FACULTY
OF SCIENCES

ENTR SEMINAR, 29 JUNE 2022

BEURLING GENERALIZED PRIMES

Frederik Broucke — fabrouck.broucke@ugent.be

GHENT
UNIVERSITY

Introduction

Let $\pi(x) = \#\{p \leq x, p \text{ prime}\}$.

Theorem (de la Vallée Poussin, Hadamard, 1896)

The prime number theorem (PNT): $\pi(x) \sim x/\log x$.

Introduction

Let $\pi(x) = \#\{p \leq x, p \text{ prime}\}$.

Theorem (de la Vallée Poussin, Hadamard, 1896)

The prime number theorem (PNT): $\pi(x) \sim x/\log x$.

Beurling's question: minimum requirements for proving the PNT?

Abstract setting: generalized primes and integers.

INTRODUCTION

Let $\pi(x) = \#\{p \leq x, p \text{ prime}\}$.

Theorem (de la Vallée Poussin, Hadamard, 1896)

The prime number theorem (PNT): $\pi(x) \sim x/\log x$.

Beurling's question: minimum requirements for proving the PNT?

Abstract setting: generalized primes and integers.

$$\mathcal{P} = (p_j)_{j\geq 1}, \qquad 1 < p_1 \leq p_2 \leq \dots, \qquad p_j \to \infty;$$
$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots, \qquad n_k = p_1^{\nu_1} \cdots p_j^{\nu_j}.$$

Introduction

Let $\pi(x) = \#\{p \leq x, p \text{ prime}\}$.

Theorem (de la Vallée Poussin, Hadamard, 1896)

The prime number theorem (PNT): $\pi(x) \sim x/\log x$.

Beurling's question: minimum requirements for proving the PNT?

Abstract setting: generalized primes and integers.

$$\mathcal{P} = (p_j)_{j\geq 1}, \qquad 1 < p_1 \leq p_2 \leq \dots, \qquad p_j \to \infty;$$
$$\mathcal{N} = (n_k)_{k\geq 0}, \qquad 1 = n_0 < n_1 \leq n_2 \leq \dots, \qquad n_k = p_1^{\nu_1} \cdots p_j^{\nu_j}.$$

Counting functions:

$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_k \leq x\}.$$

Examples

- $(\mathcal{P}, \mathcal{N}) = (\mathbb{P}, \mathbb{N}_{>0})$, the classical primes and integers.

$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

Examples

- $(\mathcal{P}, \mathcal{N}) = (\mathbb{P}, \mathbb{N}_{>0})$, the classical primes and integers.

$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

- $\mathcal{P} = (2.5, 3, 5, 7, \dots), \quad \mathcal{N} = (1, 2.5, 3, 5, 6.25, 7, 7.5, \dots).$

$$\pi_{\mathcal{P}}(x) = \pi(x) \text{ for } x \geq 2.5, \quad \pi_{\mathcal{P}}(x) = 0 \text{ for } x < 2.5,$$

$$N_{\mathcal{P}}(x) = \sum_{j \geq 0} \left(\lfloor x(2/5)^j \rfloor - \lfloor (x/2)(2/5)^j \rfloor \right) = \frac{5}{6}x + O(\log x).$$

Examples

- $(\mathcal{P},\mathcal{N}) = (\mathbb{P},\mathbb{N}_{>0})$, the classical primes and integers.

$$\pi_{\mathbb{P}}(x) = \pi(x), \quad N_{\mathbb{P}}(x) = \lfloor x \rfloor.$$

- $\mathcal{P} = (2.5, 3, 5, 7, \dots), \quad \mathcal{N} = (1, 2.5, 3, 5, 6.25, 7, 7.5, \dots)$.

$$\pi_{\mathcal{P}}(x) = \pi(x) \text{ for } x \geq 2.5, \quad \pi_{\mathcal{P}}(x) = 0 \text{ for } x < 2.5,$$

$$N_{\mathcal{P}}(x) = \sum_{j\geq 0}\left(\left\lfloor x(2/5)^j\right\rfloor - \left\lfloor (x/2)(2/5)^j\right\rfloor\right) = \frac{5}{6}x + O(\log x).$$

- $\mathcal{O}_K$ the ring of integers of a number field K.

$$\mathcal{P} = (|P|\,, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}),$$
$$\mathcal{N} = (|I|\,, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}).$$

$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = \rho_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

Beurling's PNT

Theorem (Beurling, 1937)

Let $(\mathcal{P}, \mathcal{N})$ be a g-number system. If $N(x) = \rho x + O(x/\log^{\gamma} x)$ for some $\rho > 0$ and $\gamma > 3/2$, then

$$\pi(x) \sim \frac{x}{\log x}.$$

Beurling's PNT

Theorem (Beurling, 1937)

Let $(\mathcal{P}, \mathcal{N})$ *be a g-number system. If* $N(x) = \rho x + O(x/\log^{\gamma} x)$ *for some* $\rho > 0$ *and* $\gamma > 3/2$*, then*

$$\pi(x) \sim \frac{x}{\log x}.$$

Critical exponent $\gamma = 3/2$ is sharp: $\exists\, (\mathcal{P}, \mathcal{N})$:

$$N(x) = \rho x + O\left(\frac{x}{\log^{3/2} x}\right), \quad \pi(x) \not\sim \frac{x}{\log x}.$$

The Beurling zeta function

Define

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s}, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

The Beurling zeta function

Define

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s}, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

We have

$$\begin{aligned}
\zeta_{\mathcal{P}}(s) &= \prod_{j=1}^{\infty} \left(1 + \frac{1}{p_j^s} + \frac{1}{p_j^{2s}} + \dots \right) \\
&= \prod_{j=1}^{\infty} \left(1 - \frac{1}{p_j^s} \right)^{-1} = \exp \sum_{j=1}^{\infty} \left\{ -\log \left(1 - \frac{1}{p_j^s} \right) \right\} \\
&= \exp \sum_{k=0}^{\infty} \frac{a_{n_k}}{n_k^s},
\end{aligned}$$

with $a_{n_k} = 1/\nu$ if $n_k = p_j^{\nu}$, $a_{n_k} = 0$ otherwise.

Prime counting functions

Denote

$$\Pi_{\mathcal{P}}(x) = \sum_{p_j^{\nu} \le x} \frac{1}{\nu}.$$

Then

$$\zeta_{\mathcal{P}}(s) = \int_{1^-}^{\infty} x^{-s} \, \mathrm{d}N(x) = \exp \int_{1}^{\infty} x^{-s} \, \mathrm{d}\Pi(x).$$

Prime counting functions

Denote

$$\Pi_{\mathcal{P}}(x) = \sum_{p_j^{\nu} \leq x} \frac{1}{\nu}.$$

Then

$$\zeta_{\mathcal{P}}(s) = \int_{1^-}^{\infty} x^{-s}\, \mathrm{d}N(x) = \exp \int_{1}^{\infty} x^{-s}\, \mathrm{d}\Pi(x).$$

We may also define

$$\Lambda(n_k) = \begin{cases} \log p_j & \text{if } n_k = p_j^{\nu}, \\ 0 & \text{otherwise,} \end{cases}$$

with counting function $\psi_{\mathcal{P}}(x)$. Then

$$\int_{1}^{\infty} x^{-s}\, \mathrm{d}\psi(x) = -\frac{\zeta_{\mathcal{P}}'(s)}{\zeta_{\mathcal{P}}(s)}.$$

Zeros of $\zeta_{\mathcal{P}}$

The error term in PNT is closely related to zeros of $\zeta_{\mathcal{P}}(s)$.
By absolutely converging Euler product, $\zeta_{\mathcal{P}}(s) \neq 0$ for $\operatorname{Re} s > 1$.

ZEROS OF $\zeta_{\mathcal{P}}$

The error term in PNT is closely related to zeros of $\zeta_{\mathcal{P}}(s)$.

By absolutely converging Euler product, $\zeta_{\mathcal{P}}(s) \neq 0$ for $\operatorname{Re} s > 1$.

Beurling's PNT:

$$\pi(x) \sim \frac{x}{\log x} \quad \text{“} \iff \text{”} \quad \zeta_{\mathcal{P}}(s) \neq 0 \text{ for } \operatorname{Re} s \geq 1.$$

Zeros of $\zeta_{\mathcal{P}}$

The error term in PNT is closely related to zeros of $\zeta_{\mathcal{P}}(s)$.
By absolutely converging Euler product, $\zeta_{\mathcal{P}}(s) \neq 0$ for $\operatorname{Re} s > 1$.
Beurling's PNT:

$$\pi(x) \sim \frac{x}{\log x} \quad \text{“} \iff \text{”} \quad \zeta_{\mathcal{P}}(s) \neq 0 \text{ for } \operatorname{Re} s \geq 1.$$

Larger zero-free regions:

Theorem (Landau, 1903, “avant la lettre”)

Suppose that $N(x) = \rho x + O(x^{\theta})$ for some $\rho > 0$ and $\theta < 1$. Then

$$\pi(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big).$$

Comes from zero-free region

$$\zeta_{\mathcal{P}}(\sigma + \mathrm{i}t) \neq 0 \text{ for } \sigma \geq 1 - \frac{c^2}{\log(2 + |t|)}.$$

Optimality

Remarkably, this is optimal:

Theorem (Diamond, Montgomery, Vorhauer, 2006)

For every $\theta > 1/2$ there exists a system $(\mathcal{P}, \mathcal{N})$ with

$$N(x) = \rho x + O(x^{\theta}) \quad \text{for some } \rho > 0,$$
$$\pi(x) = \mathrm{Li}(x) + \Omega_{\pm}\big(x \exp(-c\sqrt{\log x})\big).$$

OPTIMALITY

Remarkably, this is optimal:

Theorem (Diamond, Montgomery, Vorhauer, 2006)

For every $\theta > 1/2$ *there exists a system* $(\mathcal{P}, \mathcal{N})$ *with*

$$N(x) = \rho x + O(x^{\theta}) \quad \textit{for some } \rho > 0,$$

$$\pi(x) = \mathrm{Li}(x) + \Omega_{\pm}\left(x \exp(-c\sqrt{\log x})\right).$$

To prove the Riemann Hypothesis, more than merely multiplicative structure and $N(x) = \rho x + O(x^{1/2})$ is needed.

FROM π TO N

For the other direction, we have e.g. these two theorems.

Theorem (Diamond, 1977)

Suppose that $\pi(x) = \mathrm{Li}(x) + O(x/\log^{\gamma} x)$, *for some* $\gamma > 1$. *Then*

$$N(x) \sim \rho x, \quad \textit{for some } \rho > 0.$$

From π to N

For the other direction, we have e.g. these two theorems.

Theorem (Diamond, 1977)

Suppose that $\pi(x) = \mathrm{Li}(x) + O(x/\log^\gamma x)$, *for some* $\gamma > 1$. *Then*

$$N(x) \sim \rho x, \quad \textit{for some } \rho > 0.$$

Theorem (Hilberdink, Lapidus, 2006)

Suppose that $\pi(x) = \mathrm{Li}(x) + O(x^\theta)$ *for some* $\theta < 1$. *Then*

$$N(x) = \rho x + O\big(x \exp(-c' \sqrt{\log x \log\log x})\big),$$

for some $\rho > 0$ *and* $c' > 0$.

Optimality

The theorem of Hilberdink and Lapidus is also optimal:

Theorem (B., Debruyne,Vindas, 2020)

For every $\theta > 0$ *there exists a system* $(\mathcal{P}, \mathcal{N})$ *with*

$$\Pi(x) = \mathrm{Li}(x) + O(x^{\theta}),$$

$$N(x) = \rho x + \Omega_{\pm}\big(x \exp(-c' \sqrt{\log x \log\log x})\big) \quad \text{for some } \rho > 0.$$

Optimality

The theorem of Hilberdink and Lapidus is also optimal:

Theorem (B., Debruyne, Vindas, 2020)

For every $\theta > 0$ *there exists a system* $(\mathcal{P}, \mathcal{N})$ *with*

$$\Pi(x) = \mathrm{Li}(x) + O(x^{\theta}),$$
$$N(x) = \rho x + \Omega_{\pm}\left(x \exp\left(-c' \sqrt{\log x \log\log x}\right)\right) \quad \textit{for some } \rho > 0.$$

We also determine the optimal constant $c' = \sqrt{2(1-\theta)}$.

Well-behaved systems

We say the primes are α-well-behaved if

$$\Pi(x) = \mathrm{Li}(x) + O(x^{\alpha+\varepsilon}),$$

for every $\varepsilon > 0$, but no $\varepsilon < 0$.

WELL-BEHAVED SYSTEMS

We say the primes are α-well-behaved if

$$\Pi(x) = \mathrm{Li}(x) + O(x^{\alpha+\varepsilon}),$$

for every $\varepsilon > 0$, but no $\varepsilon < 0$. Similarly, the integers are β-well-behaved if

$$N(x) = \rho x + O(x^{\beta+\varepsilon}),$$

for every $\varepsilon > 0$, but no $\varepsilon < 0$.

Well-behaved systems

We say the primes are α-well-behaved if

$$\Pi(x) = \mathrm{Li}(x) + O(x^{\alpha+\varepsilon}),$$

for every $\varepsilon > 0$, but no $\varepsilon < 0$. Similarly, the integers are β-well-behaved if

$$N(x) = \rho x + O(x^{\beta+\varepsilon}),$$

for every $\varepsilon > 0$, but no $\varepsilon < 0$.
Under RH, $(\mathbb{P}, \mathbb{N}_{>0})$ is a $[1/2, 0]$-system.
Under generalizations of RH: many more examples of $[1/2, \beta]$-systems.

UNCONDITIONAL EXAMPLES

Theorem (Zhang, 2007)

There exists an $[\alpha, \beta]$ *system for some* α *and* β *with* $\max\{\alpha, \beta\} \leq 1/2$.

Theorem (B., Vindas, 2021)

There exists a $[0, 1/2]$*-system.*

Unconditional examples

Theorem (Zhang, 2007)

There exists an $[\alpha, \beta]$ *system for some* α *and* β *with* $\max\{\alpha, \beta\} \leq 1/2$.

Theorem (B., Vindas, 2021)

There exists a $[0, 1/2]$*-system.*

Theorem (Hilberdink, 2005)

For an $[\alpha, \beta]$*-system one has* $\max\{\alpha, \beta\} \geq 1/2$.

Conjecture

Let $\alpha, \beta \in [0, 1)$ *with* $\max\{\alpha, \beta\} \geq 1/2$. *Then there exists an* $[\alpha, \beta]$*-system.*

QUESTIONS?

GHENT
UNIVERSITY