TORSION FREE GROUPS GENERATED BY A PAIR OF RATIONAL PARABOLIC MÖBIUS TRANSFORMATIONS

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Abstract. Let T be a subgroup of $PSL(2,\mathbb{Q})$ generated by a pair of rational parabolic matrices P_1, P_2, and let $\mathcal{J} = \mathcal{J}(P_1, P_2)$ be the Jørgensen number. We prove that T has a non-trivial element of finite order if and only if $\mathcal{J} = \frac{4}{n^2}$ or $\mathcal{J} = \frac{9}{n^2}$ for some non-zero integer n.

Recall that a matrix $A \in SL(2,\mathbb{Q})$ is *parabolic* if $Tr(A) = \pm 2$ and $A \neq \pm I$. In 1975, Allen Charnow proved that if m is rational, then the group Γ_m, generated by the parabolic matrices $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, has an element of finite order if and only if m is the reciprocal of an integer [1]. The aim of this note is to observe that Charnow's proof can be slightly modified to give the following more general result.

Theorem. *Let T be a subgroup of $PSL(2,\mathbb{Q})$ generated by a pair of rational parabolic elements P_1, P_2, and let $\mathcal{J} = |Tr[P1, P2] - 2|$ be the Jørgensen number. Then T has a non-trivial element of finite order if and only if $\mathcal{J} = \frac{4}{n^2}$ or $\mathcal{J} = \frac{9}{n^2}$ for some natural number n.*

Proof. Let $\mu\colon SL(2,\mathbb{Q}) \to PSL(2,\mathbb{Q})$ be the natural quotient map. Choose parabolic matrices $P_1^+, P_2^+ \in SL(2,\mathbb{Q})$ with positive trace such that $\mu(P_1^+) = P_1$ and $\mu(P_2^+) = P_2$ and let T^+ be the subgroup of $SL(2,\mathbb{Q})$ generated by P_1^+ and P_2^+. First notice that T has a non-trivial element of finite order if and only if T^+ has an element of finite order not in the centre $\{\pm I\}$ of $SL(2,\mathbb{Q})$. Secondly, it is well known and easy to prove (cf. [2]) that T^+ is conjugate in $SL(2,\mathbb{C})$ to the group G_x generated by the matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, where $x = \frac{1}{2}Tr(P_1^+ P_2^+) - 1$. Note that $4x^2 = \mathcal{J}$. So it remains to show that G_x has an element of finite order not in $\{\pm I\}$ if and only if $x = \frac{1}{n}$ or $x = \frac{3}{2n}$ for some non-zero integer n.

Let $n \in \mathbb{Z}\backslash\{0\}$ and $C = A^{-1}B^n$. If $x = \frac{1}{n}$, then $C = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$ and $C^4 = I$. If $x = \frac{3}{2n}$, then $C = \begin{pmatrix} -2 & -2 \\ 3/2 & 1 \end{pmatrix}$ and $C^3 = I$. So in both cases, G_x has an element of finite order not equal to $\pm I$.

Conversely, assume G_x has a non-trivial element of finite order. So G_x has an element C whose order is a prime, p say. Recall that $SL(2,\mathbb{Q})$ only has elements of prime order p for $p = 2$ and $p = 3$. Indeed, if C has order p, then the eigenvalues λ, λ^{-1} of C are primitive p^{th} roots of unity. In particular, the degree of λ over $\mathbb{Q}$ is

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$p-1$. But as the characteristic polynomial of C is quadratic, λ has degree at most 2. Hence $p \leq 3$.

It is not difficult to show that since $C \in G_x$, C can be written in the form

$$C = \begin{pmatrix} 1+2xf_1(x) & 2f_2(x) \\ xf_3(x) & 1+2xf_4(x) \end{pmatrix},$$

where $f_1, \ldots, f_4$ are polynomials with integer coefficients. Let $x = \frac{m}{n}$, where $m \in \mathbb{N}$, $n \in \mathbb{Z} \backslash \{0\}$ and $(m,n) = 1$.

If $p = 2$, $C = -I$. In particular, $1 + 2xf_1(x) = -1$, and so $xf_1(x) + 1 = 0$. Applying the Rational Roots Test (see for example [3]), one obtains $m = 1$.

If $p = 3$, $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ and so $Tr(C) = -1$. This gives

$$2x(f_1(x) + f_4(x)) + 3 = 0. \qquad (*)$$

So by the Rational Roots Test, $m = 1$ or $m = 3$. Finally, if $m = 3$ then $(*)$ gives $2(f_1(x) + f_4(x)) + 3n = 0$, which implies that n is even. This completes the proof. □

Remark. The theorem does not hold in $SL(2, \mathbb{Q})$. For example, consider the parabolic matrices $A = -\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 5/14 & 1 \end{smallmatrix}\right)$. Here $\mathcal{J} = \frac{25}{49}$, which is evidently not of the form $\frac{4}{n^2}$ or $\frac{9}{n^2}$. However $A^7BA^{-1}B^{-7}A^{-1}B = -I$ and hence $\langle A, B \rangle$ has an element of order 2.

References

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