FACULTY
OF SCIENCES

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CONNECTION BETWEEN ZERO-FREE REGIONS AND ERROR TERM PNT

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Ingham (1932): if $\zeta(s)$ has no zeros for $\sigma > 1 - \eta(|t|)$, then

$$\Delta(x) \ll_\varepsilon \exp\big(-(1/2 - \varepsilon)\omega_\eta(x)\big), \quad \omega_\eta(x) := \inf_{t \ge 1}\big(\eta(t) \log x + \log t\big).$$

Pintz's theorems

Theorem (Pintz, 1980)

Suppose $\zeta(s)$ *has no zeros in* $\sigma > 1 - \eta(|t|)$*, then*

$$\Delta(x) \ll_\varepsilon \exp\big(-(1-\varepsilon)\omega_\eta(x)\big), \quad \forall \varepsilon > 0.$$

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Note:

$$\varepsilon_\eta(x) \to 0, \quad \varepsilon_\eta(x)\omega_\eta(x) \text{ unbounded if } \sqrt{\log x} = o(\omega_\eta(x)).$$

PINTZ'S THEOREMS

Theorem (Pintz, 1980)

Suppose $\zeta(s)$ has infinitely many zeros in $\sigma \geq 1 - g(|t|)$, then

$$\Delta(x) = \Omega_{\pm,\varepsilon}\Big(\exp\big(-(1+\varepsilon)\omega_g(x)\big)\Big), \quad \forall \varepsilon > 0.$$

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Idea: explicit formula:

$$\Delta(x) \approx -\sum_{\rho} \frac{x^{\rho-1}}{\rho}, \quad \left|\frac{x^{\rho-1}}{\rho}\right| \approx \exp\Big(-\big((1-\beta)\log x + \log\gamma\big)\Big).$$

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For fixed x, size of $\Delta(x)$ is determined by biggest term in above sum.

Beurling number systems

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$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots , \qquad n_k = p_1^{\alpha_1} \cdots p_j^{\alpha_j}.$$

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Counting functions:

$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_k \leq x\};$$
$$\psi_{\mathcal{P}}(x) = \sum_{p_j^{\alpha} \leq x} \log p_j.$$

Beurling number systems

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$$\begin{aligned} \mathcal{P} &= (p_j)_{j\geq 1}, & 1 < p_1 \leq p_2 \leq \dots, & \quad p_j \to \infty; \\ \mathcal{N} &= (n_k)_{k\geq 0}, & 1 = n_0 < n_1 \leq n_2 \leq \dots, & \quad n_k = p_1^{\alpha_1} \cdots p_j^{\alpha_j}. \end{aligned}$$

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Beurling zeta function

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}.$$

LANDAU'S PNT

Theorem (Landau, 1924)

Let $(\mathcal{P}, \mathcal{N})$ *be a Beurling number system with* $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$ *for some* $A > 0$ *and* $\theta \in [0,1)$. *Then*

$$\zeta_{\mathcal{P}}(s) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{c(1-\theta)}{\log|t|},$$

and

$$\Delta_{\mathcal{P}}(x) := \frac{\psi_{\mathcal{P}}(x) - x}{x} \ll_{\varepsilon} \exp\left(-(1-\varepsilon)2\sqrt{c(1-\theta)\log x}\right).$$

Pintz's theorems in Beurling setting

Theorem (Révész, 2024)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$*. If* $\zeta_{\mathcal{P}}(s)$ *has no zeros in* $\sigma > 1 - \eta(|t|)$*, then*

$$\Delta_{\mathcal{P}}(x) \ll_{\varepsilon} \exp\bigl(-(1-\varepsilon)\omega_{\eta}(x)\bigr), \quad \forall \varepsilon > 0.$$

If $\zeta_{\mathcal{P}}(s)$ *has infinitely many zeros in* $\sigma \geq 1 - g(|t|)$*, then*

$$\Delta_{\mathcal{P}}(x) = \Omega_{\pm,\varepsilon}\Bigl(\exp\bigl(-(1+\varepsilon)\omega_{g}(x)\bigr)\Bigr), \quad \forall \varepsilon > 0.$$

My assumptions

Set $f(u) := \eta(\mathrm{e}^u)$. We consider those η such that

- f regularly varying of index $-\alpha$, $\alpha \in (0, 1]$:

$$\frac{f(\lambda u)}{f(u)} \to \lambda^{-\alpha}, \quad u \to \infty, \quad \lambda > 0 \text{ fixed.}$$

or

- f slowly varying:

$$\frac{f(\lambda u)}{f(u)} \to 1, \quad u \to \infty, \quad \lambda > 0 \text{ fixed.}$$

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Vinogradov–Korobov zero-free region:

$$\eta(t) = \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}} \rightsquigarrow f \text{ regularly varying of index } -2/3.$$

$$\eta(t) = \frac{c}{\log\log t} \rightsquigarrow f \text{ slowly varying.}$$

My refinement

Theorem (B., 2025)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$*, and suppose* $\zeta_{\mathcal{P}}(s)$ *has no zeros in* $\sigma > 1 - \eta(|t|)$*,* η *as above. Then*

$$\Delta_{\mathcal{P}}(x) \ll \exp\bigl(-\omega_{\eta}(x) + \varpi_{\eta}(x)\bigr).$$

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Here,

$$\varpi_{\eta}(x) := Cf(u_0(x))u_0(x), \quad \text{any } C > \frac{4}{1-\theta},$$

and $u_0(x)$ is such that $\omega_{\eta}(x) = f(u_0(x)) \log x + u_0(x)$ (the minimizer).

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Theorem (B., 2025)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$, and suppose $\zeta_{\mathcal{P}}(s)$ has no zeros in $\sigma > 1 - \eta(|t|)$, η as above. Then

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and $u_0(x)$ is such that $\omega_\eta(x) = f(u_0(x)) \log x + u_0(x)$ (the minimizer).
For Riemann zeta refines Johnston's result:

$$\text{always } \varpi_\eta(x) < \varepsilon_\eta(x)\omega_\eta(x), \quad \text{in some cases } \varpi_\eta(x) = o(\varepsilon_\eta(x)\omega_\eta(x)).$$

THE EXAMPLES

Theorem (B., 2025)

Let η be as before. Then there exists a Beurling number system $(\mathcal{P}, \mathcal{N})$ such that

1. $N_{\mathcal{P}}(x) = Ax + O_{\varepsilon}(x^{1/2+\varepsilon})$ *for all* $\varepsilon > 0$;
2. $\zeta_{\mathcal{P}}(s)$ *has infinitely many zeros on* $\sigma = 1 - \eta(|t|)$, *none to the right;*
3. $$\Delta_{\mathcal{P}}(x) = \Omega_{\pm}\Big(\exp\big(-\omega_{\eta}(x) + (1/8)\varpi_{\eta}(x)\big)\Big).$$

The examples

Theorem (B., 2025)

Let η be as before. Then there exists a Beurling number system $(\mathcal{P}, \mathcal{N})$ such that

1. $N_{\mathcal{P}}(x) = Ax + O_{\varepsilon}(x^{1/2+\varepsilon})$ *for all* $\varepsilon > 0$;
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One can also construct examples with 1. and 2. and with $\Delta_{\mathcal{P}}(x) \ll \exp(-\omega_{\eta}(x))(\log x)^{-\nu}$ for some small $\nu > 0$.

Landau's method

Theorem (Landau, 1924)

Let $\varphi(t)$ *and* $\psi(t)$ *be non-decreasing functions with* $\psi(t) \ll \mathrm{e}^{\varphi(t)}$. *If*

$$\zeta_{\mathcal{P}}(\sigma + \mathrm{i}t) \ll \mathrm{e}^{\varphi(t)} \quad \text{for} \quad \sigma \geq 1 - \frac{1}{\psi(t)},$$

then there exists a constant $c > 0$ *such that*

$$\zeta_{\mathcal{P}}(\sigma + \mathrm{i}t) \neq 0 \quad \text{for} \quad \sigma \geq 1 - \frac{c}{\varphi(2t+1)\psi(2t+1)}.$$

Sharpness

This is sharp up to the value of the constant c:

Proposition (B., 2025)

For every η as before, there exists a Beurling zeta $\zeta_{\mathcal{P}}(s)$ satisfying 1. and 2., and admissible functions $\varphi(t)$ and $\psi(t)$ for which

$$\zeta_{\mathcal{P}}(\sigma + \mathrm{i}t) \ll \mathrm{e}^{\varphi(t)} \quad \textit{for} \quad \sigma \geq 1 - \frac{1}{\psi(t)},$$

$$\eta(t) \asymp \frac{1}{\varphi(2t+1)\psi(2t+1)}.$$

QUESTIONS?

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