OMEGA-RESULTS FOR BEURLING GENERALIZED INTEGERS

FREDERIK BROUCKE AND TITUS HILBERDINK

ABSTRACT. In this paper we consider Beurling number systems with very well-behaved primes, in the sense that $\psi(x) = x + O(x^{\alpha})$ for some $\alpha < 1/2$. We investigate how small the error term in the asymptotic formula for the integer-counting function $N(x)$ can be for such systems. In particular we show that

$$N(x) - \rho x = \Omega\big(\sqrt{x}\mathrm{e}^{-(\log x)^{\beta}}\big)$$

for any $\beta > \frac{2}{3}$.

1. INTRODUCTION

A *Beurling* or *g-prime* system $(\mathcal{P}, \mathcal{N})$ consists of a non-decreasing sequence $\mathcal{P} = (p_k)_{k\geq 1}$ of real numbers satisfying $p_1 > 1$ and $p_k \to \infty$ as $k \to \infty$ (called Beurling primes) and all the finite products formed from these (called Beurling integers) which we denote by $\mathcal{N} = (n_k)_{k\geq 1}$ (these include possible repetitions). They generalize the usual primes and integers and were introduced by Beurling [1] to investigate what conditions are necessary for a Prime Number Theorem to hold. Let

$$\pi(x) = \sum_{p_k \leq x} 1 \quad \text{and} \quad N(x) = \sum_{n_k \leq x} 1,$$

denote the counting functions of the Beurling primes and integers respectively. Beurling [1] showed that $\pi(x) \sim \frac{x}{\log x}$ under the assumption that $N(x) - \rho x = O(x(\log x)^{-\gamma})$ for some $\rho > 0$ and $\gamma > \frac{3}{2}$, but may fail if $\gamma \leq \frac{3}{2}$. Since then there has been much research regarding these systems, concentrating mainly on (i) obtaining weaker (or even optimal) conditions for a PNT to hold, (ii) the effects of stronger error terms, and (iii) obtaining results in the converse direction; i.e. assumptions on $\pi(x)$ leading to asymptotic behaviour of $N(x)$. For a survey of such results, see [5].

We shall also need the generalized von Mangoldt and Chebyshev functions: $\Lambda(p_k^r) = \log p_k$ if $r \geq 1$ and zero otherwise[1], while $\psi(x) = \sum_{n_k \leq x} \Lambda(n_k)$. The Beurling zeta function is defined as

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n_k^s}$$

for complex s whenever this series converges. The Dirichlet series associated to Λ is minus the logarithmic derivative of ζ:

$$\phi(s) := \sum_{k=1}^{\infty} \frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

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[1]Strictly speaking $\Lambda(n)$ is not a function in case the Beurling integers are not all distinct, but this is no problem as sums $\sum_{n_k \in A} \Lambda(n_k)$ are well-defined.

Let $\mathcal{P}$ be a g-prime system for which

$$\psi(x) = x + O(x^{\alpha}) \quad \text{for some } \alpha < \frac{1}{2}. \tag{1.1}$$

We note that in the first place, (1.1) (just with $\alpha < 1$) implies that

$$N(x) = \rho x + O(x\mathrm{e}^{-c\sqrt{\log x \log\log x}}), \quad \text{for some } \rho > 0 \text{ and any } c < \sqrt{2(1-\alpha)}.$$

Further, this is optimal, as was recently shown in [2].

Here we are interested in how *small* $N(x) - \rho x$ can be on the assumption that (1.1) holds with $\alpha < \frac{1}{2}$. In [6] it was shown that under this condition, $N(x) - \rho x = \Omega_{\varepsilon}(x^{1/2-\varepsilon})$ for any $\varepsilon > 0$. Here we improve this result. We note that the condition $\alpha < \frac{1}{2}$ is crucial. For larger α, $N(x) - \rho x$ can (presumably) be much smaller. For example, on the Riemann Hypothesis the g-prime system $(\mathbb{P}, \mathbb{N}_{>0})$ consisting of the rational primes and integers satisfies (1.1) for every $\alpha > \frac{1}{2}$ while $N(x) - x = O(1)$.

Our main result is:

Theorem 1.1. *Let $\mathcal{P}$ be a g-prime system for which* (1.1) *holds. Then*

$$N(x) - \rho x = \Omega\big(\sqrt{x}\mathrm{e}^{-(\log x)^{\beta}}\big)$$

for every $\beta > \frac{2}{3}$.

This can be compared to [3, Theorem 3.1], giving the existence of a Beurling prime system satisfying

$$\psi(x) = x + O(\log x \log\log x) \quad \text{and} \quad N(x) = x + O\big(\sqrt{x}\mathrm{e}^{c(\log x)^{2/3}}\big)$$

for some $c > 0$. Note that the exponent in the error term of N is positive here. Combining Theorem 1.1 with this result gives substantial progress to answering the question of how small $N(x) - \rho x$ can be under the assumption (1.1), although there is not yet a definitive answer. One might for example conjecture there are Beurling systems satisfying (1.1) and with $N(x) - \rho x \ll \sqrt{x}$, and that this is best possible.

There is a closely related problem concerning a system with very well behaved integers, namely when $N(x) = \rho x + O(x^{\alpha})$ with $\alpha < \frac{1}{2}$. Under this assumption, it was shown in [6] that $\psi(x) - x = \Omega(x^{1/2-\varepsilon})$ for all $\varepsilon > 0$. A natural question is to ask how small $\psi(x) - x$ can be. This was recently considered in [7], where this was improved to

$$\psi(x) - x = \Omega\big(\sqrt{x}\mathrm{e}^{-c\sqrt{\log x \log\log x}}\big)$$

for some $c > 0$. Although there is some overlap in ideas with the present paper, the actual details of the proof are quite different.

Notation The symbols $\sim$, $\asymp$, $\prec$, $\ll$, and $\Omega(\cdot)$ have their usual meaning: namely $f(x) \sim g(x)$ as $x \to \infty$ means $f(x)/g(x) \to 1$ as $x \to \infty$, $f(x) \asymp g(x)$ means there exist $a, b > 0$ such that $a < f(x)/g(x) < b$ for all x sufficiently large, while $f(x) \prec g(x)$ means $f = o(g)$ or $f(x)/g(x) \to 0$; $f \ll g$ has the same meaning as $f = O(g)$; i.e. $|f(x)| \le Cg(x)$ for some constant C and all x in the range; $f = \Omega(g)$ means $f \ne o(g)$; in other words, there exist $a > 0$ and $x_n \to \infty$ such that $|f(x_n)| > ag(x_n)$.

Finally, for sums over Beurling integers, we typically omit the index k and write e.g. $\sum_{n\in\mathcal{N}} f(n)$ to denote $\sum_{k\ge1} f(n_k)$, where it is understood any multiplicities are included.

2. Preliminary bounds

We begin with a simple lemma providing bounds on the Mellin–Stieltjes transform of certain functions $A(x)$.

Lemma 2.1. *Let $A(x)$ be a non-decreasing function, supported on $[1,\infty)$, and which satisfies*

$$A(x) = ax + O(x^{\theta}), \quad \text{for some } a > 0 \text{ and } \theta < 1.$$

Then its Mellin–Stieltjes transform $F(s) = \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}A(x)$ has analytic continuation, apart for a simple pole at $s = 1$, to $\sigma > \theta$. For $|t| \ge 2$ we have the bounds

$$F(s) \ll \Bigl(\frac{|t|}{\sigma-\theta}\Bigr)^{\frac{1-\sigma}{1-\theta}}, \qquad \text{for } \theta < \sigma < \frac{\theta+1}{2};$$
$$F(s) \ll |t|^{\frac{1-\sigma}{1-\theta}} \log|t| + 1, \qquad \text{for } \sigma \ge \frac{\theta+1}{2}.$$

Proof. We write $s = \sigma + it$ and $A(x) = ax + R(x)$. Let $X > 1$ be a parameter to be determined later. Then integration by parts yields, for $\sigma > 1$,

$$\begin{aligned} F(s) - \frac{a}{s-1} &\\ &= \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}A(x) - a\int_{1}^{\infty} x^{-s}\,\mathrm{d}x = \int_{1^-}^{X} x^{-s}\,\mathrm{d}A(x) - a\int_{1}^{X} x^{-s}\,\mathrm{d}x + \int_{X}^{\infty} x^{-s}\,\mathrm{d}R(x)\\ &= \int_{1^-}^{X} x^{-s}\,\mathrm{d}A(x) - a\frac{X^{1-s}-1}{1-s} - \frac{R(X)}{X^{s}} + s\int_{X}^{\infty} x^{-s-1}R(x)\,\mathrm{d}x. \end{aligned}$$

The right hand side yields the analytic continuation to $\sigma > \theta$. Since A is non-decreasing and $R(x) \ll x^{\theta}$, we can bound the left hand side as

$$F(s) - \frac{a}{s-1} \ll \int_{1^-}^{X} x^{-\sigma}\,\mathrm{d}A(x) + \left|\frac{X^{1-s}-1}{1-s}\right| + |s|\,\frac{X^{\theta-\sigma}}{\sigma-\theta}.$$

Integration by parts gives

$$\int_{1^-}^{X} x^{-\sigma}\,\mathrm{d}A(x) = a\frac{X^{1-\sigma}-1}{1-\sigma} + O\Bigl(\frac{\sigma}{\sigma-\theta}\Bigr),$$

where the first term is set to be $a\log X$ if $\sigma = 1$. Substituting this in the above yields, for $|t| \ge 2$,

$$F(s) \ll X^{1-\sigma} + |t|\,\frac{X^{\theta-\sigma}}{\sigma-\theta}, \qquad \text{for } \theta < \sigma < \frac{\theta+1}{2},$$
$$F(s) \ll X^{1-\sigma}\log X + |t|\,X^{\theta-\sigma} + 1, \qquad \text{for } \sigma \ge \frac{\theta+1}{2}.$$

Selecting

$$X = \Bigl(\frac{|t|}{\sigma-\theta}\Bigr)^{\frac{1}{1-\theta}} \quad \text{and} \quad X = |t|^{\frac{1}{1-\theta}},$$

respectively, yields the desired result. □

Suppose now that $N(x) = \rho x + O(\sqrt{x})$ for some $\rho > 0$ and $\psi(x) = x + O(x^{\alpha})$ for some $\alpha \in (0, 1/2)$. Then $\zeta(s)$ has analytic continuation to $\sigma > \alpha$ except for a simple pole at $s = 1$ with residue ρ, and has no zeroes there. The lemma provides bounds on $\zeta(s)$ for $\sigma > 1/2$ and bounds on $\phi(s) = -\zeta'(s)/\zeta(s)$ for $\sigma > \alpha$. From these bounds we can derive bounds on $\log\zeta(s)$, which has analytic continuation to $\{s : \sigma > \alpha\} \setminus (-\infty, 1]$.

Let $s = \sigma + it$ where $\sigma \in (1/2, 1)$ and $|t| \ge 2$. The bounds on $\zeta(s)$ provided by the lemma give upper bounds on $\log|\zeta(s)| = \operatorname{Re}\log\zeta(s)$. Applying the Borel–Carathéodory lemma with circles with centre $2 + it$ and radii $2 - \sigma$ and $2 - \frac{\sigma+1/2}{2}$, we see that

$$\log\zeta(s) \ll \frac{1}{\sigma-1/2}\Bigl(\log|t| + \log\frac{1}{\sigma-1/2}\Bigr), \quad \text{for } 1/2 < \sigma < 1. \tag{2.1}$$

On the other hand, by integrating the bounds for $\phi(s)$, we get

$$\log \zeta(s) \ll \Big(\frac{|t|}{\sigma - \alpha}\Big)^{\frac{1-\sigma}{1-\alpha}}, \quad \text{for } \alpha < \sigma < 1/2. \tag{2.2}$$

We obtain bounds in the region $\sigma \geq 1/2 - a\frac{\log\log|t|}{\log|t|}$, $a > 0$, by interpolating these two bounds.

Lemma 2.2. *There exists a constant c such that, for any $a > 0$, we have the bounds*

$$\log \zeta(s) \ll (\log|t|)^{2+ac}, \quad \textit{for } |t| \geq t_0 = t_0(a) \textit{ and } \sigma \geq 1/2 - a\frac{\log\log|t|}{\log|t|}.$$

Furthermore, if there exists some $a_0 > 0$ and some $b > 0$ such that $\log\zeta(s) \ll (\log|t|)^b$ holds for $\sigma \geq 1/2 - a_0\frac{\log\log|t|}{\log|t|}$, then we have for any $a \geq a_0$

$$\log \zeta(s) \ll (\log|t|)^{b+(a-a_0)c}, \quad \textit{for } |t| \geq t_0 \textit{ and } \sigma \geq 1/2 - a\frac{\log\log|t|}{\log|t|}.$$

From the proof it will follow that we can take any $c > 2/(1-2\alpha)$.

Proof. Let $t > 0$ be sufficiently large and set $\sigma = 1/2 - a\frac{\log\log t}{\log t} > \alpha$. We set $\sigma' = 1/2 + 1/\log t$ and $\tilde{\sigma} = \alpha + \delta$ for some small $\delta > 0$. We apply Hadamard's three circles theorem to the circles C_i, $i = 1, 2, 3$ with centre $\log t + \mathrm{i}t$ and passing through $\sigma' + \mathrm{i}t$, $\sigma + \mathrm{i}t$, and $\tilde{\sigma} + \mathrm{i}t$ when $i = 1$, 2, and 3, respectively. Denoting by $M_i = \sup_{z\in C_i}\bigl|\log\zeta(z)\bigr|$, we have

$$M_2 \leq M_1^{1-\kappa}M_3^{\kappa}, \quad \text{with } \kappa = \frac{\log\frac{\log t - \sigma}{\log t - \sigma'}}{\log\frac{\log t - \tilde{\sigma}}{\log t - \sigma'}}.$$

A small calculation gives

$$\kappa = \frac{a}{1/2 - \alpha - \delta}\frac{\log\log t}{\log t}\biggl\{1 + O\biggl(\frac{1}{\log\log t}\biggr)\biggr\}.$$

By (2.1), we have $M_1 \ll (\log t)^2$, while (2.2) gives $M_3 \ll t/\delta$. Hence we get

$$M_2 \ll_\delta (\log t)^{2+ac}, \quad \text{with } c = \frac{1}{1/2 - \alpha - \delta}.$$

The proof of the second part of the lemma is analogous: we now select $\sigma' = 1/2 - a_0\frac{\log\log t}{\log t}$, so that

$$\kappa = \frac{a - a_0}{1/2 - \alpha - \delta}\frac{\log\log t}{\log t}\biggl\{1 + O\biggl(\frac{1}{\log\log t}\biggr)\biggr\}.$$

□

3. Improving the bounds

We now assume that

$$N(x) = \rho x + O\bigl(x^{1/2}\exp\bigl(-k(x)\bigr)\bigr)$$

for some function $k(x)$ which is non-decreasing, tending to ∞, and for which $k'(x)x$ is decreasing to zero; the typical example we have in mind is $k(x) = \log^\beta x$ for some $\beta \in (0, 1)$. In this section, we show how to exploit the better error term in the asymptotic relation for $N(x)$ to improve the bounds given by Lemma 2.2.

We assume again that t is sufficiently large, and we let $s = \sigma + \mathrm{i}t$ with $\sigma = 1/2 - a\frac{\log\log t}{\log t}$. For any $\delta \in (0, 1)$ and $\kappa > 1 - \sigma$ we have

$$\sum_{n\in\mathcal{N}} n^{-s}\mathrm{e}^{-\delta n} = \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty}\zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w.$$

By the exponential decay of the Gamma-function, we may shift the contour of integration to the left. Let γ denote the contour consisting of the oriented lines L_i, $i=1,\dots,5$, defined respecitvely as $(-\mathrm{i}\infty, -2\mathrm{i}t]$, $[-2\mathrm{i}t, -1/\log t - 2\mathrm{i}t]$, $[-1/\log t - 2\mathrm{i}t, -1/\log t + 2\mathrm{i}t]$, $[-1/\log t + 2\mathrm{i}t, 2\mathrm{i}t]$, and $[2\mathrm{i}t, \mathrm{i}\infty)$. Moving the contour to γ, we cross two poles: one at $w = 1-s$ with residu $\rho\Gamma(1-s)\delta^{s-1}$, and one at $w=0$ with residu $\zeta(s)$. We get

$$\zeta(s) = \sum_n n^{-s}\mathrm{e}^{-\delta n} - \rho\Gamma(1-s)\delta^{s-1} - \frac{1}{2\pi\mathrm{i}}\int_\gamma \zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w$$

$$= \int_{1^-}^\infty x^{-s}\mathrm{e}^{-\delta x}\,\mathrm{d}R(x) - \frac{1}{2\pi\mathrm{i}}\int_\gamma \zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w + O(1), \tag{3.1}$$

where we have written $R(x) = N(x) - \rho x$. Integrating by parts, the first integral can be written as

$$I \coloneqq \int_1^\infty R(x)x^{-s}\mathrm{e}^{-\delta x}\Bigl(\frac{s}{x}+\delta\Bigr)\,\mathrm{d}x.$$

To estimate this integral, we split it into two parts I_1 and I_2, with domain of integration $[1, 1/\delta]$ and $[1/\delta, \infty)$ respectively. For I_1, we perform a dyadic splitting to get

$$I_1 \ll t\int_1^{1/\delta} x^{-1/2-\sigma}\mathrm{e}^{-k(x)}\,\mathrm{d}x \ll t\sum_{j=0}^{\lfloor \log(1/\delta)/\log 2\rfloor}\int_{2^j}^{2^{j+1}} x^{-1/2-\sigma}\mathrm{e}^{-k(x)}\,\mathrm{d}x$$

$$\ll t\sum_j \exp\bigl(-k(2^j)\bigr)\frac{(2^{j+1})^{1/2-\sigma} - (2^j)^{1/2-\sigma}}{1/2-\sigma}$$

$$\ll t\sum_j \exp\Bigl(-k(2^j) + (1/2-\sigma)\log 2^j\Bigr).$$

Now

$$\bigl(-k(x) + (1/2-\sigma)\log x\bigr)' = \frac{1}{x}\bigl(-xk'(x) + (1/2-\sigma)\bigr),$$

so that by the assumptions on k

$$\max_{x\in[1,X]}\bigl(-k(x) + (1/2-\sigma)\log x\bigr) \le \max\bigl\{-k(1); (1/2-\sigma)\log X - k(X)\bigr\}.$$

Hence the first part is bounded as

$$I_1 \ll t\bigl(1 + \delta^{\sigma-1/2}\mathrm{e}^{-k(1/\delta)}\bigr)\log\frac{1}{\delta}.$$

For I_2 we get

$$I_2 \ll \int_{1/\delta}^\infty \mathrm{e}^{-k(x)}x^{1/2-\sigma}\mathrm{e}^{-\delta x}\Bigl(\frac{t}{x}+\delta\Bigr)\,\mathrm{d}x$$

$$\ll \mathrm{e}^{-k(1/\delta)}\int_1^\infty \mathrm{e}^{-u}\bigl(tu^{-1/2-\sigma} + u^{1/2-\sigma}\bigr)\delta^{\sigma-1/2}\,\mathrm{d}u \ll t\delta^{\sigma-1/2}\mathrm{e}^{-k(1/\delta)}.$$

Collecting both estimates yields

$$I \ll t\bigl(1 + \delta^{\sigma-1/2}\mathrm{e}^{-k(1/\delta)}\bigr)\log\frac{1}{\delta}.$$

Let us now estimate the contour integral in (3.1). For the lines L_i, $i\neq 3$, we use the bound (2.2) and the exponential decay of the Gamma-function to see that

$$\int_{L_1\cup L_5}\zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w \ll \int_{2t}^\infty \exp\Bigl\{O\Bigl(v^{\frac{1}{2-2\alpha}}(\log v)^{\frac{a}{1-\alpha}}\Bigr) - \frac{\pi}{2}v\Bigr\}\,\mathrm{d}v \ll 1,$$

$$\int_{L_2\cup L_4}\zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w \ll 1.$$

Hence we get

$$\int_\gamma \zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w \ll \delta^{\frac{1}{\log t}} \int_{-2t}^{2t} \Big|\zeta\Big(s-\frac{1}{\log t}+\mathrm{i}v\Big)\Big|\Big|\Gamma\Big(-\frac{1}{\log t}+\mathrm{i}v\Big)\Big|\,\mathrm{d}v + 1. \tag{3.2}$$

For $w \in L_3$, we have that

$$\operatorname{Re}(s+w) \geq \frac{1}{2} - a'\frac{\log\log|\operatorname{Im}(s+w)|}{\log|\operatorname{Im}(s+w)|}, \quad a' = a + O\Big(\frac{1}{\log\log t}\Big),$$

so that by Lemma 2.2

$$\begin{aligned}\int_\gamma \zeta(s+w)\Gamma(w)\delta^{-w}\,\mathrm{d}w &\ll \delta^{\frac{1}{\log t}} \exp\big(A(\log t)^{2+ac}\big) \int_{-2t}^{2t}\Big|\Gamma\Big(-\frac{1}{\log t}+\mathrm{i}v\Big)\Big|\,\mathrm{d}v + 1\\ &\ll \exp\Big(-\frac{1}{\log t}\log\frac{1}{\delta} + A(\log t)^{2+ac}\Big)(\log t) + 1,\end{aligned}$$

for some constant $A > 0$. We now take $\log\frac{1}{\delta} = A(\log t)^{3+ac}$, then the above is $\ll \log t$. With this choice of δ we get

$$\zeta(s) \ll |I| + \log t \ll t(\log t)^{O(1)}\Big(1 + \exp\big\{aA(\log t)^{2+ac}\log\log t - k\big(\exp(A(\log t)^{3+ac})\big)\big\}\Big).$$

Suppose now that $k(x) \geq (\log x)^\beta$ for some $\beta > 2/3$. Then we get

$$\zeta(s) \ll t(\log t)^{O(1)},$$

provided that $\beta(3+ac) > 2+ac$, or equivalently

$$a < \frac{3\beta-2}{c(1-\beta)}.$$

By an application of Borel–Carathéodory we see that

$$\log\zeta(s) \ll \frac{(\log|t|)^2}{\log\log|t|},$$

provided that

$$\sigma \geq \frac{1}{2} - a_0\frac{\log\log|t|}{\log|t|}, \quad a_0 := \frac{3\beta-2}{2c(1-\beta)}.$$

We now use the second part of Lemma 2.2 to improve the exponent $2+ac$ to $2+(a-a_0)c$ for $a \geq a_0$:

$$\log\zeta(s) \ll (\log|t|)^{2+\max\{(a-a_0)c,0\}} \quad \text{for } \sigma \geq \frac{1}{2} - a\frac{\log\log|t|}{\log|t|}.$$

This improved bound can be fed into the above argument. In order to bound the integral (3.2), we can now take a slightly larger choice for δ, which improves the range where we have polynomial bounds on $\zeta(s)$. Indeed, one verifies that we can now get

$$\log\zeta(s) \ll \frac{(\log|t|)^2}{\log\log|t|}, \quad \text{for } \sigma \geq \frac{1}{2} - 2a_0\frac{\log\log|t|}{\log|t|}.$$

This process can be iterated: for each positive integer K we get

$$\log\zeta(s) \ll_K \frac{(\log|t|)^2}{\log\log|t|}, \quad \text{for } \sigma \geq \frac{1}{2} - Ka_0\frac{\log\log|t|}{\log|t|}. \tag{3.3}$$

Remark 3.1. *To bound the integral in the right hand side of* (3.2), *we used a supremum bound for the zeta-function. One might expect to gain something if one has a good bound for ζ on average. For example, if one can show that*

$$\int_{-T}^{T}\Big|\zeta\Big(\frac{1}{2} - a\frac{\log\log T}{\log T} + \mathrm{i}t\Big)\Big|^2\,\mathrm{d}t \ll \exp\big\{O\big((\log T)^{1+(a-a_0)c}\big)\big\}$$

holds with $a_0 = 0$ *and with* $a \geq a_0$, $a_0 > 0$ *provided that*

$$\log\zeta(s) \ll (\log|t|)^2 \quad \text{for } \sigma \geq \frac{1}{2} - a_0\frac{\log\log|t|}{\log|t|},$$

then the above argument can be applied to all functions k *satisfying* $k(x) \geq (\log x)^\beta$ *with* $\beta > 1/2$, *instead of only* $\beta > 2/3$.

The bounds in (3.3) readily give a bound for $\phi(s)$ using

$$\phi(s) = -\frac{1}{2\pi i}\int_{C(s,\varepsilon)} \frac{\log\zeta(z)}{(z-s)^2}\,\mathrm{d}z,$$

where $C(s,\varepsilon)$ is the circular contour with centre s and radius ε. As such, $\left|\phi(s)\right| \leq \frac{1}{\varepsilon}\max_{|z-s|=\varepsilon}\left|\log\zeta(z)\right|$ and taking $\varepsilon = \frac{\log\log|t|}{\log|t|}$ gives

$$\phi(s) \ll_K \frac{(\log|t|)^3}{(\log\log|t|)^2}, \qquad \text{for } \sigma \geq \frac{1}{2} - Ka_0\frac{\log\log|t|}{\log|t|}. \tag{3.4}$$

4. Lower bounds for the mean square value of $\phi_N(s)$

Let

$$\phi_N(s) = \sum_{n\leq N}\frac{\Lambda(n)}{n^s},$$

where the sum is over the generalised integers of a system satisfying (1.1) and $\Lambda(n)$ is the generalised von Mangoldt function. Let $\delta = \delta_N > 0$. Then

$$\int_0^T \left|\phi_N\Big(\frac{1}{2} - \delta + \mathrm{i}t\Big)\right|^2 \mathrm{d}t = T{\sum_{n\leq N}}^{*}\frac{\Lambda(n)^2}{n^{1-2\delta}} + 2\sum_{m<n\leq N}\frac{\Lambda(m)\Lambda(n)}{(mn)^{\frac{1}{2}-\delta}}S_{m,n}(T), \tag{4.1}$$

where $S_{m,n}(T) = \frac{\sin(T\log(n/m))}{\log(n/m)}$, and the $*$ means any multiplicities are to be squared. If $\delta \ll \frac{1}{\log N}$, then $n^\delta \asymp 1$ for $n \leq N$ and $\sum_{n\leq N}\frac{\Lambda(n)^2}{n^{1-2\delta}} \asymp (\log N)^2$, but we require this sum to be larger. So we will take $\delta = \frac{\kappa_N}{\log N}$ with $1 \prec \kappa_N \prec \log N$. As such, we see that on writing $\theta(x) = x + E(x)$,

$$\begin{aligned}
&\sum_{n\leq N}\frac{\Lambda(n)^2}{n^{1-2\delta}} \geq \sum_{p\leq N}\frac{(\log p)^2}{p^{1-2\delta}} = \int_1^N \frac{\log x}{x^{1-2\delta}}\,\mathrm{d}\theta(x)\\
&= \int_0^{\log N} u\mathrm{e}^{2\delta u}\,\mathrm{d}u + \frac{E(N)\log N}{N^{1-2\delta}} + \int_1^N \frac{E(x)((1-2\delta)\log x - 1)}{x^{2-2\delta}}\,\mathrm{d}x\\
&= \frac{N^{2\delta}\log N}{2\delta} - \frac{N^{2\delta}-1}{(2\delta)^2} + O(1) \geq \frac{\mathrm{e}^{2\kappa_N}(\log N)^2}{4\kappa_N},
\end{aligned}$$

if N is sufficiently large. In the second term in (4.1) the part of the sum where $m \leq \frac{n}{2}$ (for which $\left|S_{m,n}(T)\right| \leq \frac{1}{\log 2}$), is

$$\leq \frac{2}{\log 2}\sum_{n\leq N}\frac{\Lambda(n)}{n^{\frac{1}{2}-\delta}}\sum_{m\leq\frac{n}{2}}\frac{\Lambda(m)}{m^{\frac{1}{2}-\delta}} \ll N^{1+2\delta} = N\mathrm{e}^{2\kappa_N}.$$

Put $T = 2r-1$ and sum over $r = 1,\ldots,R$. Since

$$\sum_{r=1}^{R}\sin\big((2r-1)x\big) = \frac{1-\cos(2Rx)}{2\sin x} \geq 0 \quad \text{if } 0 \leq x \leq \pi,$$

the sum over r of the terms for which $\frac{n}{2} < m < n$ is positive, so

$$\sum_{r=1}^{R} \int_0^{2r-1} \Big|\phi_N\Big(\frac{1}{2} - \delta + \mathrm{i}t\Big)\Big|^2 \,\mathrm{d}t \geq \frac{R^2 \mathrm{e}^{2\kappa_N} (\log N)^2}{4\kappa_N} - CRN\mathrm{e}^{2\kappa_N},$$

for some $C > 0$. Thus, for $R \geq \frac{8CN\kappa_N}{(\log N)^2}$,

$$\sum_{r=1}^{R} \int_0^{2r-1} \Big|\phi_N\Big(\frac{1}{2} - \delta + \mathrm{i}t\Big)\Big|^2 \,\mathrm{d}t \geq \frac{R^2 \mathrm{e}^{2\kappa_N} (\log N)^2}{8\kappa_N}.$$

This holds for a general κ_N, but for our purpose we shall choose $\kappa_N = \mu \log\log N$, for a constant $\mu > 0$. Thus,

$$\sum_{r=1}^{R} \int_0^{2r-1} \Big|\phi_N\Big(\frac{1}{2} - \delta + \mathrm{i}t\Big)\Big|^2 \,\mathrm{d}t \geq \frac{R^2 (\log N)^{2+2\mu}}{8\mu \log\log N}, \quad \text{for } R \geq \frac{8C\mu N \log\log N}{(\log N)^2}. \tag{4.2}$$

Approximating ϕ_N by ϕ
Let $c > 0$ and $N \notin \mathcal{N}$. Then, for $n \in \mathcal{N}$,

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \Big(\frac{N}{n}\Big)^w \frac{\mathrm{d}w}{w} = O\bigg(\frac{(N/n)^c}{T\big|\log N/n\big|}\bigg) + \begin{cases} 1 & \text{if } n < N, \\ 0 & \text{if } n > N, \end{cases}$$

where the implied constant is independent of n and N. Multiply through by $\Lambda(n)n^{-s} = \Lambda(n)n^{-\sigma-\mathrm{i}t}$, where $|t| < T$, and sum over all $n \in \mathcal{N}$. Thus

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \frac{\phi(s+w)N^w}{w} \,\mathrm{d}w = \phi_N(s) + O\bigg(\frac{N^c}{T} \sum_{n\in\mathcal{N}} \frac{\Lambda(n)}{n^{c+\sigma}\big|\log N/n\big|}\bigg).$$

For $n \leq \frac{N}{2}$ and $n \geq 2N$, $\big|\log N/n\big| \geq \log 2$, while for $\frac{N}{2} < n < 2N$ we have $\big|\log N/n\big| \asymp \frac{|n-N|}{N}$ so with $s = \sigma + it$ such that $c + \sigma > 1$

$$\phi_N(s) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \frac{\phi(s+w)N^w}{w} \,\mathrm{d}w + O\bigg(\frac{N^c}{T} \sum_{n\in\mathcal{N}} \frac{\Lambda(n)}{n^{c+\sigma}}\bigg) + O\bigg(\frac{N^{1-\sigma}}{T} \sum_{\frac{N}{2}<n<2N} \frac{\Lambda(n)}{|n-N|}\bigg). \tag{4.3}$$

Now as $N(x) \sim \rho x$, we can take arbitrarily large values of N such that $\operatorname{dist}(N,\mathcal{N}) \geq 1/(2\rho) = d$ say. Then for the sum on the right of (4.3) we have

$$\sum_{\substack{\frac{N}{2}<n<2N \\ n\in\mathcal{N}}} \frac{\Lambda(n)}{|n-N|} = \bigg(\int_{(N/2,N-d]} + \int_{[N+d,2N)}\bigg) \frac{\mathrm{d}\psi(x)}{|x-N|} \ll \log N + N^{\alpha},$$

using (1.1) and integration by parts. We shall be taking $\sigma = \frac{1}{2} - \delta$, so choosing $c = 1 - \sigma + 1/\log N$ gives

$$\phi_N(s) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \frac{\phi(s+w)N^w}{w} \,\mathrm{d}w + O\Big(\frac{N^{\frac{1}{2}+\alpha+\delta}}{T} \log N\Big) \tag{4.4}$$

The error is small if we take $T \geq N$. Indeed, we will take $T = N^2$.

5. Proof of Theorem 1.1

For a contradiction, assume that $N(x)-\rho x = O(\sqrt{x}\mathrm{e}^{-(\log x)^{\beta}})$ for some $\beta > \frac{2}{3}$. Then the bounds on $\phi(s)$ in (3.4) hold.

We will apply (4.4) with $s=\sigma+\mathrm{i}t$, where

$$\sigma = \frac{1}{2}-\delta = \frac{1}{2}-\frac{\mu\log\log N}{\log N}, \quad |t|\le T/2 = N^2/2.$$

We will shift the contour of integration to $\operatorname{Re} w = -\eta$. In order to exploit the bound (3.4), we will take η of the same order as δ, say $\eta = \log\log N/\log N$. As such, $N^{\eta} = \log N \asymp \log T$.

Pushing the contour in (4.4) to $\operatorname{Re} w = -\eta$, we pick up the residues at $w=0$ and $w=1-s$ (since $|t|<T/2$). These are $\phi(s)$ and $\frac{N^{1-s}}{1-s}$. Note that

$$\left|\frac{N^{1-s}}{1-s}\right| \ll \frac{N^{1-\sigma}}{|t|+1} \ll \frac{\sqrt{N}(\log N)^{\mu}}{|t|+1}.$$

Now

$$\sigma-\eta = \frac{1}{2}-(\mu+1)\frac{\log\log N}{\log N}, \quad \text{and} \quad \frac{\log\log N}{\log N} \ll \frac{\log\log|t+y|}{\log|t+y|} \quad \text{for } |y|\le T, \quad |t+y|\ge \mathrm{e}^{\mathrm{e}}.$$

Hence for fixed μ, we can take $K=K(\mu)$ large enough such that the bound (3.4) is applicable. Then the integral along $\operatorname{Re} w=-\eta$ is, in absolute value, bounded by

$$\frac{N^{-\eta}}{2\pi}\int_{-T}^{T}\frac{\big|\phi(\sigma-\eta+\mathrm{i}(t+y))\big|}{\sqrt{\eta^2+y^2}}\,\mathrm{d}y \ll \frac{(\log T)^3}{N^{\eta}(\log\log T)^2}\int_0^T\frac{\mathrm{d}y}{\sqrt{\eta^2+y^2}} \asymp \frac{(\log N)^2\log(T/\eta)}{(\log\log N)^2}.$$

The contribution along the horizontal line $[-\eta+\mathrm{i}T, c+\mathrm{i}T]$ is, in modulus, less than

$$\frac{1}{2\pi}\int_{-\eta}^{c}\frac{N^{y}\big|\phi(\sigma+x+\mathrm{i}(t+T))\big|}{\sqrt{x^2+T^2}}\,\mathrm{d}x \ll \frac{N^{c}(\log T)^3}{T(\log\log T)^2} = o(1),$$

as $c=1/2+o(1)$. Similarly on $[-\eta-\mathrm{i}T, c-\mathrm{i}T]$.

Putting these observations together (noting that $\log T/\eta \asymp \log N$) we obtain

$$\phi_N(s) = \phi(s) + O\Big(\frac{\sqrt{N}(\log N)^{\mu}}{|t|+1}\Big) + o(1) + O\Big(\frac{(\log N)^3}{(\log\log N)^2}\Big).$$

Squaring and using (3.4) gives

$$\Big|\phi_N\Big(\frac{1}{2}-\delta+\mathrm{i}t\Big)\Big|^2 \ll \frac{(\log N)^6}{(\log\log N)^4} + \frac{N(\log N)^{2\mu}}{|t|^2+1}, \quad \text{for } |t|\le\frac{N^2}{2}.$$

Integrating and summing gives, for $R\le N^2/4$,

$$\sum_{r=1}^{R}\int_0^{2r-1}\Big|\phi_N\Big(\frac{1}{2}-\delta+\mathrm{i}t\Big)\Big|^2\,\mathrm{d}t \ll R^2\frac{(\log N)^6}{(\log\log N)^4} + RN(\log N)^{2\mu}.$$

Taking $R=N^2/4$, the second term is of smaller order than the first one. As such, this contradicts (4.2) if $\mu\ge 2$.

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Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Krijgslaan 281, 9000 Gent, Belgium

Email address: fabrouck.broucke@ugent.be

Department of Mathematics, University of Reading, Reading RG6 6AX, UK

Email address: t.w.hilberdink@reading.ac.uk