STIRLING'S SERIES FOR n! MADE EASY

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A *Stirling formula* is an estimate for $n!$. In its *qualitative* form, it simply states that

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

i.e.

$$r_n := \ln \frac{n!\, e^n}{\sqrt{2\pi}\, n^{n+\frac{1}{2}}} \approx 0.$$

Quantitative Stirling formulas give estimates for r_n. Here are some examples, all having appeared in *American Mathematical Monthly.*

$$\frac{1}{12n+1} < r_n < \frac{1}{12n} \qquad \text{(H. Robbins, 1955)}$$

$$\frac{1}{12n} - \frac{1}{360n^3} < r_n < \frac{1}{12n} \qquad \text{(T. S. Nanjundiah, 1959)}$$

$$\frac{1}{12n + \frac{3}{2(2n+1)}} < r_n \qquad \text{(A. J. Maria, 1965)}$$

$$\frac{1}{12n} - \frac{1}{360n^3} - \frac{1}{120n^4} < r_n < \frac{1}{12n} \qquad \text{(R. Michel, 2002).}$$

The sharpest of these estimates are Nanjundiah's. It is our purpose to do better than that, and to prove a formula of the form

$$r_n \sim \frac{A}{n} - \frac{B}{n^3} + \frac{C}{n^5} - \frac{D}{n^7} + \frac{E}{n^9} - \dots,$$

where $\sim$ means: *lies between any two successive partial sums.* In this notation, Nanjundiah's estimates amount to

$$r_n \sim \frac{1}{12n} - \frac{1}{360n^3}.$$

Moreover, we shall prove that the constants obtained cannot be improved. This whole paper is based on repeated application of the following very simple fact:

$$\begin{aligned} f \searrow 0 &\implies f > 0 \\ f \nearrow 0 &\implies f < 0, \end{aligned}$$

in the sequel simply called (SF). Some elementary high school calculus is also required: $f' > 0 \implies f \nearrow$ and $f' < 0 \implies f \searrow$; ln and exp are mutual inverses, both $\nearrow$; $\ln 1$, $\ln e$, $\ln(xy)$, $\ln \frac{x}{y}$, $\left(\ln(1+\frac{1}{x})\right)'$;

derivative and $\lim_{x\to+\infty}$ of rational functions; Wallis' estimates (i.e. integration by parts in $\int \cos^n x\, dx$).

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