Automorphisms and characterizations of finite generalized quadrangles

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Abstract

Our paper surveys some new developments in the theory of automorphisms and characterizations of finite generalized quadrangles. It is the purpose to mention important new results which did not appear in the following standard works (or surveys) on the subject: *Collineations of finite generalized quadrangles* (S. E. Payne, 1983), *Finite Generalized Quadrangles* (S. E. Payne and J. A. Thas, 1984), *Recent developments in the theory of finite generalized quadrangles* (J. A. Thas, 1994), *Handbook of Incidence Geometry* (Edited by F. Buekenhout, 1995), 3-*Regularity in generalized quadrangles: a survey, recent results and the solution of a longstanding conjecture* (J. A. Thas, 1998), *Generalized Polygons* (H. Van Maldeghem, 1998) and *Generalized quadrangles of order* (s, s^2)*: recent results* (J. A. Thas, 1999).
We make no aim to completeness; since the subject is so widely investigated, we see ourselves committed to ignore (important) work of several authors. Also, we omit most results on flock geometry, and refer to the extensive paper *Flocks and related structures: an update* by S. E. Payne in these proceedings [46] (we also omit all results concerning 'small' examples).
We completely ignore the already historic results on embeddings of generalized polygons of J. A. Thas and H. Van Maldeghem, as we again refer to another chapter in the proceedings, namely *Generalized Polygons in Projective Spaces* by H. Van Maldeghem [100].
Finally, we mention that this paper is not entirely a survey; there is some brand new research in Section 8. We show how one can prove that every finite Moufang generalized quadrangle is classical or dual classical without the use of the deep group theoretical results of Fong and Seitz [20, 21] (the proof however uses the classification of the finite split BN-pairs of rank 1 [54, 25]).
There is an extensive bibliography on the subject of finite generalized quadrangles contained in this paper.

1 Introduction and notation on generalized quadrangles

1.1 Finite generalized quadrangles

A (finite) *generalized quadrangle (GQ) of order* (s,t) is an incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line *incidence relation* satisfying the following axioms.

(GQ1) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(GQ3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

If $s = t$, then we also say that $\mathcal{S}$ is *of order* s.
Generalized quadrangles were introduced by J. Tits [93] in his celebrated work on triality, in order to have a better understanding of the Chevalley groups of rank 2.
The main results, up to 1983, on finite generalized quadrangles are contained in the monograph *Finite Generalized Quadrangles* by S. E. Payne and J. A. Thas [47] (denoted FGQ). Surveys of 'new' developments on this subject in the period 1984-1992, can be found in *Recent developments in the theory of finite generalized quadrangles* [62], 3-*Regularity in generalized quadrangles: a survey, recent results and the solution of a longstanding conjecture* [68], and, finally, *Generalized quadrangles of order* (s, s^2)*: recent results* [69], all by the hand of J. A. Thas. On the more general subject of *generalized polygons*, see Chapter 5 of *Handbook of Incidence Geometry* (Edited by F. Buekenhout) of J. A. Thas and the monograph *Generalized Polygons* of H. Van Maldeghem.
Let $\mathcal{S} = (P, B, I)$ be a (finite) generalized quadrangle of order (s,t), $s \neq 1 \neq t$. Then $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. Also, $s \leq t^2$ [26, 27] and, dually, $t \leq s^2$, and $s+t$ divides $st(s+1)(t+1)$.
There is a point-line duality for GQ's of order (s,t) for which in any definition or theorem the words "point" and "line" are interchanged and also the parameters. Normally, we assume without further notice that the dual of a given theorem or definition has also been given. Also, sometimes a line will be identified with the set of points incident with it without further notice.
A GQ is called *thick* if every point is incident with more than two lines and if

every line is incident with more than two points. Otherwise, a GQ is called *thin.* If $\mathcal{S}$ is a thin GQ of order $(s,1)$, then $\mathcal{S}$ is also called a *grid with parameters* $s+1, s+1$. *Dual grids* are defined dually. A *flag* of a GQ is an incident point-line pair.
Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L such that $pILIq$ (so $p \not\sim q$ means that p and q are not collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$.
For $p \in P$, put $p^\perp = \{q \in P \parallel q \sim p\}$ (and note that $p \in p^\perp$). For a pair of distinct points $\{p,q\}$, the *trace* of $\{p,q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p,q\}^\perp$. Then $|\{p,q\}^\perp| = s+1$ or $t+1$, according as $p \sim q$ or $p \not\sim q$. More generally, if $A \subset P$, $A^\perp$ is defined by $A^\perp = \bigcap\{p^\perp \parallel p \in A\}$. For $p \neq q$, the *span* of the pair $\{p,q\}$ is $sp(p,q) = \{p,q\}^{\perp\perp} = \{r \in P \parallel r \in s^\perp$ for all $s \in \{p,q\}^\perp\}$. When $p \not\sim q$, then $\{p,q\}^{\perp\perp}$ is also called *the hyperbolic line* defined by p and q, and $|\{p,q\}^{\perp\perp}| = s+1$ or $|\{p,q\}^{\perp\perp}| \leq t+1$ according as $p \sim q$ or $p \not\sim q$. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p,q\}^{\perp\perp}| = t+1$, we say that the pair $\{p,q\}$ is *regular*. The point p is *regular* provided $\{p,q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is defined dually. A point p is *coregular* provided each line incident with p is regular. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points.
A *triad* of points (respectively lines) is a triple of pairwise non-collinear points (respectively pairwise disjoint lines). Given a triad T, a *center* of T is just an element of $T^\perp$. Let $\{p_1,p_2,p_3\}$ be a triad of points of a GQ of order (s,s^2), $s > 1$. Then $|\{p_1,p_2,p_3\}^\perp| = s+1$, see [9, 47], and clearly $|\{p_1,p_2,p_3\}^{\perp\perp}| \leq s+1$. If equality holds, then the triad is called 3-*regular*. Every triad of the classical GQ $\mathcal{Q}(5,q)$ is 3-regular, see Chapter 5 of FGQ.
If (p,L) is a non-incident point-line pair of a GQ (i.e. an *anti-flag*), then by *[p,L]* we denote the unique line of the GQ which is incident with p and concurrent with L[1].
Finally, if $\mathcal{S}$ is a GQ, then by $\mathcal{S}^D$ we denote its point-line dual.

1.2 The classical and dual classical generalized quadrangles

Consider a nonsingular quadric of Witt index 2, that is, of projective index 1, in PG$(3,q)$, PG$(4,q)$, PG$(5,q)$, respectively. The points and lines of the

[1] In the literature, sometimes also the notation $proj_pL$ is used.

quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(3,q)$, $\mathcal{Q}(4,q)$, $\mathcal{Q}(5,q)$, respectively, and has order $(q,1)$, (q,q), (q,q^2), respectively. Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathrm{PG}(3,q^2)$, respectively $\mathrm{PG}(4,q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3). The points of $\mathrm{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity form a GQ $W(q)$ of order q. The generalized quadrangles defined in this paragraph are the so-called *classical generalized quadrangles*, see Chapter 3 of FGQ. It is important to mention that $W(q)^D \cong \mathcal{Q}(4,q)$ and that $H(3,q^2)^D \cong \mathcal{Q}(5,q)$, see 3.2.1 and 3.2.3 of FGQ.

2 Translation generalized quadrangles and elation generalized quadrangles

2.1 Translation generalized quadrangles and elation generalized quadrangles

A *whorl about a point* p of a GQ $\mathcal{S} = (P,B,I)$ is a collineation of the GQ fixing p linewise. An *elation about the point* p is a whorl about p that fixes no point of $P \setminus p^\perp$. By definition, the identical permutation is an elation (about every point). If p is a point of the GQ $\mathcal{S}$, for which there exists a group of elations G about p which acts regularly on the points of $P \setminus p^\perp$, then $\mathcal{S}$ is said to be an *elation generalized quadrangle (EGQ) with elation point* p and *elation group* (or *base-group*) G, and we sometimes write $(\mathcal{S}^{(p)}, G)$ for $\mathcal{S}$.
Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose L is a line of $\mathcal{S}$. A *symmetry about* L is an automorphism of $\mathcal{S}$ which fixes every line concurrent with L. A line L is an *axis of symmetry* if there is a full group of size s of symmetries about L. Dually, one defines *center of symmetry*. A point of a generalized quadrangle is a *translation point* if every line through it is an axis of symmetry. If a GQ $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p, and if any line incident with p is an axis of symmetry, then we say that $\mathcal{S}$ is a *translation generalized quadrangle (TGQ) with translation point* p and *translation group* (or *base-group*) G. In such a case, G is uniquely defined; G is generated by all symmetries about every line incident with p, and G is the set of all elations about p, see Chapter 8 of FGQ.
TGQ's were introduced by J. A. Thas in [58] for the case $s = t$ and by S. E. Payne and J. A. Thas in FGQ for the general case.

Theorem 1 (FGQ, 8.3.1) *Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s,t \neq 1$. Suppose each line through some point p is an axis of symmetry, and let G*

be the group generated by the symmetries about the lines through p. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.

For the case $s = t$, there is a stronger version of Theorem 1, see Section 4.

Theorem 2 (FGQ, 8.2.3, 8.5.2 and 8.7.2) *Suppose $(\mathcal{S}^{(x)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$. Then $(\mathcal{S}^{(x)}, G)$ is a TGQ if and only if G is an (elementary) abelian group. Also in such a case there is a prime p and there are natural numbers n and k, where k is odd, such that either $s = t = p^n$ or $s = p^{nk}$ and $t = p^{n(k+1)}$. It follows that G is a p-group. Also, if $p = 2$, then $k = 1$.*

CONJECTURE (FGQ). *If $s \neq t$ and s and t are odd in Theorem 2, then $k = 1$.*

2.2 $T(n, m, q)$'s, the $T_d(\mathcal{O})$'s of Tits ($d = 2, 3$) and translation duals of TGQ's

In this paragraph, we introduce the important notion of *translation dual* of a translation generalized quadrangle, which will be necessary for understanding the sequel.
Suppose $H = \mathrm{PG}(2n + m - 1, q)$ is the finite projective $(2n + m - 1)$-space over $\mathrm{GF}(q)$, and let H be embedded in a $\mathrm{PG}(2n + m, q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n, m, q)$ of subspaces as follows: $\mathcal{O}$ is a set of $q^m + 1$ $(n - 1)$-dimensional subspaces of H every three of which generate a $\mathrm{PG}(3n - 1, q)$, denoted $\mathrm{PG}(n - 1, q)^{(i)}$, and such that the following condition is satisfied: for every $i = 0, 1, \ldots, q^m$ there is a subspace $\mathrm{PG}(n + m - 1, q)^{(i)}$ of H of dimension $n + m - 1$, which contains $\mathrm{PG}(n - 1, q)^{(i)}$ and which is disjoint from any $\mathrm{PG}(n - 1, q)^{(j)}$ if $j \neq i$. If $\mathcal{O}$ satisfies all these conditions for $n = m$, then $\mathcal{O}$ is called a *pseudo-oval* or a *generalized oval* or an *[n - 1]-oval* of $\mathrm{PG}(3n - 1, q)$; a generalized oval of $\mathrm{PG}(2, q)$ is just an oval of $\mathrm{PG}(2, q)$. For $n \neq m$, $\mathcal{O}(n, m, q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an *[n - 1]-ovoid* or an *egg* of $\mathrm{PG}(2n + m - 1, q)$; a [0]-ovoid of $\mathrm{PG}(3, q)$ is just an ovoid of $\mathrm{PG}(3, q)$. The spaces $\mathrm{PG}(n + m - 1, q)^{(i)}$ are the *tangent spaces* of $\mathcal{O}(n, m, q)$, or just the *tangents*. Generalized ovoids where introduced for the case $n = m$ by J. A. Thas in [57] for some particular cases. Then S. E. Payne and J. A. Thas prove in [58, 47] that from any egg $\mathcal{O}(n, m, q)$ there arises a GQ $T(n, m, q) = T(\mathcal{O})$ which is a TGQ of order (q^n, q^m), for some special point (∞), as follows.

- The POINTS are of three types.

1. A symbol (∞).
2. The subspaces $\mathrm{PG}(n+m,q)$ of H' which intersect H in a $\mathrm{PG}(n+m-1,q)^{(i)}$.
3. The points of $H' \setminus H$.

- The LINES are of two types.

1. The elements of the egg $\mathcal{O}(n,m,q)$.
2. The subspaces $\mathrm{PG}(n,q)$ of $\mathrm{PG}(2n+m,q)$ which intersect H in an element of the egg.

- INCIDENCE is defined as follows: the point (∞) is incident with all the lines of type (1) and with no other lines; the points of type (2) are incident with the unique line of type (1) contained in it and with all the lines of type (2) which they contain (as subspaces), and finally, a point of type (3) is incident with the lines of type (2) that contain it.

Conversely, any TGQ can be seen in this way (that is, as a $T(n,m,q)$), and hence, *the study of translation generalized quadrangles is equivalent to the study of the generalized ovoids.*
If $(n,m)=(1,1)$, respectively $(n,m)=(1,2)$, then $\mathcal{O}(1,1,q)=\mathcal{O}$, respectively $\mathcal{O}(1,2,q)=\mathcal{O}$, is an oval in $\mathrm{PG}(2,q)$, respectively ovoid in $\mathrm{PG}(3,q)$, and $T(1,1,q)=T(\mathcal{O})$, respectively $T(1,2,q)=T(\mathcal{O})$, is called a $T_2(\mathcal{O})$, respectively $T_3(\mathcal{O})$, of order q, respectively (q,q^2). The GQ $T_2(\mathcal{O})$, respectively $T_3(\mathcal{O})$, is isomorphic to $\mathcal{Q}(4,q)$, respectively $\mathcal{Q}(5,q)$, if and only if $\mathcal{O}$ is a nonsingular conic of $\mathrm{PG}(2,q)$, respectively a nonsingular elliptic quadric of $\mathrm{PG}(3,q)$. If q is odd, then by a celebrated theorem of Segre [53], respectively Barlotti [4] and Panella [38], $\mathcal{O}$ is always a (nonsingular) conic, respectively elliptic quadric. The GQ $T_d(\mathcal{O})$ with $d \in \{2,3\}$ has no translation points besides the point (∞) if and only if $T_d(\mathcal{O})$ is not classical, see FGQ. The GQ's $T_d(\mathcal{O})$ where introduced by Tits in [18].
Each TGQ $\mathcal{S}$ of order $(s, s^{\frac{k+1}{k}})$, with translation point (∞), where k is odd and $s \neq 1$, has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞), and any given point not collinear with (∞), linewise. We have $|\mathbb{K}| \leq s$, see FGQ. The field $\mathrm{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$ [47]. The TGQ $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits with $\mathcal{O}$ an ovoid of $\mathrm{PG}(3,s)$ if and only if $|\mathbb{K}|=s$.
Completely similar remarks can be made for the case $s=t$, and in that case, the TGQ is isomorphic to a $T_2(\mathcal{O})$ of Tits with $\mathcal{O}$ an oval of $\mathrm{PG}(2,s)$ if and

only if $|\mathbb{K}| = s$.
If $n \neq m$, then by 8.7.2 of [47] the q^m+1 tangent spaces of $\mathcal{O}(n, m, q)$ form an $\mathcal{O}^*(n, m, q)$ in the dual space of $\mathrm{PG}(2n+m-1, q)$. So in addition to $T(n, m, q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n, m, q)$, or $T^*(\mathcal{O})$. The TGQ $T^*(\mathcal{O})$ is called the *translation dual* of the TGQ $T(\mathcal{O})$. The GQ's $T_3(\mathcal{O})$ and $\mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock, see Section 3, are the only known TGQ's which are isomorphic to their translation dual. The TGQ $T(\mathcal{O})$ and its translation dual $T(\mathcal{O}^*)$ have isomorphic kernels. This was pointed out to us by J. A. Thas (private communication) in the following way: if we start from and egg $\mathcal{O}$ in $\mathrm{PG}(2n + m - 1, q)$, $n \neq m$, to obtain the egg $\mathcal{O}^*$ in $\mathrm{PG}(2n + m - 1, q)$, then, as $\mathcal{O}^*$ is also represented in a $\mathrm{PG}(2n + m - 1, q)$, we know that $\mathrm{GF}(q)$ is a subfield of the kernel of $T(\mathcal{O}^*)$. Interchanging the role of $\mathcal{O}$ and $\mathcal{O}^*$ then yields the result.
A TGQ $T(\mathcal{O})$ with $s \neq t$ is called *good* at an element $\pi \in \mathcal{O}$ if for every two distinct elements π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly $q^n + 1$ elements of $\mathcal{O}$ and is skew to the other elements. If the egg $\mathcal{O}$ contains a good element, then the egg is subconsequently called *good*, and for a good egg $\mathcal{O}(n, m, q)$ there holds that $m = 2n$.
For recent work on eggs and translation generalized quadrangles, see, for example, Lavrauw and Penttila [36], J. A. Thas [70, 71, 72, 73] and K. Thas [86, 88].
Problem. *Are the GQ's $T_3(\mathcal{O})$ and $\mathcal{S}(\mathcal{F})^D$, with $\mathcal{F}$ a Kantor flock, the only known TGQ's which are isomorphic to their translation dual?*

3 Nets and generalized quadrangles (with a regular point)

A (finite) *net of order $k(\geq 2)$ and degree $r(\geq 2)$* is an incidence structure $\mathcal{N} = (P, B, I)$ satisfying the following properties.

(N1) Each point is incident with r lines and two distinct points are incident with at most one line.

(N2) Each line is incident with k points and two distinct lines are incident with at most one point.

(N3) If p is a point and L a line not incident with p, then there is a unique line M incident with p and not concurrent with L.

A net of order k and degree r has k^2 points and kr lines. For any net we have $k \geq r - 1$ [7], and if $k = r - 1$ then $\mathcal{N}$ is an *affine plane* [18, 30, 28, 29].

Theorem 3 (FGQ, 1.3.1) *Let p be a regular point of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$. Then the incidence structure with point set $p^\perp \setminus \{p\}$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are non-collinear points of $p^\perp \setminus \{p\}$, and with the natural incidence, is the dual of a net of order s and degree $t + 1$.*
If in particular $s = t$, there arises a dual affine plane of order s. (Also, in the case $s = t$, the incidence structure π_p with point set $p^\perp$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are different points in $p^\perp$, and with the natural incidence, is a projective plane of order s).

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, and that p is a regular point of $\mathcal{S}$. Then by $\mathcal{N}_p$ we will denote the net which is the dual of the dual net corresponding to p, as described in Theorem 3.
A part of the following theorem is contained in Section 7 of J. A. Thas and H. Van Maldeghem [77]. However, we bring it in a slightly different and more general context in order to characterize subnets of nets which are attached to regular points or lines of generalized quadrangles.

Theorem 4 (K. Thas [84]) *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, with a regular point p. Let $\mathcal{N}_p$ be the net which arises from p, and suppose $\mathcal{N}'_p$ is a subnet of the same degree as $\mathcal{N}_p$. Then we have the following possibilities.*

1. *$\mathcal{N}'_p$ coincides with $\mathcal{N}_p$;*
2. *$\mathcal{N}'_p$ is an affine plane of order t and $s = t^2$; also, from $\mathcal{N}'_p$ there arises a proper subquadrangle of $\mathcal{S}$ of order t having p as a regular point.*

If, conversely, $\mathcal{S}$ has a proper subquadrangle containing the point p and of order (s', t) with $s' \neq 1$, then it is of order t, and hence $s = t^2$. Also, there arises a proper subnet of $\mathcal{N}_p$ which is an affine plane of order t.

This result has several interesting corollaries.

Corollary 5 *A net $\mathcal{N}$ which is attached to a regular point of a GQ contains no proper subnet, of the same degree as $\mathcal{N}$, other than (possibly) an affine plane.*

Corollary 6 *Suppose p is a regular point of the GQ $\mathcal{S}$ of order (s, t), $s, t \neq 1$, and let $\mathcal{N}_p$ be the corresponding net. If $s \neq t^2$, then $\mathcal{N}_p$ contains no proper subnet of degree $t + 1$.*

The following corollary indicates that nets which arise from a regular point of a GQ and which do not contain affine planes are very 'irregular'.

Corollary 7 *Let p be a regular point of a GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, and suppose $\mathcal{N}_p$ is the corresponding net. Moreover, suppose $s \neq t^2$. If u, v and w are distinct lines of $\mathcal{N}_p$, for which $w \not\sim u \sim v$, then these lines generate the whole net.*

Definition. A *skew translation generalized quadrangle (STGQ) with base-point p* is an EGQ with base-point p and elation group G, such that p is a center of symmetry with the property that G contains the full group of symmetries about p (of size t).

Theorem 8 (K. Thas [84]) *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, and suppose ϕ is a nontrivial whorl about a regular point p. Also, suppose ϕ fixes distinct points q, r and u of $p^\perp \setminus \{p\}$ for which $q \sim r$ and $q \not\sim u$. Then we have one of the following possibilities.*

1. *We have that $s = t^2$, $\mathcal{S}$ contains a proper subquadrangle $\mathcal{S}'$ of order t, and if ϕ is not an elation, then $\mathcal{S}'$ is fixed pointwise by ϕ.*
2. *ϕ is a nontrivial symmetry about p.*

Notation. By *(n,m)*, with $m, n \in \mathbb{N}$, we denote the greatest common divisor of m and n.

Theorem 9 (K. Thas [84]) *Let $(\mathcal{S}^{(p)}, G)$ be an EGQ of order (s,t), $s \neq 1 \neq t$, and suppose p is a regular point. Moreover, suppose that $(s-1,t) = 1$. Then we have one of the following:*

1. *there is a full group of order t of symmetries about p which is completely contained in G, and hence $\mathcal{S}^{(p)}$ is an STGQ;*
2. *$\mathcal{S}$ contains a proper subGQ of order t for which p is a regular point, and consequently $s = t^2$.*

The following theorem is a consequence of Theorem 9, and provides a very interesting characterization theorem of skew translation generalized quadrangles of order s, $s > 1$. In particular, it gets rid of the demand in the definition for STGQ's that there must be a 'full' group of symmetries about the elation point which must be contained in the elation group (of elations about p).

Theorem 10 (K. Thas [84]) *Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order s, $s > 1$, and suppose that the elation point p is regular. Then there is a full group of symmetries $\mathcal{C}$ about p, and $\mathcal{C}$ is completely contained in G, hence $\mathcal{S}^{(p)}$ is a skew translation generalized quadrangle with base-point p.*

Skew translation generalized quadrangles provide 'special' nets.

Theorem 11 (K. Thas [84]) *Suppose $(\mathcal{S}^{(p)}, G)$ is an STGQ of order (s, t), $s \neq 1 \neq t$, with elation point p. Then the net $\mathcal{N}_p$ is a translation net [7].*

Remark 12 If we put $s = t$ in Theorem 11, then $\mathcal{N}_p$ is a *translation plane*, s is the power of a prime and $G/\mathcal{C}$ is elementary abelian; see e.g. [30].

It seems a very interesting problem to focus on the connection between nets and generalized quadrangles with a regular point. As a second illustration of this idea, we summarize recent results of J. A. Thas.
First, we introduce the *Axiom of Veblen* for dual nets $\mathcal{N}^* = (P, B, I)$.
Axiom of Veblen. If $L_1 I x I L_2, L_1 \neq L_2, M_1 \not{I} x \not{I} M_2$, and if the line L_i is concurrent with the line M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2.
An example of a dual net $\mathcal{N}^*$ which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net H_q^n, $n > 2$, which is constructed as follows: the points of H_q^n are the points of $\mathrm{PG}(n, q)$ not in a given subspace $\mathrm{PG}(n-2, q) \subset \mathrm{PG}(n, q)$, the lines of H_q^n are the lines of $\mathrm{PG}(n, q)$ which have no point in common with $\mathrm{PG}(n-2, q)$, the incidence in H_q^n is the natural one. By the following theorem these dual nets H_q^n are characterized by the Axiom of Veblen.

Theorem 13 (J. A. Thas and F. De Clerck [74]) *Let $\mathcal{N}^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1 > s$. If $\mathcal{N}^*$ satisfies the Axiom of Veblen, then $\mathcal{N}^* \cong H_q^n$ with $n > 2$ (hence $s = q$ and $t + 1 = q^{n-1}$).*

Property (G). Let $\mathcal{S}$ be a generalized quadrangle of order (s, s^2), $s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has *Property (G)*, or that $\mathcal{S}$ has *Property (G) at* $\{x_1, y_1\}$, if every triad $\{x_1, x_2, x_3\}$ of points for which $y_1 \in \{x_1, x_2, x_3\}^\perp$ is 3-regular. The GQ $\mathcal{S}$ has *Property (G) at the line* L, or the line L has *Property (G)*, if each pair of points $\{x, y\}$, $x \neq y$ and $xILIy$, has Property (G). If (x, L) is a flag, then we say that $\mathcal{S}$ has *Property (G) at* (x, L), or that (x, L) has *Property (G)*, if every pair $\{x, y\}$, $x \neq y$ and yIL, has Property (G). Property (G) was introduced by S. E. Payne [44] in connection with generalized quadrangles of order (q^2, q) arising from flocks of quadratic cones in $\mathrm{PG}(3, q)$, see below.

Theorem 14 (J. A. Thas [63]) *If the TGQ $\mathcal{S}^{(\infty)}$ contains a good element π, then its translation dual satisfies Property (G) for the corresponding flag $((\infty)', \pi')$.*

Theorem 15 (J. A. Thas and H. Van Maldeghem [77]) *Let $\mathcal{S}$ be a GQ of order (q^2, q), q even, satisfying Property (G) at the point x. Then x is regular for $\mathcal{S}$ and the dual net $\mathcal{N}_x^*$ defined by x satisfies the Axiom of Veblen. Consequently $\mathcal{N}_x^* \cong H_q^3$.*

Let $\mathcal{F}$ be a *flock* of the quadratic cone $\mathcal{K}$ with vertex x of $\mathrm{PG}(3, q)$, that is, a partition of $K \setminus \{x\}$ into q disjoint irreducible conics. Then, by Thas [60], relying on results by W. M. Kantor [33] and S. E. Payne [39, 43], with $\mathcal{F}$ there corresponds a GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q). In Payne [44] it was shown that $\mathcal{S}(\mathcal{F})$ satisfies Property (G) at its point (∞).

Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathrm{PG}(3, q)$, q odd. Then the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in \mathrm{GF}(q)$, m a given non-square in $\mathrm{GF}(q)$ and σ a given automorphism of $\mathrm{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [60]. All the planes π_t contain the exterior point $(0, 0, 1, 0)$ of $\mathcal{K}$. This flock is *linear*; that is, all the planes π_t contain a common line, if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathcal{K}$ for which the planes of the q conics share a common point, is of the type just described, see [60].
The corresponding GQ $\mathcal{S}(\mathcal{F})$ was first discovered by W. M. Kantor, and is therefore called the *Kantor (flock) generalized quadrangle*. The kernel $\mathbb{K}$ is the fixed field of σ, see [51].
This quadrangle is a TGQ for some baseline, and the following was shown by Payne in [44].

Theorem 16 (S. E. Payne [44]) *Suppose a TGQ $\mathcal{S} = T(\mathcal{O})$ is the point-line dual of a GQ $\mathcal{S}(\mathcal{F})$ which arises from a Kantor flock $\mathcal{F}$. Then $T(\mathcal{O})$ is isomorphic to its translation dual $T^*(\mathcal{O})$.*

Recently, Bader, Lunardon and Pinneri [3] proved that a *TGQ which arises from a flock* (see further) is isomorphic to its translation dual if and only if it is the point-line dual of a Kantor flock GQ[2].

Theorem 17 (J. A. Thas and H. Van Maldeghem [77]) *For any GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q) arising from a flock $\mathcal{F}$, the point (∞) is regular. If q*

[2]It has to be noted however that they rely heavily on results of J. A. Thas and H. Van Maldeghem [77].

is even, then the dual net $\mathcal{N}^*_{(\infty)}$ *always satisfies the Axiom of Veblen and so* $\mathcal{N}^*_{(\infty)} \cong H^3_q$. *If* q *is odd, then the dual net* $\mathcal{N}^*_{(\infty)}$ *satisfies the Axiom of Veblen if and only if* $\mathcal{F}$ *is a Kantor flock.*

The following theorem solves a longstanding conjecture on the equivalence of GQ's satisfying Property (G) and GQ's arising from flocks of quadratic cones.

Theorem 18 (J. A. Thas [70]) *A GQ* $\mathcal{S} = (P, B, I)$ *of order* (q, q^2), *q odd and* $q \neq 1$, *satisfies Property (G) at some flag* (x, L) *if and only if* $\mathcal{S}$ *is the dual of a flock GQ.*

Let $\mathcal{N} = (\overline{P}, \overline{B}, \overline{I})$ be a net of order k and degree r. Further, let R be a line of $\mathcal{N}$ and let $\mathcal{P}$ be the parallel class of $\overline{B}$ containing R, that is, $\mathcal{P}$ consists of R and the $k-1$ lines not concurrent with R. An automorphism θ of $\mathcal{N}$ is called a *transvection with axis* R if either $\theta = \mathbf{1}$ or if $\mathcal{P}$ is the set of all fixed lines of θ and R is the set of all fixed points of θ. The net $\mathcal{N}$ is a $\mathcal{P}$*-net* if for any two nonparallel lines $M, N \in \overline{B} \setminus \mathcal{P}$ there is some transvection with axis belonging to $\mathcal{P}$ and mapping M onto N.
In particular, let $\mathcal{N} = (\overline{P}, \overline{B}, \overline{I})$ be an affine plane of order k and let $\mathcal{D}$ be the corresponding projective plane. Then $\mathcal{N}$ is a $\mathcal{P}$-net if and only if the point z of $\mathcal{D}$ defined by $\mathcal{P}$, is a translation point of $\mathcal{D}$; see Hughes and Piper [30].
If $\mathcal{N}$ is the dual of H^n_q then it is easy to check that $\mathcal{N}$ is a $\mathcal{P}$-net for any parallel class $\mathcal{P}$.

Theorem 19 (J. A. Thas [72]) *Let* $(\mathcal{S}^{(p)}, G)$ *be a TGQ of order* (s, t), $s \neq 1 \neq t$. *Then for any line* L *incident with* p, *the dual net* $\mathcal{N}^*_L$ *defined by* L *is the dual of a* $\mathcal{P}$*-net* $\mathcal{N}_L$ *with* $\mathcal{P}$ *the parallel class of* $\mathcal{N}_L$ *defined by the point* p.

Theorem 20 (J. A. Thas [72]) *Let* $\mathcal{S} = (P, B, I)$ *be a GQ of order* (s, t), $s \neq 1 \neq t$, *with coregular point* p. *If for at least one line* L *incident with* p *the dual net* $\mathcal{N}^*_L$ *is the dual of a* $\mathcal{P}$*-net* $\mathcal{N}_L$ *with* $\mathcal{P}$ *the parallel class of* $\mathcal{N}_L$ *defined by* p, *then* $\mathcal{S}$ *is a TGQ with base-point* p.

Corollary 21 (J. A. Thas [72]) *Let* $\mathcal{S} = (P, B, I)$ *be a GQ of order* s, $s \neq 1$, *with coregular point* p. *If for at least one line* L *incident with* p *the corresponding projective plane* π_L *is a translation plane with as translation line the set of all lines of* $\mathcal{S}$ *incident with* p, *then* $\mathcal{S}$ *is a TGQ with base-point* p.

Corollary 22 (J. A. Thas [72]) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, $s \neq 1$, with coregular point p. If for at least one line L incident with p the projective plane π_L is Desarguesian, then $\mathcal{S}$ is a TGQ with base point p. If in particular s is odd, then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(4, s)$ arising from a nonsingular quadric of* $\mathrm{PG}(4, s)$.

Corollary 23 (J. A. Thas [72]) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (q^2, q), $q \neq 1$, with regular point x for which the dual net $\mathcal{N}_x^*$ defined by x satisfies the Axiom of Veblen. If x is incident with a coregular line L, then $\mathcal{S}$ is a TGQ with base-line L.*

Corollary 24 (J. A. Thas [72]) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (q^2, q), q even, satisfying Property (G) at the point x. If x is incident with a coregular line L, then $\mathcal{S}$ is a TGQ with base-line L.*

Corollary 25 (J. A. Thas [72]) *Let $\mathcal{S}(\mathcal{F})$ be a GQ of order (q^2, q), q even, arising from a flock $\mathcal{F}$. If the point (∞) of $\mathcal{S}(\mathcal{F})$ is collinear with a regular point x, with $(\infty) \neq x$, then $\mathcal{S}(\mathcal{F})$ is isomorphic to the classical GQ $H(3, q^2)$ arising from a nonsingular Hermitian variety of* $\mathrm{PG}(3, q^2)$.

Corollary 26 (J. A. Thas [72]) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s \neq 1 \neq t$, having a coregular point x. If for a fixed line L, with xIL, and any two lines M, N, with $M \not\sim L$ and $L \sim N \not\sim M$, there is a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s, t'), with $t' \neq 1$, containing L, M, N, then $t' = s = \sqrt{t}$, $\mathcal{S}$ is a TGQ with base-point x, and $\mathcal{S}'$ is a TGQ with base p point x. For s even $\mathcal{S}$ satisfies Property (G) at the flag (x, L) and for s odd the translation dual $\widehat{\mathcal{S}}$ of $\mathcal{S}$ satisfies Property (G) at the flag (x', L'), with x' the base-point of $\widehat{\mathcal{S}}$ and L' the line of $\widehat{\mathcal{S}}$ corresponding to the line L of $\mathcal{S}$. For s odd the subquadrangles $\mathcal{S}'$ are isomorphic to $\mathcal{Q}(4, s)$ and $\mathcal{S}$ is isomorphic to the dual of a GQ $\mathcal{S}(\mathcal{F})$ arising from a flock $\mathcal{F}$. If s is even and if $\mathcal{S}$ is isomorphic to the dual of a GQ $\mathcal{S}(\mathcal{F})$ arising from a flock $\mathcal{F}$, then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

4 Generalized quadrangles with some concurrent axes of symmetry

In Chapter 8 of FGQ, it was proved that a GQ $\mathcal{S}$ is a TGQ $(\mathcal{S}^{(p)}, G)$ with translation point p if and only if every line through p is an axis of symmetry (that is, if p is a translation point), and then G is precisely the group generated by all symmetries about the lines incident with p, see also the foregoing sections. In [81] we noted that their proof was also valid for all lines through p minus one. This observation is one of the main motivations of this section.

PROBLEM. *What is — in general — the minimal number of distinct axes of symmetry through a point p of a GQ $\mathcal{S}$ forcing $\mathcal{S}^{(p)}$ to be a TGQ?*
If the GQ is of order s, then there is the following (strong) result of S. E. Payne and J. A. Thas.

Theorem 27 (FGQ, 11.3.5) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, with $s \neq 1$. Suppose that there are at least three axes of symmetry through a point p, and let G be the group generated by the symmetries about these lines. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Remark 28 Several short and purely geometrical proofs are contained in [81, 88]. None of these proofs uses the coordinatization method for GQ's of order s with (axes of) symmetry as S. E. Payne and J. A. Thas did in FGQ.

Theorem 27 states that for any TGQ $(\mathcal{S}^{(p)}, G)$ of order s, $s > 1$, the symmetries about any three distinct arbitrary lines incident with p generate G. If one should demand the analogous condition for GQ's of order (s, s^2) (with four lines instead of three), then it is not hard to see that $\mathcal{S}^{(p)}$ is a $T_3(\mathcal{O})$ of Tits.
PROBLEM. *Is every TGQ of order s a $T_2(\mathcal{O})$ of Tits?*
In order to study the generalized quadrangles which have some distinct axes of symmetry through the point p, we introduced *Property (T)* as follows.
Property (T). An ordered flag (L, p) satisfies *Property (T) with respect to* L_1, L_2, L_3, where L_1, L_2, L_3 are three distinct lines incident with p and distinct from L, if the following condition is satisfied: if (i, j, k) is a permutation of $(1, 2, 3)$, if $M \sim L$ and $M \not{I} p$, and if $N \sim L_i$ and $N \not{I} p$ with $M \not\sim N$, then the triads $\{M, N, L_j\}$ and $\{M, N, L_k\}$ are not both centric.
TGQ's which satisfy Property (T) for some ordered flag always have order (s, s^2) for some s, see [88], and every thick TGQ of order (s, t) which has a proper subGQ of order s through the translation point satisfies Property (T) for some ordered flag(s) containing the translation point [88].

Theorem 29 (K. Thas [88, 81]) *Suppose that the GQ $\mathcal{S}$ satisfies Property (T) for the ordered flag (L, p) w.r.t. the distinct lines L_1, L_2, L_3, all incident with p. Moreover, suppose that L, L_1, L_2, L_3 are axes of symmetry. Then $\mathcal{S}^{(p)}$ is a TGQ and the translation group G is generated by the symmetries about L, L_1, L_2, L_3.*

We have also studied the following related problem.
PROBLEM. *Given a general TGQ $\mathcal{S}^{(p)}$, what is the minimal number of lines through p such that the translation group is generated by the symmetries about these lines?*
We have showed in [88] that there is a connection between the minimal number of lines through a translation point of a TGQ such that the translation group is generated by the symmetries about these lines, and the kernel of the TGQ; in particular, we have the following, which is a considerable improvement of the best known (general) result.

Theorem 30 (K. Thas [88]) *If $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $1 \neq s \neq t \neq 1$, with $(s,t) = (q^{na}, q^{n(a+1)})$ where a is odd and where $\mathrm{GF}(q)$ is the kernel of the TGQ, and if $k+3$ is the minimum number of distinct lines through p such that G is generated by the symmetries about these lines, then*

$$k \leq n.$$

An other result reads as follows.

Theorem 31 (K. Thas [88, 81]) *Let $\mathcal{S}$ be a thick GQ of order (s,t), $t-s \geq 1$, and let p be a point of $\mathcal{S}$ incident with more than $t-s+2$ axes of symmetry. Then $\mathcal{S}^{(p)}$ is a translation generalized quadrangle.*

Let us now introduce a second property, namely *Property (T')*, as follows.
Property (T'). An ordered flag (L,p) satisfies *Property (T') with respect to* L_1, L_2, L_3, where L_1, L_2, L_3 are distinct lines incident with p and distinct from L, if the following condition holds: if $M \sim L$ and $M \not{I} p$, and if q and q' are distinct arbitrary points on M which are not incident with L, then there is a permutation (i,j,k) of $(1,2,3)$ such that there are lines M_i, M_j, M_k, with $M_r \sim L_r$ and $r \in \{i,j,k\}$, for which $M \in \{M_i, M_k, L\}^{\perp}$ and $M_j \in \{M_i, M_k, L_j\}^{\perp}$, and such that qIM_i and $q'IM_k$.

It is one of the main goals in the context of this paper to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ arises from a flock, see J. A. Thas [63], [66], [70], [68], [71] and also K. Thas [86], [88]. In [88] we showed that a combination of Property (T') and Property (T) leads to Property (G) for TGQ's $\mathcal{S}$, and hence that it is possible to prove that such a TGQ $\mathcal{S}$ is related to a flock by the main theorem of [70]. A classification result was eventually obtained, see Theorem 35.

Theorem 32 (K. Thas [88]) *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t), such that there is a line LIp so that for every three distinct lines*

L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3, and suppose that there is at least one 3-tuple (M, N, U) such that (L, p) satisfies Property (T) w.r.t. (M, N, U). Then $T(\mathcal{O})$ is good at its element L.

Theorem 33 (K. Thas [88]) *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s, t) with $s \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^{*(p')} = T(\mathcal{O}^*)$ satisfies Property (G) for the flag (p', L'), where (p', L') corresponds to (p, L).*

The following very interesting theorem is a major step towards a classification of (translation duals of) TGQ's arising from flocks in the odd case.

Theorem 34 (Blokhuis, Lavrauw and Ball [8]) *Let $T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}), where $\mathrm{GF}(q)$ is the kernel, and suppose $T(\mathcal{O})$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, with the additional condition that $q \geq 4n^2 - 8n + 2$ with q odd. Then $T(\mathcal{O})$ is isomorphic to the point-line dual of a Kantor flock GQ.*

We now come to the aforementioned classification result; combining Theorems 33, 18 and 34, we obtain the following result; for details we refer to K. Thas [88].

Theorem 35 *Suppose that $\mathcal{S}^{(p)}$ is a thick TGQ of order (s, t), $s \neq 1 \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3. Then we have the following classification.*

1. *$s = t$ and $\mathcal{S}$ is a TGQ with no further restrictions.*

2. *$t = s^2$, s is an even prime power, and $\mathcal{O}$ is good at its element π which corresponds to L, where $\mathcal{S} = T(\mathcal{O})$. Also, $\mathcal{S}$ has precisely $s^3 + s^2$ subGQ's of order s which contain the line L, and if one of these subquadrangles is classical, i.e. isomorphic to the GQ $\mathcal{Q}(4, s)$, then $\mathcal{S}$ is classical, that is, is isomorphic to the GQ $\mathcal{Q}(5, s)$.*

3. *$t = s^2$ and $s = q^n$, q odd, where $\mathrm{GF}(q)$ is the kernel of the TGQ $\mathcal{S}^{(p)}$ with $q \geq 4n^2 - 8n + 2$, and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock.*

4. *$t = s^2$ and $s = q^n$, q odd, where* $\mathrm{GF}(q)$ *is the kernel of the TGQ* $\mathcal{S}^{(p)}$ *with* $q < 4n^2 - 8n + 2$, *and* $\mathcal{S}$ *is the translation dual of the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *for some flock* $\mathcal{F}$.

Finally, we state a divisibility condition for GQ's which have 'some' concurrent axes of symmetry.

Theorem 36 (K. Thas [88, 81]) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and let L, M and N be three different axes of symmetry incident with the same point p. Then $s|t$ and $\frac{t}{s} + 1|(s+1)t$.*

PROBLEM. *Are there GQ's of order (s,t), $s \neq 1 \neq t$, for which there is a point incident with precisely $k+1$ axes of symmetry with $k = 1$ if $s = t$, and with $1 \leq k \leq t - s + 1$ if $s \neq t$?*
By K. Thas [104], each such $\mathcal{S}$ would be a new GQ.
PROBLEM. *Suppose that $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$. Define minimal hypotheses for $\mathcal{S}$ such that s and t have the same parity.*
There are no examples for which $t - s \not\equiv 0 \mod 2$.

Note. In K. Thas [81] we proved that a thick GQ $\mathcal{S} = (P, B, I)$ of order (s,t) which has a point p incident with some axes of symmetry such that the group generated by the symmetries about these lines acts transitively on $P \setminus p^{\perp}$ has the property that s and t have the same parity.

PROBLEM. *Suppose $\mathcal{S}$ is an EGQ of order (s,t), $s \neq 1 \neq t$. Show that s and t are powers of the same prime.*
Frohardt solved the problem completely if $s \leq t$ [22]. For similar problems (and solutions), we also refer to Hachenberger [23].

5 Generalized quadrangles as group coset geometries and translation generalized quadrangles

5.1 Generalized quadrangles as group coset geometries and symmetries

Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s,t), $s,t \neq 1$, with elation point p and elation group G, and let q be a point of $P \setminus p^{\perp}$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $H_i = \{\theta \in G \parallel M_i^{\theta} = M_i\}$, $H_i^* = \{\theta \in G \parallel r_i^{\theta} = r_i\}$, and $\mathcal{J} = \{H_i \parallel 0 \leq i \leq t\}$.

Then $|G| = s^2t$ and $\mathcal{J}$ is a set of $t+1$ subgroups of G, each of order s. Also, for each i, H_i^* is a subgroup of G of order st containing H_i as a subgroup. Moreover, the following two conditions are satisfied:

(K1) $H_iH_j \cap H_k = \{\mathbf{1}\}$ for distinct i, j and k;

(K2) $H_i^* \cap H_j = \{\mathbf{1}\}$ for distinct i and j.

Conversely, if G is a group of order s^2t, $s \neq 1 \neq t$, and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), and if the conditions (K1) and (K2) are satisfied, then the H_i^* are uniquely defined by the H_i, and $(\mathcal{J}, \mathcal{J}^*)$ is said to be a 4-*gonal family of type* (s,t) *in* G (sometimes, for convenience, we will also say that $\mathcal{J}$ is a 4-*gonal family of type* (s,t)).
Let $(\mathcal{J}, \mathcal{J}^*)$ be a 4-gonal family of type (s,t) in the group G of order s^2t, $s \neq 1 \neq t$. Define an incidence structure $\mathcal{S}(G, \mathcal{J})$ as follows.

- Points of $\mathcal{S}(G, \mathcal{J})$ are of three kinds:
  - (i) elements of G;
  - (ii) right cosets H_i^*g, $g \in G$, $i \in \{0, \ldots, t\}$;
  - (iii) a symbol (∞).
- Lines are of two kinds:
  - (a) right cosets H_ig, $g \in G$, $i \in \{0, \ldots, t\}$;
  - (b) symbols $[H_i]$, $i \in \{0, \ldots, t\}$.
- Incidence. A point g of type (i) is incident with each line H_ig, $0 \leq i \leq t$. A point H_i^*g of type (ii) is incident with $[H_i]$ and with each line H_ih contained in H_i^*g. The point (∞) is incident with each line $[H_i]$ of type (b). There are no further incidences.

It is straightforward to check that the incidence structure $\mathcal{S}(G, \mathcal{J})$ is a GQ of order (s,t). Moreover, if we start with an EGQ $(\mathcal{S}^{(p)}, G)$ to obtain the family $\mathcal{J}$ as above, then we have that $(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G, \mathcal{J})$; for any $h \in G$ let us define θ_h by

$$g^{\theta_h} = gh, \qquad (H_ig)^{\theta_h} = H_igh,$$

$$(H_i^*g)^{\theta_h} = H_i^*gh, \ [H_i]^{\theta_h} = [H_i], \ (\infty)^{\theta_h} = (\infty),$$

with $g \in G$, $H_i \in \mathcal{J}$, $H_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G, \mathcal{J})$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h \parallel h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (1).
Hence, a group of order s^2t, $s \neq 1 \neq t$, admitting a 4-gonal family of type (s, t) can be represented as an elation group of a suitable elation generalized quadrangle. This was first noted by W. M. Kantor [32].
We now have the following interesting property.

Theorem 37 (FGQ, 8.2.2) *H_i is a group of symmetries about the line L_i if and only if $H_i \lhd G$ (and hence $\mathcal{S}$ is a TGQ if and only if $H_i \lhd G$ for each i), only if L_i is a regular line if and only if $H_iH_j = H_jH_i$ for all $H_j \in \mathcal{J}$.*

The following two rather group theoretical results show that for EGQ's, only few axes of symmetry (with an additional hypothesis) are needed through the elation point forcing it to be a translation point.

Theorem 38 (X. Chen and D. Frohardt [17]) *Let G be a group of order s^2t admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s, t), $s \neq 1 \neq t$. If there exist two distinct members in $\mathcal{J}$ which are normal subgroups of G, then s and t are powers of the same prime number p and G is an elementary abelian p-group.*

Theorem 39 (D. Hachenberger [23]) *Let G be a group of order s^2t admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s, t), $s \neq 1 \neq t$. If G is a group of even order, and if there exists a member of $\mathcal{J}$ which is a normal subgroup of G, then s and t are powers of 2 and G is an elementary abelian 2-group.*

In geometrical terms, Theorem 38 reads as follows: *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s, t), $s, t \neq 1$, and suppose that there are at least two axes of symmetry L and M through the elation point x, such that the full groups of symmetries about L and M are completely contained in G. Then s and t are powers of the same prime number p and G is an elementary abelian p-group.* Hence $\mathcal{S}^{(x)}$ is a translation generalized quadrangle. In geometrical terms, we have the following for Theorem 39: *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s, t), with $s, t \neq 1$ and s or t even, and suppose that there is at least one axis of symmetry L through the elation point x, such that the full group of symmetries about L is completely contained in G. Then s and t are powers of 2 and G is an elementary abelian 2-group.* Thus, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle.
Definition. A *panel* of a generalized quadrangle $\mathcal{S} = (P, B, I)$ is an element (p, L, q) of $P \times B \times P$ for which $pILIq$ and $p \neq q$. Dually, one defines *dual panels.*

Theorem 40 (K. Thas [82], see FGQ for (1)) *1. Suppose (p, L, p') is a panel of the GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, and let θ be a (p, L, p')-collineation (this is a whorl about p, L and p'). Then θ is a symmetry about L if L is regular.*

2. *Suppose that $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, with a whorl θ about a point p, and that L and M are lines such that $M \sim L \mathrm{I} p \not\mathrm{I} M$, for which $M^\theta = M$. Moreover, suppose that L is a regular line, and let q be a point on L, different from p and not on M.*
   Then we have one of the following possibilities:

   (a) *q is not fixed by θ;*

   (b) *q is fixed by θ, and then also every line through q.*

   Moreover, if we are in case (b), then θ is a symmetry about L.

3. *Let $\mathcal{S}$ be a GQ of order (s, t), $s, t \neq 1$, and suppose $\theta \neq \mathbf{1}$ is a whorl about distinct collinear points p and q. If moreover the line pq is regular, then θ is a symmetry about L.*

Corollary 41 *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that L is a line which contains a panel (p, L, p') for which there is a full group of (p, L, p')-collineations of size s. Then L is an axis of symmetry if and only if L is regular.*

Now suppose that $\mathcal{S}$ is a GQ with order (s, t), $s, t \neq 1$. Also, assume that p is a point and $L \mathrm{I} p$ a regular line, and again that $M \sim L$ is a line not through p. Suppose θ is a whorl about p which fixes M, and such that $\langle\theta\rangle$ acts semiregularly on the points of M not on L. Hence $|\langle\theta\rangle|$ is a divisor of s. Consider an arbitrary nontrivial element ϕ of $\langle\theta\rangle$ of prime power order, and consider the action of $\langle\phi\rangle$ on the points of $X = L \setminus [\{p\} \cup \{L \cap M\}]$. Since $|X| = s - 1$ and because of the fact that $s - 1$ and s are coprime, there follows immediately that there is a point $x \in X$ for which $x^{\langle\phi\rangle} = \{x\}$ (so every element of $\langle\phi\rangle$ fixes x). By Theorem 40, this implies that $\langle\phi\rangle$ is a group of symmetries about L. Since a finite group is generated by its elements of prime power order and since ϕ was arbitrary, there follows that also $\langle\theta\rangle$ is a group of symmetries about L (the product of two symmetries about the same line is clearly again a symmetry about this line).

Theorem 42 (K. Thas [82]) *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that L is a regular line through the point p. Let M be a line for which $L \sim M \not\mathrm{I} p$. If H is a group of whorls about p which fixes M and which acts transitively on the points of M which are not L, then H contains a full group of symmetries about L of order s (i.e. L is an axis of symmetry).*

There is a very nice corollary of Theorem 42.

Theorem 43 (K. Thas [82]) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and suppose that G is a group of whorls about p which acts transitively on the points of $P \setminus p^{\perp}$. Then we have the following.*

1. *If LIp is a regular line, then L is an axis of symmetry, and the full group of symmetries about L is completely contained in G.*

2. *If LIp is an axis of symmetry, with G_L the full group of symmetries about L, then G_L is completely contained in G.*

In particular, suppose that $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s,t), $s,t \neq 1$, with elation point p and elation group G, and suppose LIp is a regular line. Then L is an axis of symmetry.

Corollary 44 *Suppose $\mathcal{S}$ is an EGQ of order (s,t), $s,t \neq 1$, with elation point p. Then LIp is a regular line if and only if L is an axis of symmetry.*

Remark 45 The set of all symmetries about some line L of a GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, is always a group, and every symmetry about L is an elation about L and about every point on L, hence the group has at most size s. Hence, if the group of symmetries about L has size s, then this group is unique.

We are ready to complete Theorem 37.

Theorem 46 (K. Thas [82]) *Let $\mathcal{S} = (P,B,I)$ be an EGQ of order (s,t) with elation point p and where $s,t \neq 1$, and suppose q is a point of $P \setminus p^{\perp}$. Suppose $L_0, \ldots, L_t$ are the lines through p, and suppose M_i are lines such that $L_i \sim M_i I q$. Let H_i be the subgroup of the elation group G which fixes M_i, for all i, and put $\mathcal{J} = \{H_0, \ldots, H_t\}$. Then H_i is a group of symmetries about the line L_i if and only if $H_i \lhd G$ (and hence $\mathcal{S}$ is a TGQ if and only if $H_i \lhd G$ for each i), if and only if L_i is a regular line if and only if $H_i H_j = H_j H_i$ for all $H_j \in \mathcal{J}$.*

We can now give an alternative, more geometrical, definition of translation generalized quadrangles without the use of symmetries, abelian groups or Galois geometries (see [47]), as follows: *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with a point p for which there is a group of whorls which acts transitively on $P \setminus p^{\perp}$. Then $\mathcal{S}$ is a TGQ with translation point p if and only if every line through p is a regular line.*

Theorem 47 (K. Thas [82]) *Let $\mathcal{S}$ be a GQ of order (s,t) with $s \neq 1 \neq t$, and suppose p is a point of $\mathcal{S}$.*

1. *If $s = t$, then $\mathcal{S}$ is a TGQ with translation point p if and only if p is incident with three regular lines L_1, L_2, L_3 for which there are lines M_1, M_2, M_3 such that $L_i \sim M_i \not\mathrm{I} p$ and such that there are groups G_i of whorls about p which act transitively on the points of $M_i \setminus \{M_i \cap L_i\}$.*

2. *If $s \neq t$, then $\mathcal{S}$ is a TGQ with translation point p if and only if p is incident with at least $t - s + 3$ regular lines $L_1, \ldots, L_{t-s+3}$ for which there are lines $M_1, \ldots, M_{t-s+3}$ such that $L_i \sim M_i \not\mathrm{I} p$ and such that there are groups G_i of whorls about p which act transitively on the points of $M_i \setminus \{M_i \cap L_i\}$.*

5.2 Strong elation generalized quadrangles

We now introduce the notion of *strong elation generalized quadrangle*. A *strong elation generalized quadrangle (SEGQ)* is a generalized quadrangle $\mathcal{S}$ for which each point is an elation point.

Theorem 48 (K. Thas [82]) *If an SEGQ $\mathcal{S}$ of order (s,t), $s, t \neq 1$, has a regular line, then we have one of the following cases.*

1. $\mathcal{S} \cong \mathcal{Q}(4, s)$.

2. $\mathcal{S} \cong \mathcal{Q}(5, s)$.

Part (1) is a direct corollary of [47, 5.2.1], and (2) uses Fong and Seitz [20, 21]. Following K. Thas [82], if p is a point of a GQ $\mathcal{S} = (P, B, I)$ such that there is a group of whorls about p which acts transitively on $P \setminus p^\perp$, then we call p a *center of transitivity*. If p is a point of a GQ $\mathcal{S} = (P, B, I)$ for which there is a group of automorphisms of $\mathcal{S}$ which fixes p and which acts transitively on the points of $P \setminus p^\perp$, then p is a *point of transitivity*. Dually one defines *axis of symmetry*, respectively *line of transitivity*.
Theorem 48 is actually a particular example of the following result.

Theorem 49 (K. Thas [82]) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s, t \neq 1$. Then $\mathcal{S}$ is isomorphic to either $\mathcal{Q}(4, s)$ or $\mathcal{Q}(5, s)$ if and only if $\mathcal{S}$ contains a center of transitivity p, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular pair of lines.*

PROBLEM. *Classify all SEGQ's (i.e. prove that every SEGQ is either classical or dual classical) without the classification of the finite simple groups.*

In connection with the notions of *point of transitivity* and *center of transitivity*, we refer the reader to S. E. Payne and L. A. Rogers [48] and S. E. Payne [45]. In [45], Payne and Rogers introduce a *homogeneous generalized quadrangle (HGQ)* $\mathcal{S} = (\mathcal{S}^{(p)}, G)$ as a GQ $\mathcal{S} = (P, B, I)$ which contains a point p for which there is a group G of collineations of $\mathcal{S}$ each element of which fixes p and such that G acts regularly on $P \setminus p^{\perp}$. In [45], some concrete examples $(\mathcal{S}^{(p)}, G)$ of HGQ's are investigated which are not EGQ's for the point p.

5.3 Improvements of results of X. Chen and D. Frohardt, and D. Hachenberger

In Theorem 38 and Theorem 39, the group theoretical descriptions have one major restriction, since they demand that the groups of symmetries about the lines must be **completely contained** in the elation group. By our preceding theorems and observations, we can now lose these "contained in"-conditions completely; moreover, we do not even ask that the lines are axes of symmetry, but only that they are regular!

Theorem 50 (K. Thas [82]) *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s, t), $s, t \neq 1$. If there are two distinct regular lines through the point x, then s and t are powers of the same prime number p, G is an elementary abelian p-group and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Theorem 51 (K. Thas [82]) *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s, t), $s, t \neq 1$. If there is a regular line through the point x, and G is a group of even order, then s and t are powers of 2, G is an elementary abelian 2-group and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Remark 52 By results of Chapter 8 of FGQ, there follows that if a GQ $(\mathcal{S}^{(x)}, G)$ satisfies the conditions of Theorem 50 or Theorem 51, then G is the complete set of elations about x.

PROBLEM. *State minimal conditions for (thick) EGQ's $(\mathcal{S}^{(p)}, G)$ such that G is the set of all elations about p.*
If $\mathcal{S}^{(p)}$ is a TGQ with translation point p, then the problem is completely solved, see Chapter 8 of FGQ. Also, if $\mathcal{S}^{(p)}$ is a thick EGQ of order (s, t), then G is always the complete set of elations about p if $t > s^2/2$, see [47, 8.2.4,1.4.2] and K. Thas [81]. The essence here is the use of *Frobenius groups*, see e.g. [31, 35].

6 Span-symmetric generalized quadrangles and generalized quadrangles with two collinear translation points

6.1 Span-symmetric generalized quadrangles

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and suppose L and M are distinct non-concurrent axes of symmetry; then it is easy to see, by transitivity, that every line of $\{L,M\}^{\perp\perp}$ is an axis of symmetry, and $\mathcal{S}$ is called a *span-symmetric generalized quadrangle (SPGQ) with base-span* $\{L,M\}^{\perp\perp}$.
Let $\mathcal{S}$ be a span-symmetric GQ of order (s,t), $s,t \neq 1$, with base-span $\{L,M\}^{\perp\perp}$. Throughout this paper, we will continuously use the following NOTATIONS.
First of all, the base-span $\{L,M\}^{\perp\perp}$ will always be denoted by $\mathcal{L}$, and the set $\{L,M\}^{\perp}$ by $\mathcal{L}^{\perp}$. The group which is generated by all symmetries about the lines of $\mathcal{L}$ is G, and sometimes we will call this group the *base-group*. This group clearly acts 2-transitively on the lines of $\mathcal{L}$, and fixes every line of $\mathcal{L}^{\perp}$ (see for instance FGQ). The set of all the points which are on lines of $\{L,M\}^{\perp\perp}$ is denoted by Ω. We will refer to $\Gamma = (\Omega, \mathcal{L}\cup\mathcal{L}^{\perp}, I')$, with I' being the restriction of I to $[\Omega \times (\mathcal{L}\cup\mathcal{L}^{\perp})] \cup [(\mathcal{L}\cup\mathcal{L}^{\perp}) \times \Omega]$, as being *the base-grid*.

Theorem 53 (S. E. Payne [40], see FGQ) *If $\mathcal{S}$ is an SPGQ of order s, $s \neq 1$, with base-grid $(\Omega, \mathcal{L}\cup\mathcal{L}^{\perp}, I')$ and base-group G, then G acts regularly on the set of $(s+1)s(s-1)$ points of $\mathcal{S}$ not in Ω.*

In the sequel, we will come to an analogue of this result for the general case. Let $\mathcal{S}$ be an SPGQ of order $s \neq 1$ with base-span $\mathcal{L}$, and put $\mathcal{L} = \{U_0,\ldots,U_s\}$. The group of symmetries about U_i is denoted by G_i, $i = 0,1,\ldots,s$, throughout this paper. Then one notes the following properties (see [40, 47]):

1. the groups $G_0,\ldots,G_s$ form a complete conjugacy class in G and are all of order s $(s \geq 2)$;

2. if $G_i^* = N_G(G_i)$, then $G_i \cap G_j^* = \{\mathbf{1}\}$ for $i \neq j$;

3. $G_iG_j \cap G_k = \{\mathbf{1}\}$ for i,j,k distinct, and

4. $|G| = s^3 - s$.

We say that G is a group with a 4-*gonal basis* $\mathcal{T} = \{G_0,\ldots,G_s\}$ if these four conditions are satisfied.

It is possible to recover the GQ $\mathcal{S}$ of order s from the base-group G starting from 4-gonal bases in the following way, see [40, 47].
Suppose G is a group of order $s^3 - s$ with a 4-gonal basis $\mathcal{T} = \{G_0, \ldots, G_s\}$, and let $G_i^* = N_G(G_i)$ for $i = 0, 1, \ldots, s$. Define a point-line incidence structure $\mathcal{S}_\mathcal{T} = (P_\mathcal{T}, B_\mathcal{T}, I_\mathcal{T})$ as follows.

- $P_\mathcal{T}$ consists of two kinds of POINTS.
  - (a) Elements of G.
  - (b) Right cosets of the G_i^*'s.
- $B_\mathcal{T}$ consists of three kinds of LINES.
  - (i) Right cosets of G_i, $0 \leq i \leq s$.
  - (ii) Sets $M_i = \{G_i^* g \parallel g \in G\}$, $0 \leq i \leq s$.
  - (iii) Sets $L_i = \{G_j^* g \parallel G_i^* \cap G_j^* g = \emptyset, 0 \leq j \leq s, j \neq i\} \cup \{G_i^*\}$, $0 \leq i \leq s$.
- INCIDENCE. $I_\mathcal{T}$ is the natural incidence: a line $G_i g$ of type (i) is incident with the s points of type (a) contained in it, together with that point $G_i^* g$ of type (b) containing it. The lines of types (ii) and (iii) are already described as sets of those points with which they are to be incident.
  Then $\mathcal{S}_\mathcal{T} = (P_\mathcal{T}, B_\mathcal{T}, I_\mathcal{T})$ is a GQ of order s which is span-symmetric for the base-span $\{L_0, L_1\}^{\perp\perp}$ [40, 47]. Also, if $\mathcal{S}$ is an SPGQ of order s, $s \neq 1$, with base-span $\mathcal{L}$ and base-group G, and where $\mathcal{T}$ is the corresponding 4-gonal basis, then $\mathcal{S} \cong \mathcal{S}_\mathcal{T}$ [40, 47]. We thus have the following interesting theorem.

Theorem 54 (S. E. Payne [40], see FGQ) *A span-symmetric GQ of order $s \neq 1$ with given base-span $\mathcal{L}$ is canonically equivalent to a group G of order $s^3 - s$ with a 4-gonal basis $\mathcal{T}$.*

It is also important to recall the following.

Theorem 55 (S. E. Payne [40], see FGQ) *Let $\mathcal{S}$ be an SPGQ of order $s \neq 1$, with base-span $\mathcal{L}$. Then every line of $\mathcal{L}^\perp$ is an axis of symmetry.*

This theorem thus yields the fact that for any two distinct lines U and V of $\mathcal{L}^\perp$, the GQ is also an SPGQ with base-span $\{U, V\}^{\perp\perp}$. The corresponding base-group will be denoted by $G^\perp$. It should be emphasized that this property only holds for SPGQ's of order s.
Finally, we state a theorem regarding the possible orders of SPGQ's.

Theorem 56 (W. M. Kantor, see FGQ) *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s,t \neq 1$. Then $t \in \{s, s^2\}$.*

Note. It was conjectured in 1980 by S. E. Payne that a span-symmetric generalized quadrangle of order $s > 1$ is always classical, i.e. isomorphic to the GQ $\mathcal{Q}(4,s)$ arising from a quadric. There was a "proof" of this theorem as early as in 1981 by Payne in [40], but later on it was noticed by Payne himself that there was a mistake in the proof. The paper was very valuable however, since the author introduced there the 4-gonal bases and proved for instance Theorem 54 and Theorem 55.

6.2 Span-symmetric generalized quadrangles and split BN-pairs of rank 1

Recent developments in the theory of SPGQ's have shown that the notion of *(finite) split BN-pair of rank* 1 is particularly useful for various aims, as well as the theory of *perfect central extensions of (perfect) groups* [2, 19]. Let us recall that a group with a *split BN-pair of rank* 1 (see e.g. [52, 94]) is a permutation group (X, G) which satisfies the following properties.

(BN1) G acts 2-transitively on X.

(BN2) For every $x \in X$ there holds that the stabilizer of x in G has a normal subgroup which acts regularly on $X \setminus \{x\}$.

Note. Recently, J. Tits [96] has introduced the notion of *Moufang sets*, which are essentially the same objects as split BN-pairs of rank 1.
For more on (split) BN-pairs, see the contribution of K. Tent to these proceedings [56].
The following theorem classifies all finite groups with a split BN-pair of rank 1 without using the classification of the finite simple groups, see [54, 25].

Theorem 57 (Shult [54], Hering, Kantor and Seitz [25]) *Suppose (X, G) is a finite group with a split BN-pair of rank 1, and suppose $|X| = s+1$, with $s \in \mathbb{N}$. Then the group G is always one of the following list (up to isomorphism):*

(a) a sharply 2-transitive group on X;

(b) $\mathrm{PSL}_2(s)$;

(c) the Ree group $\mathrm{R}(\sqrt[3]{s})$ (also sometimes denoted by ${}^2G_2(\sqrt[3]{s})$), with $\sqrt[3]{s}$ an odd power of 3;

(d) the Suzuki group $\mathrm{Sz}(\sqrt{s})$ *(sometimes denoted by* ${}^2B_2(\sqrt{s})$*), with* $\sqrt{s}$ *an odd power of* 2*;*

(e) the unitary group $\mathrm{PSU}_3(\sqrt[3]{s^2})$*.*

For more on the structure of these (permutation) groups, see [35].

Theorem 58 ([40, 47, 83]) *Suppose* $\mathcal{S}$ *is a span-symmetric GQ of order* (s,t)*,* $s,t \neq 1$*, with base-span* $\mathcal{L}$*, and base-group* G*. Then* $(\mathcal{L}, G)$ *is a finite split BN-pair of rank* 1*.*

Span-symmetric GQ's of order s, $s > 1$, induce some particular collineations on the *projective planes*[3] which arise from each regular line of the base-span (see Theorem 3).
Suppose Π is a finite projective plane of order n, and suppose (p, L) is a point-line pair of the plane. A (p, L)*-collineation* θ of Π is a collineation of the plane which fixes p linewise and L pointwise. It is easy to show — see e.g. [30] — that a group of (p, L)-collineations of Π always acts semiregularly on the points of Π not on L and distinct from p. Suppose pIL in the plane Π of order n. Then Π is called (p, L)*-transitive* if the group of (p, L)-collineations acts regularly on the points, distinct from p, of any line through p which is different from L. This group has then order n. If $p\not{I}L$, then the plane is (p, L)*-transitive* if the group of (p, L)-collineations acts regularly on the points, distinct from p and not on L, of any line through p. In this case, the group has order $n-1$. Also, a plane of order n is (p, L)-transitive for the flag, respectively anti-flag, (p, L) if and only if the group of (p, L)-collineations has order n, respectively $n - 1$ [30].
The following theorem is a very nice step in the so called *Lenz-Barlotti classification of projective planes* [37, 5], see also [18, 104, 55, 105]. In particular, it states that the Lenz-Barlotti class III.2 is empty.

Theorem 59 (J. Yaqub [103]) *Let* Π *be a projective plane of order* n*, containing a non-incident point-line pair* (x, L)*, for which holds that* Π *is* (x, L)*-transitive, and assume that* Π *is* (y, xy)*-transitive for every point* y *on* L*. Then* Π *is Desarguesian.*

Using Theorem 57 and Theorem 59, we were able to prove the following.

Theorem 60 (K. Thas [83]) *Let* $\mathcal{S}$ *be a span-symmetric generalized quadrangle of order* s*, where* $s \neq 1$*. Then* $\mathcal{S}$ *is classical, i.e. isomorphic to* $\mathcal{Q}(4, s)$*.*

[3] Recall that a *projective plane of order* n is a 2-$(n^2+n+1, n+1, 1)$ design with $n \geq 2$, and an *affine plane of order* n is a 2-$(n^2, n, 1)$ design, see, for example, [30, 84].

There is a very nice corollary for groups.

Theorem 61 (K. Thas [83]) *A group is isomorphic to* $\mathrm{SL}_2(s)$ *for some* s *if and only if it has a* 4*-gonal basis.*

For span-symmetric generalized quadrangles of order (s,t) with $s \neq t$, $s \neq 1 \neq t$, however, a similar result as Theorem 60 cannot hold. Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathrm{PG}(3,q)$, q odd. Then, as before, the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in \mathrm{GF}(q)$, m a given non-square in $\mathrm{GF}(q)$ and σ a given automorphism of $\mathrm{GF}(q)$, define a Kantor flock $\mathcal{F}$ of $\mathcal{K}$. Recently, S. E. Payne noticed that the dual Kantor flock generalized quadrangles are span-symmetric, see also [42]. Moreover, every nonclassical dual Kantor flock GQ even contains a line L for which every line which meets L is an axis of symmetry![4].
Recall however that W. M. Kantor [32, 47] gave a partial classification theorem for span-symmetric generalized quadrangles by proving that for a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$, necessarily either $t = s$ or $t = s^2$. In [86] we found the following improvement of Kantor's theorem.

Theorem 62 (K. Thas [86]) *Let* $\mathcal{S}$ *be a span-symmetric generalized quadrangle of order* (s,t)*,* $s \neq 1 \neq t$*. Then either* $s = t$ *or* $t = s^2$*, and* s *and* t *are powers of the same prime.*

Moreover, we obtained the following strong theorem.

Theorem 63 (K. Thas [86, 89]) *Suppose* $\mathcal{S}$ *is a span-symmetric generalized quadrangle of order* (s,t)*,* $s \neq 1 \neq t$*, where* $s \neq t$*. Then* $\mathcal{S}$ *contains at least* $s+1$ *subquadrangles isomorphic to the classical GQ* $\mathcal{Q}(4,s)$*.*

In Section 8 we will illustrate this result in the context of *Moufang generalized quadrangles*.
Problem. *Classify all SPGQ's of order* (s,t)*,* $s \neq 1 \neq t$*.*
For $s = t$, the work is done, but for $s \neq t$, the problem seems very hard to handle (even when the SPGQ is known to be a TGQ, or when the SPGQ arises from a flock). Besides the results already mentioned, we refer to [89, 86, 90].

[4]In fact, in [90] we noted that this a particular result in a more general theory, but it is beyond the scope of this paper to go into details here.

6.3 Flocks, q-clans and generalized quadrangles

Let $\mathbb{F} = \mathrm{GF}(q)$, q any prime power, and put $G = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta').$$

This makes G into a group whose center is $C = \{(0, c, 0) \in G \parallel c \in \mathbb{F}\}$.
Let $\mathcal{C} = \{A_u \parallel u \in \mathbb{F}\}$ be a set of q distinct upper triangular 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is *anisotropic* whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0, 0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \qquad\qquad x_u, y_u, z_u, u \in \mathbb{F}.$$

For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{1}$$

is a nonsquare of $\mathbb{F}$ whenever $r, u \in \mathbb{F}$, $r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } \ tr((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{2}$$

whenever $r, u \in \mathbb{F}$, $r \neq u$.
Now we can define a family of subgroups of G by

$$A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(0, 0, \beta) \in G \parallel \beta \in \mathbb{F}^2\}.$$

Then put $\mathcal{J} = \{A(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$ and $\mathcal{J}^* = \{A^*(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(0, c, \beta) \parallel \beta \in \mathbb{F}^2\}.$$

With $G, A(u), A^*(u), \mathcal{J}$ and $\mathcal{J}^*$ as above, the following important theorem is a combination of results of S. E. Payne and W. M. Kantor.

Theorem 64 (S. E. Payne [39], W. M. Kantor [32]) *The pair $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan.*

6.4 The known TGQ's arising from a flock

We say that a TGQ *arises from a flock* if it is the point-line dual of a flock GQ. In this paragraph, we summarize the known translation duals of TGQ's which arise from flocks.

(a) The classical case. If $\mathcal{S}(\mathcal{F})$ is the classical GQ $H(3,q^2)$, then it is a TGQ with base-line L, L any line of $\mathcal{S}(\mathcal{F})$. The dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to $T_3(\mathcal{O})$, $\mathcal{O}$ an elliptic quadric of $\mathrm{PG}(3,q)$. Hence the kernel $\mathbb{K}$ is the field $\mathrm{GF}(q)$. Also, $\mathcal{S}(\mathcal{F})^D$ is isomorphic to its translation dual $(\mathcal{S}(\mathcal{F})^D)^*$.

(b) Kantor generalized quadrangles. See Section 3.

(c) Roman generalized quadrangles. Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathrm{PG}(3,q)$, with $q = 3^r$ and $r > 2$. Then the q planes π_t with equation $tX_0-(mt+m^{-t}t^9)X_1+t^3X_2+X_3 = 0$, $t \in \mathrm{GF}(q)$, m a given non-square in $\mathrm{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [44]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ is a TGQ for some base-line, and so the dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to some $T(\mathcal{O})$. By [51] the kernel $\mathbb{K}$ is isomorphic to $\mathrm{GF}(3)$. Payne [44] shows that $T(\mathcal{O})$ is not isomorphic to its translation dual $T(\mathcal{O}^*)$. Also, he proves that $T(\mathcal{O}^*)$ is a TGQ which is not the point-line dual of a flock GQ. The GQ's $T(\mathcal{O}^*)$ were called by Payne the *Roman generalized quadrangles*.

(d) A sporadic semifield flock generalized quadrangle. Let $q = 3^5$. The q planes π_t with equation $tX_0 - t^9X_1 + t^{27} + X_3 = 0$, $t \in \mathrm{GF}(q)$, define a flock of the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathrm{PG}(3,q)$. The flock was constructed by L. Bader, G. Lunardon and I. Pinneri in [3], starting from the Penttila-Williams ovoid of $\mathcal{Q}(4,3^5)$ defined in [49], and using the construction method developed by J. A. Thas in [61, 67]; the corresponding GQ is therefore referred to as the *(sporadic) Penttila-Williams generalized quadrangle*. The kernel of the Penttila-Williams GQ is isomorphic to $\mathrm{GF}(3)$.

We are now ready to state the main result of K. Thas [86]; in the proof we rely on a construction method of subGQ's developed in [86] with the use of split BN-pairs of rank 1 and universal central extensions of perfect groups, and Theorems 18 and 34.

Theorem 65 (K. Thas [86]) *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have one of the following:*

(i) $s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4,s)$;

(ii) $t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5,s)$;

(iii) $t = s^2$, $s = q^n$ *with* q *odd, where* $\mathrm{GF}(q)$ *is the kernel of the TGQ* $\mathcal{S} = \mathcal{S}^{(\infty)}$ *with* (∞) *an arbitrary translation point of* $\mathcal{S}$, $q \geq 4n^2 - 8n + 2$ *and* $\mathcal{S}$ *is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *where* $\mathcal{F}$ *is a Kantor flock;*

(iv) $t = s^2$, $s = q^n$ *with* q *odd, where* $\mathrm{GF}(q)$ *is the kernel of the TGQ* $\mathcal{S} = \mathcal{S}^{(\infty)}$ *with* (∞) *an arbitrary translation point of* $\mathcal{S}$, $q < 4n^2 - 8n + 2$ *and* $\mathcal{S}$ *is the translation dual of the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *for some flock* $\mathcal{F}$*.*

If a thick GQ $\mathcal{S}$ *has two non-collinear translation points, then* $\mathcal{S}$ *is always of classical type, i.e. isomorphic to one of* $\mathcal{Q}(4, s)$*,* $\mathcal{Q}(5, s)$*.*

Note. The class (iv) is not empty, see the foregoing section.
We emphasize that this theorem is rather remarkable, since we start from some very easy combinatorial and group theoretical properties, while flock generalized quadrangles are concretely described using finite fields, q-clans and 4-gonal families.
As a corollary of Theorem 65, the following theorem completely classifies all generalized quadrangles of order (s, t), $s \neq 1 \neq t$ and s even, with two distinct translation points.

Theorem 66 (K. Thas [86]) *Let* $\mathcal{S}$ *be a generalized quadrangle of order* (s, t)*,* $s \neq 1 \neq t$ *and* s *even, with two distinct translation points. Then* $\mathcal{S}$ *is classical, i.e. isomorphic to* $\mathcal{Q}(4, s)$ *or* $\mathcal{Q}(5, s)$*.*

In [86] there is also a generalization of Theorem 65.

Theorem 67 (K. Thas [86]) *Suppose* $\mathcal{S}$ *is a generalized quadrangle of order* (s, t)*,* $s \neq 1 \neq t$*, and suppose that one of the following conditions is satisfied.*

1. *There is a regular line* L *and there are points* p *and* q *such that* pIL *is a point which is incident with at least* $s + 1$ *axes of symmetry different from* L*, and* qIL *is a point different from* p *which is incident with at least one axis of symmetry which is not* L*.*

2. *There is a regular line* L *and there is a point* p *such that* pIL *is a point which is incident with at least* $s + 1$ *regular lines* $L_0, \ldots, L_s$ *different from* L*, for which there are lines* $M_0, \ldots, M_s$ *and points* $p_0, \ldots, p_s$ *such that* $p_i I L_i \sim M_i \not{I} p_i$ *for all* i*, and such that there is a group of whorls about each* p_i *which fixes* M_i *and which acts transitively on the points*

of M_i which are not on L_i. Also, there is a point qIL different from p which is incident with at least one regular line U which is not L, for which there is a line M, with qIM, and a point u such that $uIU \sim M \not{I} u$, and such that there is a group of whorls about u which fixes M and which acts transitively on the points of M which are not incident with U.

3. *$\mathcal{S}$ has two distinct centers of transitivity (see Section 5.2) p and q and a regular line which is contained in $(pq)^\perp \setminus \{pq\}$ if p and q are collinear.*

4. *s is even, $\mathcal{S}$ contains a regular line and also an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

5. *s is odd, $\mathcal{S}$ contains two distinct regular lines and also an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

Then we have the following possibilities:

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4, s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

(iii) *$t = s^2$, $s = q^n$, where $\mathrm{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, q is odd, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$, where $\mathrm{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, q is odd, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

6.5 Spreads in SPGQ's

A *spread* of a generalized quadrangle $\mathcal{S}$ is a set $\mathbf{T}$ of mutually non-concurrent lines such that every point of $\mathcal{S}$ is incident with a (necessarily unique) line of $\mathbf{T}$. If the GQ is of order (s,t), then $|\mathbf{T}| = st + 1$, and, conversely, a set of $st + 1$ mutually non-concurrent lines of a GQ of order (s,t) is a spread. Dually, one defines *ovoids* of generalized quadrangles. Most results up to 1993 on spreads and ovoids of GQ's are contained in J. A. Thas and S. E. Payne [75]; an update is contained in M. R. Brown [13].
Two spreads $\mathbf{T}$ and $\mathbf{T}'$ of a GQ $\mathcal{S}$ are said to be *isomorphic* if there is an automorphism of $\mathcal{S}$ which maps $\mathbf{T}$ onto $\mathbf{T}'$.
The following result is taken from [86, 89].

Theorem 68 (K. Thas [86]) *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with $s \neq t$. Then $\mathcal{S}$ contains at least $2(s^2-s)$ distinct spreads.*

7 Classifications of generalized quadrangles

So far, we have mentioned five classification results of TGQ's, that are, Theorem 34, Theorem 35, Theorem 65, Theorem 66 and the main result of J. A. Thas [70]. In this section, we mention two more such classification results.

7.1 Veronese varieties

There is a strong connection between TGQ's satisfying Property (G) and *the Veronesean* $\mathcal{V}_2^4$ of all conics of $\mathrm{PG}(2,q)$, as pointed out by J. A. Thas in [66, 70]. So, we include a section on Veronese varieties (a good reference is Chapter 25 of J. W. P. Hirschfeld and J. A. Thas [29]).
The *Veronese variety* of all quadrics of $\mathrm{PG}(n,\mathbb{K})$, $n \geq 1$ and $\mathbb{K}$ any commutative field, is the variety

$$\begin{aligned}\mathcal{V} \;=\; \{(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, x_0x_2, \ldots, x_0x_n, x_1x_2, \ldots, x_1x_n, \ldots, x_{n-1}x_n) \,\|\, \\ (x_0, x_1, \ldots, x_n) \text{ is a point of } \mathrm{PG}(n,\mathbb{K})\}\end{aligned}$$

of $\mathrm{PG}(N,\mathbb{K})$ with $N = n(n+3)/2$. The variety $\mathcal{V}$ has dimension n and order 2^n; for $\mathcal{V}$ we also write $\mathcal{V}_n$ or $\mathcal{V}_n^{2^n}$. It is also called the *Veronesean of quadrics* of $\mathrm{PG}(n,\mathbb{K})$, or simply the *quadric Veronesean* of $\mathrm{PG}(n,\mathbb{K})$. It can be shown that the quadric Veronesean is absolutely irreducible and nonsingular.
Let $\mathrm{PG}(n,\mathbb{K})$ consist of all points

$$(y_{00}, y_{11}, \ldots, y_{nn}, y_{01}, y_{02}, \ldots, y_{0n}, y_{12}, \ldots, y_{1n}, \ldots, y_{n-1,n});$$

for y_{ij} we also write y_{ji}. Let $\zeta : \mathrm{PG}(n,\mathbb{K}) \rightarrow \mathrm{PG}(N,\mathbb{K})$, with $N = n(n+3)/2$ and $n \geq 1$, be defined by

$$(x_0, x_1, \ldots, x_n) \mapsto (y_{00}, y_{11}, \ldots, y_{n-1,n}),$$

with $y_{ij} = x_ix_j$. Then ζ is a bijection of $\mathrm{PG}(n,\mathbb{K})$ onto the quadric Veronesean $\mathcal{V}$ of $\mathrm{PG}(n,\mathbb{K})$. It then follows that the variety $\mathcal{V}$ is rational.

Theorem 69 (see J. W. P. Hirschfeld and J. A. Thas [29]) *The quadrics of $\mathrm{PG}(n,\mathbb{K})$ are mapped by ζ onto all hyperplane sections of $\mathcal{V}$.*

Corollary 70 *No hyperplane of $\mathrm{PG}(N,\mathbb{K})$ contains the quadric Veronesean $\mathcal{V}$.*

Theorem 71 (see J. W. P. Hirschfeld and J. A. Thas [29]) *Any two distinct points of $\mathcal{V}$ are contained in a unique irreducible conic of $\mathcal{V}$.*

If $\mathbb{K} = \mathrm{GF}(q)$, then it is clear that $\mathcal{V}_n$ contains $\theta(n) = q^n + q^{n-1} + \cdots + q + 1$ points. As no three points of $\mathcal{V}$ are collinear we have the following theorem.

Theorem 72 (see J. W. P. Hirschfeld and J. A. Thas [29]) *The quadric Veronesean $\mathcal{V}_n$ is a $\theta(n)$-cap of $\mathrm{PG}(N, q)$, $N = n(n+3)/2$.*

Let $n = 2$. Then $\mathcal{V}$ is a surface of order 4 in $\mathrm{PG}(5, \mathbb{K})$. Apart from the conic, the variety $\mathcal{V}_2^4$ is the quadric Veronesean which is most studied and characterized. Assume also that $\mathbb{K} = \mathrm{GF}(q)$. To the conics (irreducible or not) of $\mathrm{PG}(2, q)$ there correspond all hyperplane sections of $\mathcal{V}_2^4$. The hyperplane is uniquely determined by the conic if and only if the latter is not a single point. If the conic of $\mathrm{PG}(2, q)$ is one line, then the corresponding hyperplane of $\mathrm{PG}(5, q)$ meets $\mathcal{V}_2^4$ in an irreducible conic; the surface $\mathcal{V}_2^4$ contains no other irreducible conics. It follows that $\mathcal{V}_2^4$ contains exactly $q^2 + q + 1$ irreducible conics, that any two distinct points of $\mathcal{V}_2^4$ are contained in a unique irreducible conic, and that any two distinct irreducible conics on $\mathcal{V}_2^4$ meet in a unique point. If the conic $\mathcal{C}$ of $\mathrm{PG}(2, q)$ consists of two distinct lines, then the corresponding hyperplane $\mathrm{PG}(4, q)$ meets $\mathcal{V}_2^4$ in two irreducible conics with exactly one point in common; if $\mathcal{C}$ is irreducible, then $\mathrm{PG}(4, q)$ meets $\mathcal{V}_2^4$ in a rational quartic curve. The planes of $\mathrm{PG}(5, q)$ which meet $\mathcal{V}_2^4$ is an irreducible conic are called the *conic planes* of $\mathcal{V}_2^4$.

Theorem 73 (see J. W. P. Hirschfeld and J. A. Thas [29]) *Any two distinct conic planes π and π' of $\mathcal{V}_2^4$ have exactly one point in common, and this common point belongs to $\mathcal{V}_2^4$.*

7.2 A classification of TGQ's by J. A. Thas (and two other theorems)

Now we come to one of the main results of J. A. Thas [66, 70].

Theorem 74 (J. A. Thas [66, 70]) *Consider a TGQ $T(n, 2n, q) = T(\mathcal{O})$, q odd, with $\mathcal{O} = \mathcal{O}(n, 2n, q) = \{\mathrm{PG}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), \ldots, \mathrm{PG}^{(q^{2n})}(n-1, q)\}$. If $\mathcal{O}$ is good at $\mathrm{PG}(n-1, q)$, then we have one of the following.*

- *(a) There exists a $\mathrm{PG}(3, q^n)$ in the extension $\mathrm{PG}(4n-1, q^n)$ of the space $\mathrm{PG}(4n-1, q)$ of $\mathcal{O}(n, 2n, q)$ which has exactly one point in common with each of the spaces $\mathrm{PG}(n-1, q^n), \mathrm{PG}^{(1)}(n-1, q^n), \ldots, \mathrm{PG}^{(q^{2n})}(n-1, q^n)$. The set of these $q^{2n} + 1$ points is an elliptic quadric of $\mathrm{PG}(3, q^n)$ and $T(\mathcal{O})$ is isomorphic to the classical GQ $\mathcal{Q}(5, q^n)$.*

(b) *We are not in Case (a) and there exists a* $\mathrm{PG}(4,q^n)$ *in* $\mathrm{PG}(4n-1,q^n)$ *which intersects* $\mathrm{PG}(n-1,q^n)$ *in a line* M *and which has exactly one point* r_i *in common with any space* $\mathrm{PG}^{(i)}(n-1,q^n)$, $i=1,2,\ldots,q^{2n}$. *Let* $\mathcal{W}=\{r_i \parallel i=1,2,\ldots,q^{2n}\}$ *and let* $\mathcal{M}$ *be the set of all common points of* M *and the conics which contain exactly* q^n *points of* $\mathcal{W}$. *Then the set* $\mathcal{W}\cup\mathcal{M}$ *is the projection of a quadric Veronesean* $\mathcal{V}_2^4$ *from a point* p *in a conic plane of* $\mathcal{V}_2^4$ *onto a hyperplane* $\mathrm{PG}(4,q^n)$; *the point* p *is an exterior point of the conic of* $\mathcal{V}_2^4$ *in the conic plane. Also, the point-line dual of* $T(\mathcal{O})$ *is isomorphic to the flock GQ* $\mathcal{S}(\mathcal{F})$ *with* $\mathcal{F}$ *a Kantor flock.*

(c) *We are not in Cases (a) and (b) and there exists a* $\mathrm{PG}(5,q^n)$ *in* $\mathrm{PG}(4n-1,q^n)$ *which intersects* $\mathrm{PG}(n-1,q^n)$ *in a plane* μ *and which has exactly one point* r_i *in common with any space* $\mathrm{PG}^{(i)}(n-1,q^n)$, $i=1,2,\ldots,q^{2n}$. *Let* $\mathcal{W}=\{r_i \parallel i=1,2,\ldots,q^{2n}\}$ *and let* $\mathcal{P}$ *be the set of all common points of* μ *and the conics which contain exactly* q^n *points of* $\mathcal{W}$. *Then the set* $\mathcal{W}\cup\mathcal{P}$ *is a quadric Veronesean in* $\mathrm{PG}(5,q^n)$.

Theorem 75 (J. A. Thas [70]) *If the GQ* $\mathcal{S}$ *of order* (q,q^2), $q\neq 1$, *satisfies Property (G) at the line* L , *then* L *is regular.*

Theorem 76 (J. A. Thas and H. Van Maldeghem [77, 70]) *Assume that the GQ* $\mathcal{S}$ *of order* (q,q^2), $q\neq 1$, *satisfies Property (G) at the line* L. *If* q *is even, then the dual net* $\mathcal{N}_L^*$ *always satisfies the Axiom of Veblen and so* $\mathcal{N}_L^*\cong H_q^3$. *If* q *is odd, then the dual net* $\mathcal{N}_L^*$ *satisfies the Axiom of Veblen if and only if* $\mathcal{S}$ *is the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *with* $\mathcal{F}$ *a Kantor flock.*

7.3 A Lenz-Barlotti classification of GQ's by K. Thas (and an application)

The main examples of finite generalized quadrangles are essentially of five types: (1) they are inside a projective space $\mathrm{PG}(n,q)$ over the Galois field $\mathrm{GF}(q)$, and these are the so-called 'classical examples', (2) they are the point-line duals of the classical examples (the dual $H(4,q^2)^D$ of the classical GQ $H(4,q^2)$ is never embedded in a projective space), (3) they are of order $(s-1,s+1)$ or, dually, of order $(s+1,s-1)$, and the examples of this type all are in some way connected to ovals or hyperovals of $\mathrm{PG}(2,q)$ (and all are related to GQ's of order s with a regular point, see Chapter 3 of FGQ), (4) they or their duals arise as translation generalized quadrangles (from either generalized ovals or generalized ovoids), and (5) they or their duals arise from

flocks of the quadratic cone in $PG(3,q)$.
The main examples of GQ's which were discovered in the past fifteen years all are of type (3), (4) and (5), and the GQ's of type (4) and (5) all have a common property: the duals of the flock GQ's all have at least one axis of symmetry, and the TGQ's of type (4) even have a point through which every line is an axis of symmetry, and in this last case, this property characterizes the examples. This motivated us to start a classification of generalized quadrangles based on the possible subconfigurations of axes of symmetry [81, 89].
It is beyond the reach of this paper to mention the precise classification result of [89]. Let us just mention the following result; it is a new characterization of $\mathcal{Q}(5,s)$ which was deduced from the classification in [89].

Theorem 77 (K. Thas [89]) *A generalized quadrangle of order (s,t), $s \neq 1 \neq t$, is isomorphic to $\mathcal{Q}(5,s)$ if and only if there are three distinct axes of symmetry U, V and W for which $U \cap \{V,W\}^{\perp\perp} = \emptyset$.*

CONJECTURE. *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, containing a subGQ $\mathcal{S}'$ of order s which is isomorphic to $\mathcal{Q}(4,s)$, and such that every axis of symmetry of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$. Then $\mathcal{S}$ is classical.*

The GQ's of order (s,t), $s \neq 1 \neq t$, containing a subGQ $\mathcal{S}'$ of order s which is isomorphic to $\mathcal{Q}(4,s)$, and such that every axis of symmetry of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$, are precisely the elements of the symmetry-class V.5, see K. Thas [89].
Note added in proof. The last conjecture was recently proved in a more general context, see K. Thas [89].
Note. In the second addendum, we will briefly give an overview of some new results on *complete arcs* in generalized quadrangles. The main results of that section have appeared to be very useful in the classification mentioned in this paragraph, especially in the determination of the symmetry-class I, see [89].

7.4 Characterizations of GQ's with one classical subGQ

In the final paragraph of this section, we state two recent theorems of L. Brouns, J. A. Thas and H. Van Maldeghem.

Theorem 78 (L. Brouns, J. A. Thas and H. Van Maldeghem [10]) *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, containing a subGQ $\mathcal{S}'$ of order s which is isomorphic to $\mathcal{Q}(4,s)$, and such that every automorphism of $\mathcal{S}'$ extends to an automorphism of $\mathcal{S}$. Then $\mathcal{S}$ is classical.*

Finally, the following theorem was already proved in J. A. Thas and S. E. Payne [75] in the even case, and by M. R. Brown [12] in the odd case. An alternative proof which included both the even and the odd case was given in [10]. Let us first recall a basic definition. Let $\mathcal{S}'$ be a subGQ of order (s, t'), $s, t' \neq 1$, of a GQ $\mathcal{S}$ of order (s, t). Then each point p of $\mathcal{S} \setminus \mathcal{S}'$ is collinear with the $st' + 1$ points of an ovoid $\mathcal{O}'_p$ of $\mathcal{S}'$, see [47, 2.2.1]. Such an ovoid is called a *subtended ovoid* of $\mathcal{S}'$, and $\mathcal{O}'_p$ is said to be *subtended by the point* p.

Theorem 79 ([75],[12],[10]) *Let $\mathcal{S}$ be a GQ of order (s, t), $s \neq 1 \neq t$, containing a classical subGQ $\mathcal{S}'$ of order s such that every subtended ovoid is an elliptic quadric. Then $\mathcal{S}$ is classical.*

8 Half pseudo Moufang generalized quadrangles

8.1 Moufang generalized quadrangles

J. Tits [95] defines a generalized quadrangle of *Moufang type* as a generalized quadrangle $\mathcal{S} = (P, B, I)$ in which the following conditions hold:

- **(M)** for any dual panel (L, p, M) of $\mathcal{S}$, the group of all automorphisms of $\mathcal{S}$ fixing L and M pointwise and p linewise is transitive on the lines which are incident with a given point xIL, $x \neq p$, and different from L;
- **(M')** for any panel (p, L, q) of $\mathcal{S}$, the group of all automorphisms of $\mathcal{S}$ fixing p and q linewise and L pointwise is transitive on the points which are incident with a given line MIp, $M \neq L$, and different from p.

Tits proved, using results of Fong and Seitz [20, 21], that a finite generalized quadrangle is of Moufang type if and only if it is a classical or a dual classical GQ. A panel for which (M') is satisfied is a *Moufang panel*, and dually one defines *dual Moufang panels*. A GQ is *half Moufang* if either every panel or every dual panel is Moufang.
To our knowledge, all the results on finite Moufang GQ's are contained in H. Van Maldeghem [99]. Let us only recall the following beautiful result.

Theorem 80 (J. A. Thas, S. E. Payne and H. Van Maldeghem [80]) *Any finite half Moufang GQ is Moufang, and hence classical or dual classical.*

PROBLEM. *Classify GQ's based on possible subconfigurations of Moufang panels (and/or dual Moufang panels).*

Let (p, L) be a flag of the thick GQ $\mathcal{S}$ of order (s, t). Then (p, L) is a *Moufang flag* if there is a group of size st of collineations of $\mathcal{S}$ which are whorls about p and L.

PROBLEM. *Classify GQ's based on possible subconfigurations of* Moufang flags *(see e.g. [101]).*

Combinations of both problems is also an option. For a related classification, see K. Thas [81] and especially [89].

For more on *Moufang generalized polygons*, see [102, 97].

8.2 Introducing (half) pseudo Moufang GQ's

Let (p, L, q) be a panel of a thick (finite) generalized quadrangle $\mathcal{S} = (P, B, I)$ of order (s, t). Then this panel is called *pseudo Moufang* if there is a group $H(p, L, q)$ of elations about both p and q of order s. A GQ $\mathcal{S}$ is called *half pseudo Moufang* if either every panel or every dual panel is pseudo Moufang, and we say that $\mathcal{S}$ is an *HPMGQ*. A GQ is called *pseudo Moufang* if every panel and every dual panel is pseudo Moufang.

Note. Using the classification of finite simple groups, Buekenhout and Van Maldeghem [15] have showed that any finite *distance-transitive generalized polygon* is Moufang, and hence classical by Fong and Seitz [20, 21]. Part of the motivation of [91] is to aim for weaker hypotheses than for instance (M) and (M') that still give a common characterization of all the finite classical and dual classical generalized quadrangles without relying on the classification of the finite simple groups.

Using Van Maldeghem [98], see also [11], it is straightforward to show the following (without the aforementioned classification).

Theorem 81 (K. Thas [91]) *Any thick pseudo Moufang generalized quadrangle is classical or dual classical.*

8.3 Groups of whorls and Property (H)

Suppose p and q are two non-collinear points of the GQ $\mathcal{S} = (P, B, I)$. Then we put *cl(p,q)* $= \{z \in \mathcal{S} \parallel z^\perp \cap \{p, q\}^{\perp\perp} \neq \emptyset\}$. A point x has *Property (H)* provided that $r \in cl(p, q)$ if and only if $p \in cl(q, r)$ whenever (p, q, r) is a triad of points in $x^\perp$ (the dual notion is usually also denoted by *Property (H)*).

Theorem 82 (FGQ, 5.6.2) *Suppose every point of a GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, has Property (H). Then we have one of the following possibilities:*

1. $\mathcal{S} \cong H(4,s)$;

2. *every span of non-collinear points has size* 2;

3. *every point is regular.*

The following observation is easy.

Observation 83 (K. Thas [91]) *Suppose* $\mathcal{S} = (P,B,I)$ *is a GQ of order* (s,t), $s,t \neq 1$, *having for each point* x *a group* $G^{(x)}$ *of whorls about* x *which acts transitively on the points of* $P \setminus x^{\perp}$ *(i.e. every point is a center of transitivity). Then every point of* $\mathcal{S}$ *has Property (H).*

There is an easy but important corollary.

Theorem 84 *Suppose every point of the GQ* $\mathcal{S}$ *of order* (s,t), $s \neq 1 \neq t$, *is a center of transitivity. Then we have one of the following possibilities:*

1. *every span of non-collinear points has size* 2;

2. *every point is regular;*

3. $\mathcal{S} \cong H(4,s)$.

In particular, the theorem applies to half pseudo Moufang generalized quadrangles.

Theorem 85 (K. Thas [91]) *Suppose* $\mathcal{S} = (P,B,I)$ *is an HPMGQ of order* (s,t), $s \neq 1 \neq t$, *and suppose* $\mathcal{S}'$ *is a thick classical subquadrangle of order* (s,t'), $t' < t$. *Then* $\mathcal{S}$ *is also classical.*

PROBLEM. *Give a proof of the following: 'Suppose* $\mathcal{S}^{(x)}$ *is a TGQ of order* (q,q^2), $q > 1$, *with translation point* x *and all lines regular. Then* $\mathcal{S}^{(x)}$ *is isomorphic to* $\mathcal{Q}(5,q)$*'.*
J. A. Thas has some strong unpublished results on this subject. Note that the solution of this problem would almost certainly lead to the solution of the following problem.
PROBLEM. *Give an 'elementary' and geometrical proof of the following theorem: 'Suppose* $\mathcal{S}^{(x)}$ *is a GQ of order* (q,q^2), $q > 1$, *each line of which is an axis of symmetry. Then* $\mathcal{S}^{(x)}$ *is isomorphic to* $\mathcal{Q}(5,q)$*'.*

CONJECTURE (J. A. THAS AND OTHERS). *Suppose* $\mathcal{S}$ *is a thick GQ of order* (s,t) *with all lines regular. Then* $\mathcal{S}$ *is of classical type.*

For $s = t$, the conjecture was answered affirmatively by C. T. Benson [6], and this is probably the oldest combinatorial characterization of a class of GQ's. For $s \neq t$, to my knowledge the only known result without additional hypotheses is a recent result of J. A. Thas and H. Van Maldeghem, saying that $t \neq s + 2$ if $s > 2$ [79]. The situation in the general case seems hopeless at present.

8.4 A new proof of the Moufang theorem

In 1978, S. E. Payne and J. A. Thas started the program to prove the Moufang theorem of Fong and Seitz [20, 21] for finite generalized quadrangles without the use of deep group theory (and hence geometrically more satisfactory), see Chapter 9 of FGQ. They came very close to obtaining a proof; their only obstacle was the one mentioned in the last problem of the preceding paragraph:

Theorem 86 (P. Fong and G. M. Seitz [20, 21], W. M. Kantor [34]) *If $\mathcal{S}^{(x)}$ is a GQ of order (q, q^2), $q > 1$, each line of which is an axis of symmetry, then $\mathcal{S}^{(x)}$ is isomorphic to $\mathcal{Q}(5, q)$.*

W. M. Kantor gave a proof of this theorem in [34], where he used the classification of the split BN-pairs of rank 1 [54, 25], and 4B,C of Fong and Seitz [20], but the proof is still not elementary in the sense of S. E. Payne and J. A. Thas. We do not finish their program here, but we give a proof of Theorem 86 without the use of any result of Fong and Seitz [20, 21]; we still need the classification of the split BN-pairs of rank 1 [54, 25]. However, as will be pointed out, we do not need this result 'intrinsically'; we will at some stage only need the order of some group (it is there that our geometrical proof fails).

Sketch of the proof of the theorem. In K. Thas [86], we showed purely geometrical that given an SPGQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$ and $s \neq t$ (and hence $t = s^2$) with base-group G and base-grid Γ, we have $|G| \geq s^3 - s$. Moreover, if $|G| = s^3 - s$, then there are $s + 1$ subGQ's of order s, all classical and mutually intersecting in Γ. Although we used K. Thas [83] to show that these subGQ's are classical, this result is not needed here since all lines of each such subGQ are regular, see [47, 5.2.1]. It is at this point that we need to prove that $|G|$ always equals $s^3 - s$. Relying on [54, 25] it is possible to show this, see [86, 89], and since $\mathcal{S}$ is an SPGQ for any two non-concurrent lines, there easily follows by e.g. [47, 5.3.5] that $\mathcal{S} \cong \mathcal{Q}(5, s)$. $\square$

A Addendum 1: Semi quadrangles

A.1 Introduction

In [81] we met a class of incidence structures $\mathcal{S}'$ which arise naturally from a set of concurrent axes of symmetry in a GQ $\mathcal{S}$, and which were proved to be a generalized quadrangle if certain additional properties are satisfied. In [81] we called these incidence structures (l,k)-*partial quadrangles*, and in some cases they were shown to be generalized quadrangles. It is easy to see that these geometries have ordinary subquadrangles and subpentagons, and that they are connected, see [81, 85]. Also, they satisfy (SQ1) with $n \leq 2$ (see the next section for the definition of (SQ1)), see [81, 85], and there are no triangles. If $\mathcal{S}'$ also satisfies (SQ2), that is, if we assume that any two non-collinear points of $\mathcal{S}'$ have a common neighbour, then it is a semi quadrangle. Also, by Theorem 90 in such a case it is a GQ.

A.2 Semi quadrangles

A *semi quadrangle (SQ)* is a point-line incidence structure of which any line is incident with a constant number of points, of which no two distinct points are incident with more than one line, which contains no ordinary triangles as minimal circuits, but contains an ordinary subquadrangle and subpentagon, and with the further condition that every two non-collinear points are always both collinear with at least one point; also, any line is incident with at least two points and any point is incident with at least two lines. It is clear that from this definition it does not necessarily follow that every point is incident with the same number of lines (such as in the case of thick generalized polygons [99]). In order to have some more information about these structures, we introduce the μ-*parameters* and the *order* of a semi quadrangle.
Suppose that s, t_i, μ_j, where $1 \leq i \leq n$ and $1 \leq j \leq m$ for nonzero natural numbers n and m, are natural numbers satisfying $s \geq 1$ and $t_i \geq 1$. Then a semi quadrangle of *order* $(s; t_1, \ldots, t_n)$ and with μ-*parameters* $(\mu_1, \ldots, \mu_m)$ is an incidence structure with the following properties.

(SQ1) The geometry is a *partial linear space*, i.e. any two distinct points are incident with at most one line and any two distinct lines are incident with at most one point. Any point is incident with $t_1 + 1$, $t_2 + 1, \ldots$, or $t_n + 1$ lines, and every line is incident with $s + 1$ points. Also, for any $i \in \{1, 2, \ldots, n\}$ there is a point incident with $t_i + 1$ lines.

(SQ2) If two points are not collinear, then there are exactly $\mu_1, \mu_2, \ldots,$ or μ_m points collinear with both, and any of the cases occurs.

(SQ3) For any two non-collinear points there is at least one point which is collinear with both (i.e. for any $i = 1, 2, \ldots, m$ there holds that $\mu_i \geq 1$).

(SQ4) The geometry contains an ordinary pentagon and an ordinary quadrangle but no ordinary triangle as subgeometry, hence there is a j for which $\mu_j \geq 2$.

We emphasize that (SQ2) and (SQ3) should be regarded as different axioms (instead of integrating (SQ3) in (SQ2) by demanding that for every $i = 1, 2, \cdots, m$ there holds that $\mu_i \geq 1$). For instance, suppose that $\mathcal{S}$ is a GQ of order (s, t) with $s, t > 2$, and suppose $\mathcal{L}$ is an arbitrary set of k lines with $0 < k < t$. Define a geometry by taking away the lines of $\mathcal{L}$ in the GQ, with the same points as $\mathcal{S}$ and with the natural incidence. Then this geometry satisfies (SQ1), (SQ2) and (SQ4), but not (SQ3).
Other motivations for this distinction will be clear from the sequel.
In the following we agree that $t_1 \leq t_2 \leq \ldots \leq t_n$ and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_m$. If there are only two possible numbers of lines through a point, then the SQ is called *near minimal*. Since the parameters t_1, t_n, μ_1, μ_m will play an important role in the following, we call $(s; t_1, t_n)$ the *extremal order* and (μ_1, μ_m) the *extremal μ-parameters*. For a near minimal semi quadrangle, the order and the extremal order coincide. A semi quadrangle is called *thick* if every point is incident with at least three lines and if every line is incident with at least three points. A thick semi quadrangle with $t_1 = \ldots = t_n = t$ and $\mu_1 = \ldots = \mu_m = \mu$ is a thick *partial quadrangle (PQ)* — as defined by Cameron in [16] — with parameters (s, t, μ) with $\mu \neq 1$ (this notation differs somewhat from that of Cameron, but in this context it is more convenient) and a thick partial quadrangle with $\mu = t + 1$ is precisely a thick generalized quadrangle with parameters (s, t). Thick generalized quadrangles always contain quadrangles and pentagons. In the case of generalized quadrangles, the condition that the GQ must contain a pentagon is equivalent with the thickness of the GQ, see [99]. This is not the case for semi quadrangles; there are geometries with only two points per line which satisfy all the SQ-conditions. For example, define the geometry $\mathcal{S} = (P, B, I)$ as follows. The point set P consists of six distinct 'letters' a_i, $i \in \{1, \ldots, 6\}$, lines are the sets $\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_5, a_6\}, \{a_6, a_2\}, \{a_1, a_5\}, \{a_5, a_4\}$, and incidence is the natural one. Then $\mathcal{S}$ is a semi quadrangle. Also, every *strongly regular graph* with parameters (v, k, λ, μ) (see e.g. Chapter 22 of [14]) and with $\mu \geq 2$ and $\lambda = 0$ is a semi quadrangle of order $(1; k - 1)$ and with μ-parameters (μ) (and hence a partial quadrangle). An example is

the unique strongly regular graph with parameters $(16, 5, 0, 2)$, namely the *Clebsch graph*, see Chapter 10 (p. 440) of [14].

Remark 87 A thick semi quadrangle $\mathcal{S}$ is a thick generalized quadrangle if and only if (GQ3) is satisfied.

Theorem 88 (K. Thas [85]) *Any anti-flag (p, L) of a semi quadrangle $\mathcal{S}$ which does not satisfy Property (GQ3) is always contained in a pentagon.*

Theorem 89 (K. Thas [85]) *A geometry $\mathcal{G}$ which satisfies all the SQ-conditions except that there must be a pentagon, automatically contains pentagons if and only if it is not a grid or a dual grid.*

Theorem 90 (K. Thas [85]) *Suppose that $\mathcal{S}' = (P', B', I')$ is a subgeometry[5] of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), with the properties that there are $s'+1$ points on a line for some s', that there is a subpentagon and that (SQ3) is satisfied. Then $s \neq 1 \neq t$, and two points of $\mathcal{S}'$ are collinear if and only if they are collinear in $\mathcal{S}$. If $s' = s$, then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s, t') with $t' \neq 1$.*

A.3 Examples of semi quadrangles

All the examples presented here are in some way related to generalized quadrangles or partial quadrangles.
We first of all emphasize again that it should be noted that (SQ3) is a very important condition. This will be clearly reflected in the following examples.
(1) Suppose that $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t) with $s, t \geq 3$, and suppose p is a point of $\mathcal{S}$ with the property that for every two non-collinear points q, q' of $P \setminus p^{\perp}$ there holds that

$$(M) \quad |\{p, q, q'\}^{\perp}| < t + 1.$$

By Theorem 1.7.1 of FGQ there follows that for every pair of non-collinear points (x, y) in $P \setminus p^{\perp}$ we have

$$(Q) \quad |\{p, x, y\}^{\perp}| < t.$$

Now define the following incidence structure $\mathcal{S}_p = (P_p, B_p, I_p)$: (a) P_p is the set of points of $P \setminus p^{\perp}$, (b) B_p is the set of all lines of $\mathcal{S}$ not incident with p, and (c) I_p is the restriction of I to $(P_p \times B_p) \cup (B_p \times P_p)$. Then $\mathcal{S}$ is a thick semi quadrangle with s points on every line and $t + 1$ lines through every

[5]This means that $P' \subseteq P$, $B' \subseteq B$ and that I' is the induced incidence.

point.
Suppose that $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t) with $s, t > 2$. Then the pair (x, y), $x \not\sim y$, is called *antiregular* provided $|z^\perp \cap \{x, y\}^\perp| \leq 2$ for all $z \in P \setminus \{x, y\}$. A point x is *antiregular* provided (x, y) is antiregular for all $y \in P \setminus x^\perp$, see [47]. Hence, if $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t) with $s, t > 2$ and p an antiregular point, then the geometry $\mathcal{S}_p$ always satisfies condition (M) and condition (Q), thus $\mathcal{S}_p$ is a semi quadrangle, of which the μ-parameters are contained in $\{t - 1, t, t + 1\}$.
Now specialize, and suppose that $\mathcal{S}^{(x)}$ is a TGQ of order (s, t) with $s, t > 2$ and with translation point x. If $s = t$, we also assume that s is odd. Then by Chapter 8 of FGQ, the conditions above are satisfied, and hence any TGQ of order (s, t) with $s, t > 2$ yields a thick semi quadrangle with a constant number of lines through a point.
The semi quadrangles which arise from translation generalized quadrangles in the way described above all have the property that there is an elementary abelian group which acts regularly on the points of the semi quadrangle. Also, s and t are powers of the same prime p, and there is an odd natural number a and an integer n for which $t = p^{n(a+1)}$ and $s = p^{na}$ if $s \neq t$. If $s = t$ with s odd, then by Chapter 1 of [47] the μ-parameters are given by $(s - 1, s + 1)$; if $s \neq t$ and if p and a are as above, then the (possible) μ-parameters are $(p^{n(a+1)} - p^n, p^{n(a+1)})$ by Chapter 8 of [47], and $\mathcal{S}_x$ is a partial quadrangle if and only if $a = 1$, and then $\mu = p^{2n} - p^n$.
Let $\mathcal{S}$ be a GQ of order (s, s^2) with $s > 2$. Then by Bose and Shrikhande [9] there follows that every triad of points has $s + 1$ centers. Now take an arbitrary point p of $\mathcal{S}$, and consider the geometry $\mathcal{S}_p$. Then $\mathcal{S}_p$ is a partial quadrangle with parameters $(s - 1, s^2, s^2 - s)$.
(2) Let $\mathcal{S}$ be a GQ of order (s, t) with $s, t > 2$, and suppose that $\mathcal{S}'$ is a subGQ of order $(s, t/s)$, with the property that for every two non-collinear points x and y of $\mathcal{S} \setminus \mathcal{S}'$ there holds that $|\{x, y\}^\perp \cap \mathcal{S}'| < t + 1$. By Chapter 2 of [47], there holds that every line of $\mathcal{S}$ intersects $\mathcal{S}'$ in 1 or $s + 1$ points. Next, define a geometry $\mathcal{S}_{\mathcal{S}'} = (P_{\mathcal{S}'}, B_{\mathcal{S}'}, I_{\mathcal{S}'})$ where $B_{\mathcal{S}'}$ is the set of lines of $\mathcal{S}$ which are not contained in $\mathcal{S}'$, $P_{\mathcal{S}'}$ is the set of points of $\mathcal{S} \setminus \mathcal{S}'$, and where $I_{\mathcal{S}'}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{S}'}$ is a thick semi quadrangle of order $(s - 1; t)$.
(3) Suppose $\mathcal{S}$ is a GQ of order (s, t) with $s, t > 2$, and suppose that $\mathcal{O}$ is an *ovoid* (see further) with the property that for every two non-collinear points x and y of $\mathcal{S} \setminus \mathcal{O}$ there holds that $|\{x, y\}^\perp \cap \mathcal{O}| < t + 1$. Define a geometry $\mathcal{S}_{\mathcal{O}} = (P_{\mathcal{O}}, B_{\mathcal{O}}, I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the line set of $\mathcal{S}$, $P_{\mathcal{O}} = \mathcal{S} \setminus \mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{O}}$ is a thick semi quadrangle of order $(s - 1; t)$.
Suppose $\mathcal{O}$ is an ovoid of the classical GQ $W(q)$ of order q, $q > 2$. Then every

point of $\mathcal{S}$ is regular, see [47]. By Theorem 1.8.4 of [47], there follows that $\mathcal{S}_{\mathcal{O}}$ is a semi quadrangle of order $(q-1;q)$ and with μ-parameters $(q-1,q+1)$.
(4) Suppose $\Gamma=(P,B,I)$ is a partial quadrangle with parameters (s,t,μ) where $s,t\geq 3$, and let $\Gamma'=(P',B',I')$ be a partial subquadrangle of Γ with parameters (s,t',μ'). Then a simple counting argument shows that every line of Γ intersects Γ' if and only if $|P'|\times(t-t')=|B|-|B'|$, that is, if and only if

$$(t-t')(s+1)(1+(t'+1)s(1+\frac{st'}{\mu'}))=(1+(t+1)s(1+\frac{st}{\mu}))(t+1)-$$
$$(1+(t'+1)s(1+\frac{st'}{\mu'}))(t'+1). \tag{3}$$

Note that if we interchange the words 'PQ' and 'GQ', that this condition can be simplified to $t'=t/s$, see Example (2). Assume condition (3) is satisfied. Furthermore, we suppose that $\mathcal{S}$ has the property that (1) for every two non-collinear points q,q' of $P\setminus p^{\perp}$ there holds that $|\{p,q,q'\}^{\perp}|<\mu$, and that (2) there is a pair of non-collinear points (x,y) in $P\setminus p^{\perp}$ for which $|\{p,x,y\}^{\perp}|<\mu-1$. Define a geometry $\Gamma_{\Gamma'}=(P_{\Gamma'},B_{\Gamma'},I_{\Gamma'})$ where $B_{\Gamma'}$ is the line set of $\Gamma\setminus\Gamma'$, $P_{\Gamma'}$ is the set of points of $\Gamma\setminus\Gamma'$, and where $I_{\Gamma'}$ is the natural incidence. Then there follows that $\Gamma_{\Gamma'}$ is a semi quadrangle with parameters $(s-1;t)$.
(5) A *partial ovoid* of a partial quadrangle is a set of mutually non-collinear points. An *ovoid* $\mathcal{O}$ of a partial quadrangle Γ with parameters (s,t,μ) is a set of non-collinear points such that every line is incident with exactly one point of the set[6]. By counting the point-line pairs (p,L) of Γ for which $p\in\mathcal{O}$, pIL with L a line of Γ, in two ways, there follows that $|\mathcal{O}|=\frac{s^2t(t+1)/\mu+(t+1)s+1}{s+1}$. Suppose Γ is a PQ with parameters (s,t,μ) with $s,t>2$, and suppose that $\mathcal{O}$ is an ovoid with the property that for every two non-collinear points x and y of $\Gamma\setminus\mathcal{O}$ there holds that $|\{x,y\}^{\perp}\cap\mathcal{O}|<\mu$. Also, we demand that there is a pair of non-collinear points (x,y) in $P\setminus p^{\perp}$ for which $|\{p,x,y\}^{\perp}|<\mu-1$. Define a geometry $\Gamma_{\mathcal{O}}=(P_{\mathcal{O}},B_{\mathcal{O}},I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the line set of Γ, $P_{\mathcal{O}}=\Gamma\setminus\mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\Gamma_{\mathcal{O}}$ is a thick semi quadrangle with parameters $(s-1;t)$.

Remark 91 Let $\mathcal{S}$ be a GQ of order (s,s^2) with $s>2$. Consider an arbitrary point p of $\mathcal{S}$, and consider the partial quadrangle $\mathcal{S}_p$ with parameters $(s-1,s^2,s^2-s)$ as described in Example (1). Then $\mathcal{S}_p$ cannot have ovoids.

[6] In the same way, one could define *(partial) ovoids* for semi quadrangles, and, dually, *(partial) spreads*.

Note. By 2.7.1 and 1.8.3 of FGQ there easily follows that if $\mathcal{O}$ is a partial ovoid of the PQ $\mathcal{S}_p$, there holds that $|\mathcal{O}| \leq s^3 - s - 1$.
(6) Suppose $\mathcal{K}$ is a *complete* $t+1$-*cap* of PG$(n-1,q)$ (see Paragraph A.6), and embed PG$(n-1,q)$ in PG(n,q). Suppose P is the set of points of PG(n,q) which are not contained in PG$(n-1,q)$, that B is the set of lines L of PG(n,q) which are not contained in PG$(n-1,q)$ and for which $|\mathcal{K} \cap L| = 1$. Then the geometry $\mathcal{S} = (P,B,I)$, with I the natural incidence, is a semi quadrangle with parameters $(q-1;t)$. If $n=4$ and $\mathcal{K}$ is an *ovoid* of PG$(3,q)$, then $\mathcal{S}$ is a partial quadrangle, see [16]. For more details, see further.

Remark 92 The first three constructions given above all arise by taking away a so-called '*geometric hyperplane*' of a GQ. In the terminology of Pralle [50], this means that these geometries are all *affine generalized quadrangles*, see also. Also, any of the examples (1),(2),(3),(4),(5) can clearly be generalized in a natural way by considering geometric hyperplanes of semi quadrangles (instead of geometric hyperplanes of partial quadrangles).

A.4 Divisibility conditions, constants and inequalities

If a point p of a semi quadrangle is incident with t_i+1 lines, then we denote this by $p \in P_i$, and p is said to have *degree* t_i.
If we write $\lfloor x \rfloor$, with $x \in \mathbb{R}$, then we mean the greatest natural number which is at most x, and with $\lceil x \rceil$ we mean the smallest natural number which is at least x.
Suppose $\mathcal{S}$ is a thick semi quadrangle of order $(s;t_1,\ldots,t_n)$ and with μ-parameters $(\mu_1,\ldots,\mu_m)$ which is not a generalized quadrangle. Then there is a point-line pair (p,L) such that $p\not\mathrm{I}L$ and for which there is no line M for which $pIM \sim L$. Suppose $p \in P_j$. By counting the points which are collinear with p and with a point of L in two ways, we obtain $(s+1)\mu_1 \leq (t_j+1)s$, from which follows that $\mu_1 + \frac{\mu_1}{s} \leq t_j + 1$.
Now suppose that $t_i = t$ for all i, and fix a point q. By N_k, we denote the number of points x of $P \setminus q^\perp$ for which there are μ_k points collinear with both x and q. We have the following theorem.

Theorem 93 (K. Thas [85]) *Suppose that $\mathcal{S}$ is a semi quadrangle with μ-parameters $(\mu_1,\ldots,\mu_m)$ and with a constant number, $t+1$, of lines through a point. Suppose the N_i are as above. Then $(t+1)ts^2 = \sum_i N_i\mu_i$.*

Definition. Suppose $\mathcal{S}$ is a semi quadrangle. If A is a set of two by two non-collinear points, respectively non-concurrent lines, then by $A^\perp$ we denote the set of points, respectively lines, which are all collinear, respectively concurrent, with each element of A. As in GQ's, a *triad* is a set of three points, respectively lines, two by two non-collinear, respectively non-concurrent. A *center* of a triad $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$, where $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ are all points or all lines, is an element of $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}^{\perp}$.

Theorem 94 (K. Thas [85]) *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequality.*

$$[(t_1 - 1)s\mu_1]^2 \leq \mu_m[(\mu_m - 1)(\mu_m - 2) + (t_n - 1)s](\frac{\lfloor (t_n + 1)t_n s^2 \rfloor}{\mu_1}$$
$$-s(t_1 + 1) + \mu_m - 1). \tag{4}$$

If equality holds, then there is a constant $x_0 = \frac{(t_1-1)s\mu_1}{\lfloor \frac{(t_n+1)t_n s^2}{\mu_1} \rfloor - s(t_1+1) + \mu_m - 1}$ such that each triad of points has exactly x_0 centers.
Conversely, if each triad of points has a constant number of centers, then

$$[(t_n - 1)s\mu_m]^2 \geq \mu_1[(t_1 - 1)s + (\mu_1 - 1)(\mu_1 - 2)](\frac{\lceil (t_1 + 1)t_1 s^2 \rceil}{\mu_m}$$
$$-s(t_n + 1) + \mu_1 - 1). \tag{5}$$

Corollary 95 (P. J. Cameron [16]) *Suppose $\mathcal{S}$ is a partial quadrangle with parameters (s, t, μ). Then*

$$\mu(t - 1)^2 s^2 \leq [s(t - 1) + (\mu - 1)(\mu - 2)][\frac{(t + 1)ts^2}{\mu} - (t + 1)s + \mu - 1]. \tag{6}$$

Equality holds if and only if the number of points collinear with each of any three pairwise non-collinear points is a constant; if this occurs, the constant is $1 + \frac{(\mu-1)(\mu-2)}{s(t-1)}$.

Corollary 96 (D. C. Higman [26, 27]) *Suppose $\mathcal{S}$ is a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$. Then*

$$t \leq s^2. \tag{7}$$

Equality holds if and only if the number of points collinear with every three pairwise non-collinear points is a constant, and if this occurs, the constant is $s + 1$.

Now suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m), and assume that $s \leq t_1$. Suppose b is the number of lines and v is the number of points. Counting the number θ of flags of $\mathcal{S}$, we get that

$$v(t_n + 1) \geq \theta = b(s + 1) \geq v(t_1 + 1) \tag{8}$$

Note. If $t_1 = t_n = t$ in (8), then $v(t + 1) = b(s + 1)$. If $v = b$, then $t_n \geq s \geq t_1$.
In the case $v = b$, some refinement is possible.

Theorem 97 (K. Thas [85]) *Suppose $\mathcal{S} = (P, B, I)$ is a semi quadrangle of order $(s; t_1, \ldots, t_n)$ and with μ-parameters $(\mu_1, \ldots, \mu_m)$. If $\mathcal{S}$ has the property that $v := |P| = |B| =: b$, then we have that either $t_1 = t_n = s$ or $t_1 < s < t_n$.*

Theorem 98 (K. Thas [85]) *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequalities*

$$s^2 t_1(t_1 + 1) \leq (v - (t_1 + 1)s - 1)\mu_m, \tag{9}$$

and

$$s^2 t_n(t_n + 1) \geq (v - (t_n + 1)s - 1)\mu_1, \tag{10}$$

and $\mathcal{S}$ is a PQ if and only if equality holds in both (9) and (10).

A.5 Semi quadrangles and their point graphs

A *graph* is an incidence structure in which lines are called *edges* and points are called *vertices*, and in which any edge is incident with two points and any two distinct points are incident with at most one edge. Two distinct points incident with an edge are called *adjacent*, and a graph is *complete* if any two vertices are adjacent. If a vertex v is incident with t edges, then t is called the *valency* of v. The *μ-values* of a graph $\mathcal{G}$ are numbers $\mu_1, \ldots, \mu_m$ such that any two non-adjacent vertices are both adjacent with μ_i vertices for some $1 \leq i \leq m$. The *λ-values* of a graph are numbers $\lambda_1, \ldots, \lambda_{m'}$ such that any two adjacent points are both adjacent with λ_j points for some $1 \leq j \leq m'$. An *induced subgraph* consists of a subset of points of the point set, together with all the edges joining two points in the subset, and a (maximal) *clique* is a (maximal) complete induced subgraph of a graph. The *point graph* of an incidence geometry is the graph in which two distinct points are adjacent if and only if they are collinear.

Theorem 99 (P. J. Cameron [16]) *The point graph of a partial quadrangle is strongly regular and has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Conversely, a strongly regular graph with this property is the point graph of a partial quadrangle.*

There is a similar theorem for semi quadrangles.

Theorem 100 (K. Thas [85]) *A graph is the point graph of a semi quadrangle if and only if (a) every μ-value is strictly positive (i.e. the diameter of the graph is at most* 2*), (b) there is only one λ-value, and (c) the graph contains an induced quadrangle, respectively pentagon, and it has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Moreover, if the SQ has order $(s; t_1, \ldots, t_n)$ and μ-parameters $(\mu_1, \ldots, \mu_m)$, then the λ-value of the graph is $s-1$, the possible μ-values are $\mu_1, \ldots, \mu_m$, and $\{(t_1+1)s, \ldots, (t_n+1)s\}$ is the set of valencies, and, conversely, a graph which satisfies properties (a), (b) and (c), and which has these parameters is the point graph of a semi quadrangle of order $(s; t_1, \ldots, t_n)$ and with μ-parameters $(\mu_1, \ldots, \mu_m)$.*

A.6 Linear representations

A *linear representation* of a semi quadrangle $\mathcal{S} = (P, B, I)$ is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of the affine space $\mathrm{AG}(n, q)$, in such a way that P^θ is the set of all points of $\mathrm{AG}(n, q)$, that B^θ is a union of parallel classes of lines of $\mathrm{AG}(n, q)$, and that each point of L^θ is the image of some point of L for any line L in B. Usually we identify $\mathcal{S}$ with its image $\mathcal{S}^\theta$. Note that any parallel class of lines partitions the point set of $\mathrm{AG}(n, q)$. Since parallel classes of lines in an $\mathrm{AG}(n, q)$ correspond to points of $\mathrm{PG}(n-1, q)$ in a natural way, such a representation $\mathcal{S}^\theta$ defines a set of points $\mathcal{K}$ in $\mathrm{PG}(n-1, q)$. An *r-cap* in $\mathrm{PG}(n-1, q)$ (usually called *r-arc* if $n = 3$) is a set of r points, no three of which are collinear. A line is *secant*, respectively *tangent*, to an r-cap according as it meets the cap in two points, respectively one point.

Theorem 101 (P. J. Cameron [16])

1. *A subset $\mathcal{K}$ of the point set of* $\mathrm{PG}(n-1, q)$ *provides a linear representation of a partial quadrangle with parameters $(q-1, t, \mu)$ if and only if it is a $(t+1)$-cap with the property that any point not in $\mathcal{K}$ lies on $t-\mu+1$ tangents to $\mathcal{K}$.*

2. *A subset $\mathcal{K}$ of the point set of* $\mathrm{PG}(n-1, q)$ *provides a linear representation of a generalized quadrangle $\mathcal{S}$ if and only if one of the following occurs:*

(a) $n = 2$ *and* $|\mathcal{K}| = 2$*;*

(b) $n = 3$*,* q *is even and* $\mathcal{K}$ *is a hyperoval (a* $(q + 2)$*-arc);*

(c) $q = 2$ *and* $\mathcal{K}$ *is the complement of a hyperplane.*

Remark 102 If $n = 2$ and $|\mathcal{K}| = 2$, then $\mathcal{S}$ is a grid. If $q = 2$ and $\mathcal{K}$ is the complement of a hyperplane, then $\mathcal{S}$ is a dual grid. If $n = 3$, $q > 2$, and $\mathcal{K}$ is a hyperoval, then $\mathcal{S}$ is neither a grid nor a dual grid.

Theorem 103 (K. Thas [85]) *1. A subset* $\mathcal{K}$ *of the point set of* $\mathrm{PG}(n-1, q)$*,* $n \geq 3$*, provides a linear representation of a semi quadrangle with* μ*-parameters* $(\mu_1, \ldots, \mu_k)$ *if and only if the following conditions are satisfied:*

(a) it is a complete $(t + 1)$*-cap for a certain* t *with the property that any point off* $\mathcal{K}$ *in* $\mathrm{PG}(n - 1, q)$ *lies on* $t - \mu_j + 1$ *tangents to* $\mathcal{T}$ *for some* $\mu_j \in \{\mu_1, \ldots, \mu_k\}$*;*

(b) If $q = 2$*, then* $\mathcal{K}$ *is not the complement of a hyperplane.*

2. If a $(t+1)$*-cap* $\mathcal{K}$ *of* $\mathrm{PG}(n-1, q)$ *provides a linear representation of the semi quadrangle* $\mathcal{S}$*, then every point of* $\mathcal{S}$ *is incident with* $t + 1$ *lines.*

3. Suppose $\mathcal{S} = (P, B, I)$ *is an SQ with* μ*-parameters* $(\mu_1, \ldots, \mu_k)$ *which has a linear representation in* $\mathrm{AG}(n, q)$*, and define* P_j *by* $P_j = \{\{x, y\} \in P \times P, x \not\sim y \parallel |\{x, y\}^{\perp}| = \mu_j\}$*. Then for all* j*,* $|P_j| \equiv 0$ *mod* $(q(q - 1)/2)$*.*

Remark 104 If $n = 3$ and q is even, then the r-cap $\mathcal{K}$ is a hyperoval if r equals $q + 2$, and a hyperoval is always complete. Suppose $q \geq 4$. If we consider the semi quadrangle $\mathcal{S}$ which corresponds with $\mathcal{K}$, then $\mathcal{S}$ is a generalized quadrangle of order $(q - 1, q + 1)$; this quadrangle is usually denoted by $T^*(\mathcal{K})$ and is due to R. W. Ahrens and G. Szekeres [1, 24] (see also Theorem 101). If $n = 4$, $q > 2$, and $\mathcal{K}$ is an *ovoid* (a (complete) (q^2+1)-cap) of $\mathrm{PG}(3, q)$, then the associated semi quadrangle is a partial quadrangle with parameters $(q - 1, q^2, q^2 - q)$ (see e.g. [28, 64]).

Remark 105 By Theorem 103, SQ's seem to be fit to study complete caps of $\mathrm{PG}(n, q)$, see [85]. We will come back to this issue in the future.

Note. A semi quadrangle contains no substructure isomorphic to an ordinary 2-gon or 3-gon. With this property in mind, we could define a *semi* $2N$*-gon* of order $(s; t_1, \ldots, t_n)$ and with μ-parameters $(\mu_1, \ldots, \mu_m)$ to be an incidence structure satisfying (SQ1), and also the following conditions.

1. There is no substructure isomorphic to an ordinary M-gon, for $2 \leq M \leq 2N-1$.

2. If two distinct points are not contained in a path of length $N-1$ or less, then they are contained in exactly $\mu_1, \mu_2, \ldots$, or μ_m paths of length N, where $\mu_j \geq 1$ for every j. Also, any of the cases occurs.

3. There are substructures isomorphic to an ordinary $2N$-gon and an ordinary $(2N+1)$-gon.

With this definition, a *thick partial* $2N$*-gon* [16] is just a semi $2N$-gon with $s > 1$, $t_1 = t_2 = \ldots = t_n > 1$ and $\mu_1 = \mu_2 = \ldots = \mu_m$. Also, a generalized $2N$-gon is precisely a semi $2N$-gon with $t_i = \mu_j$ for arbitrary i and j.

B Addendum 2: Complete arcs in generalized quadrangles

B.1 k-Arcs in generalized quadrangles

A *k-arc* $\mathcal{K}$ of a GQ $\mathcal{S}$ is a set of k mutually non-collinear points. Then $k \leq st + 1$, see [47] (and if $k = st + 1$, then $\mathcal{K}$ is an ovoid of $\mathcal{S}$). A k-arc is a *partial ovoid* with k points. Dually, one defines *partial spreads*, and *isomorphic partial spreads* are defined in a similar way as isomorphic spreads (see Paragraph 6.5). A k-arc is *complete* if it is not contained in a k'-arc with $k' > k$. The following theorem is an important observation.

Theorem 106 (FGQ, 2.7.1) *An $(st-m)$-arc in a GQ of order (s,t), where $-1 \leq m < t/s$ and $s \neq 1 \neq t$, is always contained in a uniquely defined ovoid.*

Considering Theorem 106, it is a natural question to ask whether or not complete $(st - t/s)$-arcs exist.
Let us first recall some notions and results concerning complete $(st - t/s)$-arcs.
Let $\mathcal{K}$ be a complete $(st - t/s)$-arc in the GQ $\mathcal{S} = (P, B, I)$ of order (s,t), $s \neq 1 \neq t$. If B' is the set of lines incident with no point of $\mathcal{K}$, P' the set of points on (at least) one line of B', and I' the restriction of I to points of P' and lines of B', then $\mathcal{S}' = (P', B', I')$ is a subquadrangle of $\mathcal{S}$ of order $(s, t/s)$ (see FGQ [47, 2.7.2]). We denote this subGQ by $\mathcal{S}(\mathcal{K})$.
The following result is taken from J. A. Thas [59], see also 1.4.2 (ii) of [47].

Theorem 107 (J. A. Thas [59], FGQ) *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s,t \neq 1$ and $s \neq t$, and let $\{x,y\}^{\perp\perp}$ be a hyperbolic line of order $p+1$, where $pt = s^2$. Then every point of $\mathcal{S}$ not in $cl(x,y)$ is collinear with $t/s + 1 = s/p + 1$ points of $\{x,y\}^{\perp}$.*

B.2 Non-existence of complete $(st - t/s)$-arcs

The following theorem is an important step in the determination of complete $(st - t/s)$-arcs in GQ's of order (s,t), $s \neq 1 \neq t$.

Theorem 108 (K. Thas [87]) *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose $\mathcal{K}$ is a complete $(st - t/s)$-arc of $\mathcal{S}$. If $\{x,y\}^{\perp\perp} = \mathrm{H}$ is a hyperbolic line of $\mathcal{S}$ of size $p+1$ with $pt = s^2$, then $|\mathrm{H} \cap \mathcal{K}| \in \{0,1\}$, or $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,2)$ or to $\mathcal{Q}(5,2)$.*

This theorem has a lot of interesting corollaries.

Theorem 109 (K. Thas [87]) *The classical generalized quadrangle* $H(4, q^2)$ *has no complete* $(q^5 - q)$*-arcs.*

As a nice corollary, we have the following upper bound for partial ovoids of the hermitian quadrangle $H(4, q^2)$.

Theorem 110 (K. Thas [87]) *If* $\mathcal{K}$ *is a partial ovoid of* $H(4, q^2)$*, then we have that* $|\mathcal{K}| \leq q^5 - q - 1$*.*

The following theorem completely solves the problem under consideration for all GQ's of order (s, s^2), $s > 1$.

Theorem 111 (K. Thas [87]) *Let* $\mathcal{S}$ *be a GQ of order* (s, s^2)*,* $s \neq 1$*. Then* $\mathcal{S}$ *has no complete* $(s^3 - s)$*-arcs unless* $s = 2$*, i.e.* $\mathcal{S} \cong \mathcal{Q}(5, 2)$*. The GQ* $\mathcal{Q}(5, 2)$ *has a unique complete* 6*-arc up to isomorphism.*

Theorem 112 (K. Thas [87]) *Suppose* $\mathcal{S}$ *is a GQ of order* s*,* $s > 2$*, with a regular point* p*. Then* $\mathcal{S}$ *contains no complete* $(s^2 - 1)$*-arcs. The unique GQ of order* 2 *has a unique complete* 3*-arc.*

Corollary 113 (K. Thas [87]) *The dual of the GQ* $T_2(\mathcal{O})$ *of order* q *has no complete* $(q^2 - 1)$*-arcs,* $q > 2$*. In particular, the classical GQ* $W(q)$ *has no complete* $(q^2 - 1)$*-arcs if* $q \neq 2$*. A* $T_2(\mathcal{O})$ *of Tits of order* q *has no complete* $(q^2 - 1)$*-arcs if* q *is even and* $q > 2$*.*

Finally, the following theorem is a direct corollary of the preceding considerations.

Theorem 114 (K. Thas [87]) *Let* $\mathcal{S}$ *be a known GQ of order* (s, t) *with* $s \neq 1 \neq t$*, and suppose* $\mathcal{S}$ *has a complete* $(st - t/s)$*-arc* $\mathcal{K}$*. Then we necessarily have one of the following possibilities.*

1. $\mathcal{S} \cong \mathcal{Q}(4, 2)$ *and up to isomorphism there is a unique complete* 3*-arc.*
2. $\mathcal{S} \cong \mathcal{Q}(5, 2)$ *and up to isomorphism there is a unique complete* 6*-arc.*
3. $\mathcal{S} \cong \mathcal{Q}(4, q)$ *with* q *odd.*

In the last case, there is an example for $q = 3$. It is one of our goals in [92] to solve this case completely.

Remark 115 There is a strong connection between *complete dual grids* with parameters $s + 1, s + 1$ and complete $(s^2 - 1)$-arcs in GQ's of order s [92].

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