Some combinatorial properties of finite generalized octagons

Kris Coolsaet

Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9, B–9000 Gent, Belgium,
e-mail: Kris.Coolsaet@rug.ac.be

Abstract

Of all finite generalized polygons the generalized octagons have been studied least. Whether this is because there is currently only one known series of examples (the Ree–Tits octagons), because few of the proofs of properties of generalized quadrangles and hexagons generalize to octagons, or because the smallest example of a thick octagon has a number of points (1755) which is too large for a thorough investigation by current computers, we cannot tell.

In this text we discuss some combinatorial properties of finite generalized octagons of order (s, t) related to size and structure of special subsets (d-cliques, d-cocliques and suboctagons). We also investigate the combinatorial consequences of the condition $s^2 = t$ (related to the so-called Krein conditions).

1 Introduction

A (finite) *generalized octagon* $\mathcal{O}$ of order (s, t) is a point–line incidence geometry with the following properties :

- Every line of $\mathcal{O}$ is incident with exactly $s + 1$ points, every point is incident with exactly $t + 1$ lines.
- $\mathcal{O}$ does not contain an ordinary k-gon as a subgeometry for $2 \leq k < 8$.
- Every pair of elements of $\mathcal{O}$ (points or lines) is contained in at least one ordinary octagon.

(In what follows we shall often omit the qualifier 'generalized'.) If $s = 1$ or $t = 1$, the octagon is called *thin*, otherwise it is called *thick*. A thick octagon is called *slim* if $s = 2$. $\mathcal{O}$ consists of $v = (1 + s)(1 + st)(1 + s^2t^2)$ points and $b = (1+t)(1+st)(1+s^2t^2)$ lines. We obtain the *dual* octagon of $\mathcal{O}$ by interchanging the roles of lines and points. The dual of $\mathcal{O}$ has parameters (t, s).

We define *distance* in $\mathcal{O}$ using the point–line incidence graph : the distance $d(x, y)$ between two elements x, y of $\mathcal{O}$ is the length d of the shortest sequence $x_0, x_1, \cdots, x_d$ with $x = x_0, y = x_d$ such that every two consecutive elements x_i, x_{i+1} in this sequence are incident. For instance, $d(p, L) = 1$ if and only if the point p is incident with the line L. Elements at mutual distance 8 (i.e., at maximal distance) are called *opposite*.

The only finite thick octagons known to date belong to the family of *Ree–Tits octagons* related to the twisted Chevalley groups of type ${}^2\mathsf{F}_4$ over a finite field K of even characteristic. In that case s is an odd power of 2 and $t = s^2$. We shall denote this octagon by $\mathsf{O}(s)$.

2 Preliminary results

(We refer to chapters 2 and 3 of [1] for proofs of the results cited in this section and for more information on the techniques used.)

When studying combinatorial properties of $\mathcal{O}$ it appears convenient to introduce the following $v \times v$ matrices A_d, with $d = 0, 2, 4, 6$ or 8 : rows and columns of A_d are indexed by the points of $\mathcal{O}$ and the entry at position (p, q) satisfies

$$A_d(p,q) = \begin{cases} 1, & \text{when } d(p,q) = d \text{ in } \mathcal{O}, \\ 0, & \text{otherwise.} \end{cases}$$

(Hence, A_0 is the identity matrix and A_2 is the collinearity matrix of $\mathcal{O}$.)

From the defining properties of the generalized octagon it is possible to compute the different eigenvalues $\lambda_i(A_d)$ of A_d (cf. table 1) and the corresponding multiplicities μ_i (cf. table 2). Note that the eigenvalues $\lambda_0(A_d)$ with multiplicity 1 denote the number of points at distance d of a given point of $\mathcal{O}$.

We obtain several numerical restrictions on the possible parameters of a finite octagon from the fact that the multiplicities should be integers. We see for instance that $2st$ must be a perfect square whenever $s > 1$ and $t > 1$. (This also follows from a theorem of Feit & Higman [4].)

	A_0	A_2	A_4	A_6	A_8
λ_0	1	$s(t+1)$	$s^2t(t+1)$	$s^3t^2(t+1)$	s^4t^3
λ_1	1	$s-1+\sqrt{2st}$	$s(t-1)+(s-1)\sqrt{2st}$	$st(s-1)-s\sqrt{2st}$	$-s^2t$
λ_2	1	$s-1$	$-s(t+1)$	$-st(s-1)$	s^2t
λ_3	1	$s-1-\sqrt{2st}$	$s(t-1)-(s-1)\sqrt{2st}$	$st(s-1)+s\sqrt{2st}$	$-s^2t$
λ_4	1	$-t-1$	$t(t+1)$	$-t^2(t+1)$	t^3

Table 1: Eigenvalues $\lambda_i(A_d)$ of the matrices A_d.

μ_0	1
μ_1	$\dfrac{st(s+1)(t+1)(1+st)[\,(1+st)(s+t)-2st-(s-1)(t-1)\sqrt{2st}\,]}{4(s^2+t^2)}$
μ_2	$\dfrac{st(s+1)(t+1)(1+s^2t^2)}{2(s+t)}$
μ_3	$\dfrac{st(s+1)(t+1)(1+st)[\,(1+st)(s+t)-2st+(s-1)(t-1)\sqrt{2st}\,]}{4(s^2+t^2)}$
μ_4	$\dfrac{s^4(1+st)(1+s^2t^2)}{(s+t)(s^2+t^2)}$

Table 2: Multiplicities μ_i of the eigenvalues $\lambda_i(A_d)$.

	A_0	A_2	A_4	A_6	A_8
$u_0(A_d)$	1	1	1	1	1
$u_1(A_d)$	1	$\frac{s-1+\sqrt{2st}}{s(t+1)}$	$\frac{s(t-1)+(s-1)\sqrt{2st}}{s^2t(t+1)}$	$\frac{t(s-1)-\sqrt{2st}}{s^2t^2(t+1)}$	$-\frac{1}{s^2t^2}$
$u_2(A_d)$	1	$\frac{s-1}{s(t+1)}$	$-\frac{1}{st}$	$-\frac{s-1}{s^2t(t+1)}$	$\frac{1}{s^2t^2}$
$u_3(A_d)$	1	$\frac{s-1-\sqrt{2st}}{s(t+1)}$	$\frac{s(t-1)-(s-1)\sqrt{2st}}{s^2t(t+1)}$	$\frac{t(s-1)+\sqrt{2st}}{s^2t^2(t+1)}$	$-\frac{1}{s^2t^2}$
$u_4(A_d)$	1	$-\frac{1}{s}$	$\frac{1}{s^2}$	$-\frac{1}{s^3}$	$\frac{1}{s^4}$

Table 3: Values of $u_i(A_d)$.

Lemma 1 *If the parameters (s,t) of a generalized octagon satisfy $s = t+1$, then $s = 2$ and $t = 1$.*

Proof : The multiplicity μ_4 of eigenvalue $\lambda_0(A_2)$ must be an integer, hence $s^4(1+st)(1+s^2t^2)$ must be divisible by s^2+t^2. Now $s^2+t^2 = 2t^2+2t+1$ and therefore $t^2 = -t-\frac{1}{2} = -s^2 \pmod{s^2+t^2}$. We easily compute the following identities (mod s^2+t^2) :

$$\begin{aligned} s^4 &= (t+\frac{1}{2})^2 = t^2+t+\frac{1}{4} = -\frac{1}{4} \\ 1+st &= t^2+t+1 = \frac{1}{2} \\ 1+s^2t^2 &= 1-(t+\frac{1}{2})^2 = \frac{5}{4} \end{aligned}$$

Hence $s^4(1+st)(1+s^2t^2) = -5/32$, and this can only be zero (mod s^2+t^2) if s^2+t^2 divides 5, i.e., if $t = 0$ or 1. ∎

With every $i = 0,\dots,4$ we may associate a matrix R_i defined as follows :

$$R_i \stackrel{\text{def}}{=} 1 + u_i(A_2)A_2 + u_i(A_4)A_4 + u_i(A_6)A_6 + u_i(A_8)A_8,$$

with $u_i(A_d) \stackrel{\text{def}}{=} \lambda_i(A_d)/\lambda_0(A_d)$. Values of $u_i(A_d)$ are listed in table 3. It can be proven that R_i has rank μ_i and that $R_i^2 = (v/\mu_i)R_i$, hence R_i is positive semidefinite. The related matrix $E_i = (\mu_i/v)R_i$ is a so-called *minimal idempotent* for $\mathcal{O}$ [1].

We shall need the following

Lemma 2 *Let D be positive semidefinite matrix. Consider any symmetric partition of D into blocks, i.e.,*

$$D = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1k} \\ D_{21} & D_{22} & \cdots & D_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ D_{k1} & D_{k2} & \cdots & D_{kk} \end{pmatrix}, \qquad \text{such that } D_{ij} = D_{ji}^T \text{ for all } i, j.$$

Let $\overline{D_{ij}}$ denote the arithmetic mean of all entries of the block D_{ij}. Then the $k \times k$ matrix

$$\begin{pmatrix} \overline{D_{11}} & \overline{D_{12}} & \cdots & \overline{D_{1k}} \\ \overline{D_{21}} & \overline{D_{22}} & \cdots & \overline{D_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{D_{k1}} & \overline{D_{k2}} & \cdots & \overline{D_{kk}} \end{pmatrix}$$

is also positive semidefinite.

It follows that the sum of the elements of any submatrix obtained from D by removing rows and columns of the same index, must be nonnegative.

(A proof can be found in [1, 3.7.1].)

3 Cliques

Let $d = 2, 4, 6$ or 8. Define a *d-clique* of $\mathcal{O}$ to be a set $\mathcal{S}$ of points such that $d(x, y) = d$ for every $x, y \in \mathcal{S}, x \neq y$. The following lemma allows us to determine an upper bound for the size of $\mathcal{S}$.

Lemma 3 *Consider a matrix D of the form $1 - x_2A_2 - x_4A_4 - x_6A_6 - x_8A_8$. If D is positive semidefinite, and $x_d > 0$, then a d-clique $\mathcal{S}$ of $\mathcal{O}$ can contain at most $1/x_d + 1$ elements.*

Moreover, if $|\mathcal{S}| = 1/x_d + 1$, then for any point p of $\mathcal{O}$ that does not belong to $\mathcal{S}$ the following property holds :

$$x_2N_2(p) + x_4N_4(p) + x_6N_6(p) + x_8N_8(p) = 0, \qquad (1)$$

where $N_i(p)$ denotes the number of points of $\mathcal{S}$ at distance i of p.

Proof : Write $|\mathcal{S}| = N+1$. Let D' be the $(N+1)\times(N+1)$ submatrix of D with rows and columns corresponding to the points of $\mathcal{S}$. D' has elements 1 on the diagonal and elements $-x_d$ otherwise. By lemma 2 we have $\overline{D'} \geq 0$ and hence $1 - Nx_d \geq 0$, which is equivalent to $N \leq 1/x_d$ when $x_d > 0$.

If we extend D' with a single row and column corresponding to a point $p \notin \mathcal{S}$, we obtain a submatrix of the following form :

$$\left(\begin{array}{c|c} D' & C \\ \hline C^T & 1 \end{array}\right)$$

where C consists of a single column and $N+1$ rows. Applying lemma 2 to the indicated partition we obtain a positive semidefinite 2×2 matrix whose determinant $\overline{D'} - \overline{C}^2$ must be nonnegative. When $\overline{D'} = 0$ this implies $\overline{C} = 0$. Note that $\overline{C} = -(x_2N_2(p) + x_4N_4(p) + x_6N_6(p) + x_8N_8(p))/(N+1)$. ■

Theorem 1 *An* 8*-clique* $\mathcal{S}$ *of a generalized octagon* $\mathcal{O}$ *can contain at most* s^2t^2+1 *points. If* $|\mathcal{S}| = s^2t^2+1$ *then a point* p *of* $\mathcal{O}$ *not in* $\mathcal{S}$ *satisfies one of the following*

- $N_2(p) = 1$, *i.e., p is collinear with exactly one point of $\mathcal{S}$, and then*

$$N_4(p) = 0,\ N_6(p) = st^2,\ N_8(p) = st^2(s-1).$$

- $N_2(p) = 0$, *i.e., p is not collinear with any point of $\mathcal{S}$, and then*

$$N_4(p) = t+1,\ N_6(p) = (s-1)t(t+1),\ N_8(p) = st(st-t-1)+t^2.$$

Proof : Apply lemma 3 to $D = R_1$ and $D = R_3$, both with $x_8 = 1/s^2t^2$. This yields the upper bound of s^2t^2+1 for $|\mathcal{S}|$. In case of equality (1) translates to

$$\begin{aligned} \tfrac{s-1+\sqrt{2st}}{s(t+1)}N_2(p) + \tfrac{s(t-1)+(s-1)\sqrt{2st}}{s^2t(t+1)}N_4(p) + \tfrac{t(s-1)-\sqrt{2st}}{s^2t^2(t+1)}N_6(p) - \tfrac{1}{s^2t^2}N_8(p) &= 0 \\ \tfrac{s-1-\sqrt{2st}}{s(t+1)}N_2(p) + \tfrac{s(t-1)-(s-1)\sqrt{2st}}{s^2t(t+1)}N_4(p) + \tfrac{t(s-1)+\sqrt{2st}}{s^2t^2(t+1)}N_6(p) - \tfrac{1}{s^2t^2}N_8(p) &= 0 \end{aligned}$$

Also clearly

$$N_2(p) + N_4(p) + N_6(p) + N_8(p) = |\mathcal{S}| = s^2t^2+1$$

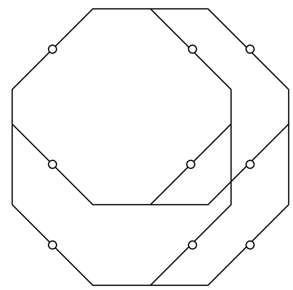

Figure 1: 9 points that form a maximal 6-clique when $s = 2, t = 1$.

Because any two points of $\mathcal{S}$ lie at distance 8, a point p cannot be collinear with two of them. Hence either $N_2(p) = 0$ or 1. Substituting these values for $N_2(p)$ in the formulas above, we may solve the 3 equations to obtain the given values for $N_4(p)$, $N_6(p)$ and $N_8(p)$. ∎

An 8-clique is also called a *partial ovoid.* An *ovoid* $\mathcal{S}$ is a partial ovoid with the extra property that every element of the octagon lies at distance at most 4 from at least one point of $\mathcal{S}$. This is equivalent to the condition $|\mathcal{S}| = s^2t^2 + 1$ (cf. [4]).

Theorem 2 *The size of a* 6*-clique* $\mathcal{S}$ *of a generalized octagon satisfies*

- $|\mathcal{S}| \leq s^3 + 1$ *when* $s \leq t + 1$,
- $|\mathcal{S}| \leq \dfrac{(st+1)(st+s-1)}{s-1}$ *when* $s \geq t + 1$,

The proof of this theorem is similar to that of theorem 1, now using $D = R_2$ and $D = R_4$. However, we obtain explicit values for the $N_i(p)$ only in the case $s = t + 1$, which, by lemma 1 implies $s = 2, t = 1$. We then find

$$N_2(p) = 0, \quad N_4(p) = 1, \quad N_6(p) = 4, \quad N_8(p) = 4, \text{ or}$$
$$N_2(p) = 1, \quad N_4(p) = 2, \quad N_6(p) = 2, \quad N_8(p) = 4.$$

The unique octagon with parameters $(s, t) = (2, 1)$ contains 10 6-cliques of size 9. The configuration of points of $\mathcal{S}$ and points with $N_2(p) = 1$ is represented in figure 1.

Finally, note that 2-cliques and 4-cliques are trivial : a maximal 2-clique consists of all points on a given line, a maximal 4-clique has $t+1$ points, one on each line through a given point.

4 Other subconfigurations

Let $d = 2, 4, 6, 8$. Define a *d-coclique* of $\mathcal{O}$ to be a set $\mathcal{C}$ of points such that $d(x, y) \neq d$ for every $x, y \in \mathcal{C}$. The following theorem provides an upper bound for the size of a d-coclique.

Theorem 3 (Hoffman) *Let Γ be a connected graph with v vertices, regular with valency $k > 0$ and smallest eigenvalue $-m$. If $\mathcal{C}$ is a coclique of Γ, then $|\mathcal{C}| \leq v/(1 + k/m)$.*

Moreover, if $|\mathcal{C}| = v/(1 + k/m)$, then every point $p \in \Gamma \setminus \mathcal{C}$ is adjacent to exactly m points of $\mathcal{C}$.

(For a proof, see [1, 3.7.2].)

Corollary 1 *Let $\mathcal{C}$ be a d-coclique of a generalized octagon $\mathcal{O}$, then*

- $|C| \leq (1 + st)(1 + s^2t^2)$ *when* $d = 2$,
- $|C| \leq (1 + s)(1 + s^2t^2)$ *when* $d = 4$,
- $|C| \leq (1 + st)(1 + s^2t^2)/(s^2 - s + 1)$ *when* $d = 6$ *and* $s \leq t + 1$,
- $|C| \leq (s^2 - 1)(1 + s^2t^2)/(st + s - 1)$ *when* $d = 6$ *and* $s \geq t + 1$,
- $|C| \leq (1 + s)(1 + st)$ *when* $d = 8$.

Proof : Apply theorem 3 to the distance d-graph of $\mathcal{O}$, whose eigenvalues are listed in table 1. ∎

For $d = 8$ the bound is trivially satisfied by the set of all points at distance ≤ 3 of a given line. No example is known of any other d-coclique which attains the given bounds (in a thick octagon).

Define a $\{d_1, d_2, \ldots, d_k\}$-*clique* to be a set of points any pair of which are at mutual distance d_1, d_2, ... or d_k. We may obtain upper bounds for the sizes of such cliques with a technique similar to that of lemma 3.

For instance, to obtain an upper bound for the size of a $\{2, 4\}$-clique, consider all matrices D of the form $D = 1 - x_2A_2 - x_4A_4 - x_6A_6 - x_8A_8$ with $x_2 = x_4$. D is positive semidefinite if and only if all its eigenvalues are nonnegative, i.e., if $1 - x_2\lambda_i(A_2) - x_2\lambda_i(A_4) - x_6\lambda_i(A_6) - x_8\lambda_i(A_8) \geq 0$ for $i = 0, \ldots, 4$. Using linear programming techniques we may now maximize x_2 under these conditions (where x_6, x_8 are allowed to take any value). This maximum gives an upper bound $1 + 1/x_2$ for the size of the coclique.

Unfortunately, computations are rather unwieldy for general s and t, and also in the smallest cases $(s,t) = (2,4)$ or $(4,2)$ they do not yield spectacular results. For $\{d_1, d_2, d_3\}$-cliques (which are really cocliques) we obtain the Hoffman bounds with this technique.

Finally, we cite the following result on the nonexistence of thick suboctagons in a thick octagon with the same parameter s :

Theorem 4 (Thas) *Assume $s > 1$. Let $\mathcal{O}'$ be a suboctagon of $\mathcal{O}$ with parameters (s, t'), then $\mathcal{O}'$ can only exist when $s \leq t$ in which case it must satisfy $t' = 1$.*

(For a proof we refer to [4, 1.8.7–1.8.8].)

5 The slim octagon $\mathsf{O}(2)$

A slim octagon must satisfy $(s,t) = (2,4)$. An example is provided by the Ree–Tits octagon $\mathsf{O}(2)$ of 1755 points and 2925 lines. It is still not known whether an other octagon with the same parameters exists.

The octagon $\mathsf{O}(2)$ contains many suboctagons with parameters $(s,t) = (2,1)$. In fact, every two opposite lines of $\mathsf{O}(2)$ are contained in exactly one such suboctagon. These suboctagons contain 10 6-cliques of size 9, hence also $\mathsf{O}(2)$ contains a lot of 6-cliques of maximal size $s^3 + 1$.

To investigate 8-cliques of slim octagons we introduce the following embedding of $\mathcal{O}$ into a projective space $\mathbf{P}$. Consider the matrix $U = 16R_4$. Note that all entries of U are integers. We may consider U as a matrix over GF(2), in which case the entry at position (p, q) of U is zero when $d(p, q) < 8$ and one when p and q are opposite.

Represent a point p of $\mathcal{O}$ by the p-th row of U (a 1755-tuple) which we denote by p^*. It can easily be proven that this representation has the property that any line $\{x, y, z\}$ of $\mathcal{O}$ satisfies $x^* + y^* + z^* = 0$. In other words, we obtain a *full embedding* of $\mathcal{O}$ in a projective space $\mathbf{P}$ over GF(2). It turns out that the dimension of this embedding (i.e., the rank of U over $GF(2)$ minus 1) provides us with an upper bound for the size of a partial ovoid (an 8-clique) :

Lemma 4 *Assume $s = 2$. Let $\mathcal{S}$ be an 8-clique of $\mathcal{O}$ and let d be the (projective) dimension of the full embedding $\mathbf{P}$ described above, then $|\mathcal{S}| \leq d + 1$ (if d is even), or $|\mathcal{S}| \leq d + 2$ (if d is odd).*

Proof : Consider the submatrix of U with rows (and columns) corresponding to all points of $\mathcal{S}$. This is a matrix with elements equal to 0 on the diagonal and to 1 in every other position. The rank of this matrix is $|\mathcal{S}|$ when $|\mathcal{S}|$ is even and $|\mathcal{S}| - 1$ when $|\mathcal{S}|$ is odd. This rank cannot be larger than the rank $d + 1$ of U. ∎

The rank of U as a real matrix is equal to $\mu_4 = 78$ (the rank of R_4). Hence the rank of U as a matrix over GF(2) is at most 78. This restricts the size of an 8-clique in a slim octagon to 79. But unfortunately this is a larger bound than the bound $1 + s^2t^2 = 65$ obtained from theorem 1.

However, for the Ree–Tits octagon, the dimension of $\mathbf{P}$ is only 25 (this is proven in [2] and can easily be verified by computer). We conclude

Theorem 5 *An 8-clique in the Ree–Tits octagon $\mathsf{O}(2)$ can contain at most 27 points. In particular, $\mathsf{O}(2)$ has no ovoids.*

It is not known whether an 8-clique of size 27 exists in $\mathsf{O}(2)$.

6 Extremal octagons

The following theorem (known as the *Krein conditions*) is a result of the general theory of association schemes.

Theorem 6 *For all* $0 \leq i,j,k \leq 4$, *we have*

$$\sum_{d=0,\ldots,8} \lambda_i(A_d) u_j(A_d) u_k(A_d) \geq 0 \tag{2}$$

with equality if and only if

$$\sum_p R_i(p,a) R_j(p,b) R_k(p,c) = 0, \tag{3}$$

for every three points a,b,c *of* $\mathcal{O}$*.(The summation is taken over all points* p *of* $\mathcal{O}$*.)*

(This is proven in [1, 2.3.2]. See also [3].)

In the case of the octagon, computing (2) for $i = j = k = 4$ yields

$$1 - \tfrac{1}{s^2}(t+1) + \tfrac{1}{s^4}t(t+1) - \tfrac{1}{s^6}t^2(t+1) + \tfrac{1}{s^8}t^3 = \tfrac{1}{s^8}(s^2 - t)(s^2 - 1)(s^4 + t^2) \geq 0.$$

Hence, $t \leq s^2$ (and dually $s \leq t^2$) for every thick octagon (this is a result of HIGMAN [4]). An octagon, which, like $\mathsf{O}(s)$, attains equality in this bound, is called *extremal.*

Similar definitions exist for generalized quadrangles and hexagons, and in those cases extremality has some important combinatorial consequences. In the case of the generalized octagon we have the following theorem.

Theorem 7 *Let* $\mathcal{O}$ *be a generalized octagon satisfying* $s^2 = t$*. If* $0 \leq i,j,k \leq 8$*, and* a,b,c *are points of* $\mathcal{O}$ *such that* $d(a,b) \leq 4$*,* $d(a,c) \leq 4$*, then the number of points* p *such that* $d(a,p) = i$*,* $d(b,p) = j$ *and* $d(c,p) = k$ *only depends on the mutual position of* a*,* b *and* c*.*

We devote the remainder of this section to the proof of this theorem in the 'most difficult' case where $d(a,b) = d(a,c) = 4$. In other cases the proof runs along similar lines and does not depend on the condition $s^2 = t$.

We first introduce some notations: consider k points $a_1,\ldots,a_k$ of $\mathcal{O}$ and let $d_1,\ldots,d_k \in \{0,2,4,6,8\}$. Let the symbol $\left[\begin{smallmatrix} a_1 & \ldots & a_k \\ d_1 & \ldots & d_k \end{smallmatrix}\right]$ denote the number of points p in $\mathcal{O}$ for which $d(a_i,p) = d_i$ for all $i = 1,\ldots,k$.

This definition immediately implies the following properties :

1. The symbol $\left[\begin{smallmatrix} a_1 & \dots & a_k \\ d_1 & \dots & d_k \end{smallmatrix}\right]$ is invariant under permutations of its columns.

2. $\sum \left[\begin{smallmatrix} a_1 & \dots & a_m & \dots & a_k \\ d_1 & \dots & d_m & \dots & d_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_1 & \dots & a_m \\ d_1 & \dots & d_m \end{smallmatrix}\right]$ where summation is carried out over all possible values $d_{m+1}, \dots, d_k \in \{0, 2, 4, 6, 8\}$.

3. $\left[\begin{smallmatrix} a_1 & a_2 & \dots & a_k \\ 0 & d_2 & \dots & d_k \end{smallmatrix}\right]$ equals 1 if and only if $d(a_1, a_i) = d_i$ for all $i, 2 \le i \le k$, and equals 0 otherwise.

The triangle inequality implies

4. If there is a pair $i, j, 1 \le i, j \le d$ such that $d(a_i, a_j) > d_i + d_j$ or $d(a_i, a_j) < |d_i - d_j|$, then $\left[\begin{smallmatrix} a_1 & \dots & a_k \\ d_1 & \dots & d_k \end{smallmatrix}\right]$ is zero.

Also, the general theory of distance regular graphs and generalized polygons in particular [1, 4] implies

5. The value of $\left[\begin{smallmatrix} a & b \\ i & j \end{smallmatrix}\right]$ depends only on i and j and on the distance $d(a, b)$.

It is our aim to establish a similar property for the symbol $\left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right]$.

In what follows we will refer to these properties by number. We will assume $s^2 = t$ for the remainder of this section.

Consider three points a, b, c of $\mathcal{O}$. We may apply the summation formula (property 2) three times to obtain

$$\sum_{i=0,\dots,8} \left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right] = \left[\begin{smallmatrix} b & c \\ j & k \end{smallmatrix}\right], \qquad \sum_{j=0,\dots,8} \left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & c \\ i & k \end{smallmatrix}\right], \qquad \sum_{k=0,\dots,8} \left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ i & j \end{smallmatrix}\right]. \tag{4}$$

Because of property 5, every 2-column symbol on the right hand side of these formulas is 'known' (i.e., can be expressed as a polynomial in s) when the distances $d(a, b)$, $d(a, c)$ and $d(b, c)$ are given. Hence, we may regard (4) as a set of linear equations in which the 3-column symbols are variables.

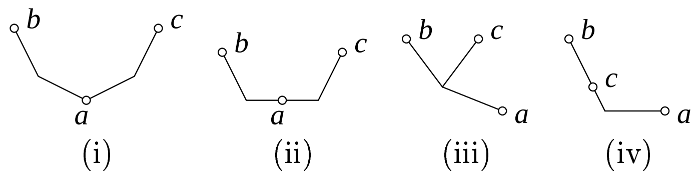

Figure 2: Configurations of points a, b, c with $d(a,b) = d(a,c) = 4$.

Unfortunately, these equations are not all independent and do not always have a unique solution.

For several configurations of a, b and c it is however possible to compute the values of some of these 3-column symbols and thus sufficiently reduce the number of unknowns. For example, property 3 gives us the value of all symbols $\left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right]$ with at least one of i, j or k equal to zero.

In what follows we shall no longer be interested in the exact value of the right hand side of these equations, only in the fact that they can always be unambiguously computed. We use the generic notation 'Cst.' to denote such a 'known' right hand side. We shall also write $[ijk]$ for $\left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right]$.

Applying property 3 as explained above, yields :

$$\sum_{i=2,4,6,8} [ijk] = \text{Cst.}, \qquad \sum_{j=2,4,6,8} [ijk] = \text{Cst.}, \qquad \sum_{k=2,4,6,8} [ijk] = \text{Cst.} \qquad (5)$$

The following lemmas may be used to further reduce the number of unknowns. We consider triples a, b, c such that $d(a,b) = d(a,c) = 4$ and we distinguish between 4 different 'configurations' as depicted in figure 2.

Lemma 5 *Consider three different points a, b, c such that $d(a,b) = d(a,c) = 4$. Then the values of $[2jk]$ only depend on s, on j, k and on the configuration of a, b and c.*

Proof : Consider all points p at distance 2 from a. If p lies on the shortest path from a to b, then $d(b,p) = 2$ (for exactly one point p) or $d(b,p) = 4$. If p does not lie on this shortest path, then $d(b,p) = 6$. A similar property can be proven for $d(c,p)$. To count the number of points p with given $d(b,p)$

and $d(c,p)$, we must distinguish between the four possible configurations of figure 2. The reader may easily verify that the following table contains the correct numbers for the different configurations. All values of $[2jk]$ not listed are understood to be zero.

(i)	$[226]=[262]=1$	$[246]=[264]=s-1$	$[266]=s(s^2-1)$
(ii)	$[224]=[242]=1$	$[244]=s-2$	$[266]=s^3$
(iii)	$[222]=1$	$[244]=s-1$	$[266]=s^3$
(iv)	$[222]=1$	$[244]=s-1$	$[266]=s^3$

∎

Lemma 6 *Consider three different points a,b,c such that $d(a,b)=d(a,c)=4$. Then the values of $[4jk]$ and of $[64m]=[6m4]$ only depend on s, on j,k,m and on the configuration of a, b and c.*

Proof : As in the proof of the previous lemma we need to distinguish between the four configurations of figure 2. Again we leave it to the reader to verify the values in the following table. All values of $[4jk]$ and $[64m]=[6m4]$ not listed are understood to be zero.

(i)	$[408]=[480]=1$ $[468]=[486]=(s-1)s^3$ $[646]=[664]=(s-1)s^2$	$[428]=[482]=s-1$ $[488]=s^4(s^2-1)$ $[648]=[684]=(s-1)^2s^2$	$[448]=[484]$ $=s(s^2-1)$
(ii)	$[406]=[460]=1$ $[466]=(s-2)s^3$ $[648]=[684]=(s-1)s^3$	$[426]=[462]=s-1$ $[488]=s^6$	$[446]=[464]$ $=s(s^2-1)$
(iii)	$[404]=1$ $[466]=(s-1)s^3$ $[646]=[664]=(s-1)s^3$	$[424]=s-1$ $[488]=s^6$	$[444]=s(s^2-2)$
(iv)	$[402]=[420]=1$ $[466]=(s-1)s^3$ $[642]=[624]=s^3$	$[422]=s-2$ $[488]=s^6$ $[644]=(s-2)s^3$	$[444]=s(s^2-1)$

∎

We now return to the specific case $d(a,b)=d(a,c)=4$ of theorem 7 which we have set out to prove. Lemmas 5 and 6 prove that every symbol $[ijk]$ with $i\le 4$, and every symbol of the form $[64m]$ or $[6m4]$ has a 'known' value. Hence $[84m]=[8m4]$ can be determined from the summation formula

$$[04m]+[24m]+[44m]+[64m]+[84m]=\text{Cst.},$$

and $[62m] = [6m2]$ from a similar summation formula, if you note that $[82m] = [8m2] = 0$ because of property 4.

We may conclude that the only 'unknown' values belong to symbols $[ijk]$ with $i, j, k \geq 6$. This allows us to reduce the system of equations (5) to the following :

$$\begin{array}{ccccc} [666] & + & [668] & = & \text{Cst.} \\ [666] & + & [686] & = & \text{Cst.} \\ [666] & + & [866] & = & \text{Cst.} \end{array} \qquad \begin{array}{ccccc} [888] & + & [886] & = & \text{Cst.} \\ [888] & + & [868] & = & \text{Cst.} \\ [888] & + & [688] & = & \text{Cst.} \end{array}$$

These equations can be solved in terms of the symbol [666] :

$$\begin{array}{ccc} [668] & = & \text{Cst.} - [666] \\ [686] & = & \text{Cst.} - [666] \\ [866] & = & \text{Cst.} - [666] \\ & & \end{array} \qquad \begin{array}{ccc} [886] & = & \text{Cst.} + [666] \\ [868] & = & \text{Cst.} + [666] \\ [688] & = & \text{Cst.} + [666] \\ [888] & = & \text{Cst.} - [666] \end{array} \tag{6}$$

The Krein conditions (theorem 7) give us one further equality. Observe that the (x, y)th entry of R_4 only depends on the distance between the points x and y. We find $R_4(x, y) = (-s)^{-d(x,y)/2}$. Hence (3) can be rewritten as

$$\sum_p (-s)^{-d(a,p)/2-d(b,p)/2-d(c,p)/2} = \sum_{i,j,k\in\{0,\ldots,8\}} (-s)^{-(i+j+k)/2}[ijk]$$

As a consequence we find

$$-\frac{1}{s^9}[666]+\frac{1}{s^{10}}([668]+[686]+[866])-\frac{1}{s^{11}}([886]+[868]+[688])+\frac{1}{s^{12}}[888] = \text{Cst.}$$

After substituting (6) and simplifying the result, we finally obtain

$$-\frac{1}{s^{12}}(s+1)^3[666] = \text{Cst.}$$

which proves that [666], and hence the value of every other symbol, is 'known'. This completes the proof of theorem 7. ■

Note : The same technique can be applied to extremal generalized *hexagons*. The resulting theorem in that case is even stronger as it requires no extra conditions on $d(a, b)$, $d(a, c)$ or $d(b, c)$.

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