Generalizing Moufang sets to a local setting

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What is a Moufang set?

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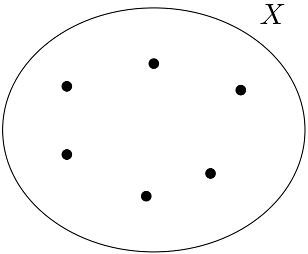
A set X with $|X| \geqslant 3$



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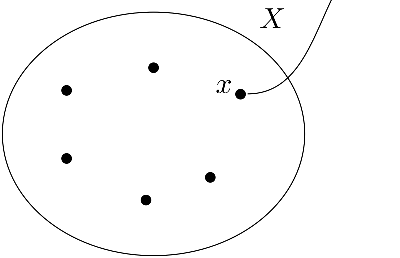

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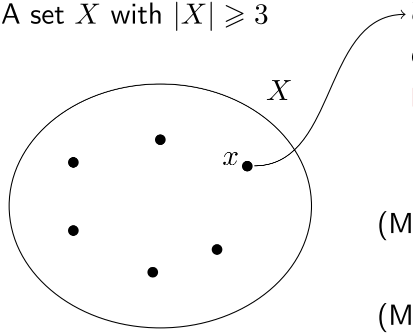

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(M1) U_x fixes x and acts regularly on $X \setminus \{x\}$;

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then we call $(X, \{U_x\}_{x \in X})$ a Moufang set.

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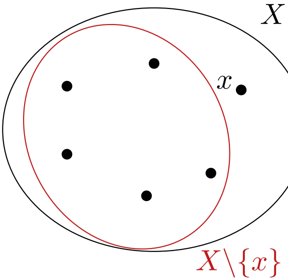

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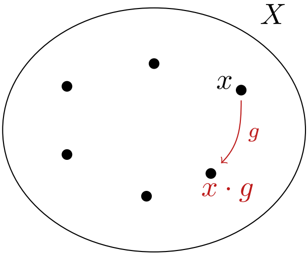

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An example: $\mathrm{PSL}_2(K)$

Let K be a field, then denote the projective line over K

$$\mathbb{P}^1(K) := \{[a,b] \mid a,b \in K, aK + bK = K\}.$$

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Denote

$$\alpha_{[1,k]} := \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } U_{[0,1]} := \left\{ \alpha_{[1,k]} \middle| k \in K \right\}.$$

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We define

$$U_{[1,0]} := U_{[0,1]}^{\tau} \text{ and } U_{[1,k]} := U_{[1,0]}^{\alpha_{[1,k]}}.$$

With these definitions, $\left(\mathbb{P}^1(K), \{U_{[a,b]}\}\right)$ is a Moufang set.

The little projective group is $G = \mathsf{PSL}_2(K)$.

Constructing Moufang sets

The same construction works for general $U \leqslant \mathrm{Sym}X$ and $\tau \in \mathrm{Sym}X$ such that

(i) U has a fixed point ∞ and acts regularly on $X \backslash \{\infty\}$;

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$$U_\infty := U \qquad U_0 := U^\tau \qquad U_x := U_0^{\alpha_x} \text{ for } x \neq 0, \infty \ .$$

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When does this construction give a Moufang set?

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Theorem (T. De Medts, R. Weiss, '06)

Denote $h_x := \tau\alpha_x\tau^{-1}\alpha^{-1}_{x\tau^{-1}}\tau\alpha^{-1}_{(-x\tau^{-1})\tau}$ *for* $x \neq 0, \infty$. *Then* $(X, \{U_x\})$ *as described above is a Moufang set if and only if* $U^{h_x} = U$ *for all* $x \neq 0, \infty$.

Generalizing to $\mathsf{PSL}_2(R)$

Instead of a field K, we can consider a local ring R:

(i) R unital, commutative;

(ii) R has a unique maximal ideal $\mathfrak{m}$.

We can consider

$$\mathsf{PSL}_2(R) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in R, ad - bc = 1 \right\}.$$

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Goal

Find a structure generalizing Moufangs sets, that includes 'local versions', for example $\mathsf{PSL}_2(R)$.

Projective line over a local ring

Let R be a local ring with unique maximal ideal $\mathfrak{m}$. We define the projective line over R

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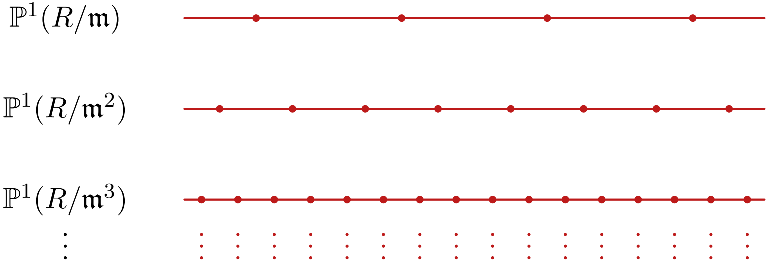

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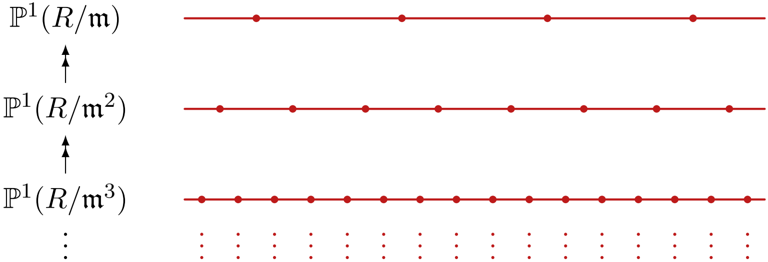

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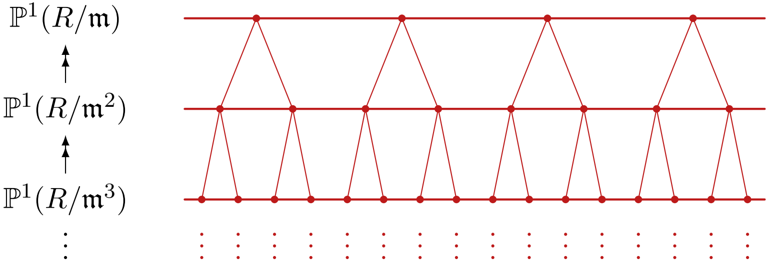

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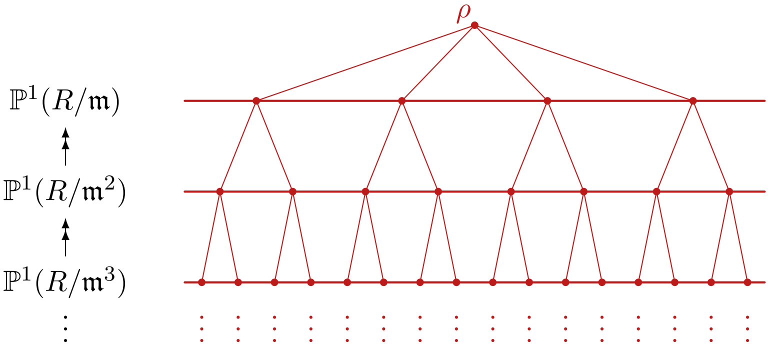

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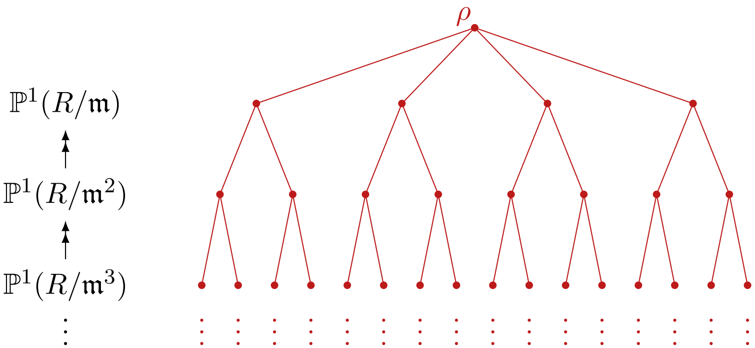
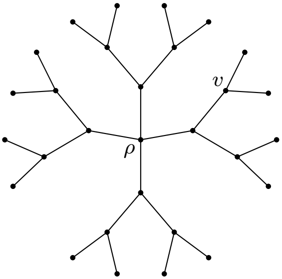


A rooted tree $\mathcal{T}$



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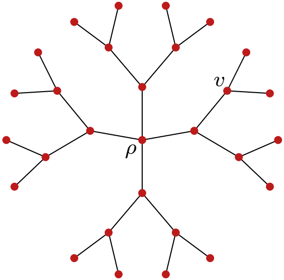

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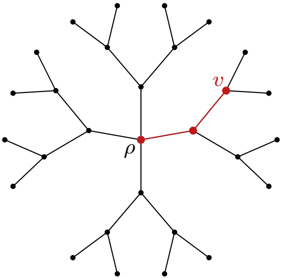

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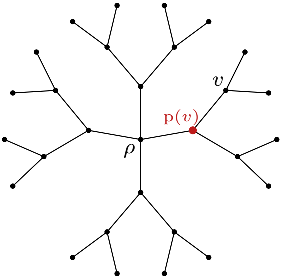

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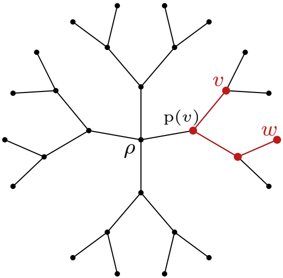

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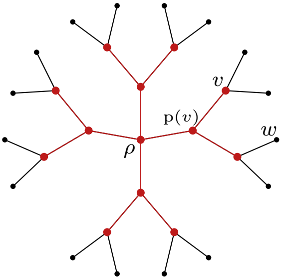

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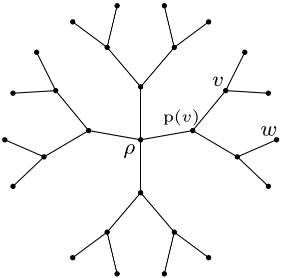

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Defining a local Moufang set

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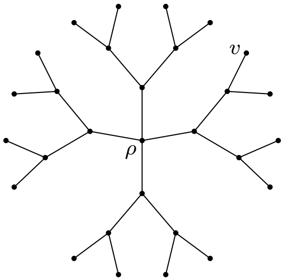

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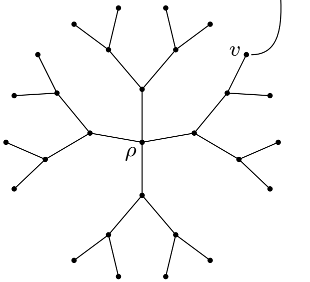

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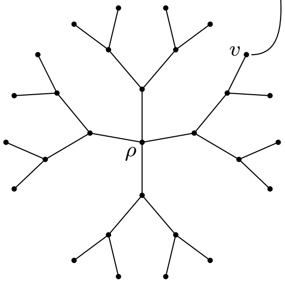

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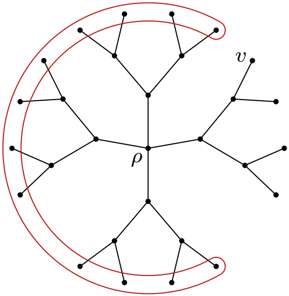

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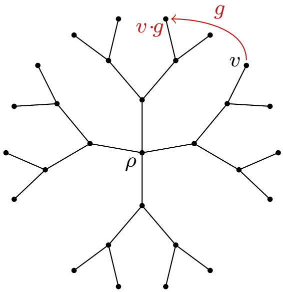

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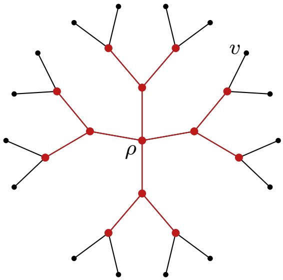

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Building blocks of a local Moufang sets

Let $\mathcal{T}$ be a rooted tree (finite depth), $U \leqslant \operatorname{Aut}\mathcal{T}$ and $\tau \in \operatorname{Aut}\mathcal{T}$ such that:

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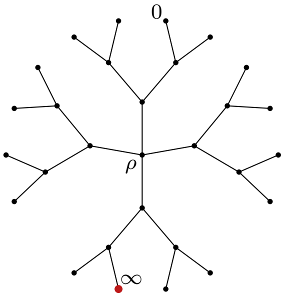
Note

There is little control on what U does on $\{v \in V(\mathcal{T}) \mid v \sim \infty\}$.

The construction



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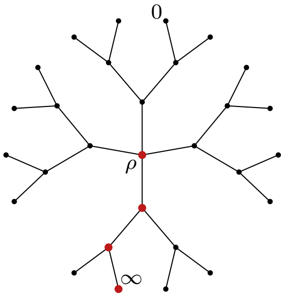

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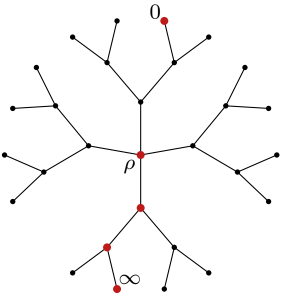

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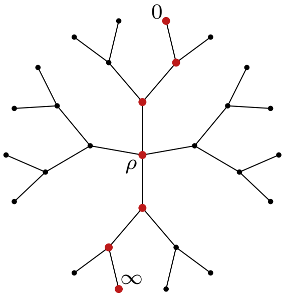

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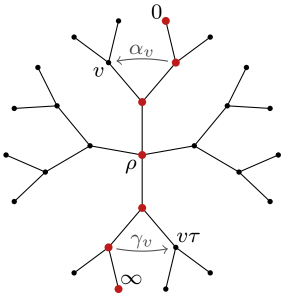

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- $U_{\mathrm{p}^i(\infty)} := \mathrm{Im}(U \to \mathsf{Aut}\, B(N-i))$;
- $U_0 := U_\infty^\tau$;
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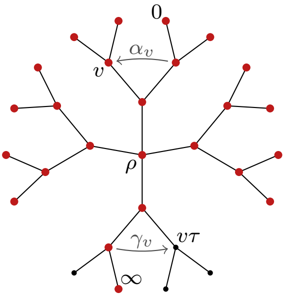
For $v \not\sim \infty$ s.t. $\mathrm{dp}(v) = N - i$, denote $\alpha_v \in U_{\mathrm{p}^i(\infty)}$ the unique element mapping $\mathrm{p}^i(0)$ to v. Denote $\gamma_v := \alpha_v^\tau$, so $\mathrm{p}^i(\infty)\gamma_v = v\tau$.

The construction



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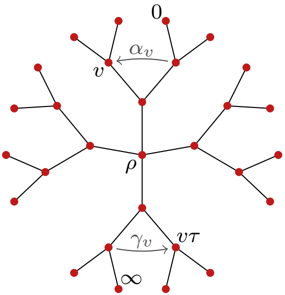

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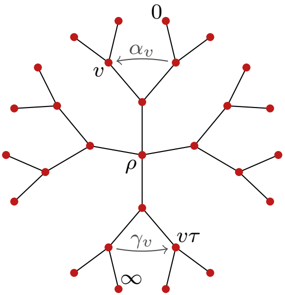

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Question

When does this construction give a local Moufang set?

Conditions to get a local Moufang set

Some terminology and notation:

- we call $v \in V(\mathcal{T})$ a unit if $v \not\sim 0$ and $v \not\sim \infty$;
- $-v := 0 \cdot \alpha_v^{-1}$ for $v \not\sim \infty$ and $\mathrm{dp}(v) = N$;
- $h_v := \tau\alpha_v\tau^{-1}\alpha_{v\tau^{-1}}^{-1}\tau\alpha_{(-v\tau^{-1})\tau}^{-1}$ the Hua map for v a unit and $\mathrm{dp}(v) = N$ (Hua maps fix 0 and ∞);
- $H := \langle h_v \mid v \text{ a unit}, \mathrm{dp}(v) = N\rangle$, the Hua subgroup;
- $U_0^\circ := \{\gamma_v \in U_0 \mid v \sim 0, \mathrm{dp}(v) = N\}$.

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Proposition (ER)

If $U^{h_v} = U$ for all units v with $\mathrm{dp}(v) = N$, and $U_0^\circ U_\infty \subseteq U_\infty H U_0^\circ$, then the given construction gives a local Moufang set.

An example: $\mathsf{PSL}_2(R)$

Let R be a local ring with maximal ideal $\mathfrak{m}$, and take

$$U = \left\{ \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \in \mathsf{PSL}_2(R) \;\middle|\; r \in R \right\} \qquad \tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

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With $\infty = [0,1]$ and $0 = [1,0]$, we get

$$v \text{ is a unit} \Leftrightarrow v = [1,r] \text{ with } r \text{ invertible in } R$$

$$U_0^\circ = \left\{ \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \;\middle|\; m \in \mathfrak{m} \right\} \qquad h_{[1,r]} = \begin{bmatrix} r^{-1} & 0 \\ 0 & r \end{bmatrix}.$$

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Without too much trouble, one can check that

$$U_0^\circ U_\infty \subseteq U_\infty H U_0^\circ \text{ and } U^{h_{[1,r]}} = U \ ,$$

so using the construction, we get a local Moufang set.

A small lemma

Lemma

Let v *be a unit with* $\mathrm{dp}(v) = N$*, then*

$$U_\infty^{h_v} = U_\infty \iff U_v = U_\infty^{\gamma_v \tau^{-1}} .$$

Proof.

A small lemma

Lemma

Let v *be a unit with* $\mathrm{dp}(v) = N$, *then*

$$U_\infty^{h_v} = U_\infty \iff U_v = U_\infty^{\gamma_{v\tau^{-1}}} \, .$$

Proof.

$$U_\infty^{h_v} = U_\infty \Leftrightarrow U_\infty^{\tau\alpha_v\tau^{-1}\alpha_{v\tau^{-1}}^{-1}\tau\alpha_{(-v\tau^{-1})\tau}^{-1}} = U_\infty \qquad \text{(definition } h_v\text{)}$$

$$\Leftrightarrow U_0^{\alpha_v} = U_\infty^{\alpha_{(-v\tau^{-1})\tau}\tau^{-1}\alpha_{v\tau^{-1}}\tau} \qquad \text{(definition } U_0\text{)}$$

$$\Leftrightarrow U_v = U_\infty^{\alpha_{v\tau^{-1}}^{\tau}} \qquad \text{(definition } U_v,\ \alpha_{(\dots)} \in U_\infty\text{)}$$

$$\Leftrightarrow U_v = U_\infty^{\gamma_{v\tau^{-1}}} \qquad \text{(definition } \gamma_{v\tau^{-1}}\text{)} \quad \square$$

Related work in progress

- Studying the Hua subgroup.
- Describing $\mathsf{PSU}_3(R)$ as a local Moufang set.
- Characterizing $\mathsf{PSL}_2(R)$ by group-theoretical conditions.

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Thank you for your attention!

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Ghent University

Extra: Sketch of the proof

Proposition (ER)

If $U^{h_v} = U$ *for all units* v *with* $\mathrm{dp}(v) = N$*, and* $U_0^\circ U_\infty \subseteq U_\infty H U_0^\circ$*, then the given construction gives a local Moufang set.*

Proof sketch.

The key of the proof is to show (LM2) for any v with $\mathrm{dp}(v) = N$:

(LM2) $U_v^g = U_{v\cdot g}$ for all $g \in G_N$.

We can find an element $\mu \in G_N$ that swaps 0 and ∞ such that

$$U_0^\mu = U_\infty \text{ and } U_\infty^\mu = U_0 \text{ , so } G_N = \langle U_\infty, \mu \rangle \text{ .}$$

We need to show:

$(\star_1)$ $U_v^\mu = U_{v\cdot\mu}$ $\qquad$ $(\star_2)$ $U_v^{\alpha_w} = U_{v\alpha_w}$ for all $w \not\sim \infty$

in two cases: $v \sim \infty$ and $v \not\sim \infty$. □