Moufang Polygons

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Abstract

In [7], Tits classified spherical buildings of rank at least three. In the addenda of [7], he introduced the Moufang property for buildings and observed that with only the results of Chapter 4 of [7] one could show that an irreducible spherical building of rank at least three and all of its irreducible residues of rank at least two must satisfy the Moufang property. Tits and the author have recently completed the classification of irreducible spherical buildings of rank two having the Moufang property [11]. We give a brief overview of this classification.

An irreducible spherical building of rank two is the same thing as a generalized polygon. A generalized polygon is simply a bipartite graph whose diameter is half the length of a shortest circuit. To avoid certain trivialities, we assume as well that every vertex has at least three (but possibly infinitely many) neighbors and that the diameter is at least three. (The diameter is not allowed to be infinite.) A generalized n-gon is a generalized polygon of diameter n.

Let Γ be a generalized n-gon for some $n \geq 3$ and let $G = \mathrm{Aut}(G)$. A circuit of length $2n$ in Γ is called an apartment. A root of Γ is an undirected path of length n. For each vertex x of Γ, let Γ_x denote the set of neighbors of x. For each root $\alpha = (x_0, x_1, \ldots, x_n)$ of Γ, we denote by U_α the pointwise stabilizer in G of the set $\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_{n-1}}$. The group U_α is called the root group associated with α.

Definition. A generalized n-gon satisfies the Moufang property if for each root α of Γ, the root group U_α acts transitively on the set of apartments containing α.

A Moufang n-gon is a generalized n-gon satisfying the Moufang property.

A generalized 3-gon (or triangle) is the same thing as the incidence graph of a projective plane. The notion of a Moufang generalized n-gon generalizes the notion of a Moufang projective plane first introduced in [3].

We now assume that Γ is a Moufang n-gon for some $n \geq 3$. We choose an apartment Σ and label the vertices of Σ by the integers modulo $2n$ so that i is adjacent to $i+1$ and different from $i+2$ for all i. Let U_i be the root group corresponding to the root $(i, i+1, \ldots, i+n)$ for all i and let U_+ denote the subgroup of G generated by the subgroups $U_1, U_2, \ldots, U_n$. The $(n+1)$-tuple $(U_+, U_1, U_2, \ldots, U_n)$ is called the root group sequence associated with

Γ; it is unique up to conjugation in G and up to the re-numbering $U_i \mapsto U_{n+1-i}$ of the root groups $U_1, \ldots, U_n$. In [11], we show:

Theorem. *A Moufang n-gon is uniquely determined by the associated root group sequence*

$$(U_+, U_1, U_2, \ldots, U_n).$$

As an example, we consider the case $n = 3$. Let A be an alternative division ring. This is a ring satisfying all the axioms of a skew-field except the law of associativity for multiplication, but with a strengthened law of inverses: For each non-zero $u \in A$, there exists an element u' such that $u' \cdot uv = v$ and $vu \cdot u' = v$ for all $v \in A$. Now let U_1, U_2, U_3 be three groups parameterized by the additive group of A. By this we mean that we can choose isomorphisms $u \mapsto x_i(u)$ from the additive group of A to the (multiplicative) group U_i for all $i \in [1, 3]$. We now impose the relations $[U_1, U_2] = [U_2, U_3] = 1$ and $[x_1(u), x_3(v)] = x_2(uv)$ for all $u, v \in A$. These equations determine the structure of a group $U_+ = \langle U_1, U_2, U_3 \rangle$. It turns out that (U_+, U_1, U_2, U_3) is the root group sequence associated with a generalized triangle which we denote by $\mathcal{T}(A)$. If A is a skew-field, $\mathcal{T}(A)$ is the incidence graph of the projective plane associated with a 3-dimensional right vector space over A. Moufang showed in [3] that every Moufang projective plane is parameterized by an alternative division ring. This result can be reformulated as follows: Every Moufang triangle is isomorphic to $\mathcal{T}(A)$ for some alternative division ring A.

Alternative division rings were classified by Bruck and Kleinfeld [1, 2]: An alternative division ring is either a skew-field (possibly commutative) or a kind of 8-dimensional non-associative algebra over a commutative field K called a Cayley-Dickson division algebra.

This case is typical. For each n, we find an algebraic system (in some general sense) with which we can parameterize the groups $U_1, \ldots, U_n$ and give formulas for all the commutators in $[U_i, U_j]$ for all distinct i, j in $[1, n]$ expressed in terms of the parameters. These formulas determine the root group sequence $(U_+, U_1, \ldots, U_n)$ and thus Γ. In each case, there then remains the problem of classifying the relevant algebraic systems. That this strategy has any chance of success rests on the following result [9, 12]:

Theorem. *Moufang n-gons exist only for n* = 3*, 4, 6 and 8.*

We indicate the conclusions in each case beginning with the largest value of n.

Moufang octagons are parameterized (see [10]) by pairs (K, σ) where K is a commutative field of characteristic two and σ is an endomorphism of K such that σ^2 is the Frobenius map of K, i.e. $(x^\sigma)^\sigma = x^2$ for all $x \in K$.

Moufang hexagons are parameterized (see [8]) by certain triples $(J, F, \#)$, where F is a commutative field, J a vector space over F and $\#$ a map from J to itself satisfying a certain list of properties. These triples are closely related to certain Jordan algebras which have been closely studied by Albert, Jacobson and several of Jacobson's students. They were classified by Petersson and Racine [5, 6]. We give two examples: Let J be a commutative field containing F and suppose either that $J^3 \subseteq F$ or that $[J : F] = 3$ and J/F is separable. We set $x^\# = x^2$ for all $x \in J$ in the first case and $x^\# = x/N(x)$ for all $x \in J^*$ in the second, where N is the norm of the extension J/F. In all the other cases, the dimension of J over F is 9 or 27.

There are three distinct classes of Moufang quadrangles: classical, indifferent and exceptional. The classical quadrangles are parameterized by pseudo-quadratic forms (see 8.2 of [7]). The indifferent quadrangles are parameterized by algebraic systems involving certain purely inseparable field extensions in characteristic two. The exceptional quadrangles (of which there are four families) are parameterized by pairs of vector spaces and several maps connecting these vector spaces and the fields over which they are defined. The parameter systems for the first three families involve the even Clifford algebra of a certain type of quadratic form. The parameter systems for the fourth family is still more exotic; these quadrangles (like the indifferent quadrangles and the Moufang octagons) exist only in characteristic two. See [4].

The Moufang triangles $\mathcal{T}(A)$ for A a field or a skew-field and the classical quadrangles are the spherical buildings associated with certain classical groups. The remaining Moufang triangles (those parameterized by a Cayley-Dickson division algebra), the remaining quadrangles (except those defined only in characteristic two) and all the Moufang hexagons (except those defined over a purely inseparable field extension in characteristic three) are the spherical buildings associated with k-forms of absolutely simple algebraic groups of k-rank two. All other Moufang polygons, namely those which are defined only in characteristic two or three, are related to groups of mixed type as defined in (10.3.2) of [7].

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