FACULTY OF SCIENCES

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ZERO-DENSITY ESTIMATES FOR BEURLING ZETA FUNCTIONS

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RIEMANN ζ

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Analytic on $\mathbb{C} \setminus \{1\}$, simple pole with residue 1.
Trivial zeros at $s = -2n$, $n \in \mathbb{N}_{>0}$.
All other zeros located in $0 \leq \sigma \leq 1$, symmetric around real axis and $s = 1/2$.

Basic facts on zeros

- de la Vallée-Poussin, 1899: zeros $\rho = \beta + \mathrm{i}\gamma$ satisfy

$$\beta \leq 1 - \frac{c}{\log(|\gamma| + 2)}, \quad \text{some } c > 0.$$

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- Riemann, 1859, von Mangoldt, 1905: let $N(T) = \#\{\rho = \beta + \mathrm{i}\gamma : 0 < \gamma \leq T\}$. Then

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

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- Riemann, 1859, “Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind.”
  RH: all non-trivial zeros $\rho = \beta + \mathrm{i}\gamma$ satisfy $\beta = 1/2$.

Approximating RH: zero-density estimates

Idea: number of exceptions to RH is "small". Set

$$N(\alpha, T) = \#\{\rho = \beta + \mathrm{i}\gamma : \beta \geq \alpha \text{ and } 0 < \gamma \leq T\}.$$

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- Titchmarsh, 1930, $N(\alpha, T) \ll_\varepsilon T^{\frac{4(1-\alpha)}{3-2\alpha}+\varepsilon}$.

PNT IN SHORT INTERVALS

Consider PNT with error term

$$\psi(x) = x + O(E(x)).$$

This implies $\psi(x+h) - \psi(x) \sim h$ if $E(x) = o(h)$.

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Idea:
$$\frac{\psi(x+h) - \psi(x)}{h} \approx 1 - \frac{1}{h} \sum_{\rho : |\gamma| \leq T} \frac{(x+h)^\rho - x^\rho}{\rho}.$$

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Ingham, 1940, $N(\alpha, T) \ll_\varepsilon T^{\frac{3(1-\alpha)}{2-\alpha}+\varepsilon}$,
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DH is known to hold for $\alpha \geq 0.78...$ (Bourgain, 2000).

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$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_j \leq x\}.$$

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Beurling zeta function

$$\zeta_{\mathcal{P}}(s) = \sum_{j=1}^{\infty} \frac{1}{n_j^s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}.$$

Examples

- $$\mathcal{P} = \{\sqrt{2}, 3, 5, 7, 11, \dots\}, \quad \mathcal{N} = \{1, \sqrt{2}, 2, 2\sqrt{2}, 3, 4, \dots\}.$$
  $$\pi_{\mathcal{P}}(x) = \pi(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{P}}(x) = \left(1 + \frac{1}{\sqrt{2}}\right)x + O(1).$$

EXAMPLES

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- $\mathcal{O}_K$ the ring of integers of a number field K.

$$\mathcal{P} = (|P|, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}),$$
$$\mathcal{N} = (|I|, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}).$$

$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = A_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

WELL-BEHAVED INTEGERS

We assume that for some $A > 0$ and $\theta < 1$:

$$N_{\mathcal{P}}(x) = Ax + O(x^{\theta}).$$

Then $\zeta_{\mathcal{P}}(s) - \frac{A}{s-1}$ has analytic continuation to $\sigma > \theta$.

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Theorem (Landau)

Under above assumptions we have zero-free region

$$\sigma > 1 - \frac{C(1-\theta)}{\log|t|}, \quad |t| \geq T_0.$$

Consequently,

$$\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O\Big(x \exp\big(-C'\sqrt{(1-\theta)\log x}\big)\Big).$$

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For any $\alpha > \theta$, one also has

$$N(\alpha, T) = N(\zeta_{\mathcal{P}}; \alpha, T) \ll_{\alpha} T \log T.$$

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- no analytic continuation beyond $\sigma = \theta$.

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In general, one has for Beurling zeta functions $\zeta_{\mathcal{P}}$ with well-behaved integers:

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- no functional equation.
- no larger zero-free region (Diamond, Montgomery, Vorhauer, 2006).

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- no analytic continuation beyond $\sigma = \theta$.
- no functional equation.
- no larger zero-free region (Diamond, Montgomery, Vorhauer, 2006).
- no Riemann–von-Mangoldt formula for $N(T)$.

Zero-density estimates for $\zeta_{\mathcal{P}}$

Révész, 2021, assuming $\mathcal{N} \subseteq \mathbb{N}$:

$$N(\alpha, T) \ll_\varepsilon T^{\frac{(6-2\theta)(1-\alpha)}{1-\theta}+\varepsilon}.$$

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Révész 2022, B., Debruyne, 2022

$$N(\alpha, T) \ll_{\varepsilon} T^{\frac{12(1-\alpha)}{1-\theta}+\varepsilon}, \quad \ll_{\varepsilon} T^{c(\alpha)\frac{1-\alpha}{1-\theta}+\varepsilon},$$

with $c(\frac{2+\theta}{3}) = 3$ and $c(1) = 4$.

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with $c(\frac{2+\theta}{3}) = 3$ and $c(1) = 4$.

Theorem (B., 2024)

Uniformly for $\alpha \geq \frac{1+\theta}{2}$:

$$N(\alpha, T) \ll T^{\frac{4(1-\alpha)}{3-2\alpha-\theta}}(\log T)^{9}.$$

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- Applied to Riemann-zeta: recover Titchmarsh $N(\alpha, T) \ll T^{\frac{4(1-\alpha)}{3-2\alpha}}(\log T)^9$, simply from $\lfloor x \rfloor = x + O(1)$.
- For $\theta \geq 1/2$ (or $\theta = 0$), exist Beurling zeta functions with $N(\frac{1+\theta}{2}, T) \gg T \log T$.

Main tools

1. Mean value estimate for Dirichlet polynomials over $\mathcal{N}$ (B., Debruyne 2022):
   Let $D(\mathrm{i}t) = \sum_{n_j \le N} a_j n_j^{-\mathrm{i}t}$, $a_j \in \mathbb{C}$. Then

$$\int_0^T \left|D(\mathrm{i}t)\right|^2 \mathrm{d}t \ll \left(TN^{\theta} + N\right) \sum_{n_j \le N} \left|a_j\right|^2 .$$

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2. Second moment $\zeta_{\mathcal{P}}$ (B., Hilberdink 2024):

$$\int_0^T \left|\zeta_{\mathcal{P}}\Big(\frac{1+\theta}{2} + \mathrm{i}t\Big)\right|^2 \mathrm{d}t \ll T(\log T).$$

PROOF SKETCH

- Mollification: multiply $\zeta_{\mathcal{P}}$ with M_X

$$M_X(s) = \sum_{n_j \leq X} \mu_{\mathcal{P}}(n_j) n_j^{-s},$$

  $\mu_{\mathcal{P}}$ Möbius function of $(\mathcal{P}, \mathcal{N})$.
- Smoothing: multiply coefficients a_j of $\zeta_{\mathcal{P}}(s)M_X(s)$ with $\mathrm{e}^{-n_j/Y}$ for some large $Y > X$.
- $$\zeta_{\mathcal{P}}(s)M_X(s) \approx 1 + \sum_{X < n_j \leq Y} \frac{a_j \mathrm{e}^{-n_j/Y}}{n_j^s} + \left(\int \text{involving } \zeta_{\mathcal{P}}, M_X, \Gamma, Y \right) + \text{ small error.}$$

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Optimize parameters X and Y.

Conditional improvements

Assuming more analytic information of $\zeta_{\mathcal{P}}$, we can get improvements.

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- Higher moments: suppose e.g.

$$\int_0^T \left|\zeta_{\mathcal{P}}\left(\frac{1+\theta}{2} + \mathrm{i}t\right)\right|^4 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon},$$

  then

$$N(\alpha, T) \ll_\varepsilon T^{\frac{3(1-\alpha)}{2-\alpha-\theta}+\varepsilon} \quad \text{(c.f. Ingham).}$$

CONDITIONAL IMPROVEMENTS

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- Subconvexity bounds: $\zeta_{\mathcal{P}}(\frac{1+\theta}{2}+\mathrm{i}t) \ll |t|^B$ for some $B < 1/2$.
  Suppose e.g. “LH”: any $B > 0$, then

$$N(\alpha, T) \ll_\varepsilon T^{\frac{2(1-\alpha)}{1-\theta}+\varepsilon}, \quad N\left(\frac{3+\theta}{4}+\delta, T\right) \ll_{\varepsilon,\delta} T^\varepsilon.$$

Montgomery-style conjecture

First tool: MVT for Dirichlet polynomials.

Suppose now $\left|a_j\right| \leq 1$, so that $\sum_{n_j \leq N} \left|a_j\right|^2 \ll N$.

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If generalization

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Theorem (B., Debruyne, 2022)

Suppose $N(\alpha, T) \ll T^{c(1-\alpha)}(\log T)^L$ *for* $\alpha \geq 1 - 1/c$*, and zero-free region*

$$\sigma > 1 - d\frac{\log\log|t|}{log|t|}.$$

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PNT in short interval fails for DMV example (no larger zero-free region).

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This is fulfilled for Riemann zeta and many L-functions, but seems to require either larger zero-free region or sieve methods.

GENERALIZATIONS

The techniques from the proof apply in great generality.
Let $F(s) = \sum_{j=1}^{\infty} a_j n_j^{-s}$ be Dirichlet series over $\mathcal{N}$ with $1/F(s) = G(s) = \sum_{j=1}^{\infty} b_j n_j^{-s}$ satisfying

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then

$$N(F; \alpha, T) \ll_\varepsilon T^{\frac{4(1-\alpha)}{3-2\alpha-\theta}+\varepsilon}.$$

Concluding remark

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CONCLUDING REMARK

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  $\implies$ techniques seem quite optimal (apart from the constant in the exponent).
- To "inch closer towards RH", we have to leverage the specific structure / symmetry of Riemann ζ in a very significant way.

THANK YOU FOR LISTENING!

GHENT UNIVERSITY