FACULTY OF SCIENCES

FNRS Functional Analysis meeting – 4 July 2025

joint work with G. Debruyne and J. Vindas

The asymmetric Beurling–Selberg extremal problem

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BEST APPROXIMATION

$$E_{2\pi} = \{G \text{ entire of exponential type } \leq 2\pi, \text{ real-valued on } \mathbb{R}\}$$

$$|G(z)| \leq C_\varepsilon e^{(2\pi+\varepsilon)|z|}.$$

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$$\int_{-\infty}^{\infty} |\mathfrak{s}(x) - G(x)| \, \mathrm{d}x, \quad G \in E_{2\pi}.$$

Best approximation

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If $\varphi = \hat{\mathfrak{s}}|_{\mathbb{R}\setminus[-2\pi,2\pi]} \in C_0(\mathbb{R} \setminus [-2\pi, 2\pi])$, then equivalent to finding minimal extrapolation $\hat{f} \in A(\mathbb{R})$:

$$\text{minimize } \|g\|_{L^1} \text{ among all } g \in L^1 \text{ with } \hat{g}|_{\mathbb{R}\setminus[-2\pi,2\pi]} = \varphi.$$

$\hat{g}$ minimal extrapolation $\iff$ $G = \mathfrak{s} - g$ best approximation.

$\mathfrak{s}(x) = \operatorname{sgn}(x)$

$$\mathfrak{s}(x) = \operatorname{sgn} x = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

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In this case, the solution can be found by interpolating $\operatorname{sgn} x$:

$$F(z) = \frac{\sin(2\pi z)}{2\pi} \sum_{k \in \mathbb{Z}} \left(\frac{\operatorname{sgn} k}{z - k} - \frac{\operatorname{sgn}(k + 1/2)}{z - (k + 1/2)} \right).$$

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Theorem

If $G \in E_{2\pi}$, then

$$\int_{-\infty}^{\infty} \left|\operatorname{sgn} x - G(x)\right| \mathrm{d}x \geq 1/2,$$

with equality if and only if $G = F$.

Application

Theorem (finite form Ingham–Karamata, Debruyne, Vindas, 2018)

Let $\tau : (0, \infty) \to \mathbb{R}$ *be differentiable with* $\left|\tau'(x)\right| \leq M$. *Suppose that* $\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x) \mathrm{e}^{-sx} \, \mathrm{d}x$ *converges for* $\operatorname{Re} s > 0$ *and admits continuous extension to a segment* $[-\mathrm{i}\lambda, \mathrm{i}\lambda]$ *for some* $\lambda > 0$. *Then*

$$\limsup_{x \to \infty} \left|\tau(x)\right| \leq \frac{\pi}{2} \cdot \frac{M}{\lambda}.$$

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$$\limsup_{x \to \infty} \left|\tau(x)\right| \leq \frac{\pi}{2} \cdot \frac{M}{\lambda}.$$

Proof sketch:

W.l.o.g. $\lambda = 2\pi$.

$$\begin{aligned} 2\tau(x) &= \int_{-\infty}^{\infty} \tau(x+y)\,\mathrm{d}\operatorname{sgn} y \\ &= \int_{-\infty}^{\infty} \tau(x+y)\,\mathrm{d}\big(\operatorname{sgn} y - F(y)\big) + \int_{-\infty}^{\infty} \tau(x+y) F'(y)\,\mathrm{d}y \end{aligned}$$

Proof sketch continued

Integrating by parts, first term is

$$-\int_{-\infty}^{\infty} \tau'(x+y)\bigl(\operatorname{sgn} y - F(y)\bigr)\,\mathrm{d}y, \quad |\ldots| \leq \frac{M}{2}.$$

Proof sketch continued

Integrating by parts, first term is

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Second term, change variables and consider for $\sigma > 0$:

$$\int_{-\infty}^{\infty} \tau(y)\mathrm{e}^{-\sigma y} F'(y-x)\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}\{\tau, \sigma + \mathrm{i}t\}\widehat{F'}(t)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t.$$

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We have $\operatorname{supp} \widehat{F'} \subseteq [-2\pi, 2\pi]$, so letting $\sigma \to 0$:

$$\int_{-\infty}^{\infty} \tau(y)F'(y-x)\,\mathrm{d}y = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \widehat{\tau}(t)\widehat{F'}(t)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t.$$

Proof sketch continued

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By Riemann–Lebesgue, the above $\to 0$ as $x \to \infty$.

Beurling–Selberg problem

Let $\mathfrak{s} \in L^1_{loc}$. Goal: minimize

$$\int_{-\infty}^{\infty} \big(G(x) - \mathfrak{s}(x)\big)\,\mathrm{d}x, \quad G \in E_{2\pi}, \quad \mathfrak{s}(x) \leq G(x), \forall x.$$

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Solved by Beurling for $\mathfrak{s}(x) = \operatorname{sgn} x$ in late 1930's,
"Popularized" by Selberg in 1970's, when $\mathfrak{s} = \chi_{[a,b]}$ in connection with large sieve.

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"Popularized" by Selberg in 1970's, when $\mathfrak{s} = \chi_{[a,b]}$ in connection with large sieve.
Interpolating sgn x and the derivative:

$$B(z) = \left(\frac{\sin(\pi z)}{\pi}\right)^2 \left(\sum_{n\ge 0} \frac{1}{(z-n)^2} - \sum_{n<0} \frac{1}{(z-n)^2} + \frac{2}{z}\right).$$

Beurling's function

Theorem (Beurling)

If $G \in E_{2\pi}$ with $G(x) \geq \operatorname{sgn} x$, for all $x \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \big(G(x) - \operatorname{sgn} x\big)\,\mathrm{d}x \geq 1,$$

with equality if and only if $G = B$.

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Theorem (one-sided finite form Ingham–Karamata, Debruyne, Vindas, 2018)

Let $\tau : (0,\infty) \to \mathbb{R}$ be differentiable with $\tau'(x) \leq M$. Suppose that $\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x)\mathrm{e}^{-sx}\,\mathrm{d}x$ converges for $\operatorname{Re} s > 0$ and admits continuous extension to a segment $[-\mathrm{i}\lambda, \mathrm{i}\lambda]$ for some $\lambda > 0$. Then

$$\limsup_{x\to\infty} \big|\tau(x)\big| \leq \pi \cdot \frac{M}{\lambda}.$$

APPLICATION: MEAN VALUE THEOREM

Theorem

Let $a_n \in \mathbb{C}$, $\lambda_n \in \mathbb{R}$ *with* $|\lambda_n - \lambda_m| \geq \delta$ *if* $n \neq m$. *Then*

$$\int_{T_0}^{T_0+T} \left| \sum_{n=1}^{N} a_n \mathrm{e}^{\mathrm{i}\lambda_n t} \right|^2 \mathrm{d}t = \left(T + \vartheta \frac{2\pi}{\delta} \right) \sum_{n=1}^{N} |a_n|^2 ,$$

for some $\vartheta \in [-1, 1]$.

Proof MVT

Set

$$\varphi_\delta(t) = \frac{1}{2}\Big\{B\Big(\frac{\delta}{2\pi}(t - T_0)\Big) + B\Big(\frac{\delta}{2\pi}(T_0 + T - t)\Big)\Big\}.$$

Then $\varphi_\delta(t) \geq \chi(t)$, $\int_{-\infty}^{\infty} \varphi_\delta(t)\,\mathrm{d}t = T + \frac{2\pi}{\delta}$, and $\operatorname{supp}\widehat{\varphi_\delta} \subseteq [-\delta, \delta]$.

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$$\int_{T_0}^{T_0+T} \left|f(t)\right|^2 \mathrm{d}t \leq \int_{-\infty}^{\infty} \varphi_\delta(t)\left|f(t)\right|^2 \mathrm{d}t = \sum_{n,m} a_n \overline{a_m}\, \widehat{\varphi_\delta}(\lambda_m - \lambda_n).$$

Here, $\widehat{\varphi_\delta}(\lambda_m - \lambda_n) = 0$ if $n \neq m$.

PROOF MVT

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Then $\varphi_\delta(t) \geq \chi(t)$, $\int_{-\infty}^{\infty} \varphi_\delta(t)\,\mathrm{d}t = T + \frac{2\pi}{\delta}$, and supp $\widehat{\varphi_\delta} \subseteq [-\delta, \delta]$.

$$\int_{T_0}^{T_0+T} \big|f(t)\big|^2\,\mathrm{d}t \leq \int_{-\infty}^{\infty} \varphi_\delta(t)\big|f(t)\big|^2\,\mathrm{d}t = \sum_{n,m} a_n\overline{a_m}\,\widehat{\varphi_\delta}(\lambda_m - \lambda_n).$$

Here, $\widehat{\varphi_\delta}(\lambda_m - \lambda_n) = 0$ if $n \neq m$. To prove the reverse inequality, use the optimal minorant $B_-(x)$.

Further applications

Proporties of B can applied to obtain:

- Bohr's inequality
- Hilbert's inequality
- Erdős–Turán inequality
- Large sieve inequality
- Berry–Esseen inequality

FURTHER APPLICATIONS

Proporties of B can applied to obtain:

- Bohr's inequality
- Hilbert's inequality
- Erdős–Turán inequality
- Large sieve inequality
- Berry–Esseen inequality

Solutions for other $\mathfrak{s}(x)$: applied in estimates for Riemann zeta $\zeta(s)$.

Asymmetric Beurling–Selberg

Let $\eta \in (0,1)$. For $g \in L^1$, set

$$\mathcal{I}_\eta(g) = (1-\eta)\int_{-\infty}^{\infty} g_+(x)\,\mathrm{d}x + \eta\int_{-\infty}^{\infty} g_-(x)\,\mathrm{d}x.$$

Here $g_+(x) = \max\{g(x), 0\}$, $g_-(x) = \max\{-g(x), 0\}$.

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Let again $\mathfrak{s} \in L^1_{loc}$. We call the *asymmetric Beurling–Selberg problem* for $\mathfrak{s}(x)$ the problem of minimizing $\mathcal{I}_\eta(s - G)$ among $G \in E_{2\pi}$.

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Case $\eta = 1/2$ corresponds to best approximation problem.
Limits $\eta \to 0$ or 1 correspond to majorants or minorants (Beurling–Selberg problem)?

Generalities

Suppose there exists $G \in E_{2\pi} \cap \mathcal{S}'$ with $\mathfrak{s} - G \in L^1$. Then

$$\varphi := \hat{\mathfrak{s}}|_{\mathbb{R}\setminus[-2\pi,2\pi]} \in C_0(\mathbb{R} \setminus [-2\pi, 2\pi]).$$

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Equivalent problem: given $\varphi \in C_0(\mathbb{R} \setminus [-2\pi, 2\pi])$, minimize $\mathcal{I}_\eta(g)$ among all $g \in L^1$ with $\hat{g} = \varphi$ on $\mathbb{R} \setminus [-2\pi, 2\pi]$ ($g = \mathfrak{s} - G$).

GENERALITIES

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Using compactness argument:

Proposition (B., Debruyne, Vindas, 2025)

Let $\varphi \in C_0(\mathbb{R} \setminus [-2\pi, 2\pi])$. If there exists an extrapolation $\hat{g}$, then there exists an η-minimal extrapolation.

Dual problem

For $h \in L^\infty$, set

$$\mathcal{I}_\eta^*(h) := \sup_{\substack{g \in L^1 \\ \mathcal{I}_\eta(g)=1}} \int_{-\infty}^{\infty} g(x)h(x)\,\mathrm{d}x = \max\left\{\frac{\|h_+\|_\infty}{1-\eta}, \frac{\|h_-\|_\infty}{\eta}\right\}.$$

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Using Hahn–Banach:

Proposition (B., Debruyne, Vindas, 2025)

Let $f \in L^1$. Then $\hat{f}$ is an η-minimal extrapolation of itself (i.e. of $\hat{f}|_{\mathbb{R}\setminus[-2\pi,2\pi]}$) if and only if there is $\chi_0 \in L^\infty$ with

$$\mathcal{I}_\eta^*(\chi_0) = 1, \quad \operatorname{supp}\widehat{\chi_0} \subseteq \mathbb{R}\setminus(-2\pi,2\pi),$$
$$\mathcal{I}_\eta(f) = \int_{-\infty}^{\infty} f(x)\chi_0(x)\,\mathrm{d}x.$$

DUAL PROBLEM

Theorem (B., Debruyne, Vindas, 2025)

Let $f \in L^1$. Then

$$\min_{\substack{g \in L^1 \\ \hat{g}|_{\mathbb{R}\setminus[-2\pi,2\pi]} = \hat{f}|_{\mathbb{R}\setminus[-2\pi,2\pi]}}} \mathcal{I}_\eta(g) = \max_{\substack{\mathcal{I}^*_\eta(\chi_0)=1 \\ \operatorname{supp}\widehat{\chi_0} \subseteq \mathbb{R}\setminus(-2\pi,2\pi)}} \int_{-\infty}^{\infty} f(x)\chi_0(x)\,\mathrm{d}x.$$

Observations

If f only vanishes on a null set, then $\mathcal{I}_\eta^*(\chi_0) = 1$ and $\mathcal{I}_\eta(f) = \int_{-\infty}^{\infty} f(x)\chi_0(x)\,\mathrm{d}x$ forces

$$\chi_0(x) = \begin{cases} 1-\eta & \text{if } f(x) > 0, \\ -\eta & \text{if } f(x) < 0, \end{cases} \quad \text{for almost every } x.$$

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OBSERVATIONS

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In this case, f is an η-minimal extrapolation
$\iff$ $\operatorname{supp} \widehat{\chi_0} \subseteq \mathbb{R} \setminus (-2\pi, 2\pi)$.
If $\chi_0(x)$ is 1-periodic with zero mean, then $\operatorname{supp} \widehat{\chi_0} \subseteq \mathbb{R} \setminus (-2\pi, 2\pi)$.

General ansatz

Let $\mathfrak{s} \in L^1_{loc}$. We want to find the minimizer of $\mathcal{I}_\eta(\mathfrak{s} - G)$ among $G \in E_{2\pi}$.

- Consider the dual problem. Anticipate that a solution is given by a translate of
$$\chi_\eta(x) = \begin{cases} 1 - \eta & \text{if } x \in [0, \eta) + \mathbb{Z}, \\ -\eta & \text{if } x \in [\eta, 1) + \mathbb{Z}. \end{cases}$$
Note that $\mathcal{I}_\eta^*(\chi_\eta) = 1$ and $\operatorname{supp} \widehat{\chi_\eta} \subseteq \mathbb{R} \setminus (-2\pi, 2\pi)$.

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  Note that $\mathcal{I}^*_\eta(\chi_\eta) = 1$ and $\operatorname{supp} \widehat{\chi_\eta} \subseteq \mathbb{R} \setminus (-2\pi, 2\pi)$.

- Maximize

$$\int_{-\infty}^{\infty} \chi_\eta(x + t)\mathfrak{s}(x)\, \mathrm{d}x.$$

  Suppose maximum is attained for $t = t_0$.

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Note that $\mathcal{I}_\eta^*(\chi_\eta) = 1$ and $\operatorname{supp} \widehat{\chi_\eta} \subseteq \mathbb{R} \setminus (-2\pi, 2\pi)$.
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Suppose maximum is attained for $t = t_0$.
- Construct $F \in E_{2\pi}$ by interpolating $\mathfrak{s}$ at the points $-t_0 + \mathbb{Z}$, $-t_0 + \eta + \mathbb{Z}$.
- Verify that a.e.
$$\mathfrak{s}(x) - F(x) \begin{cases} > 0 & \text{if } x \in (-t_0, -t_0+\eta) + \mathbb{Z}, \\ < 0 & \text{if } x \in (-t_0+\eta, -t_0+1) + \mathbb{Z}. \end{cases}$$

SOLUTION FOR $\mathfrak{s}(x) = \operatorname{sgn} x$

Following the above steps, we maximize

$$\int_{-\infty}^{\infty} \chi_\eta(x+t)\operatorname{sgn} x\,\mathrm{d}x = -\int_{-\infty}^{\infty} \chi_\eta^{(-1)}(x+t)\,\mathrm{d}\operatorname{sgn} x = -2\chi_\eta^{(-1)}(t).$$

Here, $\chi_\eta^{(-1)}$ is the primitive with mean zero. The maximum is attained at $t = t_0 = 0$.

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$$S_{0,\eta}(x) = -\frac{\sin(\pi x)\sin(\pi(x-\eta))}{\pi \sin(\pi\eta)},$$

$$F_\eta(z) = S_{0,\eta}(z)\left(\sum_{k\neq 0}\left(\frac{\operatorname{sgn} k}{z-k} - \frac{\operatorname{sgn} k}{z-(k+\eta)}\right) + \frac{1-2\eta}{z} - \frac{1}{z-\eta}\right).$$

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$$S_{0,\eta}(x) = -\frac{\sin(\pi x)\sin(\pi(x-\eta))}{\pi \sin(\pi\eta)},$$

$$F_\eta(z) = S_{0,\eta}(z)\left(\sum_{k\neq 0}\left(\frac{\operatorname{sgn} k}{z-k} - \frac{\operatorname{sgn} k}{z-(k+\eta)}\right) + \frac{1-2\eta}{z} - \frac{1}{z-\eta}\right).$$

Via Euler–Maclaurin summation, one may check that $\operatorname{sgn} x - F_\eta(x)$ displays the correct sign pattern.

UNIQUENESS

For $a, b \in \mathbb{R}$ with $a - b \notin \mathbb{Z}$, we set $S_{a,b}(x)$ trig polynomial vanishing at $a + \mathbb{Z}$, $b + \mathbb{Z}$, with derivative $+1$ resp. -1:

$$S_{a,b}(x) = -\frac{\sin(\pi(x-a))\sin(\pi(x-b))}{\pi \sin(\pi(b-a))}.$$

Uniqueness

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Theorem

If $G \in E_{2\pi}$ with $G|_{\mathbb{R}} \in L^2(\mathbb{R})$, then

$$G(z) = S_{a,b}(z)\sum_{k\in\mathbb{Z}}\left(\frac{G(a+k)}{z-(a+k)} - \frac{G(b+k)}{z-(b+k)}\right).$$

Asymmetric IK

Theorem (Asymmetric finite form of Ingham–Karamata, B., Debruyne, Vindas, 2025)

Let $\tau : (0,\infty) \to \mathbb{R}$ *be differentiable with* $-N \le \tau'(x) \le M$*. Suppose that* $\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x) e^{-sx}\, dx$ *converges for* $\operatorname{Re} s > 0$ *and admits continuous extension to a segment* $[-i\lambda, i\lambda]$ *for some* $\lambda > 0$*. Then*

$$\limsup_{x\to\infty} |\tau(x)| \le \pi \cdot \frac{MN}{\lambda(M+N)}.$$

Moreover, the above constant is sharp.

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The proof is as before, now utilizing $F_\eta(x)$ with $\eta = \frac{M}{M+N}$.

ASYMMETRIC IK

Theorem (Asymmetric finite form of Ingham–Karamata, B., Debruyne, Vindas, 2025)

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Moreover, the above constant is sharp.

The proof is as before, now utilizing $F_\eta(x)$ with $\eta = \frac{M}{M+N}$.
Note that we recover the symmetric and one-sided forms by taking $N = M$ and $N \to \infty$ respectively ($\eta = 1/2$ and $\eta \to 0$).

The signed powers

We also consider $\mathfrak{s}(x) = x^n \operatorname{sgn} x / n!$, $n \in \mathbb{N}$. Same ansatz: maximize

$$\int_{-\infty}^{\infty} \chi_\eta(x+t) \frac{x^n \operatorname{sgn} x}{n!} \, \mathrm{d}x.$$

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The maximum is attained at $t_0 = \frac{1+\eta}{2}$ if $n \equiv 1 \bmod 4$, $t_0 = \frac{\eta}{2}$ if $n \equiv 3 \bmod 4$. In case $n > 0$ even, the maximum is attained at a certain point related to Bernoulli functions, but having no explicit expression.

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The solution is given by interpolating $\mathfrak{s}(x)$ at the points $-t_0 + \mathbb{Z}$, $-t_0 + \eta + \mathbb{Z}$, and by prescribing the correct derivatives at $z = 0$:

$$F_{\eta,n}(z) = S_{-t_0,-t_0+\eta}(z) \frac{z^n}{n!} \left(\sum_{k \in \mathbb{Z}} \left(\frac{\operatorname{sgn}(k-t_0)}{z-(k-t_0)} - \frac{\operatorname{sgn}(k+\eta-t_0)}{z-(k+\eta-t_0)} \right) + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} \right).$$

Asymmetric IK

Theorem

Let $\tau : (0, \infty) \to \mathbb{R}$ *be n times differentiable with* $-N \leq \tau^{(n)}(x) \leq M$*. Suppose that* $\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x) \mathrm{e}^{-sx} \, \mathrm{d}x$ *converges for* $\operatorname{Re} s > 0$ *and admits continuous extension to a segment* $[-\mathrm{i}\lambda, \mathrm{i}\lambda]$ *for some* $\lambda > 0$*. Then there are sharp constants* $c_n(M, N)$*,* $C_n(M, N)$ *so that*

$$-\frac{c_n(M,N)}{\lambda^n} \leq \liminf_{x\to\infty} \tau(x) \leq \limsup_{x\to\infty} \tau(x) \leq \frac{C_n(M,N)}{\lambda^n}.$$

ASYMMETRIC IK

Theorem

Let $\tau : (0, \infty) \to \mathbb{R}$ be n times differentiable with $-N \leq \tau^{(n)}(x) \leq M$. Suppose that $\mathcal{L}\{\tau; s\} = \int_0^\infty \tau(x)\mathrm{e}^{-sx}\,\mathrm{d}x$ converges for $\operatorname{Re} s > 0$ and admits continuous extension to a segment $[-\mathrm{i}\lambda, \mathrm{i}\lambda]$ for some $\lambda > 0$. Then there are sharp constants $c_n(M, N)$, $C_n(M, N)$ so that

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One always has $c_n(M, N) = C_n(N, M)$.
If n is even, then $c_n(M, N) = C_n(M, N)$

The limit case

It holds that

$$\lim_{\eta \to 0} F_{\eta,0}(z) = B(z).$$

THE LIMIT CASE

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$$\lim_{\eta \to 0} F_{\eta,0}(z) = B(z).$$

Similarly,

$$\lim_{\eta \to 0} F_{\eta,n}(z) = B_n(z),$$

where $B_n(z)$ is the optimal majorant of $x^n \operatorname{sgn} x / n!$, first found by Littmann (2006).

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QUESTIONS?

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