Recent results on near polygons: a survey

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Abstract

The aim of this paper is to give an account of the most important results regarding near polygons. A brief survey can be found in Chapter 10 of the Handbook of Incidence Geometry ([24]), but since the publication several new results concerning these incidence structures were obtained. The author of this survey recently completed a Ph.D. on the subject ([15]).

1 Definitions

1.1 Near polygons

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ that satisfies the following property.

(NP) For every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point y on L nearest to x (w.r.t. the distance $\mathrm{d}(\cdot, \cdot)$ in the *collinearity graph* or *point graph* Γ).

If d is the diameter of Γ, then the near polygon is called a near $2d$-gon. A near 0-gon consists of only one point, a near 2-gon consists of one line with a number of points on it, and the class of the near quadrangles coincides with the class of the *generalized quadrangles* introduced by Tits in [33]. Near polygons themselves were introduced by Shult and Yanushka while studying the so-called *tetrahedrally closed systems of lines* in Euclidean spaces ([30]). A brief survey about near polygons can be found in [24]. Our aim is to complete this survey with the most important results obtained since the appearance of the Handbook (1995).

1.2 Direct product

If $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ are two partial linear spaces, then the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ and $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$. The point (x, y) is incident

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with the line $(a, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = a$ and $y \; \mathrm{I}_2 \; L$, and it is incident with the line $(M, b) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $y = b$ and $x \; \mathrm{I}_1 \; M$. We denote $\mathcal{S}$ also with $\mathcal{S}_1 \times \mathcal{S}_2$. If $\mathcal{S}_i$ ($i \in \{1, 2\}$) is a near $2d_i$-gon then the direct product $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(d_1 + d_2)$-gon. Since $\mathcal{S}_1 \times \mathcal{S}_2 \simeq \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \simeq \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$, also the direct product of $k \geq 1$ partial linear spaces $\mathcal{S}_1, \ldots, \mathcal{S}_k$ is well-defined. To finish this paragraph, let us mention the following theorem.

Theorem 1 ([8]) *Let* $\mathbf{S}$ *be a near polygon with the property that every two points at distance* 2 *have at least two common neighbours. If lines of different length occur, then* $\mathbf{S}$ *is the direct product of a number of near polygons, each of which has a constant length for the lines.*

1.3 Parallel lines

One of the following two cases occurs for any two lines L and M of a near polygon $\mathcal{S}$.

(1) There exists a unique point p on L and a unique point q on M such that $\mathrm{d}(l, m) = \mathrm{d}(l, p) + \mathrm{d}(p, q) + \mathrm{d}(q, m)$ for all points l on L and m on M.

(2) There exists an $i \in \mathbb{N}$ such that $\mathrm{d}(l, M) = \mathrm{d}(m, L) = i$ for all points l on L and m on M.

Two lines L and M are called *parallel* when they satisfy property (2). If $\mathcal{S}$ is a generalized quadrangle, then every two disjoint lines are parallel.

1.4 The near polygons under consideration here

Not all near polygons are interesting: some of them do not satisfy "nice" properties. Constructing near polygons is not very difficult. The paragraph about the direct product illustrates how to construct near polygons from other ones. Starting with a near polygon, one can add points and lines to obtain other near polygons, as described in Section 1.3 of [22]. Theorem 4 of [22] gives a way to construct many near polygons, e.g. all near polygons having a point at distance at most 2 from all the other points can be constructed by successive application of this theorem.

In the sequel, we will mainly restrict to those near polygons which satisfy at least one of the following ("nice") properties (see later sections for definitions).

(1) The near polygon is regular.

(2) Every two points at distance 2 are contained in a unique quad.

We already met near polygons that satisfy (1) and (2), namely all nondegenerate generalized quadrangles different from nonsymmetrical grids and nonsymmetrical dual grids. The generalized quadrangles form a subclass of the class of the generalized polygons which themselves are also near polygons. It is not my intention to discuss (recent) results concerning these geometries, this is done in other parts of this book. For undefined notions concerning these geometries, I also refer to these parts or to the literature ([27], [32], [35]).

2 Sub near polygons

2.1 Definitions

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near $2d$-gon. A subset $X \subseteq \mathcal{P}$ is called a *subspace* if every line $L \in \mathcal{L}$, which has at least two points in X, has all its points in X. Every subspace X induces a subgeometry $\mathcal{S}_X = (X, \mathcal{L}_X, \mathrm{I}_X)$. The set $\mathcal{L}_X$ consists of those lines of $\mathcal{L}$ which are completely contained in X and I_X is the restriction of I to the sets X and $\mathcal{L}_X$. A subspace X is called *geodetically closed* if all the points on a shortest path between two points of X are as well contained in X. Every geodetically closed subspace induces a sub near polygon. If this sub near polygon is a nondegenerate generalized quadrangle, then it is called a *quad*. Concerning the existence of quads, we have the following theorem.

Theorem 2 ([30]) *Let x and y be two points of $\mathcal{S}$ at mutual distance two. If x and y have at least two common neighbours c and d such that xc contains at least three points, then x and y are contained in a unique quad.*

Near polygons with quads satisfy some nice properties, like the one in the following theorem.

Theorem 3 ([8]) *Let $\mathcal{S}$ be a near polygon. If every two points at distance 2 are contained in a unique quad which is not a dual grid, then every point of $\mathcal{S}$ is incident with the same number of lines.*

Concerning the existence of other geodetically closed subspaces one can say the following.

Theorem 4 ([8]) *Let $\mathcal{S}$ be a near polygon that satisfies the following properties:*

(1) *every line is incident with at least* 3 *points,*

(2) *every two points at distance* 2 *have at least two common neighbours,*

then every two points at distance i ($i \in \{0, \ldots, d\}$) are contained in a unique geodetically closed sub near $2i$-gon.

For every two points u and v of $\mathcal{S}$, we define $S(u, v)$ as the set of all lines through u which contain a point at distance $\mathrm{d}(u, v) - 1$ from v. The unique geodetically closed sub near polygon through u and v (see the previous theorem) consists then of all the points w for which $S(u, w) \subseteq S(u, v)$. The geodetically closed sub near hexagons arising this way are called *hexes*. Condition (2) in Theorem 4 implies that the numerical girth of the collinearity graph Γ of $\mathcal{S}$ is equal to 4. Recently (see [26]) the existence of geodetically closed sub near polygons is proved if $\mathcal{S}$ satisfies (1) and (2') $d \geq g$.

2.2 The local space

With every point x of a near polygon $\mathcal{S}$, there is associated an incidence structure $\mathcal{S}_x$. The points, respectively lines, of $\mathcal{S}_x$ are the lines, respectively quads, through x, and incidence is the natural one. As an immediate corollary of the following theorem, $\mathcal{S}_x$ is a partial linear space.

Theorem 5 ([30]) *Let Q_1 and Q_2 be two different quads of a near polygon $\mathcal{S}$, then $Q_1 \cap Q_2$ is either empty, a point, or a line of $\mathcal{S}$.*

In most of the cases when local spaces are in consideration, the near polygon satisfies the property that every two points at distance 2 are contained in a unique quad. In this case $\mathcal{S}_x$ is even a linear space. As we will see later, it is sometimes possible to characterize near hexagons by means of their local spaces, e.g. when these local spaces are projective spaces (Theorem 10) or (h, k)-crosses, $h, k \geq 2$ (Theorem 25). (An (h, k)-*cross* is a linear space whose points are on two incident lines, one of length h and one of length k.)

At this point, we can already give the following characterization.

Theorem 6 ([17]) *Let $\mathcal{S}$ be a near hexagon satisfying the following properties:*

- *every two points at distance* 2 *are contained in a quad,*
- *if all lines of $\mathcal{S}$ are thin, then all quads are symmetrical dual grids,*

- *there exists a point x of* $\mathbf{S}$ *such that* $\mathbf{S}_x$ *is a* $(2,r)$*-cross for some* $r \in \mathbb{N} \setminus \{0,1\}$,

then $\mathbf{S}$ *is the direct product of a line with a nondegenerate generalized quadrangle.*

2.3 The relation between a point and a quad

The following theorem describes the possible relations between a point and a quad.

Theorem 7 ([29] and [30]) *Exactly one of the following cases occurs for every point-quad pair* (p, Q) *of a near polygon.*

(1) *There is a unique point* $\pi(p) \in Q$ *such that* $d(p, r) = d(p, \pi(p)) + d(\pi(p), r)$ *for all points* $r \in Q$*. In this case* (p, Q) *is called* classical.

(2) *The points of* Q *which are nearest to* p *form an ovoid of* Q*. In this case* (p, Q) *is called* ovoidal.

(3) *The quad* Q *induces a dual grid. Let* A *be the set of points of* Q *at smallest distance* k *from* p*. Let* B*, respectively* C*, denote those points of* Q*, that have distance* $k+1$*, respectively* $k+2$ *to* p*. Then*

   (*a*) $|A| \geq 2$ *and* $|C| \geq 1$,

   (*b*) B *and* $A \cup C$ *are the two maximal cocliques of the point graph of* Q.

   In this case (p, Q) *is called* thin ovoidal.

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon such that

(1) every line is incident with at least three points,

(2) every two points at distance 2 have at least two common neighbours.

Every two points at distance 2 are contained in a unique quad and only possibilities (1) and (2) of the previous theorem can occur. For a fixed quad Q, we define the following sets:

$$
\begin{aligned}
N_i &:= \{x \in \mathcal{P} | \, \mathrm{d}(x, Q) = i\}, \\
N_{i,C} &:= \{x \in N_i | x \text{ is classical with respect to } Q\}, \\
N_{i,O} &:= \{x \in N_i | x \text{ is ovoidal with respect to } Q\}.
\end{aligned}
$$

Clearly $N_0 = Q$, $N_{1,O} = \emptyset$, $N_{d-1,C} = \emptyset$ and $N_d = \emptyset$.

Theorem 8 ([8]) (1) *If L is a line contained in $N_i \cup N_{i+1}$ for some i, then $|L \cap N_i| = 1$.*

(2) *There are no edges between $N_{i,O}$ and $N_{i,C}$ for all i.*

(3) *If L is a line contained in $N_{i,C}$ for some i, then $\pi(L)$ is a line of Q parallel to L.*

(4) *If L is a line contained in $N_{i,O}$ for some i, then the points of L determine a fan of ovoids in Q.*

(5) *If L is contained in $N_{i,O} \cup N_{i+1}$ for some i, then L is even contained in $N_{i,O} \cup N_{i+1,O}$. In this case all points of L determine the same ovoid of Q.*

(6) *If L is contained in $N_{i,C} \cup N_{i+1,C}$ for some i, then all points of L determine the same point in Q.*

(7) *If L is contained in $N_{i,C} \cup N_{i+1,O}$ for some i, then the points of $L \cap N_{i+1,O}$ determine a rosette of ovoids in Q. The common point of all these ovoids is the point of Q determined by $L \cap N_{i,C}$.*

Theorems 7 and 8 are extremely helpful to derive classification results, e.g. the classification results concerning near hexagons with three or four points on every line, and quads through every two points at distance 2 (see Chapters 8 and 9).

3 Classical near polygons

A near polygon $\mathcal{S}$ is called *classical* if the following properties are satisfied.

(1) Every two points at distance two are contained in a unique quad.

(2) Every point-quad relation is classical.

One can easily verify that a near hexagon is classical if and only if all local spaces are (possible degenerate) projective planes. *Dual polar spaces* are defined as follows: the points are the maximal subspaces of a polar space, the lines are the next-to-maximal subspaces and incidence is reverse containment. By Cameron ([11]) the class of the classical near polygons coincides with the class of the dual polar spaces. One can prove that every geodetically closed subspace of a classical near polygon induces again a classical near polygon. The direct product of two classical near polygons is again a classical near polygon. The *Hamming near polygons* form a class of classical near polygons obtained by taking the direct product of $k(\geq 0)$ lines. The Hamming near polygons can be characterized in the following way.

Theorem 9 ([22]) *If $\mathcal{S}$ is a near polygon, then the following conditions are equivalent.*

(1) *$\mathcal{S}$ is a near polygon of Hamming type.*

(2) *Parallelism is an equivalence relation and every two points at mutual distance 2 have at least two common neighbours.*

(3) *For every point x and every line L, there is a unique line through x which is parallel to L.*

Classical near polygons can be characterized in the following way.

Theorem 10 ([5]) *Let $\mathcal{S}$ be a near polygon satisfying the following properties:*

(1) *every line is incident with at least three points,*

(2) *every two points at distance 2 have at least two common neighbours,*

(3) *all hexes are classical,*

then $\mathcal{S}$ itself is also classical.

As mentioned before, every dual polar space is associated to a polar space. We use the following notations ($n \geq 2$):

(A) $W^D(2n-1,q)$ for the dual polar space related to a symplectic polarity in $\mathrm{PG}(2n-1,q)$,

(B) $Q^D(2n,q)$ for the dual polar space related to a nonsingular quadric in $\mathrm{PG}(2n,q)$,

(C) $[Q^-(2n+1,q)]^D$ for the dual polar space related to nonsingular elliptic quadric in $\mathrm{PG}(2n+1,q)$,

(D) $[Q^+(2n-1,q)]^D$ for the dual polar space related to nonsingular hyperbolic quadric in $\mathrm{PG}(2n-1,q)$,

(E) $H^D(2n,q)$ for the dual polar space related to a nonsingular Hermitian variety in $\mathrm{PG}(2n,q)$,

(F) $H^D(2n-1,q)$ for the dual polar space related to a nonsingular Hermitian variety in $\mathrm{PG}(2n+1,q)$.

For further discussions on dual polar spaces regarded as near polygons, we refer to Section 3.6 of [15] which also contains the relevant references to the literature (e.g. [10] and [34]).

4 Regular near polygons

A near $2d$-gon $\mathcal{S}$ is *regular* if and only if there exist constants s, t_i ($i \in \{0,\ldots,d\}$) such that

(1) every line is incident with $s+1$ points,

(2) if x and y are two points at distance i, then there are exactly t_i+1 points collinear with x and at distance $i-1$ from y.

Clearly $t_0=-1$, $t_1=0$ and every point is incident with $t+1 := t_d+1$ lines. As a consequence $\mathcal{S}$ has order (s,t). A near polygon is regular if and only if its point graph is a distance regular graph ([7]). About regular near polygons which are also classical, one can say the following.

Theorem 11 ([8]) *A near polygon with parameters $s \geq 1$, t_i ($i \in \{0,\ldots,d\}$) is classical if and only if $t_{i+1} = t_2(t_i+1)$ for all $i \in \{0,\ldots,d-1\}$.*

The paper [29] contains several classification results concerning regular near polygons. Let us only mention those results concerning regular near hexagons with 3 or 4 points on every line.

Theorem 12 ([29]) *Let $\mathcal{S}$ be a near hexagon with parameters $s=2$, t_2 and t, then we have the following possibilities for (t_2,t):*

(1) $(t_2,t)=(0,1)$,	(2) $(t_2,t)=(0,2)$,
(3) $(t_2,t)=(0,8)$,	(4) $(t_2,t)=(1,2)$,
(5) $(t_2,t)=(1,11)$,	(6) $(t_2,t)=(2,6)$,
(7) $(t_2,t)=(2,14)$,	(8) $(t_2,t)=(4,20)$.

For every of these parameters, there is a unique near hexagon, except for case (2) where there is up to duality only one (For the uniqueness of cases (5) and (7), see [3] and [4] respectively). The near hexagons with parameters as in (1), (2) or (3) are generalized hexagons. The near hexagons with parameters as in (4), (6) or (8) satisfy $t=t_2(t_2+1)$ and hence they are dual polar spaces. The near hexagon with parameters $(s,t_2,t)=(2,1,11)$ can be constructed from the *extended ternary Golay code*, but it is also isomorphic to $T_5^*(\mathcal{K})$ with $\mathcal{K}$ a Coxeter cap in PG$(5,3)$, see section 5.2. Finally, the near hexagon with parameters $(s,t_2,t)=(2,2,14)$ is the one related to the Steiner system $S(5,8,24)$: the points are the blocks of $S(5,8,24)$, the lines are all the sets of three blocks which are two by two disjoint, and incidence is the natural one.

Theorem 13 ([29]) *Let $\mathcal{S}$ be a near hexagon with parameters $s=3$, t_2 and t. Then we have one of the following possibilities for (t_2,t):*

(1) $(t_2,t)=(0,1)$,	(2) $(t_2,t)=(0,3)$,
(3) $(t_2,t)=(0,27)$,	(4) $(t_2,t)=(1,2)$,
(5) $(t_2,t)=(1,9)$,	(6) $(t_2,t)=(1,34)$,
(7) $(t_2,t)=(3,12)$,	(8) $(t_2,t)=(3,27)$,
(9) $(t_2,t)=(3,48)$,	(10) $(t_2,t)=(9,90)$.

Near polygons with parameters as in (1), (2) or (3) are generalized hexagons. Except for case (1) (unique example), it is not known whether they are uniquely determined by their parameters. Near hexagons with parameters as in (4), (7) or (10) are dual polar spaces (since $t=t_2(t_2+1)$). There is no near hexagon with parameters as in (5), see [2]. Whether there exists a near polygon with parameters as in (6) is still an open problem. It is also known that there exists no near hexagon with parameters as in (8) or (9).

For further restrictions on the parameters of regular near polygons, we refer to [7] and [8]. In the case of regular near hexagons, one of the Krein conditions yields the following inequality, also known as *Mathon's bound*.

Theorem 14 ([25]) *If $\mathcal{S}$ is a regular near hexagon with parameters $s>1$, t_2 and t, then $t\le s^3+t_2(s^2-s+1)$.*

We also have the following interesting characterization.

Theorem 15 ([8]) *Let $\mathcal{S}$ be a regular near hexagon with parameters s, t_2 and t. Suppose that $s>1$ and $t_2>0$. Then $\mathcal{S}$ is related to $S(5,8,24)$ if and only if $t+1=(t_2+1)(st_2+1)$.*

There is also a regular near octagon that is constructed from the sporadic simple Hall-Janko group of order 604800 ([13]). Let V_0 be the set of all 315 involutions, whose centralizer contains Sylow 2-subgroups, i.e. groups of order 128. Let Γ be the graph with vertex set V_0, two involutions being adjacent when they commute, or, equivalently, when their product is again an element of V_0. The graph Γ is then the point graph of a regular near octagon (the lines correspond with the maximal cliques). The parameters are $s = 2$, $t_2 = 0$, $t_3 = 3$ and $t = 4$ and the near octagon is uniquely determined by its parameters, see [14]. This near octagon has (not geodetically closed) sub near polygons isomorphic to generalized hexagons of type $G_2(2)$.

5 Projective and affine embeddings of near polygons

In this section, we only discuss full embeddings of near polygons. A near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is *(fully) embeddable* in $\mathcal{D} \in \{\mathrm{PG}(n, q), \mathrm{AG}(n, q)\}$ if there exists an injection θ from $\mathcal{P}$ to the point set of $\mathcal{D}$ such that lines of $\mathcal{S}$ are mapped to full lines of $\mathcal{D}$. We may assume that $\mathcal{P}^\theta$ generates $\mathcal{D}$. All affine and projective embeddings of generalized quadrangles were determined, see [9] and [31]. We consider now two types of embeddings. The first one is a special type of projective embedding.

5.1 Flat embeddings

Definition. For every point x of a near $2d$-gon $\mathcal{S}$ and for every $i \in \{0, \ldots, d\}$, we define $W_i(x)$ as the set of points at distance at most i from x.

The embedding of the near $2d$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ in $\mathrm{PG}(n, q)$ defined by the map θ is called *flat* whenever the following properties are satisfied.

(1) For every point $x \in \mathcal{P}$, $[W_1(x)]^\theta$ is a subspace of $\mathrm{PG}(n, q)$.

(2) For every point $x \in \mathcal{P}$ and for every $i \in \{0, \ldots, d\}$, there exists a subspace $U_i(x)$ of $\mathrm{PG}(n, q)$ such that $[W_i(x)]^\theta = \mathcal{P}^\theta \cap U_i(x)$.

Theorem 16 ([12]) *Let $\mathcal{S}$ be a near polygon which satisfies the following properties:*

(a) *every two points at distance 2 have at least two common neighbours,*

(b) *$\mathcal{S}$ has a flat embedding in a projective space over $\mathrm{GF}(q)$,*

then $\mathcal{S} \simeq Q^D(2n, q)$ for some $n \in \mathbb{N}$.

Theorem 17 ([12]) *If $\mathcal{S}$ is a regular near hexagon with a flat embedding in a projective space over $\mathrm{GF}(q)$, then $\mathcal{S}$ is either isomorphic to $Q^D(6, q)$ or to the generalized hexagon of type $G_2(q)$.*

5.2 Linear representations

A *linear representation* is a special type of affine embedding. Let Π_∞ be a $\mathrm{PG}(n,q)$, $n \geq 0$, which is embedded as a hyperplane in $\Pi = \mathrm{PG}(n+1,q)$, and let $\mathcal{K}$ be a nonempty set of points of Π_∞. The linear representation $T_n^*(\mathcal{K})$ is then the geometry with point set $\Pi \setminus \Pi_\infty$, with lines all the affine lines of Π through a point of $\mathcal{K}$, and with incidence the one derived from Π.

Theorem 18 *(A) Consider in Π_∞ two disjoint subspaces π_1 and π_2 of dimensions $n_1 \geq 0$ and $n_2 \geq 0$ respectively, such that $\Pi_\infty = \langle \pi_1, \pi_2 \rangle$. Let $\mathcal{K}_i$, $i \in \{1,2\}$, be a set of points in Π_i and put $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. If $T_{n_i}^*(\mathcal{K}_i)$, $i \in \{1,2\}$, is a near $2d_i$-gon, then $T_n^*(\mathcal{K})$ is a near $2(d_1+d_2)$-gon.*

(B) Consider in Π_∞ two subspaces π_1 and π_2 of dimensions $n_1 \geq 0$ and $n_2 \geq 0$ respectively, such that $\pi_1 \cap \pi_2 = \{p\}$ and $\Pi_\infty = \langle \pi_1, \pi_2 \rangle$. Let $\mathcal{K}_i$, $i \in \{1,2\}$, be a set of points in Π_i containing p and put $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. If $T_{n_i}^*(\mathcal{K}_i)$, $i \in \{1,2\}$, is a near $2d_i$-gon, then $T_n^*(\mathcal{K})$ is a near $2(d_1+d_2-1)$-gon.

Definition. A nonempty set of points $\mathcal{K}$ in $\mathrm{PG}(n,q)$ is called *decomposable* if it can be written as $\mathcal{K}_1 \cup \mathcal{K}_2$ with $\mathcal{K}_1$ and $\mathcal{K}_2$ as in (A) or (B) of the previous theorem. Otherwise it is called *indecomposable.*

If $q \geq 3$, then the following examples of indecomposable sets $\mathcal{K}$, for which $T_n^*(\mathcal{K})$ is a near polygon, are known.

(1) $n = 0$ and $\mathcal{K}$ is a singleton in $\mathrm{PG}(0,q)$,

(2) $n = 2$, $q = 2^h$, and $\mathcal{K}$ is a hyperoval in $\mathrm{PG}(2,2^h)$,

(3) $n = 5$, $q = 3$, and $\mathcal{K}$ is a *Coxeter cap* in $\mathrm{PG}(5,3)$, i.e. a set of points projectively equivalent to the set of twelve points determined by the columns of the following matrix over $\mathrm{GF}(3)$:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}.$$

Theorem 19 ([23]) *Let the near $2d$-gon $\mathcal{S}$ be isomorphic to $T_n^*(\mathcal{K})$, where $\mathcal{K}$ is a nonempty set of points in $\mathrm{PG}(n,q)$, $q \geq 3$.*

(1) *If* $d = 1$, *then* $\mathcal{K}$ *is a singleton in* $\mathrm{PG}(0, q)$.

(2) *If* $d = 2$ *and* $\mathcal{K}$ *is indecomposable, then* $\mathcal{K}$ *is a hyperoval in* $\mathrm{PG}(2, 2^h)$.

(3) *Suppose* $d = 3$. *If* $n \leq 6$ *and* $\mathcal{K}$ *is indecomposable, then* $\mathcal{K}$ *is the Coxeter cap in* $\mathrm{PG}(5, 3)$. *If* $n \geq 7$, *then*

   (*i*) $\mathcal{K}$ *is indecomposable,*

   (*ii*) $\mathcal{S}$ *is not regular,*

   (*iii*) $q \geq 2^h$ *with* $h \geq 4$.

Theorem 20 ([18]) *Let the near* $2d$*-gon* $\mathcal{S}$ *be isomorphic to* $T_n^*(\mathcal{K})$, *where* $\mathcal{K}$ *is a nonempty set of points in* $\mathrm{PG}(n, q)$, $q \geq 3$, *then every geodetically closed sub near* 2δ*-gon,* $\delta \neq 0$, *has also a linear representation.*

There is also a connection between near polygons and certain sets of points in projective spaces. For every $m \in \mathbb{N}\backslash\{0\}$, put $f_m(x, y) = \sum_{j=1}^{m} \binom{x-1}{j} (y-1)^{j-1}$.

Theorem 21 ([18]) *Let* V *be a set of* $k > 0$ *points of* $\mathrm{PG}(n, q)$ *with the property that no* $l = 2m + 1$ *of them are contained in an* $(l - 2)$*-flat* ($n \geq l - 2 \geq 0$), *then* $f_m(k, q) \leq \frac{q^n - 1}{q - 1}$ *and equality holds if and only if* $T_n^*(V)$ *is a near* $2(m + 1)$*-gon.*

There is also a similar property if $l = 2m$ but the characterization happens in terms of so-called near $(2m+1)$-gons. In [18] all sets V are given for which the equality in the previous theorem occurs.

6 Admissible triples

6.1 Definition, examples and characterizations

In [16] a common construction for several classes of generalized quadrangles was given. The construction makes use of the so-called admissible triples. A triple $(\mathcal{D}, K, \Delta)$ is called *admissible* if the following conditions are satisfied.

(1) $\mathcal{D}$ is a linear space of order $(s, t-1)$ with s and t some nonzero positive integers. Let $\mathcal{P}$ denote the point set.

(2) K is a group of order $s + 1$ (multiplicative notation).

(3) Δ is a map from $\mathcal{P} \times \mathcal{P}$ to K such that x, y and z are collinear if and only if $\Delta(x, y)\, \Delta(y, z) = \Delta(x, z)$.

Let Γ be the graph on the vertex set $K \times \mathcal{P}$; two vertices (k_1, x) and (k_2, y) are adjacent if and only if (i) $x = y$ and $k_1 \neq k_2$, or (ii) $x \neq y$ and $k_2 = k_1\,\Delta(x, y)$. It can be proved that Γ is the point graph of a (necessarily unique) generalized quadrangle. The following generalized quadrangles can be constructed in this way.

(1) The $(s+1) \times (s+1)$-grid. Here $\mathcal{D}$ is the line of length $s+1$ and K is an arbitrary group of order $s+1$. Put $\Delta(x, y)$ equal to the identity for all points x and y of $\mathcal{D}$.

(2) The dual GQ of the $(t+1) \times (t+1)$-grid. Here $\mathcal{D}$ is the complete graph on $t+1$ vertices and K is the group of order 2. Let $\Delta(x, y)$ be equal to the identity if and only if $x = y$.

(3) The generalized quadrangle $P(W(q), x)$ ([27]). Here $\mathcal{D} = \mathrm{AG}(2, q)$ and K is the additive group of the field $\mathrm{GF}(q)$. For any two points $r_1 = (x_1, y_1)$ and $r_2 = (x_2, y_2)$ of $\mathcal{D}$ we put $\Delta(r_1, r_2) = x_1 y_2 - x_2 y_1$.

(4) The generalized quadrangle $Q(5, q)$. Consider a nonsingular nondegenerate Hermitian form $(\cdot, \cdot)$ in $V(3, q^2)$ and let U be the corresponding unital in $\mathrm{PG}(2, q^2)$. With this unital there is associated the following linear space $\mathcal{D}$. The points of $\mathcal{D}$ are the points of U and the lines of $\mathcal{D}$ are all the sets of order $q+1$ arising as an intersection of U with lines of the projective plane. Put $K = \{x \in \mathrm{GF}(q^2) | x^{q+1} = 1\}$. Let $\alpha = \langle \bar{a} \rangle$ be a fixed point of U. For every two points $\beta = \langle \bar{b} \rangle$ and $\gamma = \langle \bar{c} \rangle$ of U, we define

$$\begin{aligned} \Delta(\beta, \gamma) &= -(\bar{a}, \bar{b})^{q-1} (\bar{b}, \bar{c})^{q-1} (\bar{c}, \bar{a})^{q-1} \text{ if } \alpha \neq \beta \neq \gamma \neq \alpha, \\ &= 1 \text{ otherwise.} \end{aligned}$$

In the examples (5) and (6) $\mathcal{D}$ is the Desarguesian affine plane $\mathrm{AG}(2, q)$ and K is the additive group of $\mathrm{GF}(q)$. In the definition for Δ a function $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ appears which is supposed to satisfy one of the following two equivalent statements.

(I) the set $\mathcal{H} := \{(1, 0, 0), (0, 1, 0)\} \cup \{(f(\lambda), \lambda, 1) | \lambda \in \mathrm{GF}(q)\}$ is a hyperoval in $\mathrm{PG}(2, q)$ (hence q is even),

(II) $$\begin{vmatrix} f(\lambda_1) & \lambda_1 & 1 \\ f(\lambda_2) & \lambda_2 & 1 \\ f(\lambda_3) & \lambda_3 & 1 \end{vmatrix} \neq 0 \Leftrightarrow \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

(5) We put $\Delta((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = (\alpha_1 - \alpha_2) f\left(\frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}\right)$ if $\alpha_1 \neq \alpha_2$ and 0 otherwise. The generalized quadrangle arising is $T_2^*(\mathcal{H})$.

(6) We put $\Delta((\alpha_1,\beta_1),(\alpha_2,\beta_2)) = (f(\alpha_1)-f(\alpha_2))\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}$ if $\alpha_1 \neq \alpha_2$ and 0 otherwise. The generalized quadrangle arising is of type $(S_{xy}^{-})^D$. These GQ's were defined in [28] using a hyperoval O and two points x and y in O. In this case we have $O = H$, $x = (1,0,0)$ and $y = (0,1,0)$.

With every admissible triple $(\mathcal{D}, K, \Delta)$ there is associated a generalized quadrangle $\mathcal{Q}$. The set $S = \{L_x | x \in \mathcal{P}\}$, with $L_x = \{(k,x)|k \in K\}$, is a spread of $\mathcal{Q}$, and it is called the *associated spread* of the admissible triple. The spread S is a spread of symmetry, i.e. a spread for which there exist at least $s+1$ automorphisms fixing each line of S.

Definition. Let S be a spread of a GQ(s,t). For every two lines $M, N \in S$ and every point x of M, let $[M,N](x)$ denote the unique point of N nearest to x. The set of all permutations $[M_\alpha, N_\alpha] \circ \cdots \circ [M_1, N_1]$ ($\alpha \geq 1$, $M_i, N_i \in S$ for all $i \in \{1,\ldots,\alpha\}$, $M_1 = N_\alpha = L$), equipped with the composition of functions as group operation, is called the *group of projectivities of L with respect to S.*

We have the following characterizations.

Theorem 22 ([16]) *Let $\mathcal{Q}$ be a generalized quadrangle of order (s,t).*

(1) *$\mathcal{Q}$ has a spread of symmetry S if and only if $\mathcal{Q}$ can be derived from an admissible triple with S as associated spread.*

(2) *$\mathcal{Q}$ has a spread of symmetry S if and only if the group of projectivities of a line $L \in S$ with respect to S has order at most $s+1$.*

(3) *If the group of projectivities of a line $L \in S$ with respect to S is commutative, then $\mathcal{Q}$ can be derived from an admissible triple.*

For more information about admissible triples and spreads of symmetry, we refer to [15] and [16].

6.2 Equivalent admissible triples

In the previous section, we gave 6 classes of admissible triples. There are a lot of other examples known, but every known one is "equivalent" to an example given above. Also, AT's from different classes can still be "equivalent". When this situation precisely occurs, we refer to Theorem 23 and the different theorems concerning isomorphisms between members of different classes of GQ's ([27]). Let us now state what we precisely mean with *equivalence* ($\sim$).

Definition. Let $T_1 = (\mathcal{L}_1, K_1, \Delta_1)$ and $T_2 = (\mathcal{L}_2, K_2, \Delta_2)$ be two admissible triples.

- If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two lines of the same length, then $T_1 \sim T_2$.
- In the other case T_1 and T_2 are called equivalent if there exist

  (A) an isomorphism from $\mathcal{L}_1$ to $\mathcal{L}_2$ determined by $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$,

  (B) an isomorphism β from K_1 to K_2,

  (C) a map γ from $\mathcal{P}_1$ to K_1.

  such that
  $$\Delta_2(\alpha(x), \alpha(y)) = \beta(\gamma^{-1}(x)\,\Delta_1(x,y)\,\gamma(y))$$
  holds for all $x, y \in \mathcal{P}_1$.

Theorem 23 *Let T_1 and T_2 be two admissible triples. Let Q_i, $i \in \{1,2\}$, respectively S_i, be the* GQ, *respectively spread, associated with T_i. Then $T_1 \sim T_2$ if and only if there exists an isomorphism from Q_1 to Q_2 mapping S_1 to S_2.*

7 Glued near polygons

Let k be a nonzero integer. For every $i \in \{1, \dots, k\}$ consider the following objects:

(A) a near polygon $\mathcal{A}_i$;

(B) a spread $S_i = \{L_1^{(i)}, \dots, L_{n_i}^{(i)}\}$ of $\mathcal{A}_i$, consisting of lines which are two by two parallel;

(C) a bijection $\theta_i : L_1^{(1)} \mapsto L_1^{(i)}$.

Conditions (B) and (C) imply that all lines $L_j^{(i)}$, $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, have the same length $s+1$. If x is a point of $\mathcal{A}_l$ and $L_m^{(l)} \in S_l$, then $p_m^{(l)}(x)$ denotes the unique point of $L_m^{(l)}$ nearest to x. The following graph Γ can now be defined. The vertices of Γ are the elements of $L_1^{(1)} \times S_1 \times \ldots \times S_k$. Two vertices $(x, L_{i_1}^{(1)}, \dots, L_{i_k}^{(k)})$ and $(y, L_{j_1}^{(1)}, \dots, L_{j_k}^{(k)})$ are adjacent if and only if

(I) there exists an $l \in \{1, \dots, k\}$ such that $i_m = j_m$ for all $m \in \{1, \dots, k\} \setminus \{l\}$, and

(II) for every l like in (I), $p_{i_l}^{(l)} \circ \theta_l(x)$ and $p_{j_l}^{(l)} \circ \theta_l(y)$ are collinear points in $\mathcal{A}_l$.

The following incidence structure $\mathcal{S}$ can then be defined: the points of $\mathcal{S}$ are the vertices of Γ, and the lines of $\mathcal{S}$ are the maximal cliques of Γ.

Theorem 24 *The incidence structure $\mathcal{S}$ is a near polygon if and only if the permutations $\theta_i^{-1} \circ p_1^{(i)} \circ p_\alpha^{(i)} \circ p_\beta^{(i)} \circ \theta_i$ and $\theta_j^{-1} \circ p_1^{(j)} \circ p_\gamma^{(j)} \circ p_\delta^{(j)} \circ \theta_j$ commute for all possible α, β, γ, δ, i and j with $i \neq j$.*

Definition. If $\mathcal{S}$ is a near polygon, then it is called a *glued near polygon*.

Consider again the geometry $T_n^*(\mathcal{K})$ as defined in section 5.2. For every point $x \in \mathcal{K}$, the set of all affine lines through x defines a spread S_x of $T_n^*(\mathcal{K})$. Every two lines of S_x are parallel. The near polygons with a linear representation and these "natural" spreads are possible candidates for the construction. It can be proved that every near polygon $T_n^*(\mathcal{K})$, with $\mathcal{K}$ a decomposable set of points in $\mathrm{PG}(n, q)$, is glued.

Near polygons arising from generalized quadrangles are studied in detail in [16], [17], [19] and [21]. Assuming $\mathcal{A}_i$, $1 \leq i \leq k$, is a GQ of order (s, t_i), we have the following conditions for every $i \in \{1, \ldots, k\}$:

(A) $s = 1$, $t_i = 1$ or $s + 2 \leq t_i \leq s^2$,

(B) $s + 1 \mid t_i(t_i - 1)$,

(C) $s + t_i \mid s(s + 1)(t_i + 1)$,

(D) if $k \geq 3$, then the group of automorphisms of $\mathcal{A}_i$ fixing each line of S_i is either commutative or isomorphic to the symmetric group S_{s+1} ($\mathcal{A}_i$ must be a grid in the latter case).

In [17] estimates are given for the number of glued near hexagons which are derivable from two fixed GQ's $\mathcal{A}_1$ and $\mathcal{A}_2$. The paper [16] contains the following characterization of glued near hexagons.

Theorem 25 *Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near hexagon satisfying the following properties:*

- *every two points at distance 2 are contained in a quad,*
- *if all lines of $\mathbf{S}$ are thin, then all quads are good,*
- *there exists a point x such that $\mathbf{S}_y$ is an (h_y, k_y)-cross for all $y \in \Gamma(x)$,*

then $\mathbf{S}$ is the direct product of a line with a nondegenerate GQ or $\mathbf{S}$ is a glued near hexagon.

8 Near polygons with 3 points on every line

Let $\mathcal{S}$ be a near $2d$-gon that satisfies the following properties:

(A) every line is incident with exactly 3 points,

(B) every two points at distance 2 have at least 2 common neighbours.

For $d \leq 1$, we only have trivial examples. For $d = 2$, we have 3 examples, namely the (3×3)-grid, and the GQ's $W(2)$ and $Q(4, 2)$. If $d = 3$ all examples were determined in [6]. There are 11 examples and all of them are finite. This latter property also holds if $d \geq 4$ (one can use similar arguments as in Proposition 7.1 of [6]). We discuss now the known examples which are not classical, not regular and not glued. For the other examples, we refer to the previous sections. We have two infinite classes and one sporadic example.

(I) The following infinite class was constructed in [8]. Let V be a set of order $2n$. The following near $2(n-1)$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ can then be constructed:

- $\mathcal{P}$ is the set of all partitions of V in n sets of order 2;
- $\mathcal{L}$ is the set of all partitions of V in $n-2$ sets of order 2 and 1 set of order 4;
- a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition determined by p is a refinement of the partition determined by L.

(II) The following infinite class was constructed by B. De Bruyn. The smallest examples ($n = 2$ and $n = 3$) were already known to exist, although in an other description, see [6]. Let $H(2n-1, 4)$, $n \geq 2$, be the nonsingular hermitian variety in $\mathrm{PG}(2n-1, 4)$ determined by the equation $X_0^3 + X_1^3 + \ldots + X_{2n-1}^3 = 0$. A point of $\mathrm{PG}(2n-1, 4)$ belongs then to $H(2n-1, 4)$ if and only its weight is even. The following incidence structure $\mathcal{S}$ is then a near polygon.

- The points of $\mathcal{S}$ are those generators of $H(2n-1, 4)$ which are generated by n points with weight 2.
- The lines are the $(n-2)$-dimensional subspaces of $H(2n-1, 4)$ which are contained in at least two points of $\mathcal{S}$.
- Incidence is reverse containment.

$\mathcal{S}$ is a sub near polygon of the dual polar space $H^D(2n-1, 4)$ but it is not geodetically closed for $n \geq 3$.

(III) This example was first constructed by Aschbacher in [1]. We give the description taken from [6]. Let the 6-dimensional vector space $V(6,3)$ be equipped with a nonsingular quadratic form of Witt-index 2. Let N be the set of 126 projective points of norm 1. The points and lines of the near hexagon are the 6-tuples and pairs, respectively, of mutually orthogonal points in N, with inclusion as incidence.

9 Near hexagons with 4 points on every line

Let $\mathcal{S}$ be a near hexagon that satisfies the following properties:

(A) every line is incident with exactly 4 points,

(B) every two points at distance 2 have at least 2 common neighbours.

By Theorem 2 every two points at distance 2 are contained in a unique quad which must necessarily be one of the five GQ's of order (3,t): (1) $L \times L$ with L a line of length 4, (2) $W(3)$, (3) $Q(4,3)$, (4) $T_2^*(H)$ with H the, up to projective equivalence, unique hyperoval in PG$(4,4)$, or (5) $Q(5,3)$. There are 10 examples known of near hexagons that satisfy (A) and (B). Besides the classical examples (I) $L \times L \times L$, (II) $L \times W(3)$, (III) $L \times Q(4,3)$, (IV) $L \times T_2^*(H)$, (V) $L \times Q(5,3)$, (VI) $W^D(5,3)$, (VII) $Q^D(6,3)$, (VIII) $H^D(5,9)$, two other examples are known.

(IX) We have a glued near hexagon obtained by glueing two GQ's isomorphic to $T_2^*(H)$; this near hexagon is also isomorphic to $T_4^*(K)$ where K is the union of two hyperovals in PG$(2,4)$ whose carrying planes meet each other in a point that belongs to both hyperovals.

(X) We have a glued near hexagon obtained by glueing two GQ's of type $Q(5,3)$. For a more explicit construction, we refer to [20].

The following classification result is known.

Theorem 26 ([20]) *Let* **S** *be a near hexagon satisfying the following properties:*

(1) *all lines of* **S** *have* 4 *points;*

(2) *every two points at distance* 2 *have at least two common neighbours.*

We distinguish between two cases.

(A) *If* **S** *is classical or glued, then it is isomorphic to one of the ten examples* (I)*-*(X)*.*

(B) *If* **S** *is not classical and not glued, then only quads isomorphic to the* (4×4)*-grid or to* $Q(4,3)$ *occur. Moreover, there are numbers* a *and* b *such that every point of* **S** *is contained in* a *grids and* b *quads isomorphic to* $Q(4,3)$*. Every point is contained in the same number of lines, say* $t+1$ *lines. We have then the following possibilities for* t*,* a*,* b *and* v *(= the number of points):*

- $v = 5848$*,* $t = 19$*,* $a = 160$*,* $b = 5$*;*
- $v = 6736$*,* $t = 21$*,* $a = 171$*,* $b = 10$*;*
- $v = 8320$*,* $t = 27$*,* $a = 120$*,* $b = 43$*;*
- $v = 20608$*,* $t = 34$*,* $a = 595$*,* $b = 0$*.*

It is still an open problem whether there exist near hexagons with parameters as in (B) of the previous theorem.

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