FACULTY OF SCIENCES

Journées Complexes du Nord – 7 June 2024

Bohr's theorem for Beurling integer systems

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Dirichlet series and Bohr's theorem

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Dirichlet series

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Power series:

$$\sum_{k=0}^{\infty} a_k (2^{-s})^k.$$

Bohr's theorem

$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$

$$\sigma_c = \inf\{\sigma : f(s) \text{ converges on } \operatorname{Re} s > \sigma\},$$
$$\sigma_u = \inf\{\sigma : f(s) \text{ converges uniformly on } \operatorname{Re} s > \sigma\},$$
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Theorem (Bohr)

If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges somewhere, and the limit function has bounded analytic extension to $\{\operatorname{Re} s > 0\}$, then $\sigma_u \leq 0$.

Bohr's point of view

For each prime p_j, write $z_j = p_j^{-s}$. Bohr: z_j's act as independent variables.

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$$\sup_{\operatorname{Re} s>0} \left| \sum_{j=1}^{\infty} \frac{a_j}{p_j^s} \right| = \sum_{j=1}^{\infty} |a_j| .$$

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If $n = p_1^{\alpha_1} \cdots p_J^{\alpha_J}$, then $n^{-s} = z_1^{\alpha_1} \cdots z_J^{\alpha_J}$.

$$P(s) = \sum_{n=1}^{N} \frac{a_n}{n^s} \leftrightarrow \mathcal{B}(P)(z) = \sum_{\alpha} c_\alpha z^\alpha,$$

$$c_\alpha = a_n \quad \text{if} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$$

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Hilbert space $\mathcal{H}^2$

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}.$$

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$\mathcal{H}^2$ is the closure of $\{P(s) = \sum_{n=1}^{N} a_n n^{-s}\}$, w.r.t norm

$$\|P\|_2 = \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^2 \, \mathrm{d}t \right)^{1/2} = \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2}.$$

Dirichlet series as power series

For Dirichlet series, we can make the same formal association with power series:

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \leftrightarrow \mathcal{B}(f)(z) = \sum_{\alpha \in \mathbb{N}^{(\infty)}} c_\alpha z^\alpha.$$

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$$\mathcal{H}^\infty = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : f \text{ converges}, \|f\|_\infty = \sup_{\operatorname{Re} s > 0} \left| f(s) \right| < \infty \right\},$$

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Theorem (Hedenmalm, Lindqvist, Seip)

$\mathcal{B} : \mathcal{H}^\infty \to H^\infty(\mathbb{D}^\infty)$ *is an isometric isomorphism.*

Harmonic analysis point of view

Set

$$\mathbb{T}^\infty = \left\{(z_1, z_2, \ldots) : z_j \in \mathbb{C}, \left|z_j\right| = 1\right\}.$$

Compact abelian group with normalized haar measure $\mathrm{d}\mu_\infty$.

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Dual group: $\widehat{\mathbb{T}^\infty} \cong \mathbb{Z}^{(\infty)}$:
given $\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots) \in \mathbb{Z}^{(\infty)}$, associate the character

$$z^\alpha : (z_1, z_2, \dots) \mapsto z_1^{\alpha_1} \cdots z_k^{\alpha_k}.$$

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Every $F \in L^1(\mathbb{T}^\infty)$ has Fourier series

$$F(z) \sim \sum_{\alpha \in \mathbb{Z}^{(\infty)}} \widehat{F}(\alpha) z^\alpha, \quad \widehat{F}(\alpha) = \int_{\mathbb{T}^\infty} F(z)\overline{z^\alpha}\, \mathrm{d}\mu_\infty(z).$$

Hardy spaces

We now set

$$H^2(\mathbb{T}^\infty) = \left\{F \in L^2(\mathbb{T}^\infty) : \widehat{F}(\alpha) = 0 \text{ if } \alpha_j < 0 \text{ for some } j\right\},$$
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Extension to $p \in [1, \infty)$ by Bayart: let $\mathcal{H}^p$ be closure of polynomials $P(s) = \sum_{n=1}^N a_n n^{-s}$ with norm

$$\|P\|_p = \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, \mathrm{d}t \right)^{1/p}.$$

Then $\mathcal{H}^p \cong H^p(\mathbb{T}^\infty)$.

General Dirichlet series

Frequency:

$$(\lambda) = (\lambda_k)_k, \quad 0 \leq \lambda_1 < \lambda_2 < \ldots\,, \quad \lambda_k \to \infty.$$

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$$f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s}, \quad a_k \in \mathbb{C}.$$

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Power series: $(\lambda) = (k)_{k \geq 0}$.
Abscissas are defined similarly.

Bohr's theorem

Recall Bohr's theorem:

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Theorem (Bohr)

Suppose that

$$\lambda_{k+1} - \lambda_k \gg e^{-c\lambda_{k+1}}, \quad \text{for some } c > 0. \tag{BC}$$

Then Bohr's theorem holds for λ-Dirichlet series

Hardy spaces

$1 \leq p < \infty$: $\mathcal{H}^p(\lambda)$ closure of polynomials $P(s) = \sum_{k=1}^{K} a_k e^{-\lambda_k s}$ w.r.t.

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Two candidates for ∞-space:

$$\mathcal{H}^\infty(\lambda) = \left\{ f(s) = \sum_{k=1}^\infty a_k e^{-\lambda_k s} : \sigma_c \le 0 \text{ and bounded on } \operatorname{Re} s > 0 \right\}$$

$$\mathcal{H}^\infty_{ext}(\lambda) = \left\{ f(s) = \sum_{k=1}^\infty a_k e^{-\lambda_k s} : \sigma_c < \infty \text{ and bounded extension to } \operatorname{Re} s > 0 \right\}$$

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$\mathcal{H}^\infty(\lambda)$ complete $\iff$ Bohr's theorem holds for λ.

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Recent theory due to Defant and Schoolmann. For frequency λ , define λ-Dirichlet group:

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HARMONIC ANALYSIS POINT OF VIEW

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$$\mathcal{B} : f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s} \mapsto F(x) \sim \sum_{k=1}^{\infty} a_k \gamma_{\lambda_k}(x).$$

Harmonic analysis point of view 2

$$1 \le p < \infty \quad \mathcal{B} : \mathcal{H}^p(\lambda) \to H^p_\lambda(G) \text{ isometric isomorphism}$$
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Bohr's theorem $\Longrightarrow$ $\mathcal{H}^\infty(\lambda) = \mathcal{H}^\infty_{ext}(\lambda)$ and $\mathcal{B} : \mathcal{H}^\infty(\lambda) \to H^\infty_\lambda(G)$ surjective.

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Then we may take $G = \mathbb{T}^\infty$ as before.

BEURLING GENERALIZED NUMBER SYSTEMS

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BEURLING NUMBER SYSTEMS

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Beurling number systems

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Associated frequency: $(\lambda) = (\log m_k)_k$

Beurling zeta function

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{m_k^s} = \prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-s}}.$$

Examples

- $$\mathcal{P} = \{3, 5, 7, 11, \dots\}, \quad \mathcal{N} = \{1, 3, 5, 7, 9, \dots\}.$$
$$\pi_{\mathcal{P}}(x) = \pi(x) - 1 \sim \frac{x}{\log x}, \quad N_{\mathcal{P}}(x) = \lfloor x/2 \rfloor.$$

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- $\mathcal{O}_K$ the ring of integers of a number field K.
$$\mathcal{P} = \left(|P|, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}\right),$$
$$\mathcal{N} = \left(|I|, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}\right).$$
$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = A_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

Beurling's PNT

Main goal: investigate relation $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$, often via $\zeta_{\mathcal{P}}(s)$

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Suppose $N_{\mathcal{P}}(x) = Ax + O(x(\log x)^{-\gamma})$ *with* $A > 0$ *and* $\gamma > 3/2$. *Then* $\pi_{\mathcal{P}}(x) \sim \mathrm{Li}(x)$.

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The threshold $\gamma = 3/2$ is sharp.

Landau's PNT, density

Theorem (Landau)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$ *with* $A > 0$ *and* $\theta < 1$*. Then*

$$\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big), \quad \textit{some } c > 0.$$

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Helson's conjecture

GHENT UNIVERSITY

Vertical limits

$(\lambda) = (\log m_k)$ frequency coming from Beurling number system.

$$\mathcal{H}^2(\lambda) = \left\{ f(s) = \sum_{k=0}^{\infty} \frac{a_k}{m_k^s} : \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\} \cong H^2(\mathbb{T}^\infty).$$

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We can interpret $(z_1, z_2, \ldots) \in \mathbb{T}^\infty$ as multiplicative character χ defined by

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Theorem (Helson)

Suppose $(\lambda) = (\log m_k)$ satisfies (BC). Given $f \in \mathcal{H}^2(\lambda)$, for almost every $\chi \in \mathbb{T}^\infty$,

$$f_\chi(s) = \sum_{k=0}^{\infty} \frac{a_k \chi(m_k)}{m_k^s}$$

converges in $\operatorname{Re} s > 0$.

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Helson: *Some doubt is thrown on the conjecture, or at least on the ease of proving it.*
$f(s) = 1/\zeta(s+u)$ is outer if $u > 1/2$. RH implies convergence in $\operatorname{Re} s + u > 1/2$, but has a zero for $s = 1 - u$.

Systems with Bohr's theorem

Theorem (B., Kouroupis, Perfekt)

There exist Beurling number systems $(\mathcal{P}, \mathcal{N})$ *such that*

1. $\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O(1)$;
2. $N_{\mathcal{P}}(x) = Ax + O_{\varepsilon}(x^{1/2+\varepsilon})$, *for some* $A > 0$ *and all* $\varepsilon > 0$;
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$1/\zeta_{\mathcal{P}}(s + u)$, $1/2 < u < 1$, counterexample to Helson's conjecture!

Proof sketch

Let x_j be such that $\mathrm{Li}(x_j) = j$. Then $d\,\mathrm{Li}|_{[x_{j-1},x_j]}$ is a probability measure. We choose q_j randomly from $[x_{j-1}, x_j]$ with this distribution.

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We consider the events

$$A_{J,m} = \left\{ (q_1, q_2, \dots) : \left| \sum_{j=1}^{J} q_j^{-im} - \int_1^{x_J} u^{-im} d\,\mathrm{Li}(u) \right| \geq C_{J,m} \right\}.$$

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If $C_{J,m}$ are chosen such that $\sum_{J,m} P(A_{J,m}) < \infty$, then by Borel–Cantelli, with probability 1 only finitely many $A_{J,m}$ occur.

PROOF SKETCH CONTINUED

To ensure sufficiently large gaps between the generalized integers, we follow a similar strategy.
We consider sets $\mathcal{M}_J(q_1, q_2, \dots, q_{J-1})$, which is "forbidden" for q_J, given the choice of the first $J - 1$ Beurling primes $q_1, \dots, q_{J-1}$.

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Again we show that $\sum_J P(B_J) < \infty$.
By Borel–Cantelli, with probability 1 only finitely many B_J occur. If only $B_{J_1}, \dots, B_{J_N}$ occur, we delete the corresponding primes:

$$\tilde{\mathcal{P}} = \mathcal{P} \setminus \{q_{J_1}, \dots, q_{J_N}\}.$$

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