Local Moufang sets

Erik Rijcken

Promotor: Prof. Dr. Tom De Medts

UNIVERSITEIT GENT

Faculteit Wetenschappen
Universiteit Gent

14 juni 2017

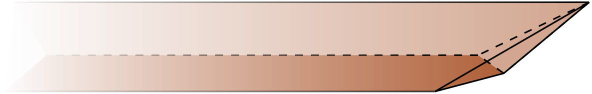
Overview

1. Group actions: using symmetry
2. Moufang sets
3. Fields
4. The connection
5. A local approach

1 Group actions: using symmetry

1 Group actions: using symmetry

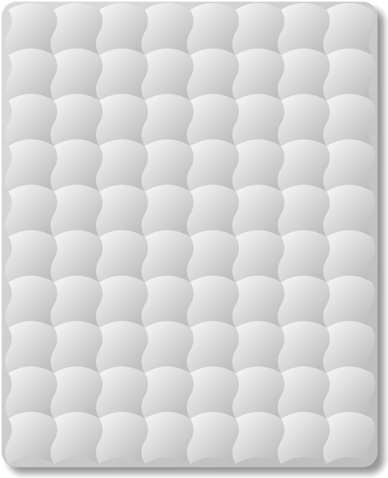
Group actions: using symmetry



Symmetries of a mattress

Four symmetries of a mattress:

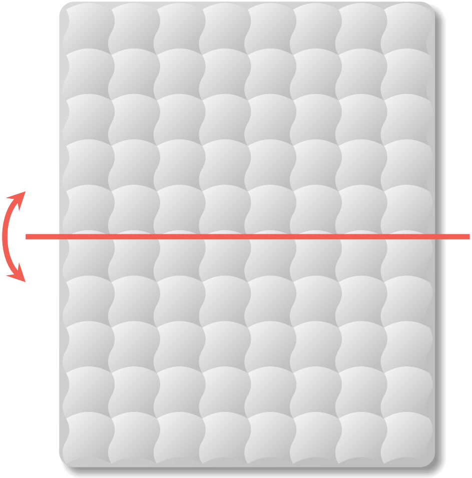
Symmetries of a mattress



Four symmetries of a mattress:

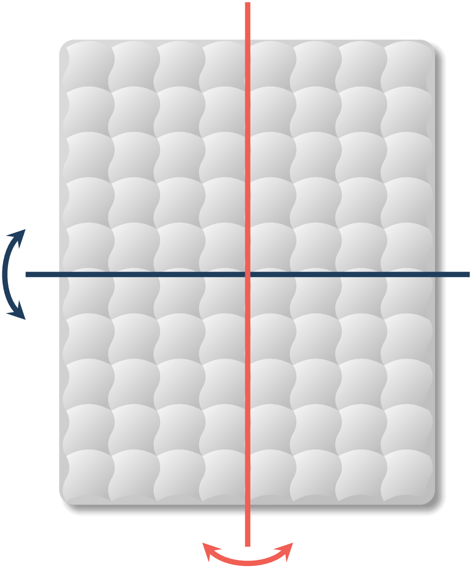
- id: do nothing

Symmetries of a mattress



Four symmetries of a mattress:

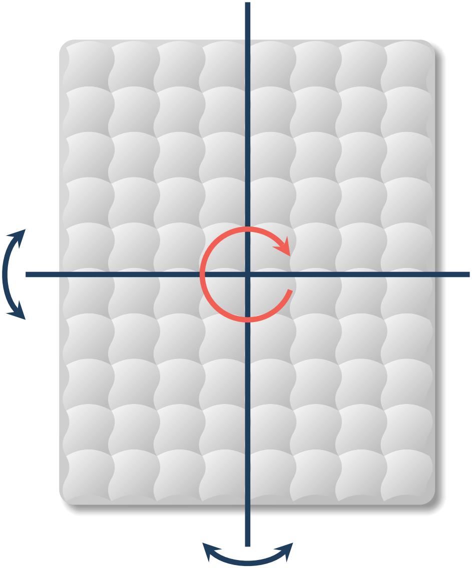
- id: do nothing
- $\updownarrow$: flip it around the short side



Four symmetries of a mattress:

- id: do nothing
- $\updownarrow$: flip it around the short side
- $\leftrightarrow$: flip it around the long side

Symmetries of a mattress



Four symmetries of a mattress:

- id: do nothing
- ↕: flip it around the short side
- ↔: flip it around the long side
- ↻: rotate 180°

What is a group action?

A group action is a list of actions you can perform on an object, with the following properties:

$\{\text{id}, \updownarrow, \leftrightarrow, \circlearrowright\}$
on a mattress

1. You can undo every action using one action from the list.

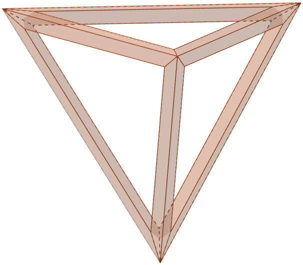
   $\updownarrow\updownarrow = \text{id}, \leftrightarrow\leftrightarrow = \text{id}\ldots$

2. Performing two actions gives the same result as one action from the list.

   $\updownarrow\leftrightarrow = \circlearrowright, \circlearrowright\leftrightarrow = \updownarrow,$
   $\leftrightarrow\text{id} = \leftrightarrow, \circlearrowright\circlearrowright = \text{id}\ldots$

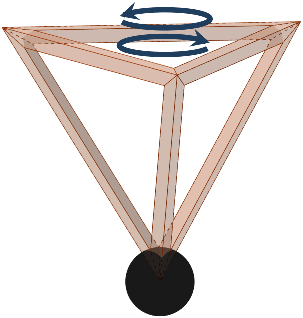
The list of actions is a group.

Rotating a tetrahedron



12 possible rotations.

Rotating a tetrahedron

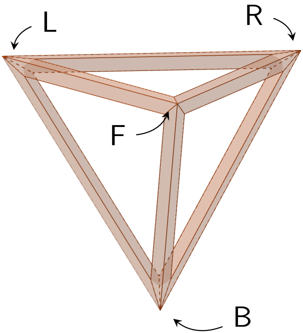


12 possible rotations.

Interesting ways?
Fix a vertex!

$$U_B = \{\text{id}, \circlearrowright, \circlearrowleft\}$$

Rotating a tetrahedron

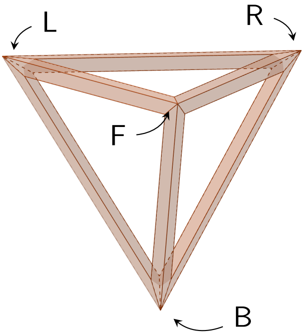


12 possible rotations.

Interesting ways?
Fix a vertex!

$$U_{\mathsf{B}} = \{\mathrm{id},\ \circlearrowright_{\mathsf{B}},\ \circlearrowleft_{\mathsf{B}}\}$$
$$U_{\mathsf{F}} = \{\mathrm{id},\ \circlearrowright_{\mathsf{F}},\ \circlearrowleft_{\mathsf{F}}\}$$
$$U_{\mathsf{L}} = \{\mathrm{id},\ \circlearrowright_{\mathsf{L}},\ \circlearrowleft_{\mathsf{L}}\}$$
$$U_{\mathsf{R}} = \{\mathrm{id},\ \circlearrowright_{\mathsf{R}},\ \circlearrowleft_{\mathsf{R}}\}$$

Rotating a tetrahedron



12 possible rotations.

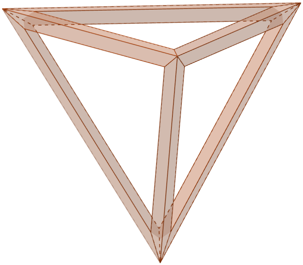
Interesting ways?
Fix a vertex!

$$U_\mathrm{B} = \{\mathrm{id},\ \circlearrowright_\mathrm{B},\ \circlearrowleft_\mathrm{B}\}$$
$$U_\mathrm{F} = \{\mathrm{id},\ \circlearrowright_\mathrm{F},\ \circlearrowleft_\mathrm{F}\}$$
$$U_\mathrm{L} = \{\mathrm{id},\ \circlearrowright_\mathrm{L},\ \circlearrowleft_\mathrm{L}\}$$
$$U_\mathrm{R} = \{\mathrm{id},\ \circlearrowright_\mathrm{R},\ \circlearrowleft_\mathrm{R}\}$$

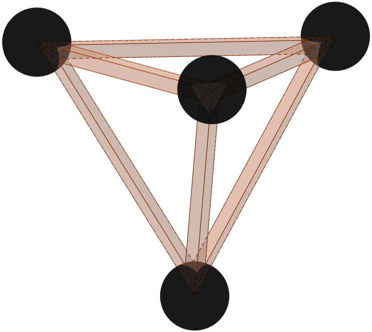
By performing multiple of these actions, we find the other rotations.

Example:
we can swap B and L by $\circlearrowright_\mathrm{B}\circlearrowleft_\mathrm{F}$.

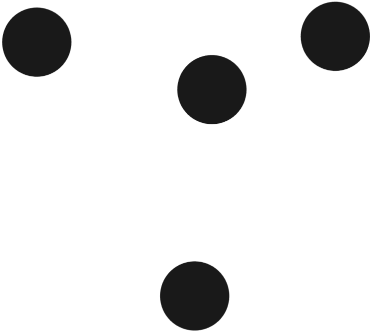
What if the tetrahedron is invisible?



What if the tetrahedron is invisible?



What if the tetrahedron is invisible?

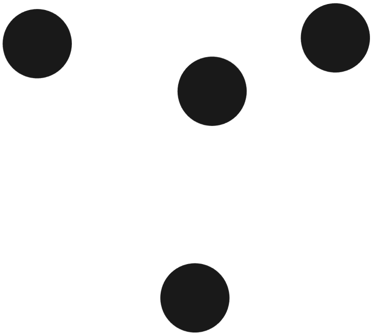


The tetrahedron may be invisible, but the possible actions remain the same!

$$U_\mathrm{B}=\{\mathrm{id}, \circlearrowright_\mathrm{B}, \circlearrowleft_\mathrm{B}\}$$
$$U_\mathrm{F}=\{\mathrm{id}, \circlearrowright_\mathrm{F}, \circlearrowleft_\mathrm{F}\}$$
$$U_\mathrm{L}=\{\mathrm{id}, \circlearrowright_\mathrm{L}, \circlearrowleft_\mathrm{L}\}$$
$$U_\mathrm{R}=\{\mathrm{id}, \circlearrowright_\mathrm{R}, \circlearrowleft_\mathrm{R}\}$$

(and 3 others)

What if the tetrahedron is invisible?



The tetrahedron may be invisible, but the possible actions remain the same!

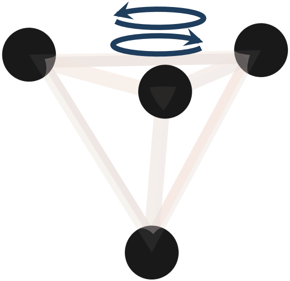
$$U_{\mathrm{B}} = \{\mathrm{id}, \circlearrowright_{\mathrm{B}}, \circlearrowleft_{\mathrm{B}}\}$$
$$U_{\mathrm{F}} = \{\mathrm{id}, \circlearrowright_{\mathrm{F}}, \circlearrowleft_{\mathrm{F}}\}$$
$$U_{\mathrm{L}} = \{\mathrm{id}, \circlearrowright_{\mathrm{L}}, \circlearrowleft_{\mathrm{L}}\}$$
$$U_{\mathrm{R}} = \{\mathrm{id}, \circlearrowright_{\mathrm{R}}, \circlearrowleft_{\mathrm{R}}\}$$

(and 3 others)

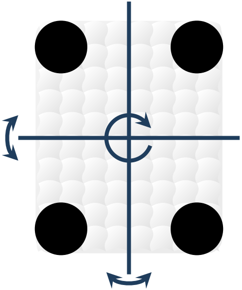
Question

How can we recognize the tetrahedron using the possible actions?

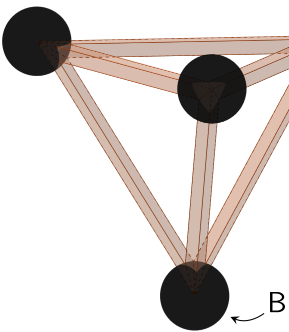
Moufang sets: a specific type of action



or



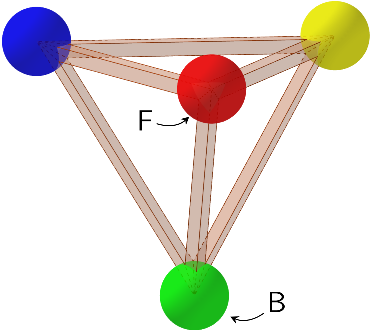
Properties of the action on the tetrahedron



Fix a vertex B.

There is a unique way from
$U_{\mathrm{B}} = \{\mathrm{id}, \circlearrowright_{\mathrm{B}}, \circlearrowleft_{\mathrm{B}}\}$
to move F to F, L or R.

We say the action of U_{B} is regular.



Fix a vertex B.

There is a unique way from
$U_{\mathrm{B}} = \{\mathrm{id}, \circlearrowright_{\mathrm{B}}, \circlearrowleft_{\mathrm{B}}\}$
to move F to F, L or R.

We say the action of U_{B} is regular.

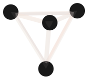
$\circlearrowright_{\mathrm{L}}$ moves B to F,
$\circlearrowleft_{\mathrm{L}}$ cancels that action

We see $\circlearrowleft_{\mathrm{L}} \circlearrowright_{\mathrm{B}} \circlearrowright_{\mathrm{L}} = \circlearrowright_{\mathrm{F}}$
In general, we get $\circlearrowleft_{\mathrm{L}} U_{\mathrm{B}} \circlearrowright_{\mathrm{L}} = U_{\mathrm{F}}$

We say U_{B} and U_{F} are conjugate.

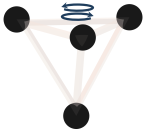
What is a Moufang set?

A Moufang set is a collection of points with for each point x an action of a root group U_x on the points, such that



$$U_\mathrm{B} = \{\mathrm{id}, \circlearrowright_\mathrm{B}, \circlearrowleft_\mathrm{B}\}$$

1. every U_x fixes the point x;
2. every U_x acts regularly on the other points;
3. the root groups are all conjugate.



$$\circlearrowleft_\mathrm{L} U_\mathrm{B} \circlearrowright_\mathrm{L} = U_\mathrm{F},$$
$$\circlearrowright_\mathrm{B} U_\mathrm{R} \circlearrowleft_\mathrm{B} = U_\mathrm{L} \ldots$$

3 Fields: 0, 1, +, −, × en ÷

A field is a structure with at least

- 0
- 1

in which we can do

- addition
- subtraction
- multiplication
- division

as you are used to.

Rational numbers (fractions)

$$0 + 5 = 5 \qquad 1 \times \frac{3}{4} = \frac{3}{4}$$

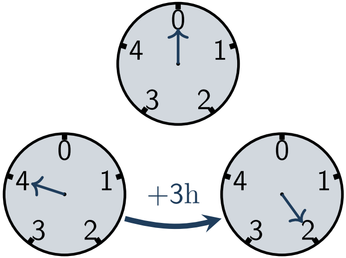
$$2 - \frac{1}{2} = \frac{3}{2} \qquad 5 \div 7 = \frac{5}{7}$$

$$2 - 2 = 0 \qquad 3 \times \frac{1}{3} = 1$$

$$\frac{3}{2} + (5 - 2) \times \frac{4}{3} \div 8 = 2$$

Modular arithmetic: a clock 5 hours

Numbers: 0, 1, 2, 3 en 4



Addition, subtraction and multiplication:
subtract 5 or add 5, until you get 0,...,4

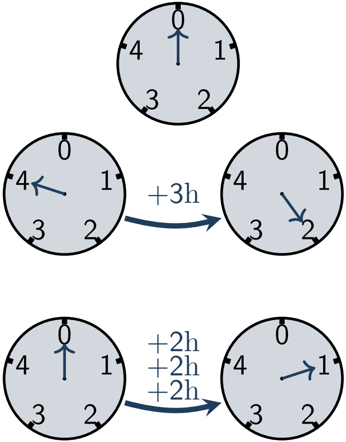
$4 + 3 \equiv 7 \equiv 2 \qquad 2 - 4 \equiv -2 \equiv 3$

Modular arithmetic: a clock 5 hours

Numbers: 0, 1, 2, 3 en 4

Addition, subtraction and multiplication:
subtract 5 or add 5, until you get 0,...,4
$4+3 \equiv 7 \equiv 2 \qquad 2-4 \equiv -2 \equiv 3$

Division: find numbers which
multiply to 1:
$2 \times 3 \equiv 6 \equiv 1$, so $2 \equiv \frac{1}{3}$
$\Longrightarrow 4 \div 3 \equiv 4 \times \frac{1}{3} \equiv 4 \times 2 \equiv 8 \equiv 3$

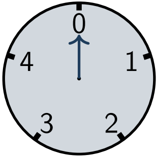


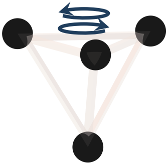
This is arithmetic modulo 5 and is a field.

construction
for every field

fields

Moufang sets



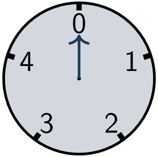


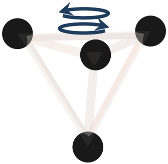
4 The connection between Moufang sets and fields

construction
for every field

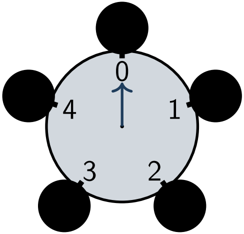
fields → Moufang sets

construction with
assumptions

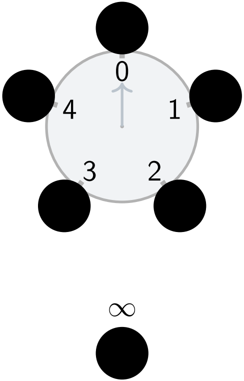




From fields to Moufang sets



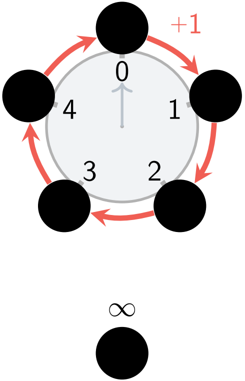
From fields to Moufang sets



Points:
the field and one extra point ∞

Root group:
$U_\infty = \{+0, +1, +2, +3, +4\}$

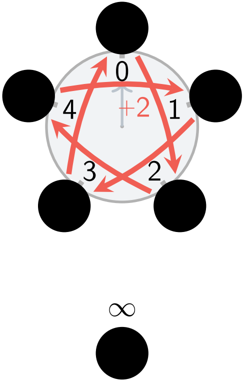
From fields to Moufang sets



Points:
the field and one extra point ∞

Root group:
$U_\infty = \{+0, +1, +2, +3, +4\}$

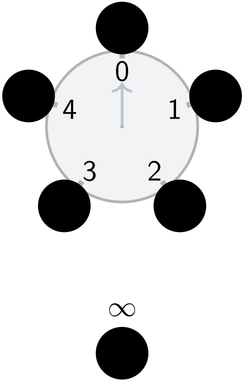
From fields to Moufang sets



Points:
the field and one extra point ∞

Root group:
$U_\infty = \{+0, +1, +2, +3, +4\}$

From fields to Moufang sets

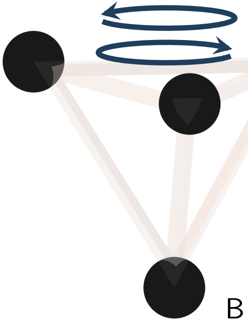


Points:
the field and one extra point ∞

Root group:
$U_\infty = \{+0, +1, +2, +3, +4\}$

Other root groups:
construct them using conjugation

From Moufang sets to fields



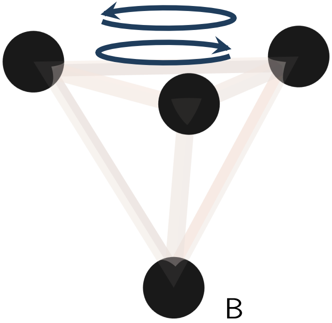
Take a root group:

$$U_{\mathrm{B}} = \{\mathrm{id}, \circlearrowright_{\mathrm{B}}, \circlearrowleft_{\mathrm{B}}\}$$

The elements become the numbers:

$$\mathrm{id} \rightsquigarrow 0 \qquad \circlearrowright_{\mathrm{B}} \rightsquigarrow 1 \qquad \circlearrowleft_{\mathrm{B}} \rightsquigarrow 2$$

From Moufang sets to fields



Take a root group:

$$U_{\mathrm{B}} = \{\mathrm{id}, \circlearrowright_{\mathrm{B}}, \circlearrowleft_{\mathrm{B}}\}$$

The elements become the numbers:

$$\mathrm{id} \rightsquigarrow 0 \qquad \circlearrowright_{\mathrm{B}} \rightsquigarrow 1 \qquad \circlearrowleft_{\mathrm{B}} \rightsquigarrow 2$$

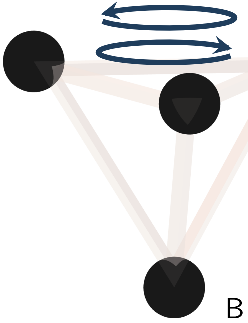
We find addition by composing the corresponding actions:

$$2 + 0 = ? \;\rightsquigarrow\; \circlearrowleft_{\mathrm{B}} \mathrm{id} = \circlearrowleft_{\mathrm{B}} \;\rightsquigarrow\; 2 + 0 = 2$$

$$1 + 1 = ? \;\rightsquigarrow\; \circlearrowright_{\mathrm{B}} \circlearrowright_{\mathrm{B}} = \circlearrowleft_{\mathrm{B}} \;\rightsquigarrow\; 1 + 1 = 2$$

$$2 + 2 = ? \;\rightsquigarrow\; \circlearrowleft_{\mathrm{B}} \circlearrowleft_{\mathrm{B}} = \circlearrowright_{\mathrm{B}} \;\rightsquigarrow\; 2 + 2 = 1$$

From Moufang sets to fields



Take a root group:

$$U_{\mathrm{B}} = \{\mathrm{id}, \circlearrowright_{\mathrm{B}}, \circlearrowleft_{\mathrm{B}}\}$$

The elements become the numbers:

$$\mathrm{id} \rightsquigarrow 0 \qquad \circlearrowright_{\mathrm{B}} \rightsquigarrow 1 \qquad \circlearrowleft_{\mathrm{B}} \rightsquigarrow 2$$

We find addition by composing the corresponding actions:

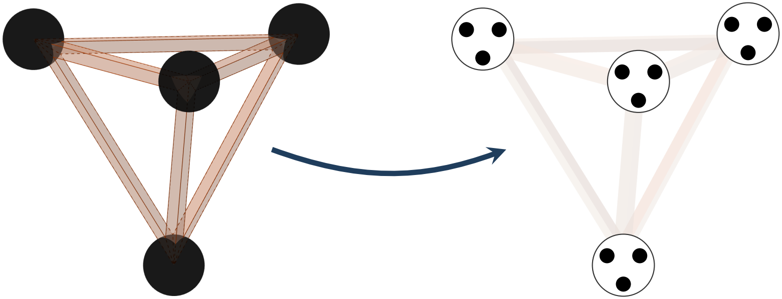
$$2 + 0 = ? \rightsquigarrow \circlearrowleft_{\mathrm{B}}\,\mathrm{id} = \circlearrowleft_{\mathrm{B}} \rightsquigarrow 2 + 0 = 2$$

$$1 + 1 = ? \rightsquigarrow \circlearrowright_{\mathrm{B}}\circlearrowright_{\mathrm{B}} = \circlearrowleft_{\mathrm{B}} \rightsquigarrow 1 + 1 = 2$$

$$2 + 2 = ? \rightsquigarrow \circlearrowleft_{\mathrm{B}}\circlearrowleft_{\mathrm{B}} = \circlearrowright_{\mathrm{B}} \rightsquigarrow 2 + 2 = 1$$

We find arithmetic modulo 3.

A local approach: close and far apart



Points are close together or far apart from each other.

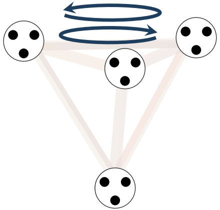
There is a local structure.

Group actions preserving ‘close’ and ‘far apart’

A group action on an object with a local structure preserves the local structure if

1. Points that are close together, stay close together after applying any action.

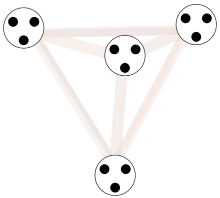
2. Points that are far apart, stay far apart after applying any action.



What is a local Moufang set?

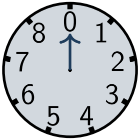
A local Moufang set is a collection of points with a local structure, with for every point x the action of a root group U_x preserving the local structure, such that

1. every U_x fixes the point x;
2. every U_x acts regularly on the points that are far apart from x;
3. every U_x acts regularly on the groups of points except for that of x;
4. the root groups are all conjugate.



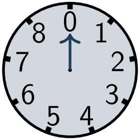
Arithmetic modulo 9: not a field, but a local ring

Numbers: 0, 1, 2, 3, 4, 5, 6, 7 en 8
Addition, subtraction and multiplication:
as before (add or subtract 9)



Arithmetic modulo 9: not a field, but a local ring

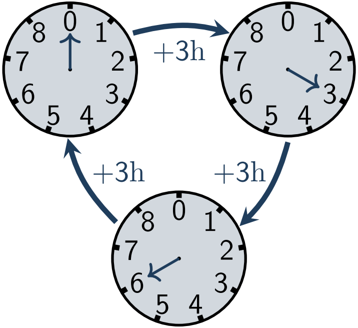
Numbers: 0, 1, 2, 3, 4, 5, 6, 7 en 8
Addition, subtraction and multiplication: as before (add or subtract 9)



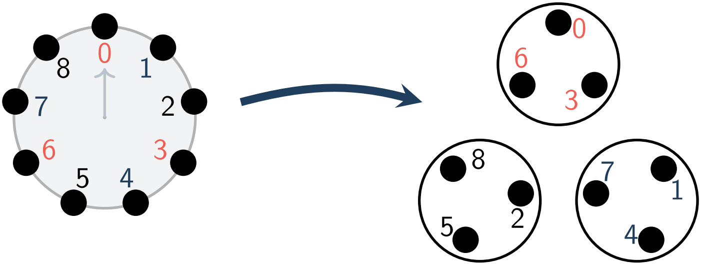
To divide by 3:
find a number such that $3 \times ? \equiv 1$,
but $3 \times ?$ is always a multiple of 3!

Dividing by 3, 6 en 0 is impossible.

Arithmetic modulo 9 is not a field,
but a local ring.

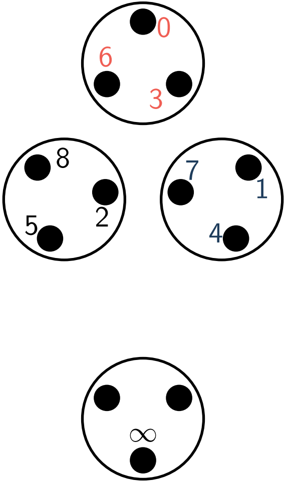


Local structure of numbers modulo 9



Numbers are close together if their difference is 0, 3 or 6.

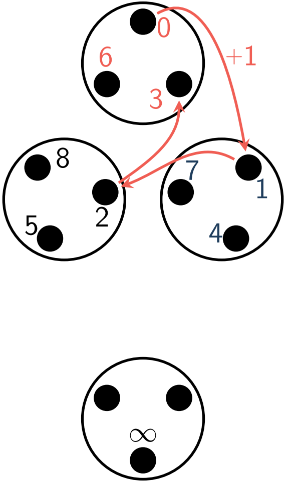
From local rings to local Moufang sets



Points:
the local ring and three extra points

Root group:
$U_\infty = \{+0, +1, +2, +3,$
$+4, +5, +6, +7, +8\}$

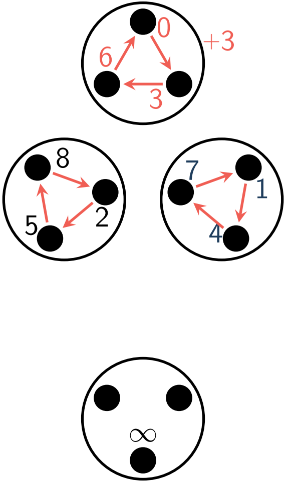
From local rings to local Moufang sets



Points:
the local ring and three extra points

Root group:

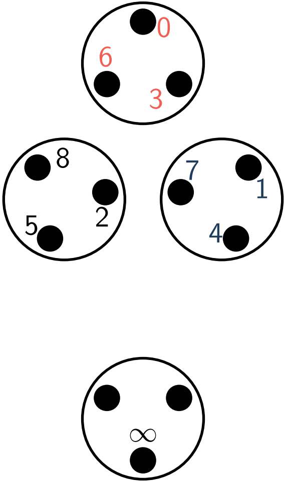
$$U_\infty = \{+0, +1, +2, +3, +4, +5, +6, +7, +8\}$$



Points:
the local ring and three extra points

Root group:
$U_\infty = \{+0, +1, +2, +3, +4, +5, +6, +7, +8\}$

From local rings to local Moufang sets



Points:
the local ring and three extra points

Root group:

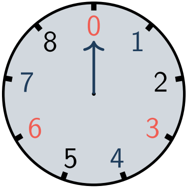
$$U_\infty = \{+0, +1, +2, +3, +4, +5, +6, +7, +8\}$$

Other root groups:
construct them by conjugation

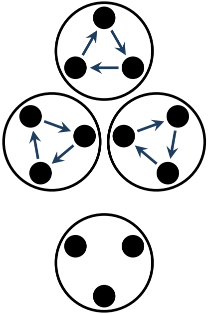
Local rings and local Moufang sets: the connection

construction for
all local rings

local rings



local Moufang sets



Local rings and local Moufang sets: the connection

construction for
all local rings

local rings → local Moufang sets

local Moufang sets → local rings

construction with
assumptions