Transitive Eggs

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Abstract

We prove that a pseudo-oval or pseudo-ovoid (that is not an oval or ovoid) admitting an insoluble transitive group of collineations is elementary and arises over an extension field from a conic, an elliptic quadric, or a Suzuki-Tits ovoid.

MSC 2000: 51E20

1 Introduction

An egg of the projective space \( \text{PG}(2n + m - 1, q) \) is a set \( E \) of \( q^m + 1 \) subspaces of dimension \((n - 1)\) such that every three are independent (i.e., span a \((3n - 1)\)-dimensional subspace), and such that each element of \( E \) is contained in a common complement to the other elements of \( E \) (i.e., each element of \( E \) is contained in an \((n + m - 1)\)-dimensional subspace having no point in common with any other element of \( E \)). The theory of eggs is equivalent to the theory of translation generalised quadrangles (see [20, Chapter 8]). If \( q \) is even, then \( m = n \) or \( m = 2n \) (see [20, 8.7.2]), and for \( q \) odd, the only known examples of eggs have \( m = n \) or \( m = 2n \). Now an ovoid of \( \text{PG}(3, q) \) is an example of an egg where \( m = 2n = 1 \); hence an egg having \( m = 2n \) is called a pseudo-ovoid. Likewise, an oval of \( \text{PG}(2, q) \) is an egg where \( m = n = 2 \), and henceforth, a pseudo-oval is an egg with \( m = n \). If \( O \) is an oval of \( \text{PG}(2, q^n) \), then by field reduction from \( \text{GF}(q^n) \) to \( \text{GF}(q) \), one obtains a pseudo-oval of \( \text{PG}(3n - 1, q) \). Such pseudo-ovals are called elementary. Likewise, field reduction of an ovoid of \( \text{PG}(3, q^n) \) yields an elementary pseudo-ovoid of \( \text{PG}(4n - 1, q) \). All known pseudo-ovals are elementary, and in even characteristic, every known example of a pseudo-ovoid is elementary. There is some conflict over the definition of a classical pseudo-ovoid. In [6] and [24], a classical pseudo-ovoid is one which arises by field reduction from an elliptic quadric. However, some authors (e.g., Cossidente and King [9]) also include the Suzuki-Tits ovoids in their definition of a classical ovoid. Such confusion will be avoided in this paper by not using the term classical at all; so we will take the perhaps cumbersome approach of stating our results explicitly.

By Segre’s Theorem [22], every oval of \( \text{PG}(2, q) \), \( q \) odd, is a conic. Similarly, every ovoid of \( \text{PG}(3, q) \), for \( q \) odd, is an elliptic quadric, and this was proved independently by Barlotti [5] and Panella [19]. In the case where \( q \) is even, there also exist the Suzuki-Tits ovoids which are inequivalent to elliptic quadrics. The second author and O’Keefe, building on the work of Abatangelo and Larato, showed that the ovals of \( \text{PG}(2, q) \), \( q \) even, which admit a transitive subgroup of \( \text{PGL}_3(q) \) are conics (see [1] and [18]). Similarly, Bagchi and Sastry [2] showed that the ovoids of \( \text{PG}(3, q) \), \( q \) even, which admit a transitive subgroup of \( \text{PGL}_4(q) \) are elliptic quadrics or Suzuki-Tits ovoids. Brown and Lavrauw [6] have shown that an egg of \( \text{PG}(4n - 1, q) \), \( q \) even, contains a pseudo-conic if and only if it is elementary and arises from an elliptic quadric. Recently, J. A. Thas and K. Thas [24] have shown that every 2-transitive pseudo-oval in even characteristic is elementary and arises from a conic. In this paper, we prove the following result:
Main Theorem:
Suppose $\mathcal{E}$ is a pseudo-oval or pseudo-ovoid (that is not an oval or ovoid) admitting an insoluble transitive group of collineations. Then $\mathcal{E}$ is elementary and arises from a conic, an elliptic quadric, or a Suzuki-Tits ovoid.

2 The Approach

A divisor $x$ of $q^d - 1$ (where $d \geq 3$) is primitive if $x$ does not divide $q^i - 1$ for each positive integer $i < d$. By a result of Zsigmondy [25], such divisors exist if $(q,d) \neq (2,6)$. Therefore, if $G$ acts transitively on a set of size $q^m + 1$ (and $(q,m) \neq (2,3)$), then a primitive prime divisor of $q^{2m} - 1$ divides the order of $G$. Such groups have an irreducible Sylow subgroup, and from this information, the structure of $G$ can be described in great detail (see [12]). The authors have used this argument to classify $m$-systems of polar spaces which admit an insoluble transitive group (see [3]). From the definitions of a pseudo-oval and pseudo-oval, we can apply a similar argument here; which is dependent on the Classification of Finite Simple Groups.

Note: Suppose $\mathcal{E}$ is a pseudo-oval (resp. pseudo-ovoid) of $\text{PG}(2n + m - 1, q)$ where $q = p^f$ for some prime $p$. Under field reduction from $\text{GF}(q)$ to $\text{GF}(p)$, there arises a pseudo-oval (resp. pseudo-ovoid) $\tilde{\mathcal{E}}$ of $\text{PG}((2n + m)f - 1, p)$. If $\mathcal{E}$ admits an insoluble transitive subgroup of $\text{PGL}_{2n+m}(q)$, then $\tilde{\mathcal{E}}$ admits an insoluble transitive subgroup of $\text{PGL}_{(2n+m)f}(p) = \text{PGL}_{(2n+m)f}(p)$. We then apply the main result of this paper to $\tilde{\mathcal{E}}$ to establish that it is elementary, from which it follows that $\mathcal{E}$ is elementary provided that it is not an oval or ovoid. Hence throughout this paper, we will assume without loss of generality that our given pseudo-oval or pseudo-ovoid admits an insoluble transitive subgroup of the homography group $\text{PGL}_{2n+m}(q)$.

3 The Pseudo-Oval Case

A pseudo-oval of $\text{PG}(d - 1, q)$ (where $d$ is a multiple of 3) is a set of $q^{d/2} + 1$ subspaces of dimension $d/3 - 1$, where $c = \frac{2}{3}d$. This phrasing makes it clear how we apply the results of [4].

3.1 Even characteristic

If $q$ is even, then the tangent spaces of a pseudo-oval $\mathcal{E}$ all have a $(d/3 - 1)$-space in common; the nucleus of $\mathcal{E}$ (see [20, pp. 182]). Since $G$ must fix the nucleus, we have that $G$ acts reducibly in this case. Let $\mathcal{N}$ be the the nucleus of $\mathcal{E}$ and consider the quotient map $\pi$ from $\text{PG}(d - 1, q)$ to $\text{PG}(d - 1, q)/\mathcal{N}$, and note that the codomain can be identified with $\text{PG}(2d/3 - 1, q)$. The image of $\mathcal{E}$ under $\pi$ is a spread $\mathcal{S}$ of $\text{PG}(2d/3 - 1, q)$ (see [20, pp. 182]). Moreover, we have that $G$ acts transitively on this spread, and by the Andre/Bruck-Bose construction, we obtain a flag-transitive affine plane admitting an insoluble group. By [7], this affine plane is Desarguesian or a Lüneburg plane, so in particular, it follows that $\mathcal{E}$ admits a 2-transitive group. So by [24, §8], we have that $\mathcal{E}$ is an elementary pseudo-oval arising from a conic of $\text{PG}(2, q^{d/3})$.

3.2 Odd characteristic

Let $\mathcal{E}$ be a pseudo-oval of $\text{PG}(d - 1, q)$, where $q$ is odd. Then each element $E$ of $\mathcal{E}$ is contained in a unique 2d/3 - 1-subspace $T_E$ of $\text{PG}(d - 1, q)$ which is called the tangent space at $E$. By [20, pp. 182], each point of $\text{PG}(d - 1, q)$ is contained in 0 or 2 tangent spaces of $\mathcal{E}$. 
Theorem 3.1. Let $q = p^f$ where $p$ is an odd prime, let $d$ be an integer divisible by $3$. If an insoluble subgroup $G$ of $\text{PGL}_d(q)$ acts transitively on a pseudo-oval $E$ of $\text{PG}(d-1,q)$, then $E$ is elementary and is obtained by field reduction of a conic of $\text{PG}(2,q^{d/3})$.

Proof. Let $E$ be a pseudo-oval of $\text{PG}(d-1,q)$ admitting a group $G \leq \text{PGL}_d(q)$ that is insoluble and acts transitively on $E$, and let $H$ be the stabiliser in $G$ of an element of $E$. Note that the number of elements of $E$ is $q^{d/2} + 1$ where $e = 2d/3$. We may assume that $q^{4/3} > 16$ as it was shown by the second author in [21] that if $q^{4/3} \leq 16$, then $E$ is elementary and is obtained by field reduction of a conic of $\text{PG}(2,q^{d/3})$. Let $\hat{G}$ be a preimage of $G$ in $\text{GL}_d(q)$. Then there exists a subgroup $\hat{H}$ of $\hat{G}$ of index $q^{e/2} + 1$ such that the image of $\hat{H}$ in $\text{PGL}_d(q)$ is $H$. So we can apply [4, Theorem 3.1] to $\hat{G}$. There are six cases to consider from this theorem: the Classical, Imprimitive, Reducible, Extension Field (case (b)), Symplectic Type, and Nearly Simple examples. Straight away, we have that the Symplectic examples do not occur as $d$ is a multiple of $3$. By [4, Lemma 13], $\hat{G}$ is not in the Classical examples case. So we are left with four families to consider: the Reducible, Imprimitive, Extension Field, and the Nearly Simple examples.

Let us first suppose we are in the Imprimitive examples case. So by [4, Theorem 3.1], we have that $d = 9$, $q \in \{3, 5\}$, and $\hat{G}$ preserves a decomposition of $V_9(q)$ into 1-spaces. So in particular, $\hat{G} \leq \text{GL}_1(q) \wr S_9$. We treat both cases, $q = 3$ and $q = 5$, simultaneously. Let $\mu$ be the natural projection map from $\text{GL}_1(q) \wr S_9$ onto $S_9$. Now $\mu(\hat{G})$ is insoluble and primitive (of degree 9), and hence $\mu(\hat{G}) \in \{\text{PSL}_2(8), \text{PGL}_2(8), A_9, S_9\}$ (see [10, Appendix B]). Moreover, $\mu(\hat{G})$ is 3-transitive in its degree 9 action. Let $B$ be the kernel of $\mu$. So $|B| = (q-1)^9 \in \{2^9, 3^{18}\}$. Now $G \cap B$ is a nontrivial normal subgroup of $G$ and hence $G \cap B$ contains the subgroup $K$ of $B$ consisting of diagonal matrices with entries $\pm 1$. Since $|G : H| \in \{28, 126\}$, we see that a subgroup $J$ of $K$ with index at most 2, is contained in $H$. The only $J$-invariant subspaces of $V_9(q)$ are the spans of vectors from the canonical basis; coordinate subspaces. Let $E$ be an element of the pseudo-oval. We may assume (up to conjugacy) that $E$ is $J$-invariant and so it is a coordinate plane. Now the action of $\mu(\hat{G})$ is 3-transitive, and so the orbit of $E$ under $\mu(\hat{G})$ is a coordinate plane. Therefore, we have that $\hat{G}$ is not in the Nearly Simple examples case does not arise.

Let us now suppose we are in the Nearly simple case. So $S \leq G \leq \text{Aut}(S)$ where $S$ is a finite nonabelian simple group, and $\hat{G}$ is irreducible. By using the fact that $q^{4/3} \geq 16$, we have only two subcases to consider: the Alternating group case and the Natural-characteristic case. In the former, we have $S = A_{10}$, $d = 9$, $q = 3$, and the vector space $V_9(3)$ can be identified with the fully deleted permutation module for $S_{10}$ over GF(3). It can be readily checked that $G$ does not have a subgroup of index $3^3 + 1$, and so this case does not arise. In the Natural-characteristic case, we have that $d = 9$ and $S = \text{PSL}_3(q^2)$ (by [4, Theorem 2.1]). Now by [8], the minimum degree of a nontrivial representation of $S$ is $(q^6 - 1)/(q^2 - 1)$. However

$$q^3 + 1 = (q^6 - 1)/(q^3 - 1) < (q^6 - 1)/(q^2 - 1)$$

so $\hat{G}$ does not have a transitive action of degree $q^d + 1$. Therefore, we have that $\hat{G}$ is not in the Nearly Simple examples case.

Now suppose we are in the Field Extension examples case. We have that $\hat{G}$ is irreducible and there is a divisor $b$ of $2d/3$ (where $b \neq 1$) such that $\hat{G}$ preserves a field extension structure $V_{d/b}(q^h)$ on $V_d(q)$. Moreover, $G \cap \text{GL}_{d/b}(q^h)$ has a subgroup of index $(q^{e/2} + 1)/x$, for some $x$, and so if $d/b > 3$, then we can apply [4, Theorem 3.2] to $G \cap \text{GL}_{d/b}(q^h)$ with parameters $q^h$, $d/b$, and $e/b$ playing the roles of $q$, $d$, and $e$ respectively. So let us assume that $d/b > 3$. Since $d/b \neq e/b$, we do not have the Classical examples case. Note that if $\hat{G}$ fixes a subspace over the field extension $q^h$, then it also fixes a subspace that is written over the field GF(q). Hence $\hat{G} \cap \text{GL}_{d/b}(q^h)$ is irreducible in its action on $\text{PG}(d/b - 1,q^h)$. We can also assume that $G \cap \text{GL}_{d/b}(q^h)$ does not preserve a field extension structure by choosing $b$ to be maximal. Since $q^h$ is not prime, we can eliminate the Imprimitive examples, Symplectic Type examples, and the Nearly Simple examples. Therefore $d/b = 3$ and $e/b = 2$. By some old work of Mitchell [17], the only absolutely irreducible insoluble maximal subgroups of $\text{PSL}_3(q^h)$ are

(i) $\text{PSL}_2(q^h)$;
(ii) $\text{PSU}_3(q^6)$ when $q^6$ is a square;

(iii) $A_6$ when $p \equiv 1, 2, 4, 7, 8, 13 \mod 15$ (and $GF(q^6)$ contains the squares of 5 and $-3$);

(iv) $\text{PSL}_2(7)$ when $p \equiv 1, 2, 4 \mod 7$.

In the case that $\text{PSU}_3(q^{d/3}) \leq G \cap \text{PGL}_3(q^{d/3}) \leq \text{PGL}_3(q^{d/3})$, we have $q^{d/3} + 1$ divides $q^{d/2}(q^{d/3} - 1)(q^{d/2} + 1)$. This is a contradiction as $q^{d/3} + 1$ is coprime to $q^{d/2}$ and $q^{d/3} - 1$ (note that $q$ is odd). So this case does not arise. In the case that $A_6 \leq G \cap \text{PGL}_3(q^{d/3}) \leq S_6$, we have $q^{d/3} + 1$ divides 6! (note that $q^{d/3} + 1$ is coprime to $|G : G \cap \text{PGL}_3(q^{d/3})|$). However, $q^{d/3} + 1$ divides 6! only if $q = 3$ and $d = 6$ (so $b = 2$). So this case does not arise as $A_6$ does not act in characteristic 3. In the case that $\text{PSL}_2(7) \leq G \cap \text{PGL}_3(q^{d/3}) \leq \text{PGL}_2(7)$, we have $q^{d/3} + 1$ divides 336. However, $q^{d/3} + 1$ divides 336 only if $q = 3$ and $d = 9$ (so $b = 3$). So this case does not arise as $\text{PSL}_2(7)$ does not embed in $\text{PGL}_3(q^6)$ in characteristic 3. Hence $\text{PSL}_2(q^6) \leq G$.

Let $J = \text{PSL}_2(q^{d/3})$. It is a classical result, but can also be found in [8], that $\text{PSL}_2(q^{d/3})$ (where $d > 2$) has a unique conjugacy class of subgroups of index $q^{d/3} + 1$. It follows from [14, Proposition 4.3.17], that there is a unique characteristic class of subgroups of $\text{PGl}_d(q)$ isomorphic to $J$ (it is not true in general that there is a unique conjugacy class of such subgroups). Let $\varphi : V_3(q^{d/3}) \rightarrow V_d(q)$ denote the natural vector space isomorphism here, and let $C$ be a conic of $V_3(q^{d/3})$ admitting $J$. Let $\alpha$ and $\beta$ be two distinct points of $C$. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are $d/3$-dimensional vector subspaces of $V_d(q)$. Note that $J$ has a unique conjugacy class of subgroups of index $q^{d/3} + 1$, and hence we can assume that the stabiliser of an element $E$ of $\mathcal{E}$ is identical to the stabiliser $J_\alpha$. Now suppose we have a third vector $v$ which is neither $\alpha$ nor $\beta$. Then

$$|v^J_\alpha| = |J_\alpha : J_\alpha \cap v^J_\alpha| = |J_\alpha : J_{\alpha,\beta}| |J_{\alpha,\beta} : J_{\alpha,\beta} \cap v^J_\alpha| = q^{d/3}|J_{\alpha,\beta} : J_{\alpha,\beta} \cap v^J_\alpha|.$$ 

Now $J$ is a Zassenhaus group (i.e., a 2-transitive group such that the stabiliser of any three points is trivial) and so $J_{\alpha,\beta} \cap v^J_\alpha = 1$. Therefore

$$|v^J_\alpha| = q^{d/3} \frac{q^{d/3} - 1}{\text{gcd}(2, q^{d/3} - 1)}$$

which is not a prime power. Now any $J_\alpha$-invariant $d/3$-subspace of $V_d(q)$ is a union of orbits of $J_\alpha$. Therefore, it follows that the only $J_\alpha$-invariant subspace of $V_d(q)$ is $\varphi(\alpha)$. Since $W$ is $J_\alpha$-invariant, we have that $W = \varphi(\alpha)$ and hence $\mathcal{E}$ is the image of $C$ under $\varphi$. Therefore, $\mathcal{E}$ is elementary and is obtained by field reduction of a conic of $\text{PG}(2, q^{d/3})$.

Reducible examples:

We have that $\hat{G}$ fixes a subspace/quotient space $U$ of $V_d(q)$ and $\dim(U) = u \geq \frac{2}{3}d$. In fact, it follows that $u = 2/3d$ by noting that a primitive divisor of $q^{(2/3)d} - 1$ also divides $|\hat{G}|$. So $\hat{G} \leq q^{u(d-u)} \cdot (\text{GL}_u(q) \times \text{GL}_{d-u}(q))$. We may assume that $U$ is a subspace, as for $q$ odd, each point of $U$ is in 0 or 2 tangent spaces of $\mathcal{E}$. Consider the set of intersections

$$\mathcal{M} = \{T_E \cap U : E \in \mathcal{E}\}.$$ 

Note that each element of $\mathcal{M}$ has a common dimension as $G$ acts transitively on $\mathcal{M}$, and thus $\dim(T_E \cap U) = d/3$ for all $E \in \mathcal{E}$. Therefore $\hat{G}^U$ acts transitively on a set of $(q^{d/3} + 1)/\delta$ subspaces of dimension $d/3$ where $\delta = 1, 2$. This implies that $\hat{G}^U$ has a subgroup of index $(q^{d/3} + 1)/\delta$, and so we can apply [4, Theorem 3.2] with $q$, $\frac{2}{3}d$, and $\frac{4}{3}d$ playing the roles of $q$, $d$, and $e$ respectively. In the following subcases, we have that $G$ has a normal insoluble subgroup $S$, which is given explicitly. Moreover, $S$ must have a union of orbits on $(d/3)$-spaces of $U$ of size $(q^{d/3} + 1)/\delta$ where $\delta = 1, 2$.

Reducible/Nearly simple examples:

In this case, $S \leq G^U \cap \text{PGL}_d(q) \leq \text{Aut}(S)$ where $S$ is a finite nonabelian simple group. Here we have four subcases.
ALTERTING GROUP CASE:

Here $S = A_r$ and the vector space $V_u(q)$ can be identified with the fully deleted permutation module for $S_r$ over $GF(q)$. We have that $u$ is $r-1$ or $r-2$ (according to whether $p$ does not or does divide $n$ respectively), and $q^u = p^u = 3^6, 5^6$. Suppose $S = A_7$, $u = 6$, and $q = 3$. Then $S$ stabilises $M$ and hence $S$ has a union of orbits on planes of PG$(5, 3)$ of size 14 or 28. Now $A_7$, in its unique irreducible representation in PG$(5, 3)$ has the following orbit lengths on planes (n.b., the exponents denote multiplicities):

\[ [35^2, 105^4, 140^3, 210^4, 315^5, 420^{10}, 630^6, 840^4, 1260^{15}] \]

Therefore this case does not arise. Now suppose $q = 5$. It can be shown using GAP [11] that the $S$-invariant sets of planes of size 63 or 126 do not cover every point either 0 or 2 times. Therefore this case does not arise.

CROSS-CHARACTERISTIC CASE: The table below lists the possibilities for this case.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$d$</th>
<th>$q$</th>
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<tbody>
<tr>
<td>PSL$_2(13)$</td>
<td>9</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>PSL$_4(13)$</td>
<td>9</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>PSU$_3(3^2)$</td>
<td>9</td>
<td>5</td>
<td>6</td>
</tr>
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</table>

Now PSL$_2(13)$ acts transitively on the points of PG$(5, 3)$, and so this case does not arise. Suppose $S = PSL_2(7)$, $u = 6$, and $q = 3$. Then $S$ stabilises $M$ and hence $S$ has a union of orbits on planes of PG$(5, 3)$ of size 14 or 28. Now by using GAP [11] and the unique irreducible representation for $S$ in PG$(5, 3)$, we have that $S$ has the following orbit lengths on planes:

\[ [7^4, 21^8, 28^{12}, 42^{18}, 56^{12}, 84^{100}, 168^{140}] \]

None of the thirteen $S$-invariant sets of planes of size 28 have each point of PG$(5, 3)$ contained in a constant number (0 or 2) of elements of the set. Likewise, of all the six $S$-invariant sets of size 14, none have each point of PG$(5, 3)$ contained in a constant number of elements of the set. Therefore, this case does not arise.

Now suppose $S = PSU_3(3^2)$, $u = 6$, and $q = 5$. Then $S$ stabilises $M$ and hence $S$ stabilises a set of points of size $(q^u - 1)/(2(q - 1)) = 1953$. However, by using GAP [11] one can calculate that $S$ has the following orbit lengths on points of PG$(5, 5)$:

\[ [189^2, 1008^2, 1512] \]

Since 1953 cannot be partitioned into these numbers, this case does not arise.

So we are left now with just two more cases: the “Classical examples” and the “Extension field” examples, which can be unified naturally.

Reducible/Classical and Extension Field examples:

We have that $\hat{G}^U$ preserves a (possibly trivial) field extension structure on $U$ as a $u/b$-dimensional subspace over $GF(b)$ where $b$ is a proper divisor of $u = (2/3)d$. So $\hat{G}^U \leq \Gamma L_{(2/3)d/6}(q^b)$ and we can apply [4, Theorem 3.2] to $\hat{G}^U \cap GL_{(2/3)d/6}(q^b)$ where $q^b$, $u/b$, and $u/b$ play the roles of $g$, $d$, and $e$ respectively. We simply have $d/b = 6$ and PSL$_2(q^{d/3}) \leq \hat{G}^U$. Let $S = PSL_2(q^{d/3})$ and note that the preimage of $S$ acts transitively on the non-zero vectors of $V_2(q^{d/3})$. However, we have here that $S$ stabilises a set of $q^{d/3} + 1$ subspaces, each of dimension $d/3 - 1$, which is impossible for $d/3 > 1$. So we conclude that $G$ is irreducible. $\Box$
4 The Pseudo-Ovoid Case

A pseudo-ovoid of PG(d − 1, q) (where d is a multiple of 4) is a set of $q^{d/2} + 1$ subspaces of dimension $d/4 − 1$. Here we can also apply the results of [4], as we did in the pseudo-oval case.

**Theorem 4.1.** Let $q = p^e$ where $p$ is a prime and let $d$ be an integer divisible by 4. If an insoluble subgroup $G$ of PGL$_d(q)$ acts transitively on a pseudo-ovoid $E$ of PG(d − 1, q), then $E$ is elementary and arises from an elliptic quadric or Suzuki-Tits ovoid.

**Proof.** Let $H$ be the stabiliser of an element of $E$ in $G$, and let $\hat{G}$ be a preimage of $G$ in GL$_d(q)$. Note that the number of elements of a pseudo-ovoid of PG(d − 1, q) is $q^{e/2} + 1$ where $e = d$. So there exists a subgroup $\hat{H}$ of $G$ of index $q^{d/2} + 1$ such that the image of $\hat{H}$ in PGL$_d(q)$ is $H$. Therefore we can apply [4, Theorem 3.2] to $\hat{G}$. First note that we can rule out the Reducible examples, Imprimitive examples, and case (a) of the Extension field examples. Recall that by [18], we can assume that $d > 4$. Hence we have ruled out the Classical and Symplectic Type examples. Also note that in the Nearly simple case, we have the following: $q = 2$, $d = 12$, and either

(a) $A_{13} \leq G \leq S_{13}$, or
(b) $S = PSp_2(25) \leq G \leq PSp_3(25)$, and $S \cap H$ is isomorphic to $S_5$ (there are two such conjugacy classes of $S$).

However in the first case, it is clear that $G$ does not have a subgroup of index 65. In the second case, we know by [13] that PSp$_2(25)$ has a unique 12-dimensional irreducible representation (up to quasi-equivalence) over GF(2) and it has the following orbit lengths on points:

$$[65, 325^2, 650, 780, 1950].$$

Let $B$ be the set of points covered by the pseudo-ovoid $E$ of PG(11, 2). Then $B$ has size $(q^{d/4} − 1)(q^{d/2} + 1) = (2^3 − 1)(2^6 + 1) = 455$ and it must be a union of orbits of $S$ as $G$ acts transitively on $E$. However, 455 cannot be partitioned into the orbit lengths displayed above, and hence this case does not arise.

That leaves us with the Extension field examples. Here we have that $\hat{G} \leq \Gamma L_{d/2}(q^b)$ where $b$ is a divisor of $d$ (where $b \neq 1$). If $d/b > 2$, We can apply [4, Theorem 3.2] (for $e/b$ odd) and [4, Theorem 3.1] (for $e/b$ even) to $\hat{G} \cap GL_{d/2}(q^b)$ with parameters $d/b$, $e/b$, and $q^b$ playing the roles of $d$, $e$, and $q$ respectively. We have the following subcases:

(i) $d/b = 4$ and $\Omega^-_4(q^{d/4}) \leq \hat{G} \cap GL_{d/2}(q^b)$;
(ii) $d/b = 4$, $q$ is even, and $Sz(q^{d/4}) \leq \hat{G} \cap GL_{d/2}(q^b)$;
(iii) $d/b = 3$, $q^{d/3}$ is a square, and $SU_3(q^{d/3}) \leq \hat{G} \cap GL_{d/2}(q^b)$.

(i) Let us suppose we have the first case above, where $d/b = 4$ and $E$ admits $\Omega^-_4(q^{d/4})$. Let $J = P\Omega^-_4(q^{d/4})$. It is a classical result, but can also be found in [8], that PSL$_2(q^{d/2})$ (where $d > 2$) has a unique conjugacy class of subgroups of index $q^{d/2} + 1$. Note that $\Omega^-_4(q^{d/4})$ is isomorphic to PSL$_2(q^{d/2})$, and by [14, Proposition 4.3.6], there is a unique conjugacy class of subgroups of PGL$_d(q)$ isomorphic to PSL$_2(q^{d/2})$. Therefore, there is a unique conjugacy class of subgroups of PGL$_d(q)$ isomorphic to $J$.

Let $\varphi : V_d(q^{d/4}) \rightarrow V_d(q)$ denote the natural vector space isomorphism here, and let $Q$ be an elliptic quadric of $V_d(q^{d/4})$ admitting $J$. Let $\alpha$ and $\beta$ be two distinct points of $Q$. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are $d/4$-dimensional subspaces of $V_d(q)$. Note that $J$ has a unique conjugacy class of subgroups of index $q^4 + 1$ (see [8]), and hence we can assume that the stabiliser of an element $E$ of $E$ is identical to the stabiliser $J_\alpha$. Now suppose we have a third vector $v$ which is neither $\alpha$ nor $\beta$. Then

$$|v^{J_\alpha}| = |J_\alpha : J_{\alpha,v}| = |J_\alpha : J_{\alpha,\beta}||J_{\alpha,\beta} : J_{\alpha,\beta,v}| = q^{d/2}|J_{\alpha,\beta} : J_{\alpha,\beta,v}|.$$
Now $J$ is a Zassenhaus group and so $J_{\alpha,\beta,v} = 1$. Therefore

$$|v^{J_{\alpha}}| = q^{d/2} \frac{q^{d/2} - 1}{\gcd(2, q^{d/2} - 1)}$$

which is not a prime power. Now any $J_{\alpha}$-invariant $d/4$-subspace of $V_{d}(q)$ is a union of orbits of $J_{\alpha}$. Therefore, it follows that the only $J_{\alpha}$-invariant subspace of $V_{d}(q)$ is $\varphi(\alpha)$. Since $W$ is $J_{\alpha}$-invariant, we have that $W = \varphi(\alpha)$ and hence $E$ is the image of $Q$ under $\varphi$. Therefore, $E$ is elementary and arises from an elliptic quadric.

(ii) By a similar argument to that above, it is not difficult to show that $E$ is the image of a Suzuki-Tits ovoid under field reduction. The key steps to note are that $Sz(q^{d/4})$ is a Zassenhaus group, there is a unique conjugacy class of subgroups of $PGL_3(q)$ isomorphic to $Sz(q^{d/4})$, and $Sz(q^{d/4})$ has a unique conjugacy class of subgroups of index $q^2 + 1$. In the seminal paper of Suzuki [23, §15], it was shown that $Sz(q^{d/4})$ is a Zassenhaus group and has a unique conjugacy class of subgroups of index $q^{d/2} + 1$ and this is the minimum non-trivial degree of $Sz(q^{d/4})$. The uniqueness of its representation in $PGL_4(q)$ needs more work. By a result of Lüneburg (see [16, 27.3 Theorem] or [15]), there is a unique conjugacy class of subgroups of $PGL_4(q^{d/4})$ isomorphic to $Sz(q^{d/4})$. Now by [14, Proposition 4.3.6], there is a unique conjugacy class of subgroups of $PGL_4(q)$ isomorphic to $PGL_4(q^{d/4})$. Therefore, there is a unique conjugacy class of subgroups of $PGL_4(q)$ isomorphic to $Sz(q^{d/4})$. Therefore, $E$ is elementary and arises from a Suzuki-Tits ovoid.

(iii) Now suppose we have the third case; $d/b = 3$, $q^{d/3}$ is a square, and $E$ admits $PSU_3(q^{d/3})$. Now the smallest orbit of $PSU_3(q^{d/3})$ on nonzero vectors consists of the non-singular vectors and has size $(q^{d/3} - 1)(q^{d/2} + 1)$. Since $E$ covers $(q^{d/4} - 1)(q^{d/2} + 1)$ vectors of $V_{d}(q)$, and this number is strictly smaller than the size of the smallest orbit of $PSU_3(q^{d/3})$, we see that this case does not arise.

Suppose now that $d/b = 2$. Since $G$ is an insoluble subgroup of $GL_2(q^{d/2})$, it follows from [4, Lemma 5] that $G$ contains $SL_2(q^{d/2})$. However, $SL_2(q^{d/2})$ is transitive on nonzero vectors and hence does not stabilise a set of $d/4$ vector subspaces of size $q^{d/2} + 1$. Hence this case does not arise.

Remark: If a (presently unknown) pseudo-oval or pseudo-ovoid over $GF(q)$ admitting a soluble transitive group $G$ exists, then $G$ is meta-cyclic; indeed $G$ is a subgroup of $\Gamma L_1(q^b)$, for an appropriate positive integer $b$.

References


Transitive Eggs

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