FACULTY OF SCIENCES

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A MEAN VALUE THEOREM FOR GENERAL DIRICHLET SERIES

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Mean value of ζ

Consider Riemann zeta function

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Also holds for $\sigma > 1/2$, while

$$\frac{1}{T} \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t \sim \log T$$

General Dirichlet series

Let $1 \leq n_1 < n_2 < \ldots$ sequence of *real* numbers, $a_j \geq 0$. General Dirichlet series

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Assume

$$A(x) := \sum_{n_j \leq x} a_j = \rho x + O(x^\theta), \quad \rho > 0, \quad \theta \in [0, 1).$$

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Then $f(s) - \rho/(s-1)$ has analytic continuation to $\operatorname{Re} s > \theta$. For $\operatorname{Re} s > 1$:

$$\frac{1}{T}\int_0^T \left|f(\sigma + it)\right|^2 \mathrm{d}t \to \sum_{j=1}^{\infty} \frac{a_j^2}{n_j^{2\sigma}}, \quad \text{as } T \to \infty.$$

Prelim. observation

We have $a_j \ll n_j^{\theta}$, so

$$\sum_{j=1}^{\infty} \frac{a_j^2}{n_j^{2\sigma}} \ll \sum_{j=1}^{\infty} \frac{a_j}{n_j^{2\sigma-\theta}},$$

abscissa of convergence $\leq \frac{1+\theta}{2}$.

PRELIM. OBSERVATION

We have $a_j \ll n_j^{\theta}$, so

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abscissa of convergence $\leq \frac{1+\theta}{2}$.

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \left|f(\sigma+it)\right|^2 \mathrm{d}t \geq \sum_{j=1}^{\infty} \frac{a_j^2}{n_j^{2\sigma}}.$$

Main result

Theorem (B., Hilberdink)

$$\frac{1}{T}\int_0^T \left|f(\sigma+it)\right|^2 \mathrm{d}t \to \sum_{j=1}^{\infty} \frac{a_j^2}{n_j^{2\sigma}}, \quad \sigma > \frac{1+\theta}{2}$$

$$\frac{1}{T}\int_0^T \left|f\left(\frac{1+\theta}{2}+it\right)\right|^2 \mathrm{d}t \ll \log T$$

$$\frac{1}{T}\int_0^T \left|f(\sigma+it)\right|^2 \mathrm{d}t \ll_\delta T^{\frac{1+\theta-2\sigma}{1-\theta}}, \quad \theta+\delta \le \sigma < \frac{1+\theta}{2}.$$

SOME REMARKS

- Related results were obtained assuming separation of n_j by e.g. Landau, and recently Drungilas, Garunkštis, Novikas, for Beurling zeta functions. We do not require any separation.

Some remarks

- Related results were obtained assuming separation of n_j by e.g. Landau, and recently Drungilas, Garunkštis, Novikas, for Beurling zeta functions. We do not require any separation.
- We obtain

$$\frac{1}{T}\int_0^T \left|\zeta(\sigma + it)\right|^2 \mathrm{d}t \to \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}}, \quad \sigma > \frac{1}{2}$$

simply from $\lfloor x \rfloor = x + O(1)$. Previous proofs use separation of integers.

PROOF IDEA

Write

$$f_N(s) = \sum_{n_j \le N} \frac{a_j}{n_j^s}.$$

Proposition

For $\frac{1+\theta}{2} < \sigma \le 1$*,*

$$\frac{1}{T}\int_0^T \left|f_N(\sigma + it)\right|^2 \mathrm{d}t \to \sum_{j=1}^{\infty} \frac{a_j^2}{n_j^{2\sigma}}$$

if $N, T \to \infty$ *such that* $T \succ N^{2-2\sigma} \log N$ *if* $\sigma < 1$ *and* $T \succ (\log N)^2$ *if* $\sigma = 1$*.*

Proof idea

Write

$$f_N(s) = \sum_{n_j \le N} \frac{a_j}{n_j^s}.$$

Proposition

Uniformly for $\sigma \le \frac{1+\theta}{2}$*:*

$$\frac{1}{T}\int_0^T \left|f_N(\sigma + it)\right|^2 \mathrm{d}t \ll \frac{N^{3-2\sigma-\theta}}{T^2} + \frac{N^{1+\theta-2\sigma} - 1}{1 + \theta - 2\sigma}.$$

Proof idea

Write

$$f_N(s) = \sum_{n_j \le N} \frac{a_j}{n_j^s}.$$

Proposition

For $\delta > 0$, uniformly for $\sigma \ge \theta + \delta$

$$\int_1^T \left| f(\sigma + it) - f_N(\sigma + it) \right|^2 \mathrm{d}t \ll_\delta \frac{T^2}{N^{2\sigma - 2\theta}} + N^{2-2\sigma}.$$

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Proposition

Let $0 < \theta < 1$. There exists an increasing sequence of reals $(n_j)_j$ with

$$\sum_{n_j \leq x} 1 = x + O(x^{\theta}),$$

and, setting $f(s) = \sum_j n_j^{-s}$, for $\theta < \sigma < \frac{1+\theta}{2}$:

$$\frac{1}{T}\int_0^T \left|f(\sigma + it)\right|^2 \mathrm{d}t = \Omega_\sigma\left(T^{\frac{1+\theta-2\sigma}{1-\theta}}\right),$$

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Convergence of $\sum_j a_j^2 n_j^{-2\sigma}$ is not sufficient for existence of the mean!

Behavior on the line $\sigma = \frac{1+\theta}{2}$

Recall that

$$\frac{1}{T}\int_0^T \left| f\left(\frac{1+\theta}{2} + it\right)\right|^2 \mathrm{d}t \ll \log T.$$

To say more, some separation is needed.

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Theorem (B., Hilberdink)

Suppose that $n_{j+1} - n_j \gg \max\{a_j, a_{j+1}\}$. *Then*

$$\frac{1}{T}\int_0^T \left| f\left(\frac{1+\theta}{2} + it\right)\right|^2 \mathrm{d}t = S(T) + O\left(S(T)^{1/2}\right),$$

$$S(T) = \sum_{n_j \le T^{\frac{1}{1-\theta}}} \frac{a_j^2}{n_j^{1+\theta}}.$$

Case $\theta = 0$, $a_j \equiv 1$

Proposition

Let $g : [0, \infty) \to [0, 1]$ *be a non-increasing function satisfying* $g(x) \succ \frac{1}{(\log x)^2}$. *Let* $(n_j)_j$ *be increasing sequence with* $n_{j+1} - n_j \gg g(n_j)$ *and* $\sum_{n_j \le x} 1 = \rho x + O(1)$. *Then*

$$\frac{1}{T} \int_0^T \left| f\left(\frac{1}{2} + it\right) \right|^2 \mathrm{d}t \sim \rho \log T.$$

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$$\frac{1}{T} \int_0^T \left| f\left(\frac{1}{2} + it\right) \right|^2 \mathrm{d}t \sim \rho \log T.$$

Proposition

For every $\delta > 0$, there exist $(n_j)_j$ with $n_{j+1} - n_j \gg n_j^{-\delta}$, $\sum_{n_j \le x} 1 = \rho x + O(1)$, and with

$$\frac{1}{T} \int_0^T \left| f\left(\frac{1}{2} + it\right) \right|^2 \mathrm{d}t \not\sim \rho \log T.$$

ASYMPTOTICS?

What is the weakest separation function g such that $n_{j+1} - n_j \ll g(n_j)$ implies the asymptotic on $\sigma = \frac{1+\theta}{2}$?

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For $\sigma < 1/2$, we have

$$\frac{1}{T}\int_0^T \left|f(\sigma+it)\right|^2 \mathrm{d}t \ll T^{1-2\sigma}.$$

Proposition

There exists $(n_j)_j$, $(a_j)_j$ with $\sum_{n_j \le x} a_j = x + O(1)$ and so that for any $0 < \sigma < 1/2$:

$$\frac{1}{T^{2-2\sigma}}\int_0^T \left|f(\sigma+it)\right|^2 \mathrm{d}t$$

does not converge.

APPLICATION TO BEURLING ZETA FUNCTIONS

Beurling number systems

Beurling generalized primes and integers: $(\mathcal{P}, \mathcal{N})$.

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$$\mathcal{N} = (n_j)_{j\geq 1}, \qquad 1 = n_1 < n_2 \leq n_3 \leq \dots, \qquad n_j = p_1^{\alpha_1} \cdots p_k^{\alpha_k}.$$

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Counting functions:

$$\pi_{\mathcal{P}}(x) = \#\{p_j \leq x\}, \quad N_{\mathcal{P}}(x) = \#\{n_j \leq x\}.$$

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Beurling zeta function

$$\zeta_{\mathcal{P}}(s) = \sum_{j=1}^{\infty} \frac{1}{n_j^s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}.$$

EXAMPLES

- $$\mathcal{P} = \{3, 5, 7, 11, \dots\}, \quad \mathcal{N} = \{1, 3, 5, 7, 9, \dots\}.$$
  $$\pi_{\mathcal{P}}(x) = \pi(x) - 1 \sim \frac{x}{\log x}, \quad N_{\mathcal{P}}(x) = \lfloor x/2 \rfloor.$$

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$$\mathcal{P} = \{3, 5, 7, 11, \dots\}, \quad \mathcal{N} = \{1, 3, 5, 7, 9, \dots\}.$$

$$\pi_{\mathcal{P}}(x) = \pi(x) - 1 \sim \frac{x}{\log x}, \quad N_{\mathcal{P}}(x) = \lfloor x/2 \rfloor.$$

- $\mathcal{O}_K$ the ring of integers of a number field K.

$$\mathcal{P} = (|P|, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}),$$
$$\mathcal{N} = (|I|, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}).$$

$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = A_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

Beurling's PNT

Main goal: investigate relation $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$, often via $\zeta_{\mathcal{P}}(s)$.

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Theorem (Landau)

Suppose $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$ with $A > 0$ and $\theta \in [0, 1)$. Then

$$\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big).$$

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The zeros of $\zeta_{\mathcal{P}}(s)$ play an important role. Landau's method yields the zero-free region

$$\sigma > 1 - \frac{C(1-\theta)}{\log|t|}, \quad |t| \geq T_0.$$

Zero-density estimates

The mean value theorem yields

$$\frac{1}{T}\int_0^T \left|\zeta_{\mathcal{P}}\left(\frac{1+\theta}{2}+it\right)\right|^2 \mathrm{d}t \ll \log T.$$

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Write

$$N(\alpha, T) = \#\{\rho = \beta + i\gamma : \zeta_{\mathcal{P}}(\rho) = 0, \quad \beta \geq \alpha, \quad |\gamma| \leq T\}.$$

“Trivially”, $N(\alpha, T) \ll T \log T$.

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Theorem (B.)

For Beurling zeta functions $\zeta_{\mathcal{P}}$ of number systems satisfying $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$, we have for $\alpha \geq \frac{1+\theta}{2}$:

$$N(\alpha, T) \ll T^{\frac{4(1-\alpha)}{3-2\alpha-\theta}} (\log T)^9.$$

Remarks

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- For $\theta \geq 1/2$ (or $\theta = 0$), exist Beurling zeta functions with $N(\frac{1+\theta}{2}, T) \gg T \log T$.

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- Applied to Riemann-zeta: $N(\alpha, T) \ll T^{\frac{4(1-\alpha)}{3-2\alpha}} (\log T)^9$ simply from $\lfloor x \rfloor = x + O(1)$.
- For $\theta \geq 1/2$ (or $\theta = 0$), exist Beurling zeta functions with $N(\frac{1+\theta}{2}, T) \gg T \log T$.
- Arithmetical application: PNT in short intervals (assuming wider zero-free region).

THANK YOU FOR LISTENING!

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