Model theory applied to generalized polygons and conversely

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Abstract

We describe the natural connection between model theory and generalized polygons and give some applications of one theory to the other.

1 Introduction

Model theory is concerned with the study of first order structures; i.e. given a fixed finitary first order language L consisting of predicate, relation and function symbols of some fixed arities, a set of constant symbols, and of variables, quantifiers and a symbol for equality, negation, conjunction etc, an L-structure $\mathfrak{A}$ is given by a set A, called *the universe*, with some interpretation of the predicate, function and constant symbols in L.

A *substructure* of an L-structure $\mathfrak{A}$ is given by a subset of A and the corresponding interpretation for the symbols in L, where we ask that A is closed under application of the function symbols in L. In particular, if L does not contain any function symbols, then any subset gives rise to a substructure.

The natural example which will be of interest here is the language $L_{pol} = (P, L, I)$ for generalized polygons with a predicate symbol for points and lines, respectively, and a binary relation symbol for the incidence. For an L_{pol}-structure to be in fact a generalized polygon, the structure also has to satisfy the *axioms* of a generalized n-gon for some fixed n. The class of generalized n-gons for fixed n forms an elementary class; i.e. there are first order axioms in the language L_{pol} such that an L_{pol}-structure is a generalized n-gon if and only if it satisfies this set of axioms: First we ask that the incidence graph is bipartite:

(0) $\forall x(P(x) \vee L(x)) \wedge \neg\exists x(P(x) \wedge L(x)) \wedge \forall x\forall y(xIy \rightarrow \ (P(x) \leftrightarrow L(y)))$

Then we ask that the distance between any two elements is at most n:

(i) $\forall x\forall y\, (\exists x_1 \ldots \exists x_{n-2}(xIx_1\wedge \ldots x_{n-2}Iy))\vee(\exists x_1 \ldots \exists x_{n-1}(xIx_1\wedge \ldots x_{n-1}Iy))$

Next we require that there are no proper cycles (i.e. without repetitions) of length less than $2n$; so we have for all $1 < k < n$:

(ii)$_k$ $\forall x_0 \dots \forall x_{2k-1}(x_0 I x_1 \wedge \dots \wedge x_{2k-1} I x_0 \rightarrow x_0 = x_2 \vee x_1 = x_3 \vee \dots \vee x_{2k-3} = x_{2k-1} \vee x_{2k-2} = x_0 \vee x_{2k-1} = x_1)$

and finally we require that every element be incident with at least three other elements:

(iii) $\forall x \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge x I x_1 \wedge x I x_2 \wedge x I x_3)$.

Notice that while the class of groups is also elementary in a suitable first order language containing a function symbol for the group operation (and possibly other symbols), the class of *nilpotent* groups for example is not first-order axiomatizable. A set of axioms or (L-)sentences is also called a(n L-)*theory*. An L-structure in which all the axioms of the theory are true is called a *model* of the theory.

While this shows that generalized polygons are a natural object for model theorists to study, mostly model theory is concerned with the study of structures which have interesting model theoretic properties. The property which will here be of greatest interest is the property of having *finite Morley rank*, which we will explain now.

Let L be a countable first order language and $\mathfrak{A}$ be an L-structure. Then a subset $X \subseteq A^n$ for some n is called a *definable set* if there is some formula $\phi = \phi(x_1, \dots x_n, \bar{b})$ in the language L with parameters $\bar{b} \subseteq A$ finite, such that X is the set of *solutions* of the formula $\phi(\bar{x}, \bar{b})$ in $\mathfrak{A}$; i.e., $X = \{(a_1, \dots a_n) \in A^n : \mathfrak{A} \models \phi(\bar{a}, \bar{b})\}$.

Definition 1 Let L be a countable first order language and $\mathfrak{A}$ be an L-structure. The *Morley rank* of a definable set X is inductively defined as follows:

(i) $RM(X) \geq 0$ if and only if $X \neq \emptyset$;

(ii) $RM(X) \geq n+1$ if and only if there are definable pairwise disjoint sets $X_i \subseteq X$ for $i < \omega$ with $RM(X_i) \geq n$;

(iii) $RM(X) \geq \omega$ if and only if $RM(X) \geq n$ for all $n < \omega$.

We say that $RM(X) = n$ if $RM(X) \geq n$ and $RM(X) \not\geq n+1$. We say that the structure $\mathfrak{A}$ has finite Morley rank if *every* definable subset of A^n for each n has this property.

Notice that for an infinite definable set, the Morley rank is always at least 1. An infinite definable set X is called *strongly minimal*, if every definable subset is either finite or co-finite, so its complement is finite. So $RM(X) = 1$ and X does not contain disjoint definable subsets of Morley rank 1. If you think of the Morley rank as measuring the complexity of the definable subsets of a given set, then the strongly minimal sets are the simplest infinite sets to be considered.

Clearly, the Morley rank of a definable set depends on the language under consideration. So from now on, we will always think of a *fixed first order language L*. In general there is no reason why the Morley rank of a definable set should be finite. Consider for example a densely ordered set X in the language $L = \{<\}$. Then any interval will have Morley rank at least 1, and it follows inductively that the Morley rank of X cannot be a finite number.

The Morley rank is a model theoretic dimension, which to some extent, behaves like the algebraic dimension in the sense of algebraic geometry. In fact, in the context of algebraic geometry, Morley rank and algebraic dimension coincide, and by a result of Macintyre, any infinite field definable in a structure of finite Morley rank is necessarily algebraically closed.

2 Applications of model theory to generalized polygons

As we just saw in the introduction, for fixed $n \geq 3$, the class of generalized n-gons is in a natural way an elementary class to which methods and results from model theory might be applied.

2.1 Projective planes with highly transitive line stabilizers

The first example of such an application is the construction of a countably infinite projective plane whose automorphism group acts highly transitively (i.e. k-transitively for any k) on the point rows. In fact, it acts much more transitively, as we will see below.

We first need to introduce some terminology.

An L-structure $\mathfrak{A}$ is said to be *finitely generated* if there are finitely many elements $b_1, \ldots b_n \in A$ such that $\mathfrak{A}$ is the smallest L-structure containing

$b_1, \dots, b_n$. A *substructure* of an L-structure $\mathfrak{A}$ is a subset B of A which together with the induced interpretation of the symbols in L forms an L-structure in its own right. Since L_{pol} does not contain any function symbols, any subset of an L_{pol}-structure is again an L_{pol}-structure. But clearly, not every substructure of a generalized polygon is itself a generalized polygon. An L_{pol}-substructure of a projective plane is called a *partial* (projective) plane; i.e. it is a bipartite graph without cycles of length less than 6, not necessarily of valency at least 3 and in which not necessarily every two lines intersect or every two points are joined by a line.

Let L be a countable language and let $\mathbf{K}$ be a set of finitely generated L-structures. Then $\mathbf{K}$ is said to be *closed under substructures*, if for $\mathfrak{A} \in \mathbf{K}$ any finitely generated substructure $\mathfrak{B}$ of $\mathfrak{A}$ is isomorphic to some structure in $\mathbf{K}$. The class $\mathbf{K}$ is said to have the *joint embedding property* **(JEP)** if for $\mathfrak{A}, \mathfrak{B} \in \mathbf{K}$, there is some $\mathfrak{C} \in \mathbf{K}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ are embeddable in $\mathfrak{C}$; and $\mathbf{K}$ is said to have the *amalgamation property* **(AP)** if for $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbf{K}$, and embeddings $e : \mathfrak{A} \longrightarrow \mathfrak{B}, f : \mathfrak{A} \longrightarrow \mathfrak{C}$, there is some $\mathfrak{D} \in \mathbf{K}$ and embeddings $g\mathfrak{B} \longrightarrow \mathfrak{D}, f : \mathfrak{C} \longrightarrow \mathfrak{D}$ such that $ge = hf$.

An L-structure D is called homogeneous is every isomorphism between finitely generated substructures of D extends to an automorphism of D.

To obtain projective planes with highly transitive line stabilizers, we will apply an old result by Fraïssé, which we can now state as follows (see e.g. [2] Theorem 6.1.2).

Theorem 2 *Let L be a countable language and let $\mathbf{K}$ be a non-empty countable set of finitely generated L-structures closed under substructures and satisfying* **(AP)** *and* **(JEP)**. *Then there is a countable and homogeneous L-structure D, unique up to isomorphism.*

We would like to apply Theorem 2 to the class of finitely generated (partial) projective planes. However, it is easy to see that this class does not satisfy **(AP)**. Let now L_{proj} be the language L_{pol} extended by a binary function symbol g. We add axioms saying that for any x, y either both lines or both points, $g(x, y)$ is the unique element incident with x and y. Otherwise $g(x, y)$ is arbitrary. Note that if $\mathfrak{P}$ is a projective plane, then it is an L_{proj}-structure in a natural way. But since substructures have to be closed under the application of function symbols, adding the function symbol g has the effect that an L_{proj}-substructure of a projective plane will be a possibly degenerate projective plane; i.e. a bipartite graph in which axioms (0),(i) and $(ii)_2$ hold, but possibly not (iii).

Let $\mathbf{K}$ be the class of finitely generated partial projective planes closed under g which are freely generated over a finite set. In particular, $\mathbf{K}$ contains all finite partial projective planes closed under g.

Clearly, **K** is closed under substructures, and it is easy to see that it satisfies **(AP)** and **(JEP)**: since the empty structure is in **K**, it suffices to consider **(AP)**. So let B and C be two finitely generated partial closed projective planes, freely generated over B_0 and C_0, say, and containing a common substructure A free over A_0. Then the free partial projective plane D generated by the disjoint union of $A_0, B_0 \setminus A_0$ and $C_0 \setminus A_0$ is again in **K** and the natural embeddings of A into B and C extend into embeddings of B and C into D.

Applying Theorem 2 to this class we obtain a countable homogeneous L_{proj}-structure, which is easily seen to be a projective plane $\mathfrak{P}$. To see that the automorphisms group of $\mathfrak{P}$ acts highly transitively on the point rows, it suffices to note that for any k, any two substructures of $\mathfrak{P}$ consisting of a line and k points on this line are isomorphic. By the homogeneity property of $\mathfrak{P}$, there is an automorphism of $\mathfrak{P}$ extending this isomorphism. Furthermore, in the same vein it is easy to see that the collineation group acts transitively on ordered ordinary quadrangles, and on any other type of finite closed partial projective planes.

As pointed out by B. Poizat, the theory of D (i.e., the collection of all L_{proj}-sentences true in D) is the model completion of the theory of projective planes and allows *quantifier elimination* (see [2] for these notions). It can be shown that D does not have finite Morley rank, but its theory is simple of SU-rank 2.

2.2 Very homogeneous generalized polygons

By a modification of Fraïssé's construction based on ideas of Hrushovski and Baldwin, we can obtain generalized n-gons of finite Morley rank with the property that the automorphism group acts transitively on the ordered ordinary $(n + 1)$-gons (see [5] for details). Rather than asking that a class **K** of finite or finitely generated structures be closed under substructures and satisfies **(AP)** and **(JEP)** we here introduce an additional relation on $\leq$ on **K**, corresponding to the structure we want to construct.

Fix $n \geq 3$. For any finite graph A we define the weighted Euler characteristic as

$$y(A) = (n - 1)|A| - (n - 2)e(A)$$

where $|A|$ denotes the number of elements in A and $e(A)$ denotes the number of edges in A.

Definition 3 If A and B are bipartite with respect to a predicate P and $A \subseteq$

B is finite, we say that A is *strong in* B and write $A \leq B$ if $P(A) = P(B) \cap A$ and if for any finite subgraph A' with $A \subseteq A' \subseteq B$ we have $y(A) \leq y(A')$.

Let **K** be the class of finite graphs A bipartite with respect to P with the following properties:

(K1) A contains no ordinary k-gons for $k < n$.

(K2) If $B \subseteq A$ contains an ordinary k-gon for $k > n$, then $y(B) \geq 2n + 2$

It can be shown that **K** has the amalgamation and joint embedding property with respect to $\leq$; i.e., **(AP)** and **(JEP)** hold whenever all embeddings are strong with the resulting embeddings also strong. Then there exists a countable $(K, \leq)$-homogeneous-universal model; i.e. a countable structure M satisfying:

(H1) If $A \in K$ is finite, then there exists an embedding $f : A \longrightarrow M$ such that $f(A) \leq M$.

(H2) If $A \subseteq M$ is finite, then $A \in K$.

(H3) If $A, B \leq M$ are finite and there exists an isomorphism $f : A \longrightarrow B$, then there exists an automorphism of M extending f.

From condition (K2) it is easy to see that the automorphism group of M then acts transitively on ordered ordinary $(n + 1)$-gons. In order to force M to have finite Morley rank, we have to modify the class **K** by introducing a certain multiplicity function which keeps the amalgams small.

2.3 Semi-finite quadrangles

While in the previous examples model theory was used to construct new models of a first order theory, thus obtaining new examples of generalized polygons, a rather different application of model theory to generalized quadrangles comes from the notion of *indiscernibles.* An ordered sequence $(a_i)_{i \in I}$ in a given L-structure $\mathfrak{A}$ is called *order indiscernible* if for any two increasing sequences $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ of elements of $(a_i)_{i \in I}$ of the same length n, there is an automorphism of $\mathfrak{A}$ taking a_i to a'_i for $i = 1, \ldots n$. If these automorphisms can be taken to fix a finite set D pointwise, then $(a_i)_{i \in I}$ is called a sequence of indiscernibles *over the set* D. The idea behind this notion is that finite parts of the sequence cannot be distinguished by any L-formula (involving parameters from D), hence by any 'finite piece of information' expressible in L. Using the compactness theorem and Ramsey's theorem one can show that if a definable set in a first order structure is infinite, then for any prescribed order type of I there is a structure satisfying the same set of first order sentences, in which the definable set contains an infinite set of order indiscernibles, (see [2] for details.)

Cherlin [1] proved the following:

Theorem 4 *Every generalized quadrangle with at most five points per line is finite.*

The facts about indiscernibles imply that if there was an infinite semi-finite quadrangle with at most five points per line, then after fixing some line l_0 and labeling the points on this line with $1, \dots k$ for $k \leq 5$, there exists such a quadrangle in which there is an infinite set $(l_i)_{i \in Q}$ of pairwise skew lines and skew to l_0, indexed by the rationals and order indiscernible over the set of points $1, \dots k$ on l_0. Using projections, all points on the other lines in $(l_i)_{i \in Q}$ obtain labels from their respective projections onto l_0 and the projectivity from l_i to l_j for $i < j$ with $i, j \neq 0$ can be presented by the corresponding permutation $\sigma \in S_k$. The sequence $(l_i)_{i \in Q}$ being indiscernible now implies in particular that σ does not depend on the choice of i and j as long as $i < j$. Notice that for $j < i$ the corresponding permutation can be presented by σ^{-1}. Furthermore it is easy to see that σ is fixed point free. By changing the sequence $(l_i)_{i \in Q}$ to some indiscernible sequence of lines pairwise skew lines skew to l_0 obtained from joining lines of the old sequence, one can furthermore show that one may assume that σ involves a transposition. For $k = 3$, this already yields a contradiction. The cases $k = 4$ and 5 are then dealt with separately.

3 Generalized polygons applied to groups of finite Morley rank

One of the big open problems in the model theory concerned with structures of finite Morley rank, is the so-called Cherlin-Zil'ber Conjecture, which states that *every infinite simple group of finite Morley rank is a linear algebraic group over an algebraically closed field.* We make use of the close connection between *buildings of spherical type* and BN-pairs, which was established by Tits [10]. In particular, the study of generalized polygons has proven to be very fruitful.

3.1 BN-pairs

If the Cherlin-Zil'ber Conjecture is true, then any infinite simple group of finite Morley rank must have a definable BN-pair, namely the standard BN-pair of an algebraic group G where B is a *Borel subgroup* of G, i.e. a maximal solvable subgroup of G, and N is the normalizer of a maximal torus of G inside B. Hence it is of particular importance to classify groups of finite Morley rank having a definable BN-pair.

It was shown in [7] that for groups of finite Morley rank (and more generally, for *stable* groups) the *Weyl group* $W = N/(B \cap N)$ is necessarily finite. (If the Weyl group is finite, the BN-pair is called *spherical*.)

Spherical buildings of Tits rank 2 are exactly the generalized polygons. In particular, a group with a BN-pair of Tits rank 2 corresponds to a generalized polygon with a strongly transitive automorphism group. This connection will be explained below. For spherical BN-pairs of Tits rank at least 3, Tits gave a complete classification of the corresponding buildings. He proved that these buildings are uniquely determined by their rank 2 residues, hence by polygons, and showed that only a very restricted class of polygons – all satisfying the Moufang condition – can come up as residues in buildings of higher rank. For polygons such a general classification is not possible. However, the Moufang condition is a heavy restriction, which allows Tits and Weiss to give a complete list, see [11]. We use these facts to show the following

Theorem 5 [4] *If G is an infinite simple group of finite Morley rank with an irreducible BN-pair of Tits rank ≥ 3, then G is (interpretably) isomorphic to a simple algebraic group over an algebraically closed field.*

Since spherical irreducible buildings of Tits rank ≥ 3 are uniquely determined by their rank 2 residues, the crucial step in the proof of Theorem 5 is in fact the following theorem which uses the classification of Moufang polygons by Tits and Weiss:

Theorem 6 [4] *If $\mathcal{P}$ is an infinite Moufang polygon of finite Morley rank, then $\mathcal{P}$ is either the projective plane, the symplectic quadrangle, or the split Cayley hexagon over some algebraically closed field.*

Remark. In particular, $\mathcal{P}$ is an algebraic polygon as classified in [3].

Theorem 6 can be restated in terms of groups as follows:

Theorem 7 *If the BN-pair of a simple group G of finite Morley rank has Tits rank 2 and the associated polygon is a Moufang polygon, then $G \cong PSL_3(K), PSp_4(K)$ or $G_2(K)$ for some algebraically closed field K.*

By Theorem 5, it is left to classify BN-pairs of Tits rank at most 2 and of finite Morley rank. BN-pairs of Tits rank 1 are just 2-transitive permutation groups, and little can so far be said about them. Thus, we now concentrate on BN-pairs of Tits rank 2. For this, we give a brief outline for the correspondence between BN-pairs of Tits rank 2 and generalized polygons.

3.2 Connection between groups and polygons

Let G be a group with an irreducible BN-pair of rank 2, and suppose that the associated Weyl group $N/B \cap N$ is finite of order $2n$, for $n \geq 3$. Let G_a and G_ℓ be the proper parabolic subgroups of G containing B. We define an incidence structure on the coset spaces $\mathcal{P} = G/G_a$ and $\mathcal{L} = G/G_\ell$ by defining a point gG_a to be incident with a line $g'G_\ell$ if and only if $gG_a \cap g'G_\ell \neq \emptyset$. The axioms of a BN-pair yield that the incidence structure defined in this way is a generalized n-gon $\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ and the group G acts on $\mathcal{P}$ as a group of automorphisms transitively on the ordered ordinary n-gons.

For the converse direction, i.e. for getting from a polygon to the group, let $\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized n-gon and suppose that a group $G \leq Aut(\mathcal{P})$ acts transitively on the set of ordered ordinary n-gons. The BN-pair of G can be seen as follows: Let (a, ℓ) be a flag, and Γ an ordinary n-gon containing (a, ℓ). Then B is the stabilizer of (a, ℓ), N is the setwise stabilizer of Γ, and $\mathcal{P}$ is isomorphic to the coset geometry $(G/G_a, G/G_\ell, \{(gG_a, gG_\ell)|\ g \in G\})$, where G_a and G_ℓ denote the stabilizer in G of the elements a and ℓ, respectively. As before G_a and G_ℓ then form the parabolic subgroups of G containing B.

This shows that polygons play an important rôle in the study of groups of finite Morley rank. However, as the examples in 2.2 show, polygons (of finite Morley rank) are not *tame*: we can summarize that section by the following theorem

Theorem 8 [5] *For all $n \geq 3$, there are generalized n-gons of finite Morley rank not interpreting any infinite group and with an automorphism group acting transitively on the set of ordered $n+1$-gons. In these examples, point rows and line pencils are strongly minimal.*

As explained above, the automorphism group of such a strongly homogeneous polygon has a BN-pair. This is a new class of examples for polygons and BN-pairs even without the condition on finite Morley rank, showing that without further restrictions one cannot expect to classify infinite groups with BN-pairs of rank 2.

In view of the close connection between generalized polygons and their automorphism groups it is natural to consider the question whether the construction can be modified in such a way to obtain *definable* automorphism groups, and thus new infinite groups of finite Morley rank. However, the following result shows that, except in the algebraic cases this is impossible, at least as long as the point rows and line pencils have Morley rank 1.

Theorem 9 [7] *If $\mathcal{P}$ is a generalized n-gon with strongly minimal point rows and line pencils, $n \geq 3$, and $G \leq Aut(\mathcal{P})$ is a group of finite Morley rank which acts transitively and definably on the set of ordered ordinary n-gons contained in $\mathcal{P}$, then one of the following holds:*

- $n = 3$ *and G is definably isomorphic to $PSL_3(K)$ for some algebraically closed field K, and $\mathcal{P}$ is the projective plane over K;*
- $n = 4$ *and G is definably isomorphic to $PSp_4(K)$ and $\mathcal{P}$ is the symplectic quadrangle over K;*
- $n = 6$ *and G is definably isomorphic to $G_2(K)$ and $\mathcal{P}$ is the split Cayley hexagon over K.*

For the proof, we make essential use of a result of Hrushovski's which characterizes transitive effective permutation groups of finite Morley rank on strongly minimal sets: by his result, the *Levi factors* of G, i.e. the permutation groups induced by G on the point rows and line pencils of $\mathcal{P}$, must be permutation equivalent to either the affine group $K^+ \rtimes K^*$ or to the simple algebraic group $PSL_2(K)$ for some algebraically closed field K. We use this information to derive the Moufang property for $\mathcal{P}$. While in light of Theorem 5 this would already be sufficient to obtain the result, we here use geometric information to show directly that $\mathcal{P}$ is isomorphic to one of the three cases mentioned in the theorem. We would like to stress the fact that our approach does not need the full classification of Moufang polygons (in particular the yet unpublished parts about quadrangles can be evaded).

The previous theorem has the following translation into the language of groups:

Theorem 10 [7] *If G is an infinite group of finite Morley rank with a definable irreducible BN-pair of (Tits-) rank 2, such that the Morley rank of the coset space P_i/B is at most 1 for the parabolic subgroups P_1 and P_2, then G is definably isomorphic to $PSL_3(K)$, $PSp_4(K)$, or $G_2(K)$ for some algebraically closed field K.*

3.3 'Split' BN-pairs

The Cherlin-Zil'ber conjecture implies in particular that any simple group must have a definable 'split' BN-pair. One of the characteristic properties of a simple algebraic group G is that a Borel subgroup B, and the normalizer N of a maximal (split) torus T form a so-called *split* BN-pair. This means that there exists a nilpotent normal subgroup U of B such that $B = U \rtimes T$. Using the connections between BN-pairs and generalized polygons we show the following partial analog of the famous paper by Fong and Seitz on finite groups with split BN-pairs:

Theorem 11 [6] *Let G be a simple group with a definable BN-pair of rank 2 where $B = U.T$ for $T = B \cap N$ and a normal subgroup U of B with $Z(U) \neq 1$. It was shown in [8] that the Weyl group $W = N/B \cap N$ has cardinality $2n$ with $n = 3, 4, 6, 8$ or 12. If G has finite Morley rank then furthermore the following holds:*

(i) If $n = 3$, then G is definably isomorphic to $PSL_3(K)$ for some algebraically closed field K.

(ii) If U is nilpotent, then $n \neq 12$, and if $n = 4$ or 6, then G is definably isomorphic to $Psp_4(K)$ and $G_2(K)$ respectively for some algebraically closed field K.

(iii) If $Z(U)$ contains a B-minimal subgroup A with $RM(A) \geq RM(P_i/B)$ for both parabolic subgroups P_1 and P_2, then $n = 3, 4$ or 6 and G is definably isomorphic to $PSL_3(K), PSp_4(K)$ or $G_2(K)$ for some algebraically closed field K.

Note that all of these conditions are necessarily satisfied if the Cherlin-Zil'ber Conjecture is true.

Most of the arguments can be modified to work in the general context and so we can almost completely delete the finiteness assumption in the theorem of Fong and Seitz:

Theorem 12 [9] *Let G be a group with an irreducible spherical BN-pair of rank 2 where $B = U.T$ for $T = B \cap N$ and a normal nilpotent subgroup U of B. Let $\mathfrak{P}$ be the generalized n-gon associated to this (B,N)-pair and let W be the associated Weyl group. Then $\mathfrak{P}$ is a Moufang quadrangle and G/R contains its little projective group, where R denotes the kernel of the action of G on $\mathfrak{P}$, except possibly if $n = 8$.*

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