Spreads and ovoids of finite generalized quadrangles

Matthew R. Brown

Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, B-9000 Gent, Belgium, e-mail: mbrown@cage.rug.ac.be

Abstract

We survey recent results on spreads and ovoids of finite generalized quadrangles. Included in the survey are results on ovoids of $\mathrm{PG}(3, q)$; translation ovoids of $Q(4, q)$; characterisations of generalized quadrangles using subquadrangles and ovoids of subquadrangles; and spreads of $T_2(\Omega)$.

1 Introduction and definitions

This paper contains an update on the study of spreads and ovoids of finite generalized quadrangles since the publication of the Handbook of Incidence Geometry ([18]) and the paper "Spreads and ovoids in finite generalized quadrangles", by Thas and Payne ([46]). Consequently our focus will be mainly on spreads and ovoids of generalized quadrangles of order q. In particular we will review a number of different topics including the connections between ovoids of $\mathrm{PG}(3, q)$ and ovoids of generalized quadrangles of order q; translation ovoids of $Q(4, q)$; generalized quadrangles with a subquadrangle of order q and spreads and ovoids of the subquadrangle; as well as spreads of $T_2(\mathcal{O})$.
To begin we give some basic definitions and results.
A (finite) generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) if X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \; I \; m \; I \; Y I \; \ell$.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have order (s, t). A GQ of order (s, s) is said to have order s. If $\mathcal{S}$ has order (s, t), then it follows that $|\mathcal{P}| = (s + 1)(st + 1)$ and $|\mathcal{B}| = (t + 1)(st + 1)$. The incidence

structure $\hat{\mathcal{S}} = (\mathcal{B}, \mathcal{P}, I)$ is a GQ of order (t, s) called the *dual* of $\mathcal{S}$. For a comprehensive introduction to GQs see the book of Payne and Thas ([33]).
An *ovoid* $\mathcal{O}$ of a GQ $\mathcal{S}$ of order (s, t) is a set of points such that each line of $\mathcal{S}$ is incident with precisely one point of $\mathcal{O}$. It follows that $|\mathcal{O}| = st + 1$. Dually, a *spread* $\mathcal{R}$ of $\mathcal{S}$ is a set of lines such that each point of $\mathcal{S}$ is incident with precisely one line of $\mathcal{R}$. Again $|\mathcal{R}| = st + 1$. Equivalently we may think of an ovoid as a set of $st + 1$ points of $\mathcal{S}$, no two of which are collinear, and a spread as a set of $st + 1$ lines of $\mathcal{S}$ no two of which are concurrent.
The study of spreads and ovoids of GQs is important for a number of reasons. They are "natural" extremal sets in GQs and generalisations of spreads and ovoids in projective spaces. They are also important in the study of subquadrangles of GQs, as well as having connections to both flock GQs and to α-flocks. We shall investigate these important applications of spreads and ovoids at later stages in the paper.
We shall now quote some general results on spreads and ovoids of GQs of order (s, t) before proceeding to concentrate on GQs of order q.
The following result due to Shult is the only result which answers the existence questions of ovoids (or spreads in the dual case) by the parameters of the GQ alone.

Theorem 1 (Shult [38], see [33, 1.8.3]) *A GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ of order (s, t) with $s > 1$ and $t > s^2 - s$, has no ovoid.*

Note that this rules out the existence of an ovoid in a GQ of order (s, s^2), a large and important class of GQs.
A *polarity* of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ of order s is an isomorphism onto $\hat{\mathcal{S}} = (\mathcal{B}, \mathcal{P}, I)$ that has order two. An *absolute point* of a polarity is a point incident with its own image under the polarity, and an *absolute line* is defined similarly.

Theorem 2 (Payne [34], see [33, 1.8.2]) *If the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ of order s admits a polarity, then $2s$ is a square. Moreover, the set of all absolute points (absolute lines, respectively) of a polarity of $\mathcal{S}$ is an ovoid (spread, respectively) of $\mathcal{S}$.*

A *subquadrangle* $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ of order (s, t) is a GQ of order (s', t') such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. In other words $\mathcal{S}'$ is a subgeometry of $\mathcal{S}$ that is also a GQ. The GQ $\mathcal{S}'$ is said to be a *proper* subquadrangle if $\mathcal{S}' \neq \mathcal{S}$, and in this case it follows that $\mathcal{P}' \neq \mathcal{P}$ and $\mathcal{B}' \neq \mathcal{B}$.
A line $\ell \in \mathcal{B} \backslash \mathcal{B}'$ meets $\mathcal{S}'$ in either one or zero points and is called accordingly a *tangent line* or an *external line*. *Tangent* points and *external* points are de-

fined dually. Given these definitions we have the following result connecting subquadrangles with spreads and ovoids in the case where $s = s'$.

Theorem 3 (see [33, 2.2.1]) *Let $\mathcal{S}'$ be a proper subquadrangle of $\mathcal{S}$, with notation as above. Then either $s = s'$ or $s \geq s't'$. If $s = s'$, then each external point is collinear with exactly $1 + st'$ points of an ovoid of $\mathcal{S}'$; if $s = s't'$, then each external point is collinear with exactly $1 + s'$ points of $\mathcal{S}'$. The dual holds, similarly.*

It is clear from this theorem that knowledge of ovoids of a GQ $\mathcal{S}'$ of order s' will give us information on the possible embeddings of $\mathcal{S}'$ into GQs of order (s', t); and dually that spreads of $\mathcal{S}'$ will give us information on possible embeddings of $\mathcal{S}'$ into GQs of order (s, s'). In particular in Section 4 we will in this way consider embeddings of a GQ of order q in GQs of order (q, q^2) and (q^2, q).

As we will mainly be considering spreads and ovoids of GQs of order q we will now review the constructions, and some basic properties, of the known GQs of that order.

The classical GQ $Q(4, q)$ consists of the points and lines on a non-singular parabolic quadric of $\mathrm{PG}(4, q)$, q odd or even. The GQ $Q(4, q)$ has the property that all lines are regular; all points are regular if and only if q is even; all points are antiregular if and only if q is odd; all points and lines are semiregular and have property (H). (For these results see [33, 3.3.1 (i)], and for definitions of terms see [33, Chapter 1].)

The second classical GQ of order q is $W(q)$, the set of absolute points and absolute lines of a symplectic polarity of $\mathrm{PG}(3, q)$, q odd or even. The GQ $W(q)$ is always dual to $Q(4, q)$, and also isomorphic to $Q(4, q)$ if and only if q is even. (See [33, 3.2.1] for details on the duality and isomorphism from $W(q)$ to $Q(4, q)$.) Consequently the combinatorial properties of $W(q)$ are the dual of those of $Q(4, q)$ above.

The only known class of non-classical GQs of order q is due to Tits and first appeared in the book of Dembowski ([20]). Let Ω be an oval of $\mathrm{PG}(2, q)$, q odd or even. Embed $\mathrm{PG}(2, q)$ in $\mathrm{PG}(3, q)$ as a hyperplane and define the following incidence structure $T_2(\Omega)$. The *points* are: (i) the points of $\mathrm{PG}(3, q) \setminus \mathrm{PG}(2, q)$, called the *affine* points, (ii) the planes of $\mathrm{PG}(3, q)$ which contain a single point of Ω and (iii) a symbol (∞). The *lines* are: (a) the lines of $\mathrm{PG}(3, q)$, not in $\mathrm{PG}(2, q)$, which meet $\mathrm{PG}(2, q)$ in a point of Ω and (b) the points of Ω. *Incidence* is inherited from $\mathrm{PG}(3, q)$ with the addition that the point (∞) is incident with all lines of type (b). Then $T_2(\Omega)$ is a GQ of order q.

The GQ $T_2(\Omega)$ is classical if and only if Ω is a conic of $\mathrm{PG}(2, q)$ in which case it is the GQ $Q(4, q)$ (see [33, 3.2.2]). In the case when $T_2(\Omega)$ is non-classical

(and hence q must be even) each line of type (b) is regular, and the point (∞) is regular (see [33, 3.3.2 (i)]).
For the existence of spreads and ovoids in the known GQs of order q it is known that $W(q)$ always has ovoids, and has spreads if and only if q is even, and for $Q(4,q)$ the dual (see [33, 3.4.1 (i)]). For the non-classical $T_2(\Omega)$ there are always ovoids (see [33, 3.4.2 (i)]) and, as we shall see in Section 5, in some cases it has spreads, in some cases it has *no* spreads and in some cases the answer is still unknown.

2 Ovoids of $\mathrm{PG}(3,q)$

In this section we review recent developments in the theory of ovoids of $\mathrm{PG}(3,q)$, q even. First, however, we take some time to show why ovoids of $\mathrm{PG}(3,q)$ are important to the study of spreads and ovoids of GQs. Indeed the ovoid of $\mathrm{PG}(3,q)$ is in some sense the "prototype" of ovoids of GQs of order q and also provides a very strong motivation for the study of ovoids of GQs.
An ovoid of $\mathrm{PG}(3,q)$ is a set of q^2+1 points no three of which are collinear. For $q>2$ this is a *maximally* sized set of points, no three collinear. It is known that a plane of $\mathrm{PG}(3,q)$ intersects an ovoid in either a single point, in which case it is called a *tangent* plane, and there is a unique tangent plane on each point of the ovoid; or in an oval, in which case it is called a *secant* plane. The classical ovoid is the non-singular elliptic quadric in $\mathrm{PG}(3,q)$. For q odd these are all of the ovoids (see [5, 32]). In the case where $q=2^{2e+1}$, $e\geq 1$, there is a non-classical ovoid, due to Tits, whose linear stabiliser is the Suzuki simple group. For q even and $q\leq 32$ these are the only ovoids of $\mathrm{PG}(3,q)$. For more details and references on ovoids see the excellent survey paper of O'Keefe ([26]).
Ovoids are important geometrical objects because of their many connections to other areas of finite geometry. These include translation planes, flocks of a quadratic cone and their many associated geometries, GQs satisfying property (G) at a pair of points, unitals, maximal arcs. In the next three sections we shall see that the strong links between ovoids of $\mathrm{PG}(3,q)$ and ovoids of GQs mean that the study of ovoids of GQs is also important.

2.1 Ovoids of $\mathrm{PG}(3,q)$ as ovoids of GQs

In the case where q is odd $W(q)$ has no ovoid. However any hyperplane section of a parabolic quadric giving rise to $Q(4,q)$ that is a non-singular elliptic quadric, is an ovoid of the hyperplane *and* of the $Q(4,q)$. Such an

ovoid also gives rise to a spread of $W(q)$ and an ovoid of $T_2(\mathcal{C})$ where $\mathcal{C}$ is a conic of $\mathrm{PG}(2,q)$.
For q even Segre showed that an ovoid of $\mathrm{PG}(3,q)$ defines a symplectic polarity of $\mathrm{PG}(3,q)$ with the absolute lines of the polarity being the tangent lines to the ovoid. Consequently any ovoid of $\mathrm{PG}(3,q)$ is an ovoid of the GQ $W(q)$ defined by its associated symplectic polarity. Conversely, Thas ([40]) showed that any ovoid of $W(q)$ is also an ovoid of $\mathrm{PG}(3,q)$.
Hence we have that for q even an ovoid of $\mathrm{PG}(3,q)$ is an ovoid (and hence a spread) of $W(q)$; an ovoid and a spread of $Q(4,q)$; and an ovoid and a spread of $T_2(\mathcal{C})$, where $\mathcal{C}$ is a conic.

2.2 Generalising ovoids of $\mathrm{PG}(3,q)$ to GQs

In this section we look at the ovoids of $\mathrm{PG}(3,q)$ as ovoids of GQs, as in the previous section, and generalise these constructions to give spreads and ovoids of GQs.
Firstly, consider the elliptic quadric as an ovoid of $T_2(\mathcal{C})$, where $\mathcal{C}$ is a conic of $\mathrm{PG}(2,q)$. Then in this setting if the elliptic quadric contains the point (∞), then it has the form $(\pi\setminus\mathrm{PG}(2,q))\cup\{(\infty)\}$ where π is a plane of $\mathrm{PG}(3,q)$ skew to $\mathcal{C}$. We can generalise this construction in the following way. Let Ω be any oval of $\mathrm{PG}(2,q)$. Construct $T_2(\Omega)$ in the usual way and let π be any plane of $\mathrm{PG}(3,q)$ skew to Ω. If we consider the set of points $(\pi\setminus\mathrm{PG}(2,q))\cup\{(\infty)\}$ we see that any point of $T_2(\Omega)$ of type (b) is incident with only (∞) and any line of type (a) meets $\pi\setminus\mathrm{PG}(2,q)$ in a unique point. Hence the set is an ovoid of $T_2(\Omega)$. Such an ovoid is called a *planar* ovoid.

Next consider an elliptic quadric ovoid of $T_2(\mathcal{C})$ that does not contain the point (∞). Then for some elliptic quadric $\mathcal{E}$, containing $\mathcal{C}$ as a plane section, the ovoid of $T_2(\mathcal{C})$ has the following form:

$$(\mathcal{E}\setminus\mathcal{C})\cup\{\text{tangent planes to } \mathcal{E} \text{ at points of } \mathcal{C}\}.$$

Generalising, let $\mathcal{O}$ be any ovoid of $\mathrm{PG}(3,q)$ and Ω any oval section of $\mathcal{O}$. Construct the GQ $T_2(\Omega)$ in $\mathrm{PG}(3,q)$ in the usual way and consider the set of points

$$(\mathcal{O}\setminus\Omega)\cup\{\text{tangent planes to } \mathcal{O} \text{ at points of } \Omega\}.$$

A point of type (b) of $T_2(\Omega)$ meets the set in the unique tangent plane to $\mathcal{O}$ at that point. Any line of type (a) is either a tangent to the ovoid $\mathcal{O}$ at a point of Ω and so incident with the tangent plane of $\mathcal{O}$ at that point, or is a secant and so incident with a unique point of $\mathcal{O}\setminus\Omega$. Thus the set is an ovoid of $T_2(\Omega)$. Such an ovoid is called a *projective* ovoid.

Finally, consider the Tits ovoid of $\mathrm{PG}(3,q)$. In the original construction ([47]) Tits constructs a polarity of $W(q)$ and then shows that the set of absolute points is an ovoid of $\mathrm{PG}(3,q)$. (Note that it is also shown that the set of absolute lines is a spread of $\mathrm{PG}(3,q)$, now known as the Lüneburg spread.) We have already seen in Theorem 2 that the set of absolute points of any GQ of order s gives rise to an ovoid, and the absolute lines a to spread. The GQs T_2(translation oval), $q = 2^h$, h odd, admit a polarity (see [33, 12.5.2]) and so also polarity ovoids and spreads.

2.3 Recent results concerning ovoids of $\mathrm{PG}(3,q)$, q even

We now review some recent results on ovoids of $\mathrm{PG}(3,q)$, q even.
Let $\mathcal{O}$ be an ovoid of $\mathrm{PG}(3,q)$, q even, and let ℓ be a tangent line to the ovoid (that is, meeting the ovoid in a unique point). Of the $q+1$ planes of $\mathrm{PG}(3,q)$ containing ℓ one will be a tangent plane while the other q will intersect $\mathcal{O}$ in an oval. Such a set of q oval sections of $\mathcal{O}$ is called a *pencil* and the line ℓ the *carrier* of the pencil.

Theorem 4 (O'Keefe and Penttila [28]) *An ovoid that has a pencil of translation ovals is an elliptic quadric or a Tits ovoid.*

A *translation hyperoval* is the completion of a translation oval to a hyperoval.

Theorem 5 (O'Keefe and Penttila [29]) *An ovoid with each oval section contained in a translation hyperoval is an elliptic quadric or a Tits ovoid.*

Both of these theorems give strong characterisations of the known ovoids of $\mathrm{PG}(3,q)$. For both results the authors made use of a representation of an ovoid of $\mathrm{PG}(3,q)$, q even, as a family of ovals in the plane (see [35, 21]) and then performed calculations in the plane. Note that Theorem 4 makes use of an earlier theorem due to Penttila and Praeger ([36]) that proves the same result with the additional hypothesis that the carrier of the pencil is an axis of at least one of the translation ovals in the pencil. (Although the result of Penttila and Praeger was published in 1997 it was, in fact, proved and the paper submitted in the late 1980's.)

The next two results due to Brown are the first to characterise ovoids of $\mathrm{PG}(3,q)$, q even, in terms of a *single* plane section.

Theorem 6 (Brown [13]) *An ovoid containing a single conic section is an elliptic quadric.*

The proof of this result made use of the construction methods outlined in Section 2.2. Let $\mathcal{O}$ be an ovoid of PG$(3, q)$ containing a conic $\mathcal{C}$. Then $\mathcal{O}$ gives rise to a projective ovoid of $T_2(\mathcal{C})$, and since, for q even, $T_2(\mathcal{C})$ and $W(q)$ are isomorphic this ovoid also gives rise to an ovoid $\overline{\mathcal{O}}$ of $W(q)$ and hence PG$(3, q)$. By some detailed calculations this is enough to be able to show that both $\mathcal{O}$ and $\overline{\mathcal{O}}$ must be elliptic quadrics.
A *pointed conic* is any oval constructed by taking a conic, removing any point and then including the nucleus of the conic. All such ovals are projectively equivalent.

Theorem 7 (Brown [14]) *If an ovoid contains a pointed conic, then either $q = 4$ and the ovoid is the elliptic quadric, or $q = 8$ and the ovoid is the Tits ovoid.*

Again the proof of this result takes an ovoid of PG$(3, q)$ containing a pointed conic $\mathcal{P}$ and constructs a projective ovoid of $T_2(\mathcal{P})$. The result is proved by applying certain regularity properties of $T_2(\mathcal{P})$ and also making use of the calculations in the proof of Theorem 6.

3 Semifield flocks and translation ovoids of $Q(4, q)$, q odd

Recently a lot of work has been done, and continues to be done, on semifields, flocks of a quadratic cone and their GQs, and translation ovoids of $Q(4, q)$. In this section we will introduce each of these terms, give the connections between them and then review the recent results on them.

3.1 Translation ovoids of $Q(4, q^n)$ and semifield flocks

Let $\mathcal{O}$ be an ovoid of $Q(4, q^n)$. If we embed $Q(4, q^n)$ in the klein quadric $Q^+(5, q^n)$, then $\mathcal{O}$ is also an ovoid of $Q^+(5, q^n)$. Using the klein correspondence, $\mathcal{O}$ gives rise to a spread $\mathcal{S}(\mathcal{O})$ of PG$(3, q^n)$. The ovoid $\mathcal{O}$ is said to be a *translation ovoid* of $Q(4, q^n)$ if $\mathcal{S}(\mathcal{O})$ is a *semifield* spread, that is using the Bruck-Bose construction of a translation plane from a spread of PG$(3, q^n)$ the spread gives a semifield plane.
Translation ovoids are best understood using a different model of the GQ $Q(4, q^n)$. This model is a generalisation of the construction of $T_2(\mathcal{C})$, where $\mathcal{C}$ is a conic in PG$(2, q^n)$. Considering the plane PG$(2, q^n)$ as a vector space we have $V(3, q^n)$ which we may then consider as the vector space $V(3n, q)$ which gives rise to the projective space PG$(3n - 1, q)$. Under this transformation

points of $\mathrm{PG}(2,q^n)$ become $(n-1)$-dimensional spaces of $\mathrm{PG}(3n-1,q)$ and lines of $\mathrm{PG}(2,q^n)$ become $(2n-1)$-dimensional spaces. Hence the q^n+1 points of the conic $\mathcal{C}$ become a set $\mathcal{C}^*$ of q^n+1 $(n-1)$-dimensional spaces of $\mathrm{PG}(3n-1,q)$ called a *generalised conic*. The q^n+1 tangent lines to the conic $\mathcal{C}$ become a set of q^n+1 $(2n-1)$-dimensional spaces called the *tangent spaces* of $\mathcal{C}^*$. It follows from this construction of $\mathcal{C}^*$ that any three elements of $\mathcal{C}^*$ span $\mathrm{PG}(3n-1,q)$ and that the tangent space of any element of $\mathcal{C}^*$ is disjoint from all other elements of $\mathcal{C}^*$.
We now construct the new model $Q(n,n,q)$ from $\mathcal{C}^*$ and its tangent spaces. Embed $\mathrm{PG}(3n-1,q)$ as a hyperplane in $\mathrm{PG}(3n,q)$. The *points* of $Q(n,n,q)$ are: (i) points of $\mathrm{PG}(3n,q)\setminus\mathrm{PG}(3n-1,q)$; (ii) the $2n$-dimensional spaces of $\mathrm{PG}(3n,q)$ meeting $\mathrm{PG}(3n-1,q)$ in a tangent space of $\mathcal{C}^*$, and (iii) a symbol (∞). The *lines* are: (a) the n-dimensional spaces of $\mathrm{PG}(3n,q)$ meeting $\mathrm{PG}(3n-1,q)$ in an element of $\mathcal{C}^*$, and (b) the elements of $\mathcal{C}^*$. The *incidence* is that inherited from $\mathrm{PG}(3n,q)$ and additionally every line of type (b) is incident with the point (∞).
This construction is an example of a more general method of representation of translation generalized quadrangles to be found in [33, Section 8.7].
In the model $Q(n,n,q)$ a translation ovoid of $Q(4,q^n)$ whose associated semifield has kernel containing $\mathrm{GF}(q)$ may be constructed in the following way. Let π be a $(2n-1)$-dimensional subspace of $\mathrm{PG}(3n-1,q)$ disjoint from all elements of $\mathcal{C}^*$. Let $\overline{\pi}$ be a $2n$-dimensional subspace of $\mathrm{PG}(3n,q)$ such that $\overline{\pi}\cap\mathrm{PG}(3n-1,q)=\pi$. Then the set

$$\mathcal{O}=(\overline{\pi}\setminus\pi)\cup\{(\infty)\}$$

is an ovoid of $Q(n,n,q)$. The group of elations of $\mathrm{PG}(3n,q)$ with axis $\mathrm{PG}(3n-1,q)$ and centre in π induces a group of automorphisms of $Q(n,n,q)$ that fixes (∞) and acts transitively on the points of $\mathcal{O}\setminus\{(\infty)\}$. In the spread of $\mathrm{PG}(3,q)$ constructed from the ovoid (of $Q(4,q^n)$) this corresponds to a group of order q^{2n} fixing one line of the spread and acting regularly on the rest. By [20, Chapter 5] this is the case if and only if the associated plane is semifield with kernel containing $\mathrm{GF}(q)$. For more details on translation ovoids see [7].

Now we introduce flocks and semifield flocks of a quadratic cone. If $\mathcal{K}$ is a quadratic cone in $\mathrm{PG}(3,q)$, then a *flock* of $\mathcal{K}$ is a set of q planes of $\mathrm{PG}(3,q)$ that partition the non-vertex points of $\mathcal{K}$. If we embed $\mathcal{K}$ in the klein quadric $Q^+(5,q)$, then the images of the elements of $\mathcal{F}$ under the polarity of $Q^+(5,q)$ form a set of q planes each of which intersects $Q^+(5,q)$ in a conic and with each of the planes containing a fixed line tangent to $Q^+(5,q)$. Furthermore, the union of the intersections of these planes with $Q^+(5,q)$ forms an ovoid of $Q^+(5,q)$. Under the klein correspondence this becomes a spread $\mathcal{S}(\mathcal{F})$ of $\mathrm{PG}(3,q)$. The flock $\mathcal{F}$ is said to be a *semifield flock* if the spread $\mathcal{S}(\mathcal{F})$ gives rise to a semifield plane. When q is even the only examples of semifield flocks are the linear flocks ([22], and see also [39] for a shorter proof of the result).

3.2 The equivalence of semifield flocks and translation ovoids for q odd: the Thas construction

In [43, Appendix B] Thas gave the following construction which shows the equivalence between semifield flocks and translation ovoids of $Q(4,q)$ for q odd.

Consider the model $Q(n,n,q)$ for the GQ $Q(4,q^n)$, q odd, and let $\mathcal{O} = \overline{\pi} \cup \{(\infty)\}$ be a translation ovoid as described in the previous section. The $2n$-dimensional space $\overline{\pi}$ meets $\mathrm{PG}(3n-1,q)$ in a $(2n-1)$-dimensional space π disjoint from each element of the generalized conic $\mathcal{C}^*$. By taking a polarity of $\mathrm{PG}(3n-1,q)$ the set $\mathcal{C}^*$ corresponds to a set $(\mathcal{C}^*)^D$ of q^n+1 $(2n-1)$-spaces which may be considered to be a dual conic $\hat{\mathcal{C}}$ in the plane $\mathrm{PG}(2,q^n)$ corresponding to $\mathrm{PG}(3n-1,q)$. Under the polarity the space π becomes a $(n-1)$-dimensional space π^D disjoint from each element of $(\mathcal{C}^*)^D$. Recall that the points of $\mathrm{PG}(2,q^n)$ are represented in $\mathrm{PG}(3n-1,q)$ as an $(n-1)$-spread and so the set π^D corresponds to a set of points $\hat{\pi}$ of $\mathrm{PG}(2,q^n)$ by non-empty intersection with this $(n-1)$-spread. Also, no element of the set $\hat{\pi}$ is incident with an element of the dual conic $\hat{\mathcal{C}}$.

Now let Π be an n-dimensional space of $\mathrm{PG}(3n,q)$ meeting $\mathrm{PG}(3n-1,q)$ in π^D. If we let $\mathrm{AG}(3n,q) = \mathrm{PG}(3n,q) \setminus \mathrm{PG}(3n-1,q)$ represent $\mathrm{AG}(3,q^n) = \mathrm{PG}(3,q^n) \setminus \mathrm{PG}(2,q^n)$ (so points of $\mathrm{AG}(3,q^n)$ are points of $\mathrm{AG}(3n,q)$ and lines of $\mathrm{AG}(3,q^n)$ are n-dimensional subspaces of $\mathrm{AG}(3n,q)$ with $(n-1)$-dimensional space at infinity corresponding to a point of $\mathrm{PG}(2,q^n)$), then $\hat{\Pi} = \Pi \setminus \mathrm{PG}(3n-1,q)$ is also a set of q^n points of $\mathrm{AG}(3,q^n)$. Furthermore, since the line (in $\mathrm{PG}(3,q^n)$) spanned by two points of Π meets $\mathrm{PG}(2,q^n)$ in a point not on an element of $\hat{\mathcal{C}}$ it follows that $\hat{\Pi}$ is a dual flock. Since the corresponding spread has a group fixing one element and acting transitively on the remaining q^{2n} elements, the flock is semifield.

Note that we can also reverse this construction to obtain a translation ovoid of $Q(4,q^n)$ from a semifield flock of a quadratic cone in $\mathrm{PG}(3,q^n)$.

Bloemen ([6]) and Lunardon ([25]) give expanded treatments of this construction by Thas.

3.3 Known translation ovoid/semifield flock pairs and new results

In this section we list the known translation ovoid/semifield flock pairs and then quote some new results concerning translation ovoids.

Known translation oval/semifield flock pairs

1. elliptic quadric ovoid/linear flock.
2. Kantor-Knuth ovoids/flocks.
3. Thas-Payne ovoid/Ganley flock.

   A *new* translation ovoid/semifield flock

4. Penttila-Williams ovoid/flock for $q = 243$ (see [37]).

Note that the Penttila-Williams flock was first *explicitly* constructed (from the ovoid) by Bader, Lunardon and Pinneri in [1].
Recently the following impressive theorem was proved which classifies the (translation ovoids/)semifield flocks as either Kantor-Knuth or linear when q is large enough compared to n.

Theorem 8 (Ball, Blokhuis and Lavrauw [4]) *Any semifield flock of a quadratic cone in* $\mathrm{PG}(3, q^n)$ *with kernel* $\mathrm{GF}(q)$*,* q *odd,* $q > 4n^2 - 8n + 1$*, is either linear or Kantor.*

Their proof essentially works by observing that if $\mathcal{F}^D$ is a dual semifield flock, then there is a corresponding subgeometry in $\mathrm{PG}(2, q^n)$, the plane of the dual conic $\hat{C}$ (by considering the points of $\mathrm{PG}(2, q^n)$ obtained by intersecting the span of points of $\mathcal{F}^D$ with $\mathrm{PG}(2, q^n)$); a subgeometry consisting of points internal with respect to the conic which has $\hat{\mathcal{C}}$ as its tangents. This gives a subplane of $\mathrm{PG}(2, q^n)$ if and only if the dual semifield flock is not linear nor Kantor (see [41, Section 1.5.6]). They then consider conditions for existence of a subplane of points internal to a given conic of $\mathrm{PG}(2, q^n)$ to obtain the result.

There is a wealth of literature on semifield flocks, translation ovoids, translation GQs and their connections. In particular the reader is referred to the paper of Thas "Generalized quadrangles of order (s, s^2), II", [44].

As this section is concerned with (translation) ovoids of $Q(4, q)$, q odd we include the following recent result of Ball.

Theorem 9 (Ball [3]) *Let $q = p^h$ for some prime p. Then an ovoid of $Q(4,q)$ meets an elliptic quadric ovoid of $Q(4,q)$ in 1 mod p points.*

The proof of the result uses a novel representation of $Q(4,q)$ and employs polynomials in this representation. The result is new for q odd but was already known for q even (in fact, Bagchi and Sastry ([2]) proved that in the even case the intersection of any ovoid of $Q(4,q)$ with any elliptic quadric or any Tits ovoid must have odd size).

4 Spreads, ovoids and subquadrangles

Recall from Theorem 3 that if $\mathcal{S}'$ is a subquadrangle of order q of a GQ $\mathcal{S}$ of order (q,q^2) and P a point of $\mathcal{S} \setminus \mathcal{S}'$, then the points of $\mathcal{S}'$ collinear with P form an ovoid of $\mathcal{S}'$. Such an ovoid is said to be *subtended by* P, or just *subtended*. Dually a subquadrangle of order q of a GQ of order (q^2,q) has subtended spreads.
By a result of Bose and Shrikhande ([8]) a triad of points of a GQ of order (s,s^2) (three points, pairwise non-collinear) has $s+1$ centers (points collinear with all three points). Consequently, a subtended ovoid of $\mathcal{S}'$ may be subtended by at most *two* points as otherwise we would have a triad with the q^2+1 points of the ovoid as centres. If an ovoid $\mathcal{O}$ of $\mathcal{S}'$ is subtended by two points, then $\mathcal{O}$ is said to be *doubly subtended*. In this case the intersection of $\mathcal{O}$ with any other subtended ovoid of $\mathcal{S}'$ must have size $q+1$ since it is the trace of a triad of points of $\mathcal{S}$.
Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two subtended ovoids of $\mathcal{S}'$, subtended by points P_1 and P_2, respectively. If P_1 and P_2 are collinear, then $\mathcal{O}_1$ and $\mathcal{O}_2$ intersect in exactly one point, the point of $\mathcal{S}'$ incident with the line of $\mathcal{S}$ spanned by P_1 and P_2. If ℓ is a line of $\mathcal{S}$, not a line of $\mathcal{S}'$ with $\ell \cap \mathcal{S}' = X$, then the q points of $\ell \setminus \{X\}$ subtend a set of q ovoids intersecting pairwise in X and partitioning the points of $\mathcal{S}'$ not collinear with X. Such a set of ovoids of a GQ of order q is called a *rosette* and the rosette is said to be *subtended* by ℓ. (Note that the definition of a rosette does not require it to be subtended.) Since each ovoid may be subtended by at most two points it follows that each subtended rosette may be subtended by at most two lines.
Since each point of $\mathcal{S} \setminus \mathcal{S}'$ corresponds to an ovoid of $\mathcal{S}'$ and each line of $\mathcal{S} \setminus \mathcal{S}'$ corresponds to a rosette of $\mathcal{S}'$ it is clear that the ovoids and rosettes of $\mathcal{S}'$ will tell us a great deal about the embeddings of $\mathcal{S}'$ as a subquadrangle of a GQ of order (q,q^2). Dualising the above discussion we see that spreads and rosettes of spreads of a GQ of order q are useful in the study of its embedding as a subquadrangle of GQs of order (q^2,q).

Such considerations have provided characterisations of $Q^-(5,q)$ in terms of a single (classical) subquadrangle and subtended ovoids. The first such result was due to Thas and Payne ([46, Theorem 7.1]), in the case where q is even and each subtended ovoid of the classical subquadrangle is an elliptic quadric.

Theorem 10 *Let $\mathcal{S}$ be a GQ of order (q,q^2), q even, having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4,q)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ consisting of all points collinear with a given point X of $\mathcal{S} \setminus \mathcal{S}'$ (that is, subtended by X) is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q^-(5,q)$.*

In the case where q is even the two classical GQs $Q(4,q)$ and $W(q)$ are isomorphic and the only known ovoids are the elliptic quadrics and the Tits ovoids. Generalising the work of Thas and Payne, Brown ([10]) proved the following result.

Theorem 11 *If a GQ $\mathcal{S}$ of order (q,q^2), q even, has a subquadrangle isomorphic to $W(q)$ and each subtended ovoid is either an elliptic quadric or a Tits ovoid, then $\mathcal{S}$ is isomorphic to $Q^-(5,q)$.*

In the case where q is odd Brown ([11]) proved the equivalent result of Theorem 10 by noting that each elliptic quadric of $Q(4,q)$ must be doubly subtended and the geometry of $\mathcal{S} \setminus \mathcal{S}'$ is a double cover of the subtended ovoid/rosette geometry. A cohomology calculation then proved the result. Brouns, Thas and Van Maldeghem ([9]) and Brown ([15]), independently, gave proofs valid for both q odd and even.

When q is even $Q^-(5,q)$ is the only known GQ of order (q,q^2) that has $Q(4,q)$ as a subquadrangle (indeed, by Theorem 11, the existence of another such GQ implies the existence of a new ovoid of $\mathrm{PG}(3,q)$). When q is odd Kantor ([23]) observed that his dual semifield flock GQs (that is the dual GQs of his semifield flock GQs) contain subquadrangles isomorphic to $Q(4,q)$, and for these subquadrangles Brown ([12]) showed that the corresponding subtended ovoids are Kantor-Knuth translation ovoids (and each subtended ovoid is doubly subtended). More generally the dual of any semifield flock GQ is a Translation Generalized Quadrangle (TGQ) and for any TGQ of order (q,q^2) there is an associated TGQ of order (q,q^2) called the *translation dual* (see Chapter 8 of [33]). Thas ([42]) showed that the translation dual of a dual semifield flock GQ contains subquadrangles isomorphic to $Q(4,q)$. In [24] Lavrauw and Penttila use the theory of eggs (see Payne and Thas [33, Chapter 8]) to give a simple representation of the dual of a semifield flock GQ and its subquadrangles. Lavrauw (personal communications) has shown that some of the subtended ovoids are the translation ovoid corresponding to the semifield flock. In general it is expected that for each $Q(4,q)$ subquadrangle all of the subtended ovoids will be of the corresponding translation ovoid type.

In the case where q is odd the above are the only known examples of subquadrangles of order q of GQs of order (q, q^2). In the q even case we also have a number of examples. If $\mathcal{O}$ is an ovoid of $\mathrm{PG}(3, q)$ then the GQ $T_3(\mathcal{O})$ of order (q, q^2) will have a subquadrangle $T_2(\Omega)$ for each oval section Ω of $\mathcal{O}$. The other known GQs of order (q, q^2) with subquadrangles of order q are the dual flock GQs in the case where q is even. If $\mathcal{F}$ is a flock of a quadratic cone in $\mathrm{PG}(3, q)$, q even, then there is associated a GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q); a family of ovals $\mathcal{H} = \{\Omega_1, \ldots, \Omega_{q+1}\}$ called a *herd*; and a family $\{T_2(\Omega_1), \ldots, T_2(\Omega_{q+1})\}$ of subquadrangles of $\mathcal{S}(\mathcal{F})$ where $\Omega_i \in \mathcal{H}$. (For more details on and references to results on flocks and flock GQs see the article of Payne in this volume.) Since $\mathcal{S}(\mathcal{F})$ has subquadrangles $T_2(\Omega_i)$ it follows that each $T_2(\Omega_i)$ has spreads subtended by $\mathcal{S}(\mathcal{F})$. The structure of these subtended spreads was determined independently by Brown, O'Keefe, Payne, Penttila and Royle ([16]) and Thas ([45]). (The paper of Thas also contains a geometric construction of the ovals of a herd from the corresponding flock.)

Theorem 12 *Each subtended spread of the subquadrangle $T_2(\Omega_i)$ of $\mathcal{S}(\mathcal{F})$ consists of the line $P \in \Omega_i$ and cones $\mathcal{K}_X$. The $\mathcal{K}_X$ have distinct vertices $X \in \Omega_i \setminus \{P\}$ and for N the nucleus of Ω_i, $\mathcal{K}_X \cup \langle P, N\rangle$ is a quadratic cone with nuclear line $\langle X, N\rangle$.*

In fact in [16] an oval contained in a herd is characterised by the existence of a spread as described in Theorem 12.
As a final note regarding classical subquadrangles of GQ of order (q^2, q), it was proved by O'Keefe and Penttila ([30]) that a herd contains a conic if and only if the corresponding flock is linear and so the corresponding flock GQ is classical. Consequently a flock GQ will never give rise to a new ovoid of $\mathrm{PG}(3, q)$. (In fact O'Keefe and Penttila proved that if a herd contains at least one translation oval, then the associated flock must either be linear or Fisher-Thas-Walker.)

5 Spreads of $T_2(\Omega)$, q even

In Section 2.2 we saw that every $T_2(\Omega)$ admits planar ovoids. In the case where Ω is a conic this construction always yields an elliptic quadric ovoid. In general for a given $T_2(\Omega)$ the planar ovoids are not all equivalent under the action of the group of the $T_2(\Omega)$. In fact the number of equivalence classes of planar ovoids is given by the number of orbits of the group of Ω on lines

external to Ω. (For more details on automorphisms of the GQs $T_2(\mathcal{O})$ see [33, Chapter 12] and [31].)
The question of existence of spreads of $T_2(\Omega)$ is not completely resolved but recently significant progress has been made towards this goal including the interesting result that there exist a number of $T_2(\Omega)$ for which there is no spread.
In [16] the authors established the projective structure of a spread of $T_2(\Omega)$.

Theorem 13 *A spread of $T_2(\Omega)$, q even, consists of some $P \in \Omega$ and q cones $\mathcal{K}_X$, $X \in \Omega \setminus \{P\}$. The vertex of the cone $\mathcal{K}_X$ is X and for N the nucleus of Ω, $\mathcal{K}_X \cup \langle P, N\rangle$ is an oval cone with nuclear line $\langle X, N\rangle$.*

Clearly not any collection of q oval cones with vertices distinct points of $\Omega \setminus \{P\}$ will form a spread of $T_2(\Omega)$. If the oval Ω is normalised to the form $\{(1, t, f(t)) \ : \ t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1)\}$ with f an o-polynomial and nucleus $(0, 1, 0)$, such that P is normalised to the point $(0, 0, 1)$, then the cones may be relabelled as $\mathcal{K}_t$ where $(1, t, f(t))$ is the vertex. Next taking a plane π meeting $\mathrm{PG}(2, q)$ in the line $\langle(0, 0, 1), (0, 1, 0)\rangle$, the intersection $\pi \cap \mathcal{K}_t$ is an oval Ω_t containing $(0, 0, 1)$ and with nucleus $(0, 1, 0)$. Further, for $s \neq t$ $\Omega_s \cap \Omega_t = \{(0, 0, 1)\}$ and also the point $(0, s+t, f(s)+f(t))$ has the property that any line incident with it that is secant to Ω_s is external to Ω_t, and vice versa. In this case the ovals Ω_s and Ω_t are said to be *compatible* at the point $(0, s + t, f(s) + f(t))$. In [16] this family $\{\Omega_t \ : \ t \in \mathrm{GF}(q)\}$ is called a *generalized f-fan* and it is shown that the existence of a spread of $T_2(\Omega)$ is equivalent to the existence of the generalized fan; that is the spread may be reconstructed from the generalized fan.
The above representation of spreads of $T_2(\Omega)$ was used in [16] to prove a number of results on spreads of $T_2(\Omega)$. One of the most important of these is the connection between spreads of $T_2(\Omega)$ and α-flocks. If α is a generator of the automorphism group of $\mathrm{GF}(q)$, then the set of points $\{(1, t, t^\alpha) \ : \ t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1)\}$ is an oval with nucleus $(0, 1, 0)$, called a *translation oval* (as it is fixed by an elation group of order q). A cone of such an oval is called an *α-cone* and a flock of such a cone is called an *α-flock.* (Note that $\alpha = 2$ gives, respectively, a conic, quadratic cone and a flock.) In [19] Cherowitzo showed algebraically that to each α-flock $\mathcal{F}$ is associated an oval $\Omega(\mathcal{F})$. In [16] it was shown that the existence of an α-flock $\mathcal{F}$ was equivalent to the existence of a particular type of generalized fan, and hence to a particular type of spread of the GQ $T_2(\Omega(\mathcal{F}))$.

Theorem 14 *Let $\mathcal{F}$ be an α-flock with associated oval $\Omega(\mathcal{F})$. Then the GQ $T_2(\Omega(\mathcal{F}))$ has a spread.*

Since many known ovals are associated with α-flocks this result provides a construction of spreads in many GQs $T_2(\Omega)$. Also as for a given oval Ω the existence of an α-flock with associated oval Ω is equivalent to the existence of a certain class of spread of $T_2(\Omega)$ this gives a geometrical characterisation of those ovals associated with α-flocks. Note that this includes the quadratic flock case as mentioned in Section 4.
Recently, Brown and Thas have extended the work of Thas in [45] to give a geometric construction of the oval $\Omega(\mathcal{F})$ associated with an α-flock $\mathcal{F}$ and also a geometric construction of the spread of $T_2(\Omega(\mathcal{F}))$.

In [17] the authors used the generalized fan representation as well as further work in [16] to calculate (with the aid of a computer) all α-flocks for $q = 32$, and all spreads of GQs $T_2(\Omega)$ for $q = 32$. As a result it was shown that there exist GQs $T_2(\Omega)$ which have no spreads.

Theorem 15 *Let* Ω *be an oval of* PG(2, 32) *that completes to the O'Keefe-Penttila hyperoval (see [27]). The GQ* $T_2(\Omega)$ *admits no spread.*

References

[1] L. Bader, G. Lunardon and I. Pinneri, A New Semifield flock, *J. Combin. Theory Ser. A*, **86** (1999), 49–62.

[2] B. Bagchi and N. S. Narasimha Sastry, Even order inversive planes, generalized quadrangles and codes, *Geom. Dedicata*, **22** (1987), No. 2, 137–147.

[3] S. Ball, On ovoids of $O(5, q)$, *in preparation.*

[4] S. Ball, A. Blokuis and M. Lavrauw, On the classification of semifield flocks, *in preparation.*

[5] A. Barlotti, Un'estensione del teorema di Segre-Kustaanheimo, *Boll. Un. Mat. Ital.*, **10** (1955), 96–98.

[6] I. Bloemen, *Substructures and Characterizations of Finite Generalized Polygons*, Ph.D. thesis, Ghent University, 1995.

[7] I. Bloemen, J. A. Thas and H. Van Maldeghem, Translation ovoids of generalized quadrangles and hexagons, *Geom. Dedicata*, **72** (1998), 19–62.

[8] R. C. Bose and S. S. Shrikhande, Geometric and pseudo-geometric graphs $(q^2 + 1, q + 1, 1)$, *J. Geom.*, **2** (1972), 75–94.

[9] L. Brouns, J. A. Thas and H. Van Maldeghem, A characterisation of $Q(5,q)$ using one subquadrangle $Q(4,q)$, *Europ. J. Combin.*, submitted.

[10] M. R. Brown, Generalized quadrangles of order (q,q^2), containing $W(q)$ as a subquadrangle, *Geom. Dedicata*, **56** (1995), No. 3, 299–306.

[11] M. R. Brown, *Generalized Quadrangles and Associated Structures*, Ph.D. thesis, University of Adelaide, 1997.

[12] M. R. Brown, Semipartial geometries and generalized quadrangles of order (r,r^2), *Bull. Belg. Math. Soc. Simon Stevin*, **5** (1998), No. 2-3, 187–205.

[13] M. R. Brown, Ovoids of $\mathrm{PG}(3,q)$, q even, with a conic section, *J. London Math. Soc. (2)*, **62** (2000), No. 2, 569–582.

[14] M. R. Brown, The determination of ovoids of $\mathrm{PG}(3,q)$ containing a pointed conic, *J. Geom.*, **67** (2000), No. 1–2, 61–72.

[15] M. R. Brown, A characterisation of the generalized quadrangle $Q(5,q)$ using cohomology, *J. Algebraic Combin.*, submitted.

[16] M. R. Brown, C. M. O'Keefe, S. E. Payne, T. Penttila and G. F. Royle, Spreads of $T_2(\mathcal{O})$, α-flocks and ovals, *in preparation*.

[17] M. R. Brown, C. M. O'Keefe, S. E. Payne, T. Penttila and G. F. Royle, The classification of α-flocks over small fields, *in preparation*.

[18] F. Buekenhout, Ed. *The Handbook of Incidence Geometry*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1995.

[19] W. Cherowitzo, α-flocks and hyperovals, *Geom. Dedicata*, **72** (1998), 221–246.

[20] P. Dembowski, *Finite Geometries*, Springer-Verlag, 1968.

[21] D. G. Glynn, Plane representations of ovoids, *Bull. Belg. Math. Soc. Simon Stevin*, **5** (1998), 275–286.

[22] N. L. Johnson, Semifield flocks of quadratic cones, *Simon Stevin*, **61** (1987), 313–326.

[23] W. M. Kantor, Some generalized quadrangles with parameters q^2,q, *Math. Z.*, **192** (1986), 45–50.

[24] M. Lavrauw and T. Penttila, On eggs and translation generalized quadrangles, *in preparation.*

[25] G. Lunardon, Flocks, Ovoids of $Q(4, q)$ and Designs, *Geom. Dedicata*, **66** (1997), 163–173.

[26] C. M. O'Keefe, Ovoids of $\mathrm{PG}(3, q)$: a survey, *Discrete Math.*, **151** (1996), No. 1–3, 175–188.

[27] C. M. O'Keefe and T. Penttila, A new hyperoval in $\mathrm{PG}(2, 32)$, *J. Geom.*, **44** (1992), 117–139.

[28] C. M. O'Keefe and T. Penttila, Ovoids with a Pencil of Translation Ovals, *Geom. Dedicata*, **62** (1996), 19–34.

[29] C. M. O'Keefe and T. Penttila, Ovals in Translation Hyperovals and Ovoids, *European J. Combin.*, **18** (1997), 667–683.

[30] C. M. O'Keefe and T. Penttila, Characterisations of flock quadrangles, *Geom. Dedicata*, **82** (2000), No. 1-3, 171–191.

[31] C. M. O'Keefe and T. Penttila, On subquadrangles of generalized quadrangles of order (q^2, q), q even, *J. Combin. Theory Ser. A*, to appear.

[32] G. Panella, Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito, *Boll. Un. Mat. Ital.* **10** (1955), 507–513.

[33] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Pitman, London, 1984.

[34] S. E. Payne, Symmetric representations of nondegenerate generalized n-gons. *Proc. Amer. Math. Soc.*, **19** (1968), 1321–1326.

[35] T. Penttila, A plane representation of ovoids, *Util. Math.*, **56** (1999), 245–250.

[36] T. Penttila and C. E. Praeger, Ovoids and translation ovals, *J. London Math. Soc. (2)*, **56** (1997), No. 3, 607–624.

[37] T. Penttila and B. Williams, Ovoids of Parabolic Spaces, *Geom. Dedicata*, **82** (2000), No. 1, 1–19.

[38] S. Shad and E. E. Shult, Near n-gons, in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 461–463, Providence, R.I., 1980. Amer. Math. Soc.

[39] L. Storme and J. A. Thas, k-arcs and partial flocks, *Linear Algebra Appl.*, **226/228** (1995), 33–45.

[40] J. A. Thas, Ovoidal translation planes, *Arch. Math.*, **23** (1972), 110–112.

[41] J. A. Thas, Generalized quadrangles and flocks of cones, *European J. Combin.*, **8** (1987), 441–452.

[42] J. A. Thas, Generalized quadrangles of order (s, s^2), I, *J. Combin. Theory Ser. A*, **67** (1994), No. 2, 140–160.

[43] J. A. Thas, Symplectic spreads in $\mathrm{PG}(3, q)$, inversive planes and projective planes, *Discrete Math.*, **174** (1997), 329–336.

[44] J. A. Thas, Generalized quadrangles of order (s, s^2), II, *J. Combin. Theory Ser. A*, **79** (1997), No. 2, 223–254.

[45] J. A. Thas, A result on spreads of the generalized quadrangle $T_2(\mathcal{O})$, with $\mathcal{O}$ an oval arising from a flock and applications, *submitted*.

[46] J. A. Thas and S. E. Payne, Spreads and ovoids in finite generalized quadrangles, *Geom. Dedicata*, **52** (1994), no. 3, 227–253.

[47] J. Tits, Ovoïdes et groupes de Suzuki, *Arch. Math.*, **13** (1968), 187–198.