Flock generalized quadrangles and related structures: an update

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Abstract

Novel approaches are given to the construction and study of flock quadrangles, i.e., q-clan geometries. Since 1997 three new infinite families have been discovered, one for $q \equiv \pm 1 \pmod{10}$ ([57]), one for $q = 3^e$, ([34]), and one for $q = 4^h$ ([17]). For q a power of 2, C. M. O'Keefe and T. Penttila ([36]) have given an interesting new "magic" action of $\mathrm{P\Gamma L}(2, q)$ on the herds of ovals associated with flock GQ. This action complements that induced by the Fundamental Theorem for q-clan geometries ([49]). With computer aided searches, many sporadic examples with q odd have been found. At the present time at least 38 different (currently sporadic) q-clans have been discovered, many producing several new flocks. (Note: All the most recently discovered constructions have been found with the collaboration of T. Penttila.) New characterization theorems for flock generalized quadrangles are given. Most importantly, J. A. Thas (cf. especially [68]) has shown that a GQ with parameters (q^2, q), q odd, is a flock GQ if and only if it has Property (G) at some point, indeed, if and only if it has Property (G) at some flag. For q even there is a similar result, but the hypotheses need to be a little stronger at present. J. A. Thas (cf. [69] and [68]) has given a geometric construction of all flock quadrangles and for q odd has given a direct isomorphism from his new model to that constructed geometrically by N. Knarr [31] (cf. [70]). Finally, J. A. Thas has made great progress towards determining all translation GQ with parameters (q, q^2) (cf. [66] and [68]).

Several other authors have made contributions to our understanding of q-clan geometries, notably in the search for monomial flocks, in the study of spreads and ovoids, and in the study of subquadrangles. Some of this work is surveyed briefly.

1 Prolegomena

Let K be any field. Let $\mathcal{C}$ be the cone $\mathcal{C} = \{(X_0, X_1, X_2, X_3) \in \mathrm{PG}(3, K) : X_2^2 = X_0X_1\}$ with vertex $P = (0, 0, 0, 1)$. A *flock* of $\mathcal{C}$ is a partition of

the points of $\mathcal{C}$ minus the vertex into disjoint (irreducible) conics, i.e., plane intersections by planes not containing the vertex. It is customary to work with the set $\mathcal{F}$ of planes whose intersections with $\mathcal{C}$ give the flock. Associated with a flock is a set of 2×2 matrices over the field K. When $K = \mathrm{GF}(q)$, there are q such matrices whose pairwise differences are anisotropic. This set of q matrices is called a *q-clan* and is used to construct a 4-gonal family of $q+1$ subgroups of a group G, which by a method introduced by W. M. Kantor [29] may be used to construct a GQ. Hence with a flock (in the finite case) there is always an associated GQ. Also associated with a flock is a linespread of $\mathrm{PG}(3,q)$ and hence a translation plane. Very recently R. D. Baker, G. L. Ebert and K. L. Wantz [6] have introduced the concept of hyperbolic fibration of $\mathrm{PG}(3,q)$, a set of $q-1$ hyperbolic quadrics and two lines which together partition the points of $\mathrm{PG}(3,q)$. These fibrations were introduced in order to study large classes of translation planes. It turns out that a special type of hyperbolic fibration corresponds to a flock of a quadratic cone. This relationship between hyperbolic fibrations and GQ seems not to have exposed new examples of q-clans at this time, but this new discovery underscores the importance of flocks and their connections with other geometric constructs and adds its own emphasis to the fact that current research on projective geometries and generalized quadrangles in particular is very much alive!
This survey is concerned with the several types of algebraic and geometric constructions associated with flocks, especially with the flock generalized quadrangles of order (q^2,q). A generalized quadrangle (GQ) is itself a special type of generalized polygon, which in the finite case may be a projective plane, GQ, generalized hexagon or generalized octagon. Before concentrating on flock GQ we give a few general remarks.
It is well known that J. Tits introduced the concept of generalized polygon in an appendix to the very celebrated paper [73]. Examples of generalized quadrangles (now known as classical) were given in [73] but played no role in that work. However, they soon became the object of study in their own right. The first book-length treatment of (finite) generalized quadrangles was the research monograph by S. E. Payne and J. A. Thas [54]. This work was a rather complete treatment of what was known up through 1983 concerning finite generalized quadrangles. As soon as that monograph was completed, W. M. Kantor [30] and S. E. Payne [44] found other new families of generalized quadrangles with parameters (q^2,q) using sets of 2×2 matrices over $\mathrm{GF}(q)$ that came to be known as q-clans (cf. [46]) and were soon shown by J. A. Thas [64] to be equivalent to flocks of a quadratic cone in $\mathrm{PG}(3,q)$. The connection between flocks of cones and linespreads of $\mathrm{PG}(3,q)$ had already been discovered by J. A. Thas and independently by M. Walker. Almost immediately N. L. Johnson explained how to recognize which spreads were

associated with conical flocks, and there began an extensive study of projective planes associated with this general family of geometries.

A large part of the theory concerning finite generalized quadrangles that was developed between 1984 and 1993 has been surveyed in the HANDBOOK OF INCIDENCE GEOMETRY [15] in the two chapters 7 and 9 written by J. A. Thas. In 1994 F. De Clerck and H. Van Maldeghem [18] extended the associated theories of conical flocks, q-clans, generalized quadrangles and related geometries to the infinite case. N. L. Johnson and coauthors investigated the infinite case in considerable detail. Much of this work, along with a great deal of recent progress in the finite case, was rather thoroughly surveyed in N. L. Johnson, and S. E. Payne [26]. In some respects the present update may be considered to be a sequel to this survey written in 1996. Because [26] is so thorough, we have tended to give a rather brief treatment of certain major themes. For example, we say relatively little about the infinite case, and we do not even give a computationally complete statement of the Fundamental Theorem of q–clan geometry (cf. [49]). For more information on infinite topological generalized quadrangles and related geometries see the monograph [61].

In a masterful sequence of papers J. A. Thas ([65], [66], [69], [68]) has solved several major problems that had remained unsolved for a long time, most notably characterizing the flock GQ in terms of Property (G) introduced in [46], characterizing translation GQ, and producing a geometric construction for all flock GQ. This latter problem had been solved with a beautiful construction by N. Knarr [31] for q odd, but the construction by J. A. Thas for all q was truly an outstanding step forward.

In 1998 the very welcome book *Generalized Polygons* [74] appeared, but it treats generalized hexagons and octagons more seriously than it does generalized quadrangles. And although it contains a good treatment of the coordinatization of generalized polygons (developed primarily by its author), it does not advance the theory of flock quadrangles and related geometrical structures beyond what is surveyed in [26]. About the same time The Subiaco Notebook [42] became available on the WEB. This gives a fairly complete treatment of q-clan geometry for $q = 2^e$, with special emphasis on the Subiaco GQ and their associated ovals. Since [42] is available on the WEB and [26] is widely available, here we give only a rather terse review of the construction of the Subiaco ovals and their collineation groups. More interesting is the new construction of an infinite family of q-clans with q a power of 4 (cf. [17]) that generalizes the examples found by computer in [52].

Our own interests lie primarily with the finite geometries, and the bulk of this report deals with geometries defined over GF(q). However, since there seems to be some general interest in the infinite case, in Section 2 we review some of the terms and constructs for arbitrary fields. The first to generalize to the infinite case were F. De Clerck and H. Van Maldeghem in [18], where the theory of coordinatization was utilized. Then L. Bader and S. E. Payne [5] developed a complementary approach using the group coset geometry construction of W. M. Kantor [29]. In the meantime N. L. Johnson (along with various coauthors) has developed the general theory rather extensively in several directions. Following Section 2 the underlying field K is assumed to be finite.

For $K = \mathrm{GF}(q)$ there are significant differences between the cases q odd and q even. Section 3 is devoted to a review of the known infinite families of examples of flocks and q-clans with q odd. The two most recently discovered families have $q \equiv \pm 1 \pmod{10}$ and $q = 3^h$, respectively. The first example was introduced by T. Penttila [57] in a form that is not yet conveniently associated with an explicit q-clan. The new form of presentation given here is also used to give a pleasing description of the classical examples as well as those associated with the so-called Fisher flocks. The examples with $q = 3^h$ were discovered by M. Law and T. Penttila and will appear in [34]. These are studied further in [41], where the full collineation group of the associated flock GQ is determined with very interesting consequences for flocks. The first known rigid flocks (i.e., with trivial automorphism groups) are discovered.

Section 4 is devoted to the known infinite families with $q = 2^h$. In addition to the classical examples, the Fisher-Thas-Walker-Kantor-Betten family, the Payne family (all included in the chapters 7 and 9 by J. A. Thas in [15]), and the Subiaco family (included in the survey [26]), there is a recently discovered family called the Adelaide family by its discoverers W. E. Cherowitzo, C. M. O'Keefe and T. Penttila in [17]. This family generalizes the examples found by computer in [52]. Its construction is by a new, quite technical method that also gives the classical and the Subiaco families. When q is even, the existence of a q-clan is equivalent to the existence of a family of $q + 1$ ovals called a *herd*. In Section 4 there is given a new "magic" action of $\mathrm{P\Gamma L}(2, q)$ on such a herd. It is still an open problem to interpret this magic action in terms of the action induced by the collineation group of the associated GQ via the Fundamental Theorem of S. E. Payne [49].

There are several known sporadic (i.e., not now known to belong to an infinite family) q-clans with q odd. Section 5 gives 38 known examples along with a minimal amount of information concerning their automorphism groups. For $q \leq 23$ it seems almost certain that all flocks are known.

Section 6 surveys some of the deep and fundamental results of J. A. Thas mentioned above, especially those characterizing GQ having property (G) at a flag as being flock GQ (with an additional hypothesis needed when q is even), characterizing translation GQ, and giving a characteristic-free

geometric construction of flock GQ. Here we also mention some very recent results of K. Thas characterizing translation GQ. Finally, we mention some very interesting results of M. Brown, for example a construction of a new family of semipartial geometries starting with the Kantor-Knuth flock GQ (example K1 in Section 3).
In Section 7 we mention the recently discovered connection with the hyperbolic fibrations of R. D. Baker, G. L. Ebert and K. L. Wantz.
We have found it necessary to ignore a large amount of work done in the infinite case, and also the extensive related work on translation planes and partial flocks. The articles (and their references) by N. L. Johnson (and his coauthors) listed in our bibliography provide a rather thorough introduction to this material.
Not all definitions and pertinent results are given in this survey. Some good general references are [54], Chapters 7 and 9 by J. A. Thas in [15], [26] and [74].
Acknowledgement. We especially thank T. Penttila for sharing much unpublished material with us, and also F. De Clerck, T. Penttila and J. A. Thas for helpful comments on an early draft of this survey.

2 Flocks and their geometries

The initial study of flock quadrangles in the infinite case in [18] was based on the coordinatization of GQ and avoided the consideration of four-gonal families (i.e., Kantor families) of subgroups of a group. The latter point of view, developed in [5], is complementary to the former and will be given here. In the meantime a great deal of work in the infinite case has been done by N. L. Johnson and his coauthors. We have nothing to add to the survey [26], so for the most part will not attempt to update that report.
Let s and t be any two cardinal numbers larger than 1. Let $\mathcal{I}$ be an index set of order $t+1$. Let G be a group with two families $\mathcal{J} = \{A_i : i \in \mathcal{I}\}$, $\mathcal{J}^* = \{A_i^* : i \in \mathcal{I}\}$ of subgroups of G with $A_i \leq A_i^*$, $|A_i| = s$, $[A_i^* : A_i] = t$ for all $i, j \in \mathcal{I}$. Put $\Omega = \cup(A_i : i \in \mathcal{I})$. Starting with the triple $(G, \mathcal{J}, \mathcal{J}^*)$ define a point-line incidence geometry $Q(G, \mathcal{J}, \mathcal{J}^*)$ as follows.

Points of $Q(G, \mathcal{J}, \mathcal{J}^*)$ are of three types:

(1) Elements g of G;

(2) Cosets $A_i^* g$, $i \in \mathcal{I}$, $g \in G$;

(3) The symbol (∞).

Lines of $Q(G, \mathcal{J}, \mathcal{J}^*)$ are of two types:

(a) Cosets $A_i g$, $i \in \mathcal{I}$, $g \in G$;

(b) Symbols $[A_i]$, $i \in \mathcal{I}$.

Incidence is defined by: the point (∞) is on all lines of type (b). The point $A_i^* g$ is on the line $[A_i]$ and on the lines of type (a) contained in $A_i^* g$. The point g of type (1) is on the lines $A_i g$ of type (a) that contain it. There are no other incidences.

Theorem 1([5]) *The point-line geometry $Q(G, \mathcal{J}, \mathcal{J}^*)$ is a generalized quadrangle* GQ(s,t) *with parameters (s,t) if and only if the triple $(G, \mathcal{J}, \mathcal{J}^*)$ satisfies the following four properties:*

K1. $A_i A_j \cap A_k = \{e\}$, *for distinct* $i, j, k \in \mathcal{I}$.

K2. $A_i^* \cap A_j = \{e\}$, *for distinct* $i, j \in \mathcal{I}$.

K3. $A_i^* A_j = G$, *for distinct* $i, j \in \mathcal{I}$.

K4. $A_i^* = A_i \cup (A_i g : A_i g \cap \Omega = \emptyset)$.

If G is finite, i.e., if s and t are finite, put $|\mathcal{J}| = |\mathcal{J}^*| = r$. If K1 and K2 hold, then K3 holds and $r \leq 1 + t$, with equality if and only if K4 holds. Hence a GQ(s,t) is obtained provided $r = 1 + t$ and both K1 and K2 hold. In this case $\mathcal{J}^*$ is completely determined by $\mathcal{J}$ (assuming that $\mathcal{J}^*$ exists), and we say that $(G, \mathcal{J})$ is a four-gonal family (or a Kantor family).

Let K be any field. Let $\mathcal{C}$ be the cone $\mathcal{C} = \{(X_0, X_1, X_2, X_3) \in \mathrm{PG}(3, K) : X_2^2 = X_0 X_1\}$ with vertex $P = (0,0,0,1)$. A *flock* of $\mathcal{C}$ is a partition of the points of $\mathcal{C}$ minus the vertex into disjoint (irreducible) conics, i.e., plane intersections by planes not containing the vertex. It is customary to work with the set $\mathcal{F}$ of planes whose intersections with $\mathcal{C}$ give the flock.

Let $\mathbf{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$ be a given set of 2×2 matrices over K indexed by the elements of K. For each $t \in K$, let π_t be the plane with equation $\pi_t : x_t X_0 + z_t X_1 + y_t X_2 + X_3 = 0$. As first noted by J. A. Thas [64], $\pi_t \cap \pi_s$ is disjoint from the cone if and only if $(X_2, X_1) \begin{pmatrix} x_t - x_s & y_t - y_s \\ 0 & z_t - z_s \end{pmatrix} \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} = 0$

implies $(X_2, X_1) = (0,0)$, i.e., if and only if the matrix $\begin{pmatrix} x_t - x_s & y_t - y_s \\ 0 & z_t - z_s \end{pmatrix}$ is *anisotropic*. If K is finite, this says that given the set $\mathbf{C}$ of 2×2 matrices indexed by K, the set $\mathcal{F}(\mathbf{C})$ of planes π_t yields a flock of $\mathcal{C}$ if and only if $\mathbf{C}$ is a *q-clan*, i.e., for $s, t \in K$ with $s \neq t$, $A_s - A_t$ is anisotropic. When K is infinite, matters are more complicated.

F. De Clerck and H. Van Maldeghem [18] defined $\mathbf{C}$ to be a *K-clan* provided the set $\mathcal{F}(\mathbf{C})$ of planes is a flock of $\mathcal{C}$. Then they proved the following theorem.

Theorem 2 ([18], Theorem 2.5) *Let K be any field of characteristic m, and let*

$$\mathbf{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$$

be given. Let $\tilde{K} = K \cup \{\infty\}$.

(i) If $m \neq 2$, then $\mathcal{F}(\mathbf{C})$ is a flock if and only if the following two conditions hold:

(a) $(y_t - y_s)^2 - 4(x_t - x_s)(z_t - z_s)$ is a non-square whenever $s, t \in K$ with $s \neq t$.

(b) The mapping $K \to K : t \mapsto x_t k^2 + y_t k + z_t$ is surjective for all $k \in \tilde{K}$. (For $k = \infty$, this means $t \mapsto x_t$ is a bijection.)

(ii) If $m = 2$, then $\mathcal{F}(\mathbf{C})$ is a flock if and only if the following two conditions hold:

(a') For every $s, t \in K$, $s \neq t$, we have $y_s \neq y_t$ and the element $(x_s + x_t)(z_s + z_t)(y_s + y_t)^{-2} \in C_1(K) = \{k \in K : x^2 + x + k = 0$ has no solution in $K\}$.

(b') The mapping $K \to K : t \mapsto x_t k^2 + y_t k + z_t$ is surjective for all $k \in \tilde{K}$.

Remark. If K is finite in the preceding theorem, condition (a) implies condition (b), and condition (a') implies condition (b').

Let K be an arbitrary field. Consider the group $G = K^2 \times K \times K^2$ with binary operation $(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta')$. For $t \in K$, put $K_t = A_t + A_t^T$. The center of G is $Z = \{(0, c, 0) : c \in K\}$. Put $A(\infty) = \{(0, 0, \beta) : \beta \in K^2\}$; for $t \in K$ put $A(t) = \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) : \alpha \in K^2\}$. For $s \in \tilde{K}$, put $A^*(s) = A(s)Z$. Put $\mathcal{J} = \{A(t) : t \in \tilde{K}\}$, and $\mathcal{J}^* = \{A^*(t) : t \in \tilde{K}\}$.

Starting with the triple $(G, \mathcal{J}, \mathcal{J}^*)$ we can construct a point-line geometry $\mathrm{GQ}(\mathbf{C}) = Q(G, \mathcal{J}, \mathcal{J}^*)$ as at the beginning of this section. It is natural to seek conditions on $\mathbf{C}$ for $\mathrm{GQ}(\mathbf{C})$ to be a generalized quadrangle. Using the notation introduced above, we state the following theorem of [5].
Theorem 3 *Let K be any field, and let*

$$\mathbf{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$$

be given with $K_t = A_t + A_t^T$. Then $\mathrm{GQ}(\mathbf{C})$ is a generalized quadrangle if and only if the following conditions hold for all distinct $s, t, u \in K$:

(i) $K_t - K_s$ is nonsingular;

(ii) $A_t - A_s$ is anisotropic;

(iii) $(K_t - K_u)^{-1}(A_t - A_u)(K_t - K_u)^{-1} + (K_u - K_s)^{-1}(A_u - A_s)(K_u - K_s)^{-1}$ is anisotropic;

(iv) The mapping $K \to K : t \mapsto a^2 x_t + a b y_t + b^2 z_t$ is surjective for all $\alpha = (a, b) \in K^2$ with $(a, b) \neq (0, 0)$;

(v) The mapping $K \setminus \{s\} \to K \setminus \{0\} : t \mapsto \frac{(x_t - x_s)a^2 + (y_t - y_s)ab + (z_t - z_s)b^2}{(y_t - y_s)^2 - 4(x_t - x_s)(z_t - z_s)}$ is surjective for all $a, b, s \in K$ with $(a, b) \neq (0, 0)$.

Flocks and GQs are connected by the following pretty result of [18].
Theorem 4 ([18], Theorem 2.6) *Let K be any field, and let*

$$\mathbf{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in K \right\}$$

be given. Then $\mathrm{GQ}(\mathbf{C})$ is a generalized quadrangle if and only if for every $s \in K$ the set $\mathbf{C}_s = \left\{ A_t^s = \begin{pmatrix} X_t^s & Y_t^s \\ 0 & Z_t^s \end{pmatrix} : t \in K \right\}$ is a K-clan, where $X_s^s = Y_s^s = Z_s^s = 0$ and $X_t^s = \frac{x_t - x_s}{D(t,s)}$, $Y_t^s = \frac{y_t - y_s}{D(t,s)}$, $Z_t^s = \frac{z_t - z_s}{D(t,s)}$ with $D(t, s) = (y_t - y_s)^2 - 4(x_t - x_s)(z_t - z_s)$ whenever $t \neq s$.

Remark. The above theorem says that a generalized quadrangle arises from a flock if and only if all the *derivations* are well defined flocks. (Derivation of a finite flock over a field of odd order was introduced in [4]. The concept will be discussed more fully later.) For examples of flocks that do not yield generalized quadrangles, see [9] and [8].
The number of constructions of infinite families of infinite flocks of quadratic cones has grown so fast (cf. [18], [24], [25], [28], [5]) that we shall not attempt

to give a census of them here. Indeed, for the remainder of this survey we shall restrict our attention to finite geometries. In this case a flock of the cone $\mathcal{C}$ always corresponds to a generalized quadrangle.

3 Infinite families of flocks with q odd

Throughout this section we assume that $K = \mathrm{GF}(q)$, q an odd prime power. Let $\mathcal{C}$ be the cone in $\mathrm{PG}(3, q)$ as above with vertex P_0. Let $Q(4, q)$ be a nonsingular quadric in $\mathrm{PG}(4, q)$ with $\mathcal{C}$ embedded in $Q(4, q)$ so that $\mathcal{C}$ is the intersection with $Q(4, q)$ of the hyperplane of $\mathrm{PG}(4, q)$ tangent to $Q(4, q)$ at P_0. Let $\mathcal{F} = \{C_1, \ldots, C_q\}$ be a flock of $\mathcal{C}$. Then (because q is odd) there exist unique points $P_1, \ldots, P_q$ of $Q(4, q)$ such that C_i is the set of points of $Q(4, q)$ collinear in $Q(4, q)$ with both P_0 and P_i, $1 \leq i \leq q$, i.e., $C_i = P_0^\perp \cap P_i^\perp$, $1 \leq i \leq q$, where $\perp$ is relative to $Q(4, q)$. Then by construction (because $\mathcal{C}$ is a flock), the set $\{P_0, \ldots, P_q\}$ has the property that for $1 \leq i < j \leq q$, the triple (P_0, P_i, P_j) is a set of pairwise noncollinear points of $Q(4, q)$ for which no point of $Q(4, q)$ is collinear in $Q(4, q)$ with all three points of the triple. A major result of [4] is that for all choices of i, j, k with $0 \leq i < j < k \leq q$, the triple (P_i, P_j, P_k) also has this property. Hence if we fix j, $0 \leq j \leq q$, the lines of $Q(4, q)$ through P_j form a cone $\mathcal{C}_j$ for which the conics $C_{j,k} = P_j^\perp \cap P_k^\perp$, $0 \leq k \leq q$, $k \neq j$, form a flock of the cone $\mathcal{C}_j$. This set of $q + 1$ pairwise noncollinear points of $Q(4, q)$ such that no three are collinear in $Q(4, q)$ with a same point of $Q(4, q)$ is called a BLT-*set*.

Two flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ of the quadratic cone $\mathcal{C}$ of $\mathrm{PG}(3, q)$ are called *isomorphic* provided there is an element of $\mathrm{P\Gamma L}(4, q)$ which fixes $\mathcal{C}$ and maps $\mathcal{F}_1$ to $\mathcal{F}_2$. In [4] it is proved that if P_1 and P_2 are two points of the BLT-set $\mathcal{P}$, then the flock $\mathcal{F}_1$ arising from P_1 is isomorphic to the flock $\mathcal{F}_2$ arising from P_2 if and only if P_1 and P_2 are in the same orbit of the stabilizer $\mathrm{P\Gamma O}(5, q)_{\mathcal{P}}$ of $\mathcal{P}$ in the appropriate orthogonal group. It was shown in [53] that each of the flocks determined by a given BLT-set (i.e., *derived* from any one of the flocks determined by that BLT-set) gives essentially the same generalized quadrangle. N. Knarr [31] has given a very nice direct geometric construction of the generalized quadrangle arising from a BLT-set. Two BLT-sets $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic, i.e., there exists an element of $\mathrm{P\Gamma O}(5, q)$ which maps $\mathcal{P}_1$ to $\mathcal{P}_2$, if and only if the generalized quadrangle $\mathcal{S}_1$ arising from $\mathcal{P}_1$ is isomorphic to the generalized quadrangle $\mathcal{S}_2$ arising from $\mathcal{P}_2$. Moreover, if $\mathcal{S}$ is the generalized quadrangle arising from the BLT-set $\mathcal{P}$, then the number of orbits of lines through the base point (∞) of $\mathcal{S}$ (under the action of the subgroup $\mathcal{G}_0$ of collineations of $\mathcal{S}$ that fix the point (∞) and some point not collinear with (∞)) is equal to the number of orbits in $\mathcal{P}$ of the stabiliser

of $\mathcal{P}$ in $\mathrm{P\Gamma O}(5,q)$. In particular, the group of the generalized quadrangle is determined (in principle) once the group of the BLT-set is determined. These facts follow from results contained in [54], [55], [56], [31] and [49].
Translation planes can be constructed from flocks by constructing an ovoid of the Klein quadric from a flock of the quadratic cone in $\mathrm{PG}(3,q)$. This ovoid corresponds to a line spread of $\mathrm{PG}(3,q)$ via the Klein correspondence, which in turn gives rise to a translation plane via the André/Bruck-Bose construction.This was independently observed by both Walker [75] and J. A. Thas. A direct construction of the line spread from the associated q-clan was given in [22]. Alternatively, the ovoid was constructed directly from the BLT-set in [33].
Traditionally, the known examples of flock quadrangles have been given in terms of the associated q-clans. In [57] T. Penttila gave a new construction for $q \equiv \pm 1 \pmod{10}$ as a BLT-set which still lacks a satisfactory description as a q-clan. The newest infinite family for q odd has $q = 3^e$ and was discovered by M. Law and T. Penttila as a generalization of an example with $q = 27$ that was studied via computer (cf. [32]). The infinite family appears in [34]. Below we give a census of all these families along with some information about the associated flocks.
As q is odd in this section, we give the q-clans as symmetric matrices of the form

$$A_t = \begin{pmatrix} t & f(t) \\ f(t) & g(t) \end{pmatrix}, \quad t \in K;\ f,g : K \to K.$$

The corresponding BLT-set $\mathcal{P}$ is then given by

$$\mathcal{P} = \{(1, t, f(t), -g(t), tg(t) - (f(t))^2)\} \cup \{(0,0,0,0,1)\}.$$

Then $\mathcal{C} = \{A_t : t \in K\}$ is a q-clan if and only if

$$-\det(A_s - A_t) = (f(s) - f(t))^2 - (s-t)(g(s) - g(t))$$

is a nonsquare of $K = \mathrm{GF}(q)$ whenever $s \neq t$. We follow the presentation and naming conventions of the recent survey by N. Johnson and S. E. Payne [26]. In the case where a BLT-set gives rise to more than one flock the family is named simply by concatenating the names given to the non-isomorphic flocks. For a given q-clan $\mathcal{C}$ (respectively, BLT-set $\mathcal{P}$), the associated generalized quadrangle is denoted $\mathrm{GQ}(\mathcal{C})$ (respectively, $\mathrm{GQ}(\mathcal{P})$). As some additional information about the flocks known then was given in [26], we indicate here only how many nonisomorphic flocks there are associated with the generalized quadrangle. No comment indicates that only one flock occurs.

3.1 Classical: For all q and with $x^2 + bx + c$ irreducible over K.

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{1}{2}bt \\ \frac{1}{2}bt & ct \end{pmatrix} : t \in K \right\}.$$

3.2 FTW: For $q \equiv -1 \pmod 3$.

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{3}{2}t^2 \\ \frac{3}{2}t^2 & 3t^3 \end{pmatrix} : t \in K \right\}.$$

3.3 K_1: For all q with m a nonsquare of K and σ an automorphism of K.

$$\mathcal{C} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -mt^\sigma \end{pmatrix} : t \in K \right\}.$$

3.4 K_2/JP: For $q \equiv \pm 2 \pmod 5$.

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{5}{2}t^3 \\ \frac{5}{2}t^3 & 5t^5 \end{pmatrix} : t \in K \right\}.$$

Two flocks arise.

3.5 K_3/BLT: For $q = 5^h$, with k a nonsquare of K.

$$\mathcal{C} = \left\{ \begin{pmatrix} t & 3t^2 \\ 3t^2 & k^{-1}t(1 + kt^2)^2 \end{pmatrix} : t \in K \right\}.$$

Two flocks arise.

3.6 G/PTJLW: For $q = 3^h$, with n a nonsquare of K.

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{1}{2}t^3 \\ \frac{1}{2}t^3 & -nt - n^{-1}t^9 \end{pmatrix} : t \in K \right\}.$$

Two flocks arise.

This is an interesting example because each entry of the matrix is an additive function of t. This means on the one hand that the associated translation plane is a semifield plane (cf. [22]), and on the other hand that the associated generalized quadrangle is the point-line dual of a translation generalized quadrangle GQ(q, q^2) (cf. Chapt. 8 of [54]). Moreover, for $q \geq 9$ this GQ has a translation dual that is not a flock quadrangle for any flock (cf. 8.7.2(iv) of [54] and [46]).

3.7 Fi: For all odd $q \geq 5$.

The Fisher flock has a q-clan representation (first discovered by J. A. Thas [64]; first given in explicit q-clan form in [45]) which appears in detail in [26]. However, it is rather involved and not as pretty as the BLT representation discovered by T. Penttila [57]. Since we need the basic setup to give the new family presented in [57] anyway, we give that here and use it to give this new description of the Fisher BLT-set.

Let $K = \mathrm{GF}(q) \subset \mathrm{GF}(q^2) = E$, q odd and $q \geq 5$. Let ζ be a primitive element for E and put $\eta = \zeta^{q-1}$, so η has multiplicative order $q+1$. Let $T(x) = x + \bar{x}$, where $\bar{x} = x^q$. Let $V = \{(x, y, a) : x, y \in E,\ a \in K\}$ considered as a 5-dimensional vector space over K in the standard way. Define a map $Q : V \to K$ by

$$Q(x, y, a) = x^{q+1} + y^{q+1} - a^2.$$

It is easy to see that Q is a quadratic form on V with polar form f given by

$$f((x, y, a), (z, w, b)) = T(x\bar{z}) + T(y\bar{w}) - 2ab.$$

Then the BLT-set $\mathcal{P}$ is given by

$$\mathcal{P} = \{(\eta^{2j}, 0, 1) : 1 \leq j \leq \frac{q+1}{2}\} \cup \{(0, \eta^{2j}, 1) : 1 \leq j \leq \frac{q+1}{2}\}.$$

3.8 Pe: For all $q \equiv \pm 1 \pmod{10}$.

This example was discovered by T. Penttila [57]. (Our choice of quadratic form is superficially different from that of [57], so the description of the BLT-set is also slightly different.)

With the same notation as in the preceding example, the BLT-set is given by

$$\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) \in V : 0 \leq j \leq q\}.$$

The present author has worked out a q-clan representation of this BLT-set, but it seems too involved to be helpful. Moreover, with the BLT-set representation it is almost trivial to show that the group of the BLT-set is transitive, implying that there arises only one flock.

3.9 LP: For all $q = 3^e$.

For $t \in K$ with $q = 3^e$ and n a fixed nonsquare of K, let

$$A_t = \begin{pmatrix} t & t^4 + nt^2 \\ t^4 + nt^2 & -n^{-1}t^9 + t^7 + n^2t^3 - n^3t \end{pmatrix}.$$

Then $\mathcal{C} = \{A_t : t \in K\}$ is a q-clan.
This construction will appear in M. Law and T. Penttila [34]. The full collineation group of GQ($\mathcal{C}$) is determined in [41], with some very interesting consequences for flocks. Recall that the orbits on the lines through the point (∞) of GQ($\mathcal{C}$) correspond to the isomorphism types of the flocks. In this case the lines $[A(\infty)]$ and $[A(0)]$ each yield orbits of size one. For each $\sigma \in \mathrm{Aut}(K)$ and each choice of ± 1 there is a collineation of GQ($\mathcal{C}$) that maps $[A(t)]$ to $[A(\bar{t})]$, where

$$\bar{t} = \pm n^{\frac{1-\sigma}{2}} t^{\sigma}.$$

If the orbit of $[A(t)]$ has size h, then the stabilizer of $[A(t)]$ has order $\frac{2e}{h}$. So the flocks corresponding to $[A(\infty)]$ and $[A(0)]$ both have stabilizers of order $2e$. In general there are many isomorphism classes of flocks, including some with stabilizers of order $2e$ and others with stabilizers of order 1. If $q = 3^e$ with e odd, there are $2 + \sum_{d|e}(3^d - 1) \cdot \phi(\frac{e}{d})$ orbits. If $e = p^h$ with p an odd prime, there are $3^{p^{h-1}} \left(\frac{3^{\phi(p^h)}-1}{2p^h} \right)$ orbits in $\mathrm{GF}(3^{p^h}) \setminus \mathrm{GF}(3^{p^{h-1}})$ with length $2e$. This means that they correspond to flocks which are *rigid*, i.e., which have trivial automorphism group. (Note that our use of the word "rigid" differs from the use in [25], where a *group* preserving a flock is called *rigid* provided it fixes each plane of the flock.)

4 Infinite families of flocks with q even

4.1 q-clans, GQ and herds of ovals

For q even, the connection between flocks, spreads, q-clans, generalized quadrangles of order (q^2, q), subquadrangles of order q and herds of ovals is given in detail in the survey [26] along with a description of the Fundamental Theorem and its consequences. Consequently we give here only the basic machinery necessary for the constructions and just enough material on collineations of the GQ to make a comparison with some recent work of C. M. O'Keefe and T. Penttila [36]. Then we set up some new machinery so that we can include in our list of known examples the new Adelaide geometries [17].
Let $K = \mathrm{GF}(q)$ with $q = 2^e$. Let tr : $K \to \mathrm{GF}(2)$ be the absolute trace function. As above, let $\mathcal{C}$ be the cone $\mathcal{C} = \{(X_0, X_1, X_2, X_3) \in \mathrm{PG}(3, K)$:

$X_2^2 = X_0X_1\}$ with vertex $P = (0,0,0,1)$. For each $t \in K$ let $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$ be a 2×2 matrix over K, and let π_t be the plane with equation $\pi_t : x_tX_0 + z_tX_1 + y_tX_2 + X_3 = 0$. As before, put $\mathbf{C} = \{A_t : t \in K\}$. Then $\mathcal{F}(\mathbf{C}) = \{\pi_t : t \in K\}$ determines a flock of $\mathcal{C}$ if and only if $\mathbf{C}$ is a q-clan, which is the case if and only if

$$y_s \neq y_t \text{ and } \operatorname{tr}\left(\frac{(x_s - x_t)(z_s - z_t)}{(y_s - y_t)^2}\right) = 1$$

for all $s, t \in K$ with $s \neq t$.

In this case (i.e., $q = 2^e$), not only is there a generalized quadrangle GQ($\mathbf{C}$) associated with the q-clan $\mathbf{C}$ as well as a spread of PG$(3,q)$, there is a family of $q+1$ subquadrangles of GQ($\mathbf{C}$) each having order (q,q) and a *herd* of ovals in PG$(2,q)$. In the published literature there are different representations of the associated elation group G. For this survey we have chosen the following normalizations and standardized notations.

Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If α and β are both 2-tuples of elements of K, put $\alpha \circ \beta = \sqrt{\alpha P \beta^T}$. Then the elation group G is given by

$$G = \{(\alpha, \beta, c) : \alpha, \beta \in K^2; c \in K\},$$

where the group operation is given by

$$(\alpha, \beta, c) \circ (\alpha', \beta', c') = (\alpha + \alpha', \beta + \beta', c + c' + \beta \circ \alpha').$$

Without loss of generality, we may assume that all q-clans have upper triangular matrices

$$A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} \text{ with } A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

with $x_1 = 1$, and with $y_t = t^{\frac{1}{2}}$ for all $t \in K$. Also put $A_\infty = A_0$.

There are two families of subgroups of G, each subgroup having order q^3 and being elementary abelian, that play important roles in q-clan geometry.

$$\text{For } \bar{0} \neq \gamma \in K^2, \quad \mathcal{L}_\gamma = \{(\gamma \otimes \alpha, c) \in G : \alpha \in K^2, c \in K\};$$
$$\text{For } \bar{0} \neq \alpha \in K^2, \quad \mathcal{R}_\gamma = \{(\gamma \otimes \alpha, c) \in G : \gamma \in K^2, c \in K\}.$$

For nonzero $\alpha,\ \gamma \in K^2$, $\mathcal{L}_\gamma = \mathcal{L}_\alpha$ (resp., $\mathcal{R}_\gamma = \mathcal{R}_\alpha$) if and only if $\{\alpha, \gamma\}$ is K-dependent. Hence we may think of the $\mathcal{L}_\gamma$ (resp., $\mathcal{R}_\gamma$) as indexed by the points of PG$(1,q)$. The scalar multiplication $d((\gamma \otimes \alpha), c) = ((d\gamma \otimes \alpha), dc) =$

$((\gamma \otimes d\alpha), dc)$ makes each $\mathcal{L}_\gamma$ and $\mathcal{R}_\alpha$ into a three-dimensional vector space over K. So there are associated projective planes $\bar{\mathcal{L}}_\gamma \cong \bar{\mathcal{R}}_\alpha \cong \mathrm{PG}(2,q)$. Clearly the center $Z = \{((\bar{0},\bar{0}),c) \in G : c \in K\}$ of G is contained in each $\mathcal{L}_\gamma$ and $\mathcal{R}_\alpha$.

When we have used a q-clan $\mathbf{C}$ to construct a four-gonal family $\mathcal{J}$ for G, and have then the associated generalized quadrangle $\mathrm{GQ}(\mathbf{C})$, it will turn out that the $\mathcal{L}_\gamma$ essentially index the lines through the point (∞), and the $\mathcal{R}_\alpha$ index the subquadrangles of order (q,q) containing the points (∞) and $(\bar{0},\bar{0},0)$ along with their associated ovals.

Identify elements of $\tilde{K} = K \cup \{\infty\}$ with points of $\mathrm{PG}(1,q)$ in the following way. For $t \in K$, write $\gamma_t = (1,t)$, and $\gamma_\infty = (0,1)$. So for $\bar{0} \neq \gamma = (a,b)$, $\gamma \equiv \gamma_{b/a}$. For $t \in \tilde{K}$, we then write $\mathcal{L}_t = \mathcal{L}_{\gamma_t}$ and $\mathcal{R}_t = \mathcal{R}_{\gamma_t}$. Now let $\mathbf{C}$ be a fixed q-clan normalized as above, including the matrix A_∞. (Here, for $t \in K$, $\gamma_{y_t} = (1, t^{\frac{1}{2}})$. For $t = \infty$, strictly speaking, $y_t = 0$, but we write $\gamma_{y_\infty} = \gamma_\infty = (0,1)$ anyway.) For $t \in \tilde{K}$ and $\alpha \in K^2$, put $g_t(\alpha) = \sqrt{\alpha A_t \alpha^T}$. It then follows easily that

$$g_t(\alpha+\beta) = g_t(\alpha) + g_t(\beta) + t^{\frac{1}{4}}(\alpha \circ \beta) \text{ and } g_t(k\alpha) = k g_t(\alpha).$$

Then for each $t \in \tilde{K}$ define a subgroup of G as follows:

$$A(t) = \{(\gamma_{y_t} \otimes \alpha, g_t(\alpha)) \in G : \alpha \in K\}$$

It follows that $A(t)$ is a subgroup of $\mathcal{L}_{\gamma_{y_t}}$ having order q^2. Specifically, for $t \in K$, $A(t) \leq \mathcal{L}_{t^{\frac{1}{2}}}$, and $A(\infty) \leq \mathcal{L}_\infty$. The point here is that $\mathcal{J}(\mathbf{C}) = \{A(t) : t \in K\}$ is the usual 4-gonal family for G with $\mathcal{L}_t$ being the *tangent space* of $\mathcal{J}(\mathbf{C})$ at $A(t^2)$.

Note that for each $k \in K$, the map $(\alpha,\beta,c) \mapsto (k\alpha, k\beta, kc)$ is an automorphism of G which fixes setwise each subgroup $A(t)$, $t \in \tilde{K}$. Put $A_\alpha(t) = \mathcal{R}_\alpha \cap A(t) = \{d(\gamma_{y_t} \otimes \alpha, g_t(\alpha)) : d \in K\}$, so $|A_\alpha(t)| = q$. It follows that $A_\alpha(t)$ determines a point $\bar{A}_\alpha(t)$ in the projective plane $\bar{\mathcal{R}}_\alpha$. Property K1 for 4-gonal families guarantees that $\bar{\mathcal{O}}_\alpha = \{\bar{A}_\alpha(t) : t \in \tilde{K}\}$ is an oval of $\bar{\mathcal{R}}_\alpha$. The isomorphism $\bar{\mathcal{R}}_\alpha \to \mathrm{PG}(2,q) : ((x,y) \otimes \alpha, z) \mapsto (x,y,z)$ gives the *standard coordinates* for $\bar{\mathcal{R}}_\alpha$. Using these standard coordinates, $\bar{A}_\alpha(\infty)$ is the point $(0,1,0)$ and $\bar{A}_\alpha(t)$ is the point $(1, t^{\frac{1}{2}}, \sqrt{\alpha A_t \alpha^T})$.

It follows easily that $\bar{\mathcal{O}}_\alpha$ is isomorphic to the oval $\mathcal{O}'_\alpha$ in $\mathrm{PG}(2,q)$ with nucleus $(0,0,1)$ given by

$$\mathcal{O}'_\alpha = \left\{\left(1, t^{\frac{1}{2}}, \sqrt{\alpha A_t \alpha^T}\right) : t \in K\right\} \cup \{(0,1,0)\}.$$

The $q+1$ distinct subgroups $\mathcal{R}_\alpha$ arise from the $q+1$ elements $\alpha = (0,1)$ and $\alpha = (1, s^{\frac{1}{2}})$ for $s \in K$. For these $q+1$ values of α, we substitute for α

and for the q-clan matrices A_t, in the set $\mathcal{O}'_\alpha$, and then apply the collineation $(x, y, z) \mapsto (x^2, y^2, z^2)$ (and if $\alpha \neq (0, 1)$ we divide the third coordinate by an appropriate constant) to obtain the $q+1$ projectively equivalent ovals $\mathcal{O}_\alpha$ where:

$$\mathcal{O}_{(0,1)} = \{(1, t, z_t) : t \in K\} \cup \{(0, 1, 0)\},$$

$$\mathcal{O}_{(1,s^{\frac{1}{2}})} = \left\{ \left(1, t, \frac{x_t + sz_t + s^{\frac{1}{2}}t^{\frac{1}{2}}}{1 + sz_1 + s^{\frac{1}{2}}}\right) : t \in K \right\} \cup \{(0, 1, 0)\}.$$

Equivalently, $\mathcal{J}_\alpha = \{A_\alpha(t) : t \in \tilde{K}\}$ is a 4-gonal family for $\mathcal{R}_\alpha$. It was shown in [50] that the associated generalized quadrangle of order q may be viewed as a subquadrangle of GQ($\mathbf{C}$) containing the points (∞) and $(0, 0, 0)$. Moreover, the subquadrangle associated with $\mathcal{R}_\alpha$ is isomorphic to $T_2(\mathcal{O}_\alpha)$ (cf. [43], [50], and [44]).
Following Cherowitzo, Penttila, Pinneri and Royle [16], a *herd* of ovals in PG$(2, q)$, q even, is a family of $q + 1$ ovals $\{\mathcal{O}_s : s \in \mathrm{GF}(q) \cup \{\infty\}\}$, each of which has nucleus $(0, 0, 1)$ and contains the points $(1, 0, 0), (0, 1, 0)$ and $(1, 1, 1)$, and such that (with $K = \mathrm{GF}(q)$ as above)

$$\mathcal{O}_\infty = \{(1, t, f_\infty(t)) : t \in K\} \cup \{(0, 1, 0)\} \text{ and}$$
$$\mathcal{O}_s = \{(1, t, f_s(t)) : t \in K\} \cup \{(0, 1, 0)\}, \quad x \in K,$$
$$f_s(t) = \frac{f_0(t) + \kappa s f_\infty(t) + s^{\frac{1}{2}}t^{\frac{1}{2}}}{1 + \kappa s + s^{\frac{1}{2}}},$$

for some $\kappa \in K$ satisfying $tr(\kappa) = 1$.
Here the functions f_∞ and f_s, $s \in K$, are all *o-polynomials*, since each gives an oval containing the prescribed points.
In [16] it is proved that such a herd of ovals gives rise to a (normalized) q-clan

$$\left\{ \begin{pmatrix} f_0(t) & t^{\frac{1}{2}} \\ 0 & \kappa f_\infty(t) \end{pmatrix} : t \in K \right\}.$$

As we have seen above, such a normalized q-clan $\mathbf{C}$ corresponds to a herd $\mathcal{H}(\mathbf{C})$ of ovals. In our notation, $f_0(t) = x_t$ and $\kappa f_\infty(t) = z_t$.

4.2 Collineations

Let $\mathcal{G}$ denote the group of collineations of GQ($\mathbf{C}$) fixing the point (∞) (which is the complete group of collineations when $\mathcal{C}$ is not classical, i.e., not linear), and let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ fixing the point $(\bar{0}, \bar{0}, 0)$. Then a major consequence of the Fundamental Theorem is that each element of $\mathcal{G}_0$ is induced by an automorphism θ of the group G with a very particular form. There is an automorphism σ of K, and there are two matrices $A \in GL(2,q)$, $B \in SL(2,q)$ for which $\theta = \theta(\sigma, A, B)$. Then

$$\theta(\sigma, A, B) : ((\alpha, \beta), c) \mapsto ((\alpha^\sigma, \beta^\sigma)(A \otimes B), --),$$

where for our present purposes we may ignore the effect of θ on the third coordinate. The important thing to notice is that

$$((\gamma \otimes \alpha), c) \mapsto (\gamma^\sigma A \otimes \alpha^\sigma B, --).$$

This means that $\theta(\sigma, A, B) : \mathcal{L}_\gamma \mapsto \mathcal{L}_{\gamma^\sigma A}$ and $\theta(\sigma, A, B) : \mathcal{R}_\alpha \mapsto \mathcal{R}_{\alpha^\sigma B}$. Hence σ and A determine the action of $\mathcal{G}_0$ on the lines of GQ($\mathbf{C}$) through the point (∞), and σ and B determine the action on the set of subquadrangles and especially on the herd of ovals. (Recall that to each line $[A(t)]$ through the point (∞) of GQ($\mathbf{C}$) there is associated a flock (cf. [2], [48], [42]), with two lines in the same $\mathcal{G}_0$-orbit if and only if their associated flocks are projectively equivalent.) Using this approach it has been possible to compute the full collineation group of GQ($\mathbf{C}$) for the q-clans (with q even) in each of the infinite families known prior to the year 2000, and then to compute the group of collineations induced on the herd, especially the induced subgroup stabilizing a given oval. This was done in the Subiaco Notebook by adapting results of several previously published papers. But it was always fairly technical, especially so for the Subiaco q-clans. Very recently a new approach to computing the group acting on a herd was given by C. M. O'Keefe and T. Penttila [36]. We describe this new approach below, but leave open the question of just how the two approaches fit together. The treatment in [36] seems a bit incomplete at this point.

Let $\mathcal{F}$ denote the collection of all functions $f : K \to K$ such that $f(0) = 0$. Note that each element of $\mathcal{F}$ can be expressed as a polynomial in one variable of degree at most $q-1$ and that $\mathcal{F}$ is a vector space over K. If $f(z) = \sum a_i z^i \in \mathcal{F}$ and $\sigma \in \mathrm{Aut}(K)$, then write $f^\sigma(z) = \sum a_i^\sigma z^i$. In notation adapted from that of [36], an element $\psi \in \mathrm{P\Gamma L}(2,q)$ acts on the projective line PG$(1,q)$ by $\psi : \bar{x} \mapsto A\bar{x}^\sigma$ for some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,q)$ and some $\sigma \in \mathrm{Aut}(K)$. Let the image of $f \in \mathcal{F}$ under ψ be the function $\psi f : K \to K$ such that

$$\psi f(z) = |A|^{-\frac{1}{2}}(bz+d) f^\sigma\left(\frac{az+c}{bz+d}\right) + |A|^{-\frac{1}{2}} bz f^\sigma\left(\frac{a}{b}\right) + |A|^{-\frac{1}{2}} d f^\sigma\left(\frac{c}{d}\right).$$

This definition yields an action of $\mathrm{P\Gamma L}(2,q)$ on $\mathcal{F}$ which O'Keefe and Penttila call the *magic action*. They prove that the magic action of $\mathrm{P\Gamma L}(2,q)$ on $\mathcal{F}$ is (projective) semi-linear and that of the subgroup $\mathrm{PGL}(2,q)$ on $\mathcal{F}$ is (projective) linear. It follows that the magic action of $\mathrm{P\Gamma L}(2,q)$ on $\mathcal{F}$ is determined by the magic action of a collection of generators for $\mathrm{P\Gamma L}(2,q)$. In particular, they use the following generators:

$$\begin{array}{lll}
\sigma_a : \bar{x} \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \bar{x}, & \sigma_a f(z) = a^{-\frac{1}{2}} f(az), & \text{for } 0 \neq a \in K; \\
\tau_c : \bar{x} \mapsto \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \bar{x}, & \tau_c f(z) = f(z+c) + f(c), & \text{for } c \in K; \\
\phi : \bar{x} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{x}, & \phi f(z) = z f(z^{-1}); & \\
\rho_\sigma : \bar{x} \mapsto \bar{x}^\sigma, & \rho_\sigma f(z) = f^\sigma(z), & \text{for } \sigma \in \mathrm{Aut}(K).
\end{array}$$

Let $o(\mathcal{F}) = \{f \in \mathcal{F} : \deg(f) \leq q-2,$ and for each $s \in K,$ the function $f_s(z) = (f(z+s)+f(s))/x,\ x \neq 0,$ is a permutation$\}$. For each $f \in o(\mathcal{F})$, define the following set of points in $\mathrm{PG}(2,q)$:

$$\mathcal{D}(f) = \{(1,t,f(t)) : t \in K\} \cup \{(0,1,0)\}.$$

Then $\mathcal{D}(f)$ is an oval with nucleus $(0,0,1)$ and containing the points $(1,0,0)$ and $(0,1,0)$ precisely because $f \in o(\mathcal{F})$, and each oval with these properties can be written in this form. Hence a polynomial in $o(\mathcal{F})$ is called an o-*permutation*. If an o-permutation f satisfies $f(1)=1$, so that $\mathcal{D}(f)$ contains the point $(1,1,1)$, then f is called an o-*polynomial*. Associated with an o-polynomial f are $q-1$ o-permutations, the non-zero scalar multiples of f. With an o-permutation f is associated a unique o-polynomial $(\frac{1}{f(1)})f$. For $f \in \mathcal{F}$ we use $\langle f \rangle$ to denote the one-dimensional subspace of $\mathcal{F}$ containing f. Clearly all nonzero members of $\langle f \rangle$ give projectively equivalent ovals. A remarkable result of [36] is that $\mathrm{P\Gamma L}(2,q)$ acts (via the magic action) on the set of one-dimensional subspaces spanned by o-permutations, and hence it acts on the ovals. Moreover, if $f, g \in o(\mathcal{F})$, then $\mathcal{D}(f)$ and $\mathcal{D}(g)$ are equivalent under the usual action of $\mathrm{P\Gamma L}(3,q)$ if and only if they are equivalent under the magic action of $\mathrm{P\Gamma L}(2,q)$. What is even more remarkable is that under the magic action $\mathrm{P\Gamma L}(2,q)$ maps q-clans to q-clans and herds to herds. This suggests that one definition of the automorphism group of a herd $\mathcal{H}(\mathbf{C}) = \{\mathcal{D}(f_s) : s \in \tilde{K}\}$ might be that it is the stabilizer of $\{\langle f_s \rangle : s \in \tilde{K}\}$ in $\mathrm{P\Gamma L}(2,q)$ under the magic action.

Until recently the known infinite families of herds were the classical herds, the FTWKB herds, the Payne herds and the Subiaco herds. The automorphism groups of these herds had been determined earlier, (in the sense that $\mathcal{G}_0$ had been determined and the induced stabilizer of each oval was shown to be the

complete stabilizer), but in [36] they were recalculated quite efficiently using the magic action. There were also some examples of herds over fields of small order (cf. [52]) which have been shown by W. E. Cherowitzo, C. M. O'Keefe and T. Penttila [17] to belong to an infinite family called the Adelaide family by their discoverers. In that paper the automorphism groups of the Subiaco and Adelaide herds were calculated in a unified manner. A consequence of this computation is that the stabilizer of the Adelaide ovals contains a cyclic group of order $2e$ (where $q = 2^e$), but it is not determined whether or not this is the complete stabilizer of these ovals.
It is important to keep in mind that the group $\mathcal{G}_0$ of collineations of GQ($\mathbf{C}$) fixing the points (∞) and $(\bar{0}, \bar{0}, 0)$ simultaneously has a (faithful) representation as a permutation group acting on the lines through (∞) and also one acting on the associated herd (i.e., on the set of subquadrangles of order q containing the points (∞) and $(\bar{0}, \bar{0}, 0)$). Of course these two permutation representations are isomorphic as abstract groups, but in general they do not have to be permutation equivalent. As we shall see a little later, the Subiaco geometries for $q = 2^e$ with $e \equiv 2 \pmod 4$ provide examples of this. It seems clear that the group induced by $\mathcal{G}_0$ on the herd should be permutation isomorphic to that induced by the magic action on the herd, but the 'proof' in [36] is quite terse.

4.3 A census of q-clans and herds

We now give a listing of the known q-clans along with an indication of how many inequivalent flocks and inequivalent ovals arise.

1. **Classical** A herd is *classical* provided the associated GQ is isomorphic to the GQ comprising the points and lines of the hermitian variety $H(3, q^2)$. In this case the flock is linear and the translation plane is Desarguesian. The normalized q-clan is

$$\mathbf{C} = \left\{ \begin{pmatrix} t^{\frac{1}{2}} & t^{\frac{1}{2}} \\ 0 & \kappa t^{\frac{1}{2}} \end{pmatrix} : t \in K \right\}$$

(for fixed $\kappa \in K$ with $tr(\kappa) = 1$). The classical herd is $\{\mathcal{D}(f_s) : s \in \tilde{K}\}$, where $f_s(x) = x^{\frac{1}{2}}$ for all $s \in \tilde{K}$. So only one flock and one oval arise.

2. **FTWKB** A herd is FTWKB provided $q = 2^e$ with e odd, and the associated normalized q-clan is given by

$$\mathbf{C} = \left\{ \begin{pmatrix} t^{\frac{1}{4}} & t^{\frac{1}{2}} \\ 0 & \kappa t^{\frac{3}{4}} \end{pmatrix} : t \in K \right\}$$

The flocks arise by the geometrical construction of J. C. Fisher and J. A. Thas [21], but the corresponding translation planes were discovered by M. Walker [75] (using flocks) and independently by D. Betten [7]. The associated GQ were discovered by W. M. Kantor [29] essentially via q-clans. This example is non-classical for $e \geq 2$, in which case still only one flock arises and all the ovals are equivalent to $\mathcal{D}(x^{\frac{1}{4}})$ [50].

3. **Payne** The *Payne* herd, for $q = 2^e$, e odd, has normalized q-clan

$$\mathbf{C} = \left\{ \begin{pmatrix} t^{\frac{1}{6}} & t^{\frac{1}{2}} \\ 0 & t^{\frac{5}{6}} \end{pmatrix} : t \in K \right\}.$$

It is classical if and only if $q = 2$ and FTWKB if and only if $q = 8$. For $q \geq 32$ two flocks arise [47], and two ovals arise, one equivalent to the Segre oval $\mathcal{D}(x^{\frac{1}{6}})$, and one equivalent to $\mathcal{D}(x^{\frac{1}{6}} + x^{\frac{1}{2}} + x^{\frac{5}{6}})$ [44].

The remaining examples are the Subiaco and Adelaide geometries. The Subiaco examples were first given by W. E. Cherowitzo et al.[16] as q-clans. They exist for all $q = 2^e$ and were new for $q \geq 32$, except that certain of the smaller examples had been found by computer and the general construction was obtained in pieces. Since their construction is rather technical, since a rather complete review of them is in [26], and since the new construction of the Adelaide geometries is via a technique that gives both the Subiaco and the Adelaide (as well as the classical) examples, we give only this new version and refer the reader to [16] (also see [51]) for the original constructions. This new construction was discovered during an attempt to generalize the cyclic construction of [52]. However, the problem of giving a direct connection between the Adelaide construction and that in [52] seems to be open.
Let $E = \mathrm{GQ}(q^2)$ be a quadratic extension of $K = \mathrm{GF}(q)$, $q = 2^e$. Let $1 \neq \beta \in E$ satisfy $\beta^{q+1} = 1$, and let $T(x) = x + x^q$ for all $x \in E$. Let $a \in K$ and $f, g : K \to K$ be defined by:

$$a = \frac{T(\beta^m)}{T(\beta)} + \frac{1}{T(\beta^m)} + 1;$$

$$f(t) = f_{m,\beta}(t) = \frac{T(\beta^m)(t+1)}{T(\beta)} + \frac{T((\beta t + \beta^q)^m)}{T(\beta)(t + T(\beta)t^{\frac{1}{2}} + 1)^{m-1}} + t^{\frac{1}{2}};$$

and

$$ag(t) = ag_{m,\beta}(t) = \frac{T(\beta^m)}{T(\beta)}t + \frac{T((\beta^2 t + 1)^m)}{T(\beta)T(\beta^m)(t + T(\beta)t^{\frac{1}{2}} + 1)^{m-1}} + \frac{1}{T(\beta^m)}t^{\frac{1}{2}}.$$

Finally, put

$$\mathbf{C} = \mathbf{C}_{m,\beta} = \left\{ \left(\begin{array}{cc} f(t) & t^{\frac{1}{2}} \\ 0 & ag(t) \end{array} \right) : t \in K \right\}.$$

Then W. E. Cherowitzo, C. M. O'Keefe and T. Penttila [17] prove the following remarkable theorem.

Theorem 5 *If $m \equiv \pm 1 \pmod{q+1}$, then* **C** *is the classical q-clan for all $q = 2^e$ and for all β. If $q = 2^e$ with e odd and $m \equiv \pm\frac{q}{2} \pmod{q+1}$, then* **C** *is the example first found as a q-clan by W. M. Kantor and which gives the Fisher-Thas-Walker Flock. If $q = 2^e$ with $m = \pm 5$, then* **C** *is the Subiaco q-clan for all β such that if λ is a primitive element of K and $\beta = \lambda^{k(q-1)}$, then $q+1$ does not divide km. If $q = 4^e > 4$ and $m \equiv \pm\frac{q-1}{3} \pmod{q+1}$, then for all β,* **C** *is a q-clan called the* Adelaide *q-clan.*

4. **Subiaco** Given $q = 2^e$, the q-clan is unique up to equivalence, giving just one flock in all cases. The situation for herds is different. For $e \equiv 2 \pmod 4$ there are two types of ovals. The Subiaco herd has $\frac{q+1}{5}$ ovals of one type, the remaining $\frac{4(q+1)}{5}$ ovals being of one other type. For all other cases there is just one flock and just one oval.

5. **Adelaide** These are constructed for $q = 2^e$ with e even. The magic action of the group of the herd is shown in [17] to be $C_{q+1} \rtimes C_{2e}$, but the treatment given of the effect of the magic action on the lines of GQ(**C**) through (∞) and of that on the ovals of the herd is a bit murky. However, for the examples with $q = 4^k$, $k \leq 8$, that were studied by computer, it is known that the group is transitive both on the lines through (∞) and on the ovals of the herd, so that only one of each arises. This certainly must be true in general, but these examples must be studied some more.

5 Sporadic q-clans with q odd

With the discovery of the Adelaide geometries [17] all q-clans known to us with q a power of 2 belong to one or more of the infinite families mentioned in the previous section. For q odd the situation is quite different. With computer aided searches at least 38 different (presently sporadic) q-clans have been discovered, many producing several new flocks. We start with one that has interesting properties and seems to be truly sporadic.

5.1 An example with $q = 3^5$

This example was found by an interesting mix of theory and computer assistance. T. Penttila and B. Williams [60], with a slight assist from a computer, discovered a translation ovoid of the quadric $Q(4, 3^5)$. Already in 1994 (see the proceedings of the Oberwolfach meeting on Designs and Codes in 1994), J. A. Thas [67] had shown how to interpret a translation ovoid of a quadric $Q(4, q)$ as a q-clan and flock whose associated GQ is the point-line dual of a translation GQ. L. Bader, G. Lunardon and I. Pinneri [3] translated the ovoid into a q-clan and flock using this method, which is also elucidated in [35]. Then L. Bader, D. Ghinelli and T. Penttila [1] gave a computer-free proof of the existence of the q-clan. The q-clan for this flock is given by

$$\mathbf{C} = \left\{ A_t = \begin{pmatrix} t & -t^{27} \\ -t^{27} & -t^9 \end{pmatrix} : t \in K \right\}.$$

The entries of this matrix are additive functions of t, so that the associated translation plane is a semifield plane and the corresponding generalized quadrangle is the point-line dual of a translation GQ, both of which are shown to be new in [3]. Exactly two flocks arise, the original semifield flock and q copies of a non-semifield flock. There is then also the translation dual GQ (as in [54]), which is shown in [3] to be new and which itself has a point-line dual that is not a flock quadrangle for any flock. This is the only such example known other than the family of examples in Section 3.6. In [1] there is made a tangential connection between this example and the sporadic simple group M_{11}.

Another observation about the above q-clan is that each entry of the given matrix is a monomial function of t, hence $\mathbf{C}$ is called a monomial q-clan. For q even, the first three families of Section 4. are monomial, and in [59] these are shown to be the only ones. For q odd, the first four families of Section 3. are monomial. In [1] a computer search is used to show that semifield monomial flocks are completely classified for $q = 3^n$ and $n \leq 100$, and for $q = p^n$ with p prime and $p \leq 100$ and $n \leq 100$. In fact, the authors do a great deal more. A collection of $q - 1$ pairwise disjoint irreducible conics on a quadratic cone is a *regular partial flock* provided there is a subgroup of $\mathrm{PGL}(4, q)$ acting regularly on the set of conics. In [55] it is shown that any partial flock of order $q-1$ can be uniquely completed to a flock. Hence regular partial conical flocks give rise to (complete) flocks. Y Hiramine and N. L. Johnson [23] have classified regular partial conical flocks into three general types. In [1] the first type is studied via computer, with no new examples turning up and rather strong conditions being imposed on the remaining possibilities.

5.2 Other sporadic examples

In the table below we list the remaining known sporadic examples found by computer. In the column of attributions, we use the following notation: DCH refers to [19], PR to [58], LP to [32] (which will appear as a sequel to [58]), and DCP to an unpublished result of F. De Clerck and T. Penttila that will appear in [32]. The representation of $\mathcal{G}_0$ on the lines through (∞) has a kernel $\mathcal{N}$ of order $q-1$. We let $\bar{\mathcal{G}}_0$ denote the quotient group $\mathcal{G}_0/\mathcal{N}$. In the column labeled 'orbits' an entry of the form $a^3.b^2$ means three orbits of length a and two orbits of length b.

q	$\lvert\bar{\mathcal{G}}_0\rvert$	orbits	who?	No. of flocks
17	144	12.6	DCH	2
17	24	12.6	PR	2
19	20	20	PR	1
19	16	$8^2.2^2$	PR	4
23	72	18.6	DCH	2
23	1152	24	PR	1
23	24	24	PR	1
23	16	$8^2.4^2$	PR	4
23	6	$6^3.3^2$	PR	5
25	16	16.8.2	PR	3
25	8	$8^3.2$	PR	4
29	720	30	LP	1
29	48	24.6	LP	2
29	8	$8^3.4.2$	LP	5
29	6	$6^4.3^2$	LP	6
29	3	3^10	LP	10
31	96	24.6.2	LP	3
31	10	$10^2.5^2.2$	LP	5
31	8	$8^3.4^2$	LP	5
31	4	4^8	LP	8
37	72	36.2	LP	2
37	4	$4^9.2$	LP	10
37	4	$4^9.1^2$	LP	11
41	60	30.12	LP	2
41	24	24.12.6	LP	3
41	8	$8^5.2$	LP	6
43	4	4^{11}	LP	11
47	2304	48	DCP	1
47	24	24^2	LP	2
47	3	3^{16}	LP	16
49	40	40.10	LP	2
49	20	$20^2.5^2$	LP	4
53	24	$24^2.6$	LP	3
53	12	$12^4.6$	LP	5
59	120	60	LP	1
59	5	5^12	LP	12
125	72	$72.24^2.6$	LP	4

6 Characterizations, spreads and a geometric construction

6.1 Property (G) and flock GQ

Let $\mathrm{GQ}(\mathcal{F})$ be a generalized quadrangle of order (q^2, q) arising from a flock $\mathcal{F}$. In [46] it was proved that $\mathrm{GQ}(\mathcal{F})$ satisfies the following condition called Property (G) at the point (∞).
Start with a $\mathrm{GQ}(q^2, q) = \mathcal{S}$, and let L_1 and M_1 be any distinct lines of $\mathcal{S}$ incident with the point p. Suppose that (L_1, L_2, L_3, L_4) are four distinct pairwise nonconcurrent lines and (M_1, M_2, M_3, M_4) are any four distinct lines for which L_i is concurrent with M_j whenever $2 \leq i + j \leq 7$. We say $\mathcal{S}$ has *Property (G) at the point* p provided it always follows that L_4 and M_4 are concurrent. Because the parameters (s, t) of the GQ have $s = t^2$, any triad of pairwise nonconcurrent lines has a perp of size $1 + q$, and there is another way to express Property (G). Alternatively, we say that $\mathcal{S}$ has Property (G) at p provided that whenever (L_1, L_2, L_3) is a triad of (pairwise nonconcurrent) lines, one of which is incident with p, and for which there is a line M_1 incident with p and concurrent also with all of L_1, L_2 and L_3, then $|\{L_1, L_2, L_3\}^{\perp\perp}| = 1 + q$.
In a truly remarkable sequence of papers, J. A. Thas ([65], [66], [69], [68]) has proved (among several other things) that when q is odd, if $\mathcal{S}$ is a GQ of order (q^2, q) and has Property (G) at a point p, then $\mathcal{S}$ is a flock GQ. When q is even an additional hypothesis must be used to reach this conclusion, and in the general case his result is slightly better than the one just mentioned. The reference [68] gives an excellent survey (along with some improvements) of these and related results. Here we give only the barest outline.
To explain the results we find it convenient to switch to the point-line dual setting. Let $\mathcal{S} = (P, B, I)$ be a $\mathrm{GQ}(q, q^2)$, so each triad of points has a perp of size $q + 1$, from which it follows that its span (or perp perp) has size at most $q + 1$. The triad (y_1, y_2, y_3) is said to be 3-*regular* provided $|\{y_1, y_2, y_3)^{\perp\perp}| = q + 1$. Let (p, L) be a flag. We say that $\mathcal{S}$ *has* Property (G) *at the flag* (p, L) provided that each triad $(y_1, y_2, y_3) \subseteq p^{\perp}$ with y_1 incident with L is 3-regular. In [68] the following theorem is proved.

6.1.1 Property G and flock GQ, q odd

Theorem 6 *If $\mathcal{S}$ has Property (G) at the flag (p, L) and q is odd, then $\mathcal{S}$ is the point-line dual of a flock GQ $\mathrm{GQ}(\mathcal{F})$ for some flock $\mathcal{F}$ of a quadratic cone. Moreover, the line L is regular, so there is a dual net $\mathcal{N}_L^*$. This dual* *net satisfies the Axiom of Veblen (cf. [68]) if and only if $\mathcal{F}$ is a Kantor Flock (i.e., $\mathcal{F}$ is of the type given in 3.3).*

Now suppose $\mathcal{S}$ has Property (G) at the flag (p, L) and y is any point of L different from p. Consider the incidence structure $S_{xy} = (P_{xy}, B_{xy}, I_{xy})$ where

- $P_{xy} = x^\perp \setminus \{x, y\}^{\perp\perp}$.
- Elements of B_{xy} are of two types: (a) the sets $\{y, y', y''\}^{\perp\perp} \setminus \{y\}$, with $\{y, y', y''\}$ a triad contained in $x^\perp$; and (b) the sets $\{x, w\}^\perp \setminus \{x\}$, with $x \sim w \not\sim y$.
- I_{xy} is containment.

It follows that the incidence structure S_{xy} is the design of points and lines of the affine space $AG(3, q)$. In particular, q is a prime power. Now let z be a point of P not on L, and collinear with a point u on L with $x \neq u \neq y$. The q^2 lines of type (b) of S_{xy} are parallel, so they define a point ∞ of the projective completion $\mathrm{PG}(3, q)$ of $AG(3, q)$. Then $(\{x, z\}^\perp \setminus \{u\}) \cup \{\infty\}$ is an ovoid O_z of $\mathrm{PG}(3, q)$. We can now state the theorem for q even.

6.1.2 Property G and flock GQ, q even

Theorem 7 *Let $\mathcal{S}$ be a GQ of order (q, q^2), $q = 2^e > 1$. If $\mathcal{S}$ has Property (G) at the flag (p, L), and if each ovoid O_z is an elliptic quadric, then $\mathcal{S}$ is the point-line dual of a flock GQ. The line L is regular, and the corresponding dual net $\mathcal{N}_L^*$ satisfies the Axiom of Veblen (so is known; cf. [68]).*

The above theorem makes it very interesting to know all the ovoids in $\mathrm{PG}(3, q)$, q even. For $q = 2, 4$ or 16 it is known that all ovoids are elliptic quadrics (cf. [38] and [39]). The only known ovoids are the elliptic quadrics that exist for all $q = 2^e$ and the Tits ovoids that exist for $q = 2^e$ with e odd and greater than 1. The problem of determining all ovoids in $\mathrm{PG}(3, q)$ remains unsolved for $q \geq 64$ and is likely to remain so for a while yet in spite of a great deal of progress on this problem. For example, M. Brown has proved that if even one secant plane section of an ovoid $\mathcal{O}$ is a conic, then the ovoid is an elliptic quadric (cf. [13]). He also proved that if even one secant plane section is a 'pointed conic' (i.e., an oval obtained by interchanging the nucleus of a conic with any point of the conic), then either $q = 4$ and the ovoid is an elliptic quadric, or $q = 8$ and the ovoid is a Tits ovoid ([12]. Even if it should turn out that the only ovoids are the known ones, the following conjecture of J. A. Thas [68] would remain open.
Conjecture. ([68]) Let $\mathcal{S}$ be a GQ of order (q, q^2), q even, and assume that $\mathcal{S}$ satisfies Property (G) at some line L. Then $\mathcal{S}$ is the dual of a flock GQ.

We note that a GQ $T_3(O)$ of J. Tits with O a Tits ovoid (cf. [54]) satisfies Property (G) at any flag (∞, L), with ∞ the unique point p of $T_3(O)$ for which each flag (p, L) has Property (G). As $T_3(O)$ is a translation GQ, it is not the dual of a flock GQ by the main result of [27].

6.2 Characterizations of translation GQ

J. A. Thas has obtained deep and beautiful characterization theorems for translation GQ, but only in some cases do his results apply to the point-line dual of a flock GQ. We refer the reader to [68] and [62] for excellent expositions of these results. For the present we mention only one result from [62]: Let GQ($\mathcal{F}$) be a GQ of order (q^2, q), q even, arising from a flock $\mathcal{F}$. If the point (∞) of GQ($\mathcal{F}$) is collinear with a regular point different from (∞), then GQ($\mathcal{F}$) is isomorphic to the classical GQ $H(3, q^2)$, i.e., the flock $\mathcal{F}$ is linear.

6.3 Subquadrangles

Any 3-regular triad of points of a GQ S of order (s, s^2), s even, defines a subquadrangle $\mathcal{S}'$ of order s (cf. 2.6.2 of [54]). Also, from the point-line dual point of view, any flock quadrangle $\mathcal{S}(\mathcal{F})$ has a herd of ovals and corresponding subquadrangles isomorphic to $T_2(O)$ for O in the herd. As noted in [63], this accounts for at least $q^3 + q^2$ subquadrangles of order q. In [37] it is shown that these subquadrangles and their images under the collineation group of $\mathcal{S}(\mathcal{F})$ are the only subquadrangles of order q containing the point (∞). It is also noted that every isomorphism between known subquadrangles $\mathcal{S}_1$ and $\mathcal{S}_2$ of order q of a known GQ $\mathcal{S}$ of order (q^2, q), q even, extends to an automorphism of $\mathcal{S}$. In fact, it is shown that the herd associated with a flock GQ (and hence also the set of subquadrangles of order q containing the point (∞)) is an invariant of the flock GQ. The article [37] is a convenient reference for a variety of results concerning the automorphisms and subquadrangles of the classical GQ $Q(5, q)$ and the Tits generalized quadrangles $T_3(\Omega)$ where Ω is the Tits ovoid.

6.4 Spreads, ovoids and generalized fans

In a GQ of order (s, t) a *spread* is a set of $1 + st$ lines that are pairwise nonconcurrent, i.e., they partition the pointset. Dually, an ovoid is a set of $1+ st$ pairwise noncollinear points. Many papers have been written concerning spreads and ovoids of GQ. In particular, any flock GQ of order (q^2, q) has ovoids (cf. [71]). In [68] this is improved for q even to say that any GQ of

order (q^2, q), q even, that satisfies Property (G) at some point must have ovoids.
Let $\mathcal{S} = \mathrm{GQ}(\mathcal{F})$ be the GQ of order (q^2, q) arising from the flock $\mathcal{F}$ of the quadratic cone $\mathcal{C}$ in $\mathrm{PG}(3, q)$, q even. Let $\mathcal{S}' = T_2(O)$, O an oval of $\mathrm{PG}(2, q)$, be one of these subquadrangles, so that the point (∞) of $T_2(O)$ is also the point (∞) of $\mathrm{GQ}(\mathcal{F})$. Let M be a line of $\mathrm{GQ}(\mathcal{F})$ not belonging to the subquadrangle $T_2(O)$. Each point of M is on a unique line of $T_2(O)$, giving a spread S_M of $T_2(O)$. Recall the model of $T_2(O)$ starting with the oval O of $\mathrm{PG}(2, q)$ embedded in $\mathrm{PG}(3, q)$. The $q^2 + 1$ lines of S_M consist of a point $y \in O$ (viewed as a line of $T_2(O)$), and q^2 lines not in $\mathrm{PG}(2, q)$ of q quadratic cones $\mathcal{C}_x$, $x \in O \setminus \{y\}$, of the space $\mathrm{PG}(3, q)$ containing $T_2(O)$, where $\mathcal{C}_x$ has vertex x, is tangent to $\mathrm{PG}(2, q)$ at xy and has nucleus line xn, with n the nucleus of O. This theorem is a main result of [63] and it also appears in [14] along with many other results. The oval O is a conic if and only if the flock $\mathcal{F}$ is linear. This result first appeared in [40], but it reappeared in [63] as an application of the previous result. The intersection of the lines of $\mathcal{C}_x$, $x \in O \setminus \{y\}$, with a fixed plane π ($\neq \mathrm{PG}(2, q)$) through the line yn, gives a collection of ovals of the plane π called a *generalized fan*.
The paper [14] makes a detailed study of the spreads of $T_2(O)$, but most its results are only indirectly related to flock GQ. For example, if α is a generator of $\mathrm{Aut}(\mathrm{GF}(q)$, $q = 2^e$, the paper [14] establishes a link between α-flocks (flocks of cones over translation ovals $\mathcal{D}(\alpha)$) and generalized fans where each oval of the fan is isomorphic to $\mathcal{D}(1/\alpha)$.
Let $\mathcal{S}' = (P', B', I')$ be a subquadrangle of order s of the GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s > 1$. If $x \in P \setminus P'$, then the $s^2 + 1$ points of P' which are collinear with x form an ovoid O_x of $\mathcal{S}'$ (cf. 2.2.1 of [54]). We say that $\mathcal{S}'$ is *doubly subtended* when for each point $x \in P \setminus P'$ there is exactly one point $x' \in P \setminus P'$, with $x \neq x'$, for which $O_x = O_{x'}$. J. A. Thas [68] proves that if $\mathcal{S}$ has a doubly subtended subquadrangle $\mathcal{S}'$, then if s is even $\mathcal{S}'$ is isomorphic to the GQ $W(q)$ arising from a symplectic polarity, and if s is odd then all points of $\mathcal{S}'$ are *antiregular*, i.e., each triad of points has a perp of size 0 or 2.
Consider the following theorem from [71]: Let $\mathcal{S}$ be a GQ of order (q, q^2), q even, having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, q)$. If in $\mathcal{S}'$ each ovoid O_x consisting of all points of $\mathcal{S}'$ collinear in $\mathcal{S}$ with a given point x of $\mathcal{S} \setminus \mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, q)$. It follows that any GQ of order (16, 4) that has a subquadrangle of order 4 must be isomorphic to $Q(5, 4)$. In his Ph.D. thesis, M. R. Brown improves this theorem by weakening the hypothesis to the assumption that the subtended ovoids be elliptic quadrics or Tits ovoids.
A *rosette* based at a point X of a GQ $\mathcal{S}$ of order (s, t) is a set $\mathcal{R}$ of ovoids

with pairwise intersection $\{X\}$ and such that $\{\theta\backslash\{X\} : \theta \in \mathcal{R}\}$ is a partition of the points of $\mathcal{S}$ not collinear with X. The point X is called the *base point* of $\mathcal{R}$. It follows that a rosette $\mathcal{R}$ has s ovoids.
If $\mathcal{S} = (P, B, I)$ is a GQ of order (r, r^2) with a subquadrangle $\mathcal{S}' = (P', B', I')$ of order r, then every line of $\mathcal{S}$ is either a line of $\mathcal{S}'$ or is incident with exactly one point of $\mathcal{S}'$. A line of $\mathcal{S}$ meeting $\mathcal{S}'$ in exactly one point is called a *tangent*. Given a tangent line l to $\mathcal{S}'$, the set of r ovoids subtended by points of l not in $\mathcal{S}'$ forms a rosette of $\mathcal{S}'$. We say that this rosette is the rosette *subtended* by the line l or that the rosette is *subtended*.
A (finite) *semipartial geometry* (SPG) is an incidence structure $\mathcal{T} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $I \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If X is a point and l is a line not incident with X, then the number of pairs $(Y, m) \in P \times B$ for which $XImIYIl$ is either a constant α ($\alpha > 0$), or 0;

(iv) For any pair of non-collinear points (X, Y) there are μ ($\mu > 0$) points Z such that Z is collinear with both X and Y.

The integers s, t, α, μ are the *parameters* of $\mathcal{T}$. For more information on SPGs see [20].
By Theorem 3.1.7 of M. Brown [11], if $\mathcal{S}$ is a GQ of order (r, r^2) and $\mathcal{S}'$ is a subGQ of order r such that each subtended ovoid of $\mathcal{S}'$ is subtended by precisely two points, then the subtended ovoid/rosette structure is an SPG with parameters $s = r - 1$, $t = r^2$, $\alpha = 2$ and $\mu = 2r(r - 1)$. M. Brown also shows that the above condition is equivalent to the existence of an involution of $\mathcal{S}$ that fixes $\int'$ pointwise. By starting with the GQ of order (q, q^2), (q odd and not prime), arising from the q-clan in Example 3.3 first given by Kantor, which has classical subquadrangle of order q, M. Brown finds SPQs not isomorphic to any previously known ones.

6.5 The geometric construction by J. A. Thas of all flock GQ

The problem of finding a geometric construction of a flock GQ was open for quite some time. As the years passed after N. Knarr [31] found such a construction for q odd, many researchers came to believe that no such construction could exist for q even. Hence it was quite surprising when J. A. Thas presented his construction in [69], with refinements of related material in [68] and [70]. This construction represents a major resolution of a long open problem, and no survey of recent results on flock GQ would be complete without it.

Let $\mathcal{C}$ be a quadratic cone with vertex x of $\mathrm{PG}(3,q)$. Further, let y be a point of $\mathcal{C} \setminus \{x\}$ and let ζ be a plane of $\mathrm{PG}(3,q)$ not containing y. Project $\mathcal{C} \setminus \{y\}$ from y onto ζ. Let τ be the tangent plane of $\mathcal{C}$ at the line xy and let $\tau \cap \zeta = T$. Then with the q^2 points of $\mathcal{C} \setminus xy$ there correspond the q^2 points of the affine plane $\zeta - T = \zeta'$; with any point of $xy \setminus \{y\}$ there corresponds the intersection ∞ of xy and ζ; with the generators of $\mathcal{C}$ distinct from xy there correspond the lines of ζ distinct from T containing ∞; with the (nonsingular) conics on $\mathcal{C}$ passing through y there correspond the affine parts of the q^2 lines of ζ not passing through ∞; and with the (nonsingular) conics on $\mathcal{C}$ not passing through y there correspond the $q^2(q-1)$ (nonsingular) conics of ζ which are tangent to T at ∞.

Let $F = \{C_1^*, C_2^*, \ldots, C_q^*\}$ be a flock of the cone $\mathcal{C}$. Now consider the set $\tilde{F} = \{C_1, C_2, \ldots, C_{q-1}, N\}$ consisting of the $q-1$ nonsingular conics $C_1, C_2, \ldots, C_{q-1}$ and the line N of ζ, which is obtained by projecting the elements of F from y onto ζ. So $C_1, C_2, \ldots, C_{q-1}$ are conics which are mutually tangent at ∞ (with common tangent line T) and N is a line of ζ not containing ∞.

Now we consider planes $\pi_\infty \neq \zeta$ and $\mu \neq \zeta$ of $\mathrm{PG}(3,q)$, respectively containing T and N. In μ we consider a point r, with $r \notin \zeta \cup \pi_\infty$. Next, let O_i be the nonsingular quadric which contains C_i, which is tangent to π_∞ at ∞ and which is tangent to μ at r, with $i = 1, 2, \ldots, q-1$. As $C_i \cap N = \emptyset$, the quadric O_i is elliptic, $1 \leq i \leq q-1$.

Next let $\mathcal{S}$ be the following incidence structure.

Points of $\mathcal{S}$

(a) The $q^3(q-1)$ nonsingular elliptic quadrics O containing $O_i \cap \pi_\infty = L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathrm{GF}(q^2)$) such that the intersection multiplicity of O_i and L at ∞ is at least three (these are O_i, the nonsingular elliptic quadrics $O \neq O_i$ containing $L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathrm{GF}(q^2)$) and intersecting O_i over $\mathrm{GF}(q)$ in a nonsingular conic containing ∞, and the nonsingular elliptic quadrics $O \neq O_i$ for which $O \cap O_i$ over $\mathrm{GF}(q^2)$ is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice), with $1 \leq i \leq q-1$.

(b) The q^3 points of $\mathrm{PG}(3,q) \setminus \pi_\infty$.

(c) The q^3 planes of $\mathrm{PG}(3,q)$ not containing ∞.

(d) The $q-1$ sets $\mathcal{O}_i$, where $\mathcal{O}_i$ consists of the q^3 quadrics O of type (a) corresponding with O_i, $1 \leq i \leq q-1$.

(e) The plane π_∞.

(f) The point ∞.

Lines of $\mathcal{S}$

(i) Let (ω, γ) be a point-plane flag of $\mathrm{PG}(3,q)$, with $\omega \notin \pi_\infty$ and $\infty \notin \gamma$. Then all quadrics O of type (a) which are tangent to γ at ω, together with ω and γ, form a line of type (i). Any two distinct quadrics of such a line have exactly two points (∞ and ω) in common. The total number of lines of type (i) is q^5.

(ii) Let O be a point of type (a) which corresponds to the quadric O_i, $1 \leq i \leq q-1$. If $O \cap \pi_\infty = O_i \cap \pi_\infty = L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathrm{GF}(q^2)$), then all points O' of type (a) for which $O' \cap O$ over $\mathrm{GF}(q^2)$ is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice, together with O and O_i, form a line of type (ii). There are $q^2(q-1)$ lines of type (ii).

(iii) A set of q parallel planes of $AG(3,q) = \mathrm{PG}(3,q) \setminus \pi_\infty$, where the line at infinity does not contain ∞, together with the plane π_∞, is a line of type (iii).

(iv) Lines of type (iv) are the lines of $\mathrm{PG}(3,q)$ not in π_∞ containing ∞.

(v) $\{\infty, \pi_\infty, \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{q-1}\}$ is the unique line of type (v).

Incidence of $\mathcal{S}$

Incidence is containment.

It is proved in [69] that $\mathcal{S}$ is a GQ isomorphic to the point-line dual of the flock GQ $\mathrm{GQ}(F)$.

7 Additional remarks

After the writing of this paper was essentially completed there came to our attention a new connection between flocks (qua q-clans) and another geometric construct. R. D. Baker, G. L. Ebert and K. L. Wantz [6] define a *hyperbolic fibration* to be a set of $q-1$ hyperbolic quadrics and two lines which together partition the points of $\mathrm{PG}(3,q)$. Such a fibration is called *regular* provided the two (skew) lines of the fibration are conjugate with respect to each of the hyperbolic quadrics. A surprising discovery is that a special type of regular hyperbolic quadric is equivalent to a flock of a quadratic cone. The interest in hyperbolic fibrations derives from their connection with linespreads of $\mathrm{PG}(3,q)$, and hence with translation planes. The interconnections are still being sorted out, but this new discovery adds its own emphasis to our sense that the theory of flocks and their related geometries is very much alive and is at the heart of a great deal of current research on finite projective geometries and their special substructures.
Other results related to flock GQ are given by K. Thas in the preprint [72]. A point p of a $\mathcal{S} = \mathrm{GQ(s,t)}$ is a *translation point* provided each line through p is an axis of symmetry. This says $\mathcal{S}$ is a translation GQ with base point p for which the associated elation group is an abelian group (whose elements are called *translations*). We close this survey with the main result from [72].

Theorem 8 (K. Thas [72]) *Let $\mathcal{S}$ be a* GQ*(s,t), $s > 1$, $t > 1$, and suppose $\mathcal{S}$ has two distinct translation points. Then one of the following must occur.*

i The two translation points are noncollinear and $\mathcal{S}$ is of classical type.

In all other cases the two translation points are collinear.

ii $s = t$ and $\mathcal{S}$ is isomorphic to $Q(4,s)$.

iii $s \neq t$, s is even and $\mathcal{S}$ is isomorphic to $Q(5,s)$.

iv $s = q^n$, $t = q^{2n}$ with q some odd prime power, $q \geq 4n^2 - 8n + 2$, and $\mathcal{S}$ is the point-line dual of a flock GQ whose flock is a Kantor flock.

v $s = q^n$, $t = q^{2n}$ with q some odd prime power, $q < 4n^2 - 8n + 2$, and $\mathcal{S}$ is the translation dual of the point-line dual of some flock GQ.

Note: The inequality in part (iv) derives from some very interesting work in progress by A. Blokhuis, M. Lavrauw and S. Ball [10].

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