	$\mathsf{W}(2r-1, q)$
v	q^{4r+2}	q^{2r+2}	q^{2r}
k	$m(q^2-1)(q^{2r+1}+1)$	$m(q-1)(q^{r+1}+1)$	$m(q-1)(q^r+1)$
λ	$m(q^2-1)(3+m(q^2-1))-q^{2r+1}$	$m(3+m(q-1))(q-1)-q^r$	$m(3+m(q-1))(q-1)-q^r$
μ	$m(q^2-1)(m(q^2-1)+1)$	$m(q-1)(m(q-1)+1)$	$m(q-1)(m(q-1)+1)$

PROOF. Here we outline some of the steps needed to observe this result. First recall that a hyperplane is the perp of a point of the ambient projective space. So we consider two types of points: those in $\mathcal{S}$ and those in the exterior of $\mathcal{S}$. In the former case, we know that for all $P \in \mathcal{S}$, the number of points of $\mathcal{S}$ collinear with P is determined by the type of $\mathcal{S}$, and we get two intersection numbers for $|P^\perp \cap \mathcal{O}|$; call them h_1 and h_2. In the latter (which does not occur for symplectic spaces), we have for all $P \notin \mathcal{S}$ that $P^\perp$ is a non-degenerate hyperplane, and so by Lemma 7, $P^\perp \cap \mathcal{S}$ is an i-tight set for some i, and hence intersects $\mathcal{O}$ in mi points. Coincidentally, for the polar spaces we are considering, the value of i that we obtain implies that $\mathcal{O}$ has precisely the two intersection numbers h_1 and h_2 that we had before:

$\mathcal{S}$	$\mathsf{H}(2r, q^2)$	$\mathsf{Q}^-(2r+1, q)$	$\mathsf{W}(2r-1, q)$
h_1	$m(q^{2r-1}+1)$	$m(q^r+1)$	$m(q^{r-1}+1)$
h_2	$m(q^{2r-1}+1)-q^{2r-1}$	$m(q^r+1)-q^r$	$m(q^{r-1}+1)-q^{r-1}$

Hence $\mathcal{O}$ gives rise to a projective set $[c, d+1, h_1, h_2]$ where $c = |\mathcal{O}| = m\theta_r$ and d is the projective dimension (see also [21]). In turn, this projective set gives rise to a strongly regular graph with parameters (v, k, λ, μ) given below (note: β is the size of the underlying field, q or q^2):

$$\begin{aligned}
v &= \beta^{d+1} \\
k &= c(\beta - 1) \\
\lambda &= k^2 + 3k - \beta(k+1)(2c - h_1 - h_2) + \beta^2(c-h_1)(c-h_2) \\
\mu &= \beta^2(c-h_1)(c-h_2)/v.
\end{aligned}$$

From these formulae, the result follows. □

We have a similar result for tight sets.

Theorem 12 *Let $\mathcal{S}$ be one of the polar spaces $\mathsf{H}(2r-1, q^2)$, $\mathsf{Q}^+(2r-1, q)$, $\mathsf{W}(2r-1, q)$ and let $\mathcal{T}$ be a proper i-tight set that spans the ambient projective space. Then the set of points covered by $\mathcal{T}$ have two intersection numbers with respect to hyperplanes and so* *defines a strongly regular graph with parameters (v,k,λ,μ) as follows:*

$\mathcal{S}$	$\mathsf{H}(2r-1,q^2)$	$\mathsf{Q}^+(2r-1,q)$	$\mathsf{W}(2r-1,q)$
v	q^{4r}	q^{2r}	q^{2r}
k	$i(q^{2r}-1)$	$i(q^r-1)$	$i(q^r-1)$
λ	$i(i-3)+q^{2r}$	$i(i-3)+q^r$	$i(i-3)+q^r$
μ	$i(i-1)$	$i(i-1)$	$i(i-1)$

PROOF. Similar to the proof of Theorem 11, except that the non-degenerate hyperplanes in the polar spaces here (if they exist), are m-ovoids for some m (and so intersect $\mathcal{T}$ in mi points). $\square$

Given that the parameters (in particular λ) of a strongly regular graph have to be non-negative integers, we now have some non-existence results concerning m-ovoids of some spaces.

Theorem 13 *Let $\mathcal{S}$ be one of the polar spaces $\mathsf{H}(2r,q^2)$, $\mathsf{Q}^-(2r+1,q)$, or $\mathsf{W}(2r-1,q)$ and let $\mathcal{O}$ be an m-ovoid. Then $m \geqslant b$ where b is given in the table below:*

$\mathsf{H}(2r,q^2)$	$(-3+\sqrt{9+4q^{2r+1}})/(2q^2-2)$
$\mathsf{Q}^-(2r+1,q)$	$(-3+\sqrt{9+4q^{r+1}})/(2q-2)$
$\mathsf{W}(2r-1,q)$	$(-3+\sqrt{9+4q^{r}})/(2q-2)$

PROOF. First observe that if $\mathcal{S}$ is $\mathsf{H}(2r,q^2)$ or $\mathsf{Q}^-(2r+1,q)$, then a non-degenerate hyperplane section $X^\perp \cap \mathcal{S}$ is i-tight where i is $q^{2r-1}+1$ and q^r+1 respectively. By Corollary 5, $\mathcal{O}$ must meet $X^\perp$ in mi points, and since $i \neq \theta_r$, we have that $\mathcal{O}$ is not contained in any non-degenerate hyperplane. Suppose $X \in \mathcal{S}$ and $\mathcal{O}$ is contained in $X^\perp$. So $|\mathcal{O}|$ is equal to $m\theta_{r-1}$ or $m\theta_{r-1}-\theta_{r-1}+1$, according to whether X is in $\mathcal{O}$ or not. However $|\mathcal{O}| = m\theta_r$, which is strictly greater than both of these values. Therefore $\mathcal{O}$ must span $\mathsf{PG}(2r,q^2)$. So by Theorem 11 and our hypothesis on m, we would have a related strongly regular graph with a negative value for λ – a contradiction. Therefore, there are no m-ovoids of $\mathcal{S}$ with $m < b$. $\square$

Note that if $\mathcal{S}$ is $\mathsf{H}(2r,q^2)$ or $\mathsf{Q}^-(2r+1,q)$, the above bounds are greater than 1, and in the case $\mathcal{S} = \mathsf{W}(2r-1,q)$, the bound is greater than 1 if $r>2$; so in these cases ovoids never exist. Hence we have proved via this method a non-existence result due to Thas [36]. Unforunately, a similar approach does not yield useful conditions on the existence of tight sets. Now we prove the non-existence of 2-ovoids in certain polar spaces.

Theorem 14 *The following polar spaces do not admit 2-ovoids:*

(i) $\mathsf{W}(2r-1,q)$ *for $r>2$ and q odd;*

(ii) $\mathsf{Q}^-(2r+1,q)$ *for* $r>2$*;*
(iii) $\mathsf{H}(2r,q^2)$ *for* $r>2$*;*
(iv) $\mathsf{Q}(2r,q)$ *for* $r>4$.

PROOF. Let $\mathcal{O}$ be a 2-ovoid of the polar space $\mathcal{S}$ and let P be a point of $\mathcal{O}$. Then every line of $\mathcal{S}$ on P contains at most one further point of $\mathcal{O}$. Hence the set of points $Q \in \mathcal{O}$ that are collinear but not equal to P, projects onto an ovoid of $P^\perp/P$, which is a polar space of the same type as $\mathcal{S}$ but of rank one less. Since $\mathsf{W}(2r-3,q)$ has no ovoids for q odd and $r>2$, it follows that $\mathsf{W}(2r-1,q)$ has no 2-ovoids for q odd and $r>2$. Similarly, since $\mathsf{Q}^-(2r-1,q)$ has no ovoids for $r>2$, it follows that $\mathsf{Q}^-(2r+1,q)$ has no 2-ovoids for $r>2$. Also, since $\mathsf{H}(2r-2,q^2)$ has no ovoids for $r>2$, it follows that $\mathsf{H}(2r,q^2)$ has no 2-ovoids for $r>2$. Finally, since $\mathsf{Q}(2r-2,q)$ has no ovoids for $r>4$, it follows that $\mathsf{Q}(2r,q)$ has no 2-ovoids for $r>4$. □

One could also extend the method of the last proof to $\mathsf{Q}(8,q)$. It was shown by Ball, Govaerts, and Storme [1] that $\mathsf{Q}(6,q)$ has no ovoids for q a prime number greater than 3, and hence $\mathsf{Q}(8,q)$ has no 2-ovoids for these values of q. Below we prove the non-existence of irreducible i-tight sets for small i, which extends the results of Drudge mentioned in Section 3.

Theorem 15 *Let q be a prime power and we will suppose that q is a square when referring to a hermitian variety. There are no irreducible i-tight sets in $\mathsf{W}(2r-1,q)$, $\mathsf{H}(2r-1,q)$, or $\mathsf{Q}^+(2r-1,q)$ for $2<i<\epsilon_q$, where $q+\epsilon_q$ is the size of the smallest non-trivial blocking set in $\mathsf{PG}(2,q)$ for $q>2$, and $\epsilon_2=2$. Moreover, there are no proper 2-tight sets in $\mathsf{H}(2r-1,q)$ or $\mathsf{Q}^+(2r-1,q)$, and for r odd there are no i-tight sets (including the reducible examples) in $\mathsf{Q}^+(2r-1,q)$ for the values of i given above.*

PROOF. Let $\mathcal{T}$ be an irreducible i-tight set in $\mathsf{W}(2r-1,q)$, $\mathsf{H}(2r-1,q)$, or $\mathsf{Q}^+(2r-1,q)$, where $2<i<\epsilon_q$. The underlying set of points of $\mathcal{T}$ is a *minihyper* of $\mathsf{PG}(2r-1,q)$ with parameters satisfying the hypotheses of [20, Theorem 13]. It follows from their theorem that $\mathcal{T}$ is the union of i subspaces, each of dimension $r-1$. If the polar space is not $\mathsf{W}(2r-1,q)$, it follows that these subspaces are maximals, and the result is a consequence. A minimal counterexample in $\mathsf{W}(2r-1,q)$ contains no maximal. Call the subspaces $U_1,\ldots,U_i$. For a point P in U_j, we have that $P^\perp\cap U_k$ is either a hyperplane of U_k or U_k itself. Hence there is a unique j for which the latter alternative occurs for P. Thus the non-empty sets amongst the set of $U_k^\perp\cap U_j$ (where $1\leqslant k\leqslant i$) partition the points of U_j not in the radical of U_j. Since there are less than q choices for k and at least q^{r-1} points to be covered, some $U_k^\perp$ equals U_j. But now $U_j\cup U_k$ is 2-tight (since the radical of U_j is now empty), and the original set is reducible. When the polar space is $\mathsf{Q}^+(2r-1,q)$ and r is odd, the largest partial spread has size 2, and so there are no reducible examples either. □

Remark:

The quantity ϵ_q is bounded below by $\sqrt{q}$, with equality when q is square (see [4]), but better bounds exist for q not a square. Moreover, these bounds are always being improved.

7 Known Intriguing Sets in Polar Spaces of Low Rank

In the last section of [2], some computer results were reported of interesting intriguing sets of classical generalised quadrangles. The same will be done here except that we will limit our scope to polar spaces of rank at most 3. It will be evident from the data below that intriguing sets are very much abundant in polar spaces of low rank, although we will be interested in knowing which type of intriguing set is more dominant. Corollary 5 seems to indicate that spaces with many m-ovoids do not harbour as many i-tight sets, and vice versa.

7.1 Symplectic spaces

For q even, there exist ovoids of $\mathsf{W}(3,q)$. Moreover, there exist sets of disjoint ovoids of $\mathsf{W}(3,q)$ and hence there are many m-ovoids arising in $\mathsf{W}(3,q)$ for q even. It was shown in [2] that for q odd, there exists a partition of the set of points of $\mathsf{W}(3,q)$ into 2-ovoids. Therefore, there exist m-ovoids of $\mathsf{W}(3,q)$ for every possible even value of m. Moreover, it has been shown recently by computer that the only m-ovoid of $\mathsf{W}(3,3)$ is the 2-ovoid constructed in [2]. In the last section of [2], there were descriptions of some computer results in $\mathsf{W}(3,7)$ where there are partitions of the set of points of $\mathsf{W}(3,7)$ into tight sets. They arise by taking A_7 acting on its fully deleted permutation module. It has two orbits on points (each is therefore intriguing) which are 15-tight and 35-tight respectively. The subgroup A_6 of A_7 has four orbits on points of $\mathsf{W}(3,7)$ of lengths 40, 120, 120, and 120. By taking a union of two of the larger orbits, one obtains a partial spread of 30 lines. The remaining orbits are 5-tight and 15-tight respectively.

Recall from the introduction that there is a unique perp-system of $\mathsf{PG}(5,3)$ giving rise to a 3-ovoid of $\mathsf{W}(5,3)$. The stabiliser of this 3-ovoid in $\mathrm{PGSp}_6(3)$ is S_5. If one takes the subgroup of S_5 isomorphic to S_4, then a union of orbits of it forms a 3-ovoid that is not a perp-system. To date, we have been unable to give a geometric description of this 3-ovoid. For q even, $\mathsf{W}(5,q)$ is isomorphic to $\mathsf{Q}(6,q)$, and hence there are many 1-systems giving rise to $(q+1)$-ovoids of $\mathsf{W}(5,q)$. By computer, we have classified the proper m-ovoids of $\mathsf{W}(5,2)$ which turn out to be 3-ovoids or complements thereof. In fact, there are 13,384 of these 3-ovoids, and they arise either as the points underlying a 1-system, or their stabiliser is $A_5 \times C_2$ or $2^{1+4}.(S_3 \times S_3)$. The m-systems of $\mathsf{W}(5,2)$ were classified in [21]. We list below a table summarising the known m-ovoids of $\mathsf{W}(5,q)$, q odd, which do not arise from the points underlying a 1-system, nor those that are known to be reducible. We also restrict m to the interval $1 \leqslant m \leqslant (q^2+q)/2$, since other m-ovoids arise as complements.

q	m	Stabiliser	q	m	Stabiliser
3	3	S_4	5	5	$(C_5 \times A_5) : C_4$
	3	S_5		10	$(C_5 \times A_5) : C_4$
	6	$C_7 : C_6$		15	$(C_5 \times A_5) : C_4$
					$(2^{1+4}.A_5).2$
					$(A_4 \times A_5).2$
					$5^2 : (C_4 \times S_3)$
					$J_2.2$

Table 2
Interesting m-ovoids of $\mathsf{W}(5, q)$, q odd.

7.2 Hermitian spaces

As was noted in [2], there exist partitions of $\mathsf{H}(3, q^2)$ into disjoint ovoids (which are called *fans*) for all q. Thus, there exist m-ovoids of $\mathsf{H}(3, q^2)$ for all possible q. For tight sets, we have the points covered by partial spreads of $\mathsf{H}(3, q^2)$, embedded symplectic spaces $\mathsf{W}(3, q)$, and those obtained by field reduction of $\mathsf{W}(1, q^4)$; or complements and unions of these. It is not known whether there exists a tight set of $\mathsf{H}(3, q^2)$ which cannot be constructed this way.

To date, there are no known m-ovoids of $\mathsf{H}(4, q^2)$, and it was shown in [2] that if an m-ovoid exists in $\mathsf{H}(4, q^2)$, then $m \geqslant \sqrt{q}$. Now we saw in Section 5 that every i-tight set of $\mathsf{H}(3, q^2)$ gives rise to an i-tight set of $\mathsf{H}(4, q^2)$ by its natural embedding. The only known tight sets of $\mathsf{H}(4, q^2)$ are contained in a non-degenerate hyperplane section, or arise as the points of a partial spread.

Recall that in $\mathsf{H}(5, q^2)$, the points lying on a non-degenerate hyperplane section form a $(q^2 + 1)$-ovoid, and by field reduction of $\mathsf{H}(2, q^6)$, we induce a $(q^2 + 1)$-ovoid of $\mathsf{H}(5, q^2)$. In $\mathsf{H}(5, 2^2)$ and $\mathsf{H}(5, 3^2)$, there exist partitions of the point sets into m-ovoids, one part of which is the classical $(q^2 + 1)$-ovoid. These are summarised below:

q	m	Stabiliser	Comments
2	5	$3.\mathrm{PSU}_5(2^2).2$	$\mathsf{H}(4, 2^2)$
	6	$\mathrm{PSL}_2(11).S_3$	–
	10	$\mathrm{PSL}_2(11).S_3$	–
3	10	$8.\mathrm{PSU}_5(3^2).2$	$\mathsf{H}(4, 3^2)$
	36	$C_{244} : C_{10}$	–
	45	$C_{244} : C_{10}$	–

Table 3
Partitions of $\mathsf{H}(5, q^2)$ into m-ovoids.

Once one looks further to higher q, there are more partitions of $\mathsf{H}(5, q^2)$ into a non-degenerate hyperplane section and at least two m-ovoids. In $\mathsf{H}(5, q^2)$, there exist $(q + 1)$-tight sets obtained by field reduction of $\mathsf{H}(1, q^6)$. We also have an interesting example in $\mathsf{H}(5, 2^2)$ of an 11-tight set admitting the almost simple group $M_{22}.2$. Note that the same group acts on a 2-dimensional dual hyperoval.

As for $\mathsf{H}(6,q^2)$, there are no nontrivial m-ovoids known. Now a non-degenerate hyperplane section of $\mathsf{H}(6,q^2)$ is (q^5+1)-tight, and if $\mathcal{I}$ is an i-tight set of $\mathsf{H}(5,q^2)$ (for some i), then the natural embedding of $\mathcal{I}$ in $\mathsf{H}(6,q^2)$ yields an i-tight set. The only known tight sets of $\mathsf{H}(6,q^2)$ are those induced by a tight set of $\mathsf{H}(5,q^2)$ (see above).

7.3 Parabolic quadrics

In Section 5 we saw that non-degenerate hyperplane sections and subfield geometries gave rise to intriguing sets of parabolic quadrics. We also found that intriguing sets of the Split Cayley hexagon $\mathcal{H}(q)$ produce intriguing sets of $\mathsf{Q}(6,q)$. In [2], there were many m-ovoids of $\mathsf{Q}(4,q)$ found by computer, however, it is still unknown whether there exist 2-ovoids of $\mathsf{Q}(4,q)$ for $q>5$. In addition, we have classified m-ovoids of $\mathsf{Q}(4,3)$ by computer, finding a unique 1-ovoid, a unique 2-ovoid and a unique 3-ovoid.

In $\mathsf{Q}(6,3)$, there are m-ovoids for every possible m (since there exists a partition of the points into ovoids). Similarly, there are many tight sets of $\mathsf{Q}(6,3)$ arising from partial spreads. In $\mathsf{Q}(6,5)$, we have a 2-tight set admitting $2.\mathrm{PSL}_3(5).2$, which is precisely the stabiliser of the weak subhexagon of order $(11,1)$ in the dual of $\mathcal{H}(5)$. In the complement of this 2-tight set lies a quadric $\mathsf{Q}^+(5,5)$ which is 26-tight. There are some interesting m-ovoids of $\mathsf{Q}(6,5)$, namely, a hemisystem admitting $3.A_7.2$, elliptic quadric $\mathsf{Q}^-(5,5)$, a 15-ovoid admitting $2^3.\mathrm{PSL}_3(2)$, a 15-ovoid admitting $\mathrm{PSU}_3(3).2$, and a 16-ovoid admitting $3.A_6.2$.

7.4 Elliptic quadrics

Recall that a parabolic quadric $\mathsf{Q}(2r,q)$ embedded in $\mathsf{Q}^-(2r+1,q)$ is (q^r+1)-tight. For all q, there exist spreads of $\mathsf{Q}^-(5,q)$, and hence many tight sets (see [28, A.3.4]). In $\mathsf{Q}^-(7,q)$, not much is known about its tight sets. There are no ovoids of $\mathsf{Q}^-(5,q)$, but there do exist interesting $(q+1)/2$-ovoids (i.e., hemisystems) of $\mathsf{Q}^-(5,q)$. Indeed, Segre [34] proved that a proper m-ovoid of $\mathsf{Q}^-(5,q)$ is a hemisystem, and that there exist no proper m-ovoids for q even. Recently, it has been shown by Cossidente and Penttila [7] that for q odd, there exists a hemisystem of $\mathsf{Q}^-(5,q)$ admitting $\mathrm{P}\Omega_4^-(q)$. In $\mathsf{Q}^-(5,5)$, there is a hemisystem admitting $3.A_7.2$, and in $\mathsf{Q}^-(5,7)$, there is a hemisystem admitting a metacyclic group of order 516 (see [7]). Below, we summarise the known hemisystems of $\mathsf{Q}^-(5,q)$ for small q. By [29] and [30], $\mathsf{Q}^-(7,q)$ has a unique 1-system for all q, which gives rise to a $(q+1)$-ovoid of $\mathsf{Q}^-(7,q)$; otherwise there are no other proper m-ovoids known.

q	Group admitted	q	Group admitted
3	$\mathrm{PSL}_3(4)$	7	$2^4.A_5$
5	$\mathrm{P}\Omega_4^-(5)$		$C_{43}:C_{12}$
	$3.A_7.2$	9	$\mathrm{P}\Omega_4^-(9)$
7	$\mathrm{P}\Omega_4^-(7)$		$C_{73}:C_{12}$

Table 4
Known hemisystems of $\mathsf{Q}^-(5,q)$ for small q.

7.5 Klein quadric

As noted by Drudge [15], tight sets of the Klein quadric $\mathsf{Q}^+(5,q)$ are exactly the images of Cameron-Liebler line classes of $\mathsf{PG}(3,q)$. These have their own literature which can be found in Govaerts and Penttila [18]. Applying the Klein correspondence to a packing of $\mathsf{PG}(3,q)$ (a partition of the lines into spreads) gives a partition of the point set of $\mathsf{Q}^+(5,q)$ into ovoids. Packings of $\mathsf{PG}(3,q)$ were first constructed in [14]. This shows that m-ovoids of $\mathsf{Q}^+(5,q)$ exist for all possible m. Irreducible 2-ovoids of $\mathsf{Q}^+(5,q)$ were constructed in Ebert [17] and Drudge [16].

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