Arcs and caps in projective spaces

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Intensive Course on Galois Geometry and Generalized Polygons

1 Arcs in projective spaces

1.1 Definitions and elementary properties

An *n-arc* in $PG(N, q)$, with $n \geq N+1$, is a set of n points, every $N+1$ of which are linearly independent. An n-arc is called *complete* when it is not contained in an $(n+1)$-arc. A point R extends an n-arc $\mathcal{K}$ to an $(n+1)$-arc if and only if $\mathcal{K} \cup \{R\}$ is an $(n+1)$-arc.

The maximum size for which there exists an n-arc in $PG(N, q)$ is denoted by $m(N, q)$. The size of the second largest complete arc in $PG(N, q)$ is denoted by $m'(N, q)$.

We start by looking at n-arcs in $PG(2, q)$.

1.2 n-Arcs in $PG(2, q)$

Here, an n-arc is a set of n points, no three of which are collinear.

The classical example of an arc is the set of rational points of a conic. This shows that there exist $(q+1)$-arcs in $PG(2, q)$.

For q odd, this is the maximum size of an arc in $PG(2, q)$. For q even, a conic plus its nucleus (the intersection point of its tangents) is a $(q+2)$-arc. Since any arc in $PG(2, q)$ has at most $q+2$ points, so, for q even, $q+2$ is the maximum size of an arc.

These results are summarized in Table 1.1.

q	$m(2, q)$	
q odd	$q+1$	[3]
q even	$q+2$	[3]

Table 1.1: $m(2, q)$

An $m(2, q)$-arc in $PG(2, q)$, q odd, is called an *oval* and an $m(2, q)$-arc in $PG(2, q)$, q even, is called a *hyperoval*. Formerly, the term *oval* was used in both cases.

For q odd, a first question that arises is whether every $(q+1)$-arc is a conic. This was decided by a result of Segre.

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Theorem 1.1 (Segre [64, 65]) *For q odd, an oval is the set of rational points of a conic.*

This result, sometimes referred to as **the fundamental result of finite geometry**, was a strong motivation for further research in this field.

There are considerable differences between q even and q odd.

For q even, a conic plus its nucleus is a hyperoval. A hyperoval of this type is called *regular*. As shown by Segre [67], for $q = 2, 4, 8$, every hyperoval is regular.

But for $q = 2^h, h \geq 4$, there exist *irregular* hyperovals, that is, hyperovals which are not the union of a conic and its nucleus. Several infinite classes of irregular hyperovals are known.

The next result gives a canonical form for hyperovals.

Theorem 1.2 (Segre [67], [31, 35, §8.4]) *Any hyperoval of $PG(2,q)$, $q = 2^h$ and $h > 1$, is projectively equivalent to a hyperoval $\mathcal{D}(F) = \{(1, t, F(t)) \,\|\, t \in \mathbf{F}_q\} \cup \{e_1, e_2\}$ where F is a permutation polynomial over $\mathbf{F}_q$ of degree at most $q-2$, satisfying $F(0) = 0$, $F(1) = 1$, and such that $F_s(X) = (F(X+s) + F(s))/X$ is a permutation polynomial for each s in $\mathbf{F}_q$, satisfying $F_s(0) = 0$.*

A polynomial $F(X)$ of this type is called an o-*polynomial* [12] and, conversely, every o-polynomial gives rise to a hyperoval. Table 1.2 presents the known infinite classes of hyperovals.

Name	$F(X)$	$q = 2^h$	Conditions	
Regular	X^2			[3]
Translation	X^{2^i}		$(h, i) = 1$	[67]
Segre	X^6	h odd		[70]
Glynn I	$X^{3\sigma+4}$	h odd	$\sigma = 2^{(h+1)/2}$	[23]
Glynn II	$X^{\sigma+\lambda}$	h odd	$\sigma = 2^{(h+1)/2}$; $\lambda = 2^m$ if $h = 4m-1$; $\lambda = 2^{3m+1}$ if $h = 4m+1$	[23]
Payne	$X^{1/6} + X^{3/6} + X^{5/6}$	h odd		[54]
Cherowitzo	$X^\sigma + X^{\sigma+2} + X^{3\sigma+4}$	h odd	$\sigma = 2^{(h+1)/2}$	[14]
Subiaco	$S_1(X)$	$h = 4r+2$	$\omega^2 + \omega + 1 = 0$	[13, 55], [56]
Subiaco	$S_2(X)$	$h = 4r+2$	$\delta = \zeta^{q-1} + \zeta^{1-q}$ ζ primitive in $GF(q^2)$	[13, 55], [56]
Subiaco	$S_3(X)$	$h \neq 4r+2$	$\mathrm{Tr}(1/\delta) = 1$	[13, 55], [56]

Table 1.2: Hyperovals in $PG(2,q), q$ even

In Table 1.2,

$$S_1(X) \quad = \quad \frac{\omega^2(X^4 + X)}{X^4 + \omega^2 X^2 + 1} + X^{1/2};$$

$$
\begin{aligned}
S_2(X) &= \frac{\delta^2X^4+\delta^5X^3+\delta^2X^2+\delta^3X}{X^4+\delta^2X^2+1}+\left(\frac{X}{\delta}\right)^{1/2};\\
S_3(X) &= \frac{(\delta^4+\delta^2)X^3+\delta^3X^3+\delta^2X}{X^4+\delta^2X^2+1}+\left(\frac{X}{\delta}\right)^{1/2}.
\end{aligned}
$$

1.3 Arcs in $PG(2,q)$ and algebraic curves

Apart from the problem of classifying hyperovals in $PG(2,q)$, q even, much attention has been paid to the problem of extending an n-arc to to an $m(2,q)$-arc.

Finding the values of n for which an n-arc is always contained in an oval, for q odd, or hyperoval, for q even, is relevant for solving problems on arcs or on other substructures in higher-dimensional spaces.

The crucial technique in obtaining results for this problem is the link between algebraic curves and n-arcs in $PG(2,q)$.

To describe this link, we will consider an *n-arc of lines*. This is a set of n lines, no three of which have a point in common.

1.3.1 The case q even

Theorem 1.3 *Let $\mathcal{K}$ be an n-arc of lines $\{l_1,\ldots,l_n\}$ in $PG(2,q)$, q even. The line l_j for $j\neq i$ intersects l_i in $n-1$ different points; the remaining $t=q+2-n$ points of $PG(2,q)$ on l_i belong to exactly one line of $\mathcal{K}$. So there are exactly tn points of $PG(2,q)$ on precisely one line of $\mathcal{K}$. These tn points belong to an algebraic curve Γ of degree t with the following properties:*
(i) *Γ is unique if $n>t$, that is, when $n>(q+2)/2$;*
(ii) *Γ contains no point P on two lines of $\mathcal{K}$ and consequently no line of $\mathcal{K}$ is a component of Γ;*
(iii) *each point of $PG(2,q)$ that belongs to exactly one line of $\mathcal{K}$ is a non-singular point of Γ.*

This curve Γ of degree t is called *the associated curve of the n-arc of lines $\mathcal{K}$.*

Theorem 1.4 *Let $\mathcal{K}$ be an n-arc of lines in $PG(2,q)$, q even, $(q+2)/2<n\leq q+1$. Suppose Γ is the curve of degree $q+2-n$ which corresponds to $\mathcal{K}$.*

If $\mathcal{K}$ is incomplete, then any line ℓ of $PG(2,q)$ which extends $\mathcal{K}$ to an $(n+1)$-arc is a component of Γ, and conversely, any line ℓ of $PG(2,q)$ which is a component of Γ extends $\mathcal{K}$ to an $(n+1)$-arc.

1.3.2 The case q odd

Theorem 1.5 *Let $\mathcal{K}=\{l_1,\ldots,l_n\}$ be an n-arc of lines in $PG(2,q)$, q odd. As for q even, a line l_i of $\mathcal{K}$ $(1\leq i\leq n)$ contains $q+2-n=t$ points of $PG(2,q)$ that do not belong to any other line of $\mathcal{K}$. These tn points of $PG(2,q)$ belong to an algebraic curve Γ of degree $2t$ with the following properties:*
(i) *Γ is unique if $n>2t$, that is, when $n>(2q+4)/3$;*

(ii) Γ *contains no point* P *that belongs to two lines of* $\mathcal{K}$ *and consequently no line of* $\mathcal{K}$ *is a component of* Γ*;*
(iii) *if* R *is a point of* $PG(2,q)$ *that belongs to just one line* l_j *of* $\mathcal{K}$*, then* l_j *and* Γ *intersect in* R *with multiplicity* 2*, that is,* $I(l_j, \Gamma; R) = 2$;
(iv) Γ *can contain components of multiplicity* 2*, but not all components of* Γ *have multiplicity* 2.

This curve Γ of degree $2(q+2-n)$ is called *the associated curve of the* n*-arc of lines* $\mathcal{K}$.

Theorem 1.6 *Let* $\mathcal{K}$ *be an* n*-arc of lines in* $PG(2,q)$*,* q *odd,* $(2q+4)/3 < n \leq q+1$*. Suppose* Γ *is the curve of degree* $2(q+2-n)$ *which corresponds to* $\mathcal{K}$*.*

If $\mathcal{K}$ *is incomplete, then any line* ℓ *of* $PG(2,q)$ *which extends* $\mathcal{K}$ *to an* $(n+1)$*-arc is a component of* Γ*, and conversely, any line* ℓ *of* $PG(2,q)$ *which is a component of* Γ *extends* $\mathcal{K}$ *to an* $(n+1)$*-arc.*

These results linking an algebraic curve to an arc allow the use of algebraic geometry methods to study arcs.

Table 1.3 surveys the known results on the extendability problem of arcs. In $PG(2,q)$, for q subject to the conditions in the second column, the third column gives an upper bound for $m'(2,q)$; the fourth column indicates when this upper bound is sharp in the case that equality does not hold in general. The fifth column gives the value of $m(2,q)$. So any n-arc with $n > m'(2,q)$ is contained in an $m(2,q)$-arc, and the latter arc is described in the last column.

	q	$m'(2,q)$	Sharpness	$m(2,q)$	$m(2,q)$-arc
(1)	$q = p^{2e}$, $p > 2,\ e \geq 1$	$\leq q - \sqrt{q}/4 + 25/16$		$q+1$	conic
(2)	$q = p^{2e+1}$, $p > 2,\ e \geq 1$	$\leq q - \sqrt{pq}/4$ $+29p/16 + 1$		$q+1$	conic
(3)	q prime, $q > 2$	$\leq 44q/45 + 8/9$		$q+1$	conic
(4)	$q = p^h,\ p \geq 5$	$\leq q - \sqrt{q}/2 + 5$		$q+1$	conic
(5)	$q = p^h,\ p \geq 3$, $q \geq 23^2,\ q \neq 5^5,\ 3^6$, h even for $p = 3$	$\leq q - \sqrt{q}/2 + 3$		$q+1$	conic
(6)	$q = 2^{2e},\ e > 1$	$= q - \sqrt{q} + 1$		$q+2$	hyperoval
(7)	$q = 2^{2e+1},\ e \geq 1$	$\leq q - \sqrt{2q} + 2$	$q = 8$	$q+2$	hyperoval

Table 1.3: Upper bounds for $m'(2,q)$

Result (1) is due to Thas [80]; (2) and (7) to Voloch [83]; (3) to Voloch [82]; (4) and (5) to Hirschfeld and Korchmáros [36, 37], and (6) to Segre [71].

Of great interest is the sharpness of the bound in (6). Segre had proved that every n-arc in $PG(2,q)$, q even, with $n > q - \sqrt{q} + 1$, is extendable to a $(q+2)$-arc. Independently, Boros and Szőnyi [2], Fisher, Hirschfeld and Thas [21], and Kestenband [41] constructed a complete $(q-\sqrt{q}+1)$-arc in $PG(2,q)$, $q=2^{2e}$, $e>1$. This example implied the determination of the precise value of $m'(2,q)$, for q an even square.

For small values of q, the uniqueness of the complete $(q-\sqrt{q}+1)$-arcs follows from Hirschfeld [31, §14.8], [35, §8.7], Lisonek [42, p.193], Chao and Kaneta [10, 11].

Theorem 1.7 *In $PG(2,q)$, $q=p^{2e}$, $q>4$, a complete $(q-\sqrt{q}+1)$-arc exists. For $q=9,16,25$, a complete $(q-\sqrt{q}+1)$-arc is projectively unique.*

It is conjectured also for q an odd square with $q>9$ that $m'(2,q)=q-\sqrt{q}+1$.

1.4 n-Arcs in $PG(N,q)$

1.4.1 $m(N,q)$-Arcs

We first present the known results on the characterization of $m(N,q)$-arcs in projective spaces $PG(N,q)$ of small dimension N. To describe these arcs, we need the following definitions.

Definition 1.8 (i) *A* normal rational curve (NRC) *in $PG(N,q)$, $2\le N\le q-2$, is a $(q+1)$-arc projectively equivalent to the $(q+1)$-arc $\{(1,t,\ldots,t^N)\,\|\,t\in \mathbf{F}_q^+\}$ where $\mathbf{F}_q^+=\mathbf{F}_q\cup\{\infty\}$ and where $t=\infty$ defines the point e_N.*
(ii) *In $PG(3,q)$, q even, $\mathcal{L}_e=\{(1,t,t^e,t^{e+1})\,\|\,t\in\mathbf{F}_q^+\}$, with $t=\infty$ again defining the point e_3.*

Table 1.4 shows the value of $m(N,q)$ for small dimensions N. The characterization of the $m(N,q)$-arcs $\mathcal{L}$ in $PG(N,q)$ is given in the last column.

	q	N	$m(N,q)$	$\mathcal{L}$
(1)	$q=p^h$	$N\ge q-1$	$N+2$	$\{e_0,\ldots,e_N,e\}$
(2)	q odd, $q>3$	3	$q+1$	NRC
(3)	$q=2^h$, $q>4$	3	$q+1$	$\mathcal{L}_e, e=2^v,(v,h)=1$
(4)	q odd, $q>5$	4	$q+1$	NRC for $q>83$
(5)	$q=2^h$, $q>4$	4	$q+1$	NRC
(6)	$q=2^h$, $q\ge 8$	5	$q+1$	NRC for $q\ge 16$
(7)	$q=2^h$, $q\ge 16$	6	$q+1$	NRC for $q>64$

Table 1.4: $m(N,q)$ and $m(N,q)$-arcs

Result (1) is due to Bush [5] and (2) to Segre [66]. The value of $m(N,q)$ in (3) is due to Casse [6], as well as to Gulati and Kounias [25], while the characterization of the $(q+1)$-arcs is by Casse and Glynn [7]. In (4), the value of $m(N,q)$ is due to Segre [66]; the characterization of L follows from Table 1.5. The bound in (5) is by Casse [6], and by Gulati and Kounias [25], while the characterization of the $(q+1)$-arcs is due to Casse and Glynn [8]. Result (6) is by Maruta and Kaneta [44], and (7) by the same authors [40, 44] for the bound and by Storme and Thas [74] for the characterization of the $(q+1)$-arcs.

The preceding results show that when q is odd and $N=2$ or $N=3$, then the $m(N,q)$-arcs always are normal rational curves. This led to the conjecture that in $PG(N,q)$, q odd, $2 \leq N \leq q-2$, every $m(N,q)$-arc is a normal rational curve.

This conjecture is however **false**. Glynn [24] found an example of an $m(N,q)$-arc in $PG(4,9)$ different from a normal rational curve. This example is of particular interest since this 10-arc $\mathcal{L}$ is the only known $(q+1)$-arc in $PG(N,q)$, q odd, $2 \leq N \leq q-2$, which is not a normal rational curve.

Theorem 1.9 (Glynn [24]) *In $PG(4,9)$, a 10-arc is one of two types; it is either a normal rational curve or is projectively equivalent to*

$$\mathcal{L} = \{(1,t,t^2+\eta t^6, t^3, t^4) \,\|\, t \in \mathbf{F}_9\} \cup \{(0,0,0,0,1)\},$$

where $\eta^4 = -1$.

1.4.2 Extendability of arcs in $PG(N,q)$

As for $N=2$, a problem that has been intensively studied is the extendability of n-arcs in $PG(N,q)$ to larger arcs. Table 1.5 summarizes the main results on this topic. An n-arc in $PG(N,q)$, satisfying the condition on n in the third column, can be extended uniquely to a $(q+1)$-arc $\mathcal{L}$, whose description is given in column 4.

N	q	$n >$	$\mathcal{L}$	
≥ 2	$q=p^{2e}$, p odd, $e \geq 1$	$q-\sqrt{q}/4+N-7/16$	NRC	[78, 80]
≥ 2	$q=p^{2e+1}$, p odd, $e \geq 1$	$q-\sqrt{pq}/4+29p/16$ $+N-1$	NRC	[83]
≥ 2	q prime, $q>2$	$44q/45+N-10/9$	NRC	[82]
≥ 2	$q=p^h$, $p \geq 5$	$q-\sqrt{q}/2+N+3$	NRC	[36]
≥ 2	$q=p^h$, $p\geq 3$, $q \geq 23^2$, h even for $p=3$, $q \neq 5^5$, 3^6	$q-\sqrt{q}/2+N+1$	NRC	[37]
3	$q=2^h, h>1$	$q-\sqrt{q}/2+9/4$	$\mathcal{L}_e$, $e=2^v$, $(v,h)=1$	[74]
≥ 4	$q=2^h$, $h>2$	$q-\sqrt{q}/2+N-3/4$	NRC	[74]

Table 1.5: Upper bounds for $m'(N,q)$

To obtain the results for q even, the following theorems on the relation between arcs and surfaces in $PG(3,q)$ and between arcs and hypersurfaces in $PG(4,q)$ are crucial.

Theorem 1.10 (Bruen, Thas and Blokhuis [4]) *Let $\mathcal{K}=\{\Pi_1,\ldots,\Pi_n\}$ be an n-arc of planes in $\Sigma = PG(3,q)$, $q=2^s$.*

For each plane Π_i, $1 \leq i \leq n$, let Z_{ij}, $i \neq j$, denote the set of those points of the line $\Pi_i \cap \Pi_j$ lying on exactly two planes of $\mathcal{K}$. Then

(i) *there exists an algebraic curve* C_i *in* Π_i *of degree* $q+3-n$ *containing all the sets* Z_{ij}, $j \neq i$, *and with* $C_i \cap \Pi_j = Z_{ij}$;
(ii) *there exists an algebraic surface* $\Phi = \Phi(\mathcal{K})$ *of degree* $t = q+3-n$ *algebraically containing the curves* C_i *and with* $\Phi \cap \Pi_i = C_i$, $1 \leq i \leq n$.

This surface Φ is called the *surface associated to the* n*-arc* $\mathcal{K}$.

Theorem 1.11 (Bruen, Thas and Blokhuis [4]) *Let* $\mathcal{K} = \{\Pi_1, \ldots, \Pi_n\}$ *be an* n*-arc of planes in* $\Sigma = PG(3,q)$, $q = 2^s$, *with associated surface* $\Phi = \Phi(\mathcal{K})$.

Assume that $n > (q+4)/2$. *Then*
(i) *this surface is unique;*
(ii) *if* $\mathcal{K}' = \{\Pi_1, \ldots, \Pi_u\}$, *with* $u > n$, *is an arc of planes in* $\Sigma = PG(3,q)$ *containing* $\mathcal{K}$, *then* $\Phi(\mathcal{K})$ *factors into the varieties* $\Phi(\mathcal{K}'), \Pi_{k+1}, \ldots, \Pi_u$ *where* $\Phi(\mathcal{K}')$ *is the surface associated to the arc* $\mathcal{K}'$;
(iii) *there is a bijection between the planes of* $\Sigma = PG(3,q)$ *extending* $\mathcal{K}$ *to an* $(n+1)$*-arc and the linear components over* $\mathbf{F}_q$ *of* Φ.

Theorem 1.12 (Bruen, Thas and Blokhuis [4]) *Let* $\mathcal{K} = \{\Pi_1, \ldots, \Pi_n\}$ *be an* n*-arc of hyperplanes in* $\Sigma = PG(4,q)$, $q = 2^s$.

For each hyperplane Π_i, $1 \leq i \leq n$, *let* Φ_i *denote the surface of degree* $q+4-n$ *corresponding to the* $(n-1)$*-arc of planes* $\Pi_i \cap \Pi_j$, $1 \leq j \leq n$, $j \neq i$, *in* Π_i. *Then there exists an algebraic hypersurface* $\Phi = \Phi(\mathcal{K})$ *of degree* $t = q+4-n$ *algebraically containing the surfaces* Φ_i *and with* $\Phi \cap \Pi_i = \Phi_i$, $1 \leq i \leq n$.

This hypersurface is called the *hypersurface associated to the* n*-arc* $\mathcal{K}$.

Theorem 1.13 (Bruen, Thas and Blokhuis [4]) *Let* $\mathcal{K} = \{\Pi_1, \ldots, \Pi_n\}$ *be an* n*-arc of hyperplanes in* $\Sigma = PG(4,q)$, $q = 2^s$, *with associated hypersurface* $\Phi = \Phi(\mathcal{K})$.

Assume that $n > (q+6)/2$. *Then*
(i) *this hypersurface is unique;*
(ii) *if* $\mathcal{K}' = \{\Pi_1, \ldots, \Pi_u\}$, *with* $u > n$, *is an arc of hyperplanes in* $\Sigma = PG(4,q)$ *containing* $\mathcal{K}$, *then* $\Phi(\mathcal{K})$ *factors into the varieties* $\Phi(\mathcal{K}'), \Pi_{k+1}, \ldots, \Pi_u$ *where* $\Phi(\mathcal{K}')$ *is the hypersurface associated to the arc* $\mathcal{K}'$;
(iii) *there is a bijection between the hyperplanes of* $\Sigma = PG(4,q)$ *extending* $\mathcal{K}$ *to an* $(n+1)$*-arc and the linear components over* $\mathbf{F}_q$ *of* Φ.

1.5 Arcs and C-dual arcs

1.5.1 The C-duality principle

Remark 1.14 The existence of an n-arc in $PG(n-k-1,q)$ is equivalent to the existence of a linear $[n,k,n-k+1]$ MDS code C. But the dual code $C^\perp$ of an $[n,k,n-k+1]$ MDS code is an $[n,n-k,k+1]$ MDS code [43, Ch.11, §2]. Hence, to $C^\perp$ also corresponds an n-arc $\hat{\mathcal{K}}$, where this is an n-arc in $PG(k-1,q)$.

This theorem, proved independently by Thas [79] and by Halder and Heise [26], makes it possible to link n-arcs in $PG(n-k-1,q)$ to n-arcs in $PG(k-1,q)$. An n-arc $\mathcal{K}$ in

$PG(n-k-1,q)$ and an n-arc $\hat{\mathcal{K}}$ in $PG(k-1,q)$ are called C-*dual arcs* if and only if they define dual MDS codes.

This C-duality principle implies also that every new result on arcs immediately leads to a second new result on arcs, but now for the C-dual arcs.

1.5.2 Applications for q odd

This C-duality principle on arcs states for instance that an n-arc of $PG(n-k-1,q)$ is contained in a normal rational curve of $PG(n-k-1,q)$ if and only if its dual n-arc in $PG(k-1,q)$ is contained in a normal rational curve of $PG(k-1,q)$ [43, Ch.10, §8]. In Table 1.5, results regarding arcs contained in normal rational curves, in projective spaces of **small** dimension, were presented. Using the C-duality principle, the results in Table 1.5 on arcs in $PG(N,q)$, with N small, can be retranslated into results on arcs in $PG(N,q)$ contained in normal rational curves, but now for dimensions N **close to** q.

In Table 1.6, in the spaces $PG(N,q)$, with N satisfying the bounds in the table, any n-arc, where n satisfies the bound in the second column, is contained in a normal rational curve. So, in these spaces, $m(N,q)=q+1$ and every $(q+1)$-arc is a normal rational curve.

Only n-arcs in $PG(N,q)$, with $n \geq N+4$, are considered since it is a known result that an $(N+3)$-arc of $PG(N,q)$, $2 \leq N \leq q-2$, is always contained in a unique normal rational curve [33, §21.1].

q	$n \geq$	$m(N,q)=q+1$
$q=p^{2e},\ p>2,\ e\geq 1$	$N+4$	$q-3 \geq N > q-\sqrt{q}/4-39/16$
$q=p^{2e+1},\ p>2,\ e\geq 1$	$N+4$	$q-3 \geq N > q-\sqrt{pq}/4+29p/16-3$
q prime, $q>2$	$N+4$	$q-3 \geq N > 44q/45-28/9$
$q=p^h,\ p\geq 5$	$N+4$	$q-3 \geq N > q-\sqrt{q}/2+1$
$q=p^h,\ p\geq 3,\ q\geq 23^2,\ q\neq 5^5,\ 3^6,$ h even for $p=3$	$N+4$	$q-3 \geq N > q-\sqrt{q}/2-1$
$q=2^h,\ h>2$	$N+6$	$q-5 \geq N > q-\sqrt{q}/2-11/4$

Table 1.6: Arcs in $PG(N,q)$, N close to q

1.5.3 Applications for q even

We have already seen fundamental differences between arcs in spaces of even characteristic and in spaces of odd characteristic, such as the existence of $(q+2)$-arcs in $PG(2,q)$, q even.

These differences are reflected when applying the C-duality principle. For instance, the $(q+2)$-arcs in $PG(2,q)$, q even, have C-dual $(q+2)$-arcs in $PG(q-2,q)$, q even.

This latter result was first discovered by Thas.

Theorem 1.15 (Thas [79]) *In $PG(q-2,q)$, q even, $m(q-2,q)=q+2$.*

In Section 1.2, we addressed the problem of classifying the hyperovals of $PG(2,q)$, q even. This problem can be immediately retranslated into the equivalent problem of classifying the C-dual $(q+2)$-arcs in $PG(q-2,q)$, q even.

In [33, §21.1], it was proved that every $(N+3)$-arc of $PG(N,q)$ is contained in a unique normal rational curve of $PG(N,q)$. The particular application of this result for $N=q-2$ implies that every $(q+1)$-arc of $PG(q-2,q)$ is a normal rational curve.

Hence, a $(q+2)$-arc of $PG(q-2,q)$, q even, can be described as the union of a normal rational curve and a point.

The necessary and sufficient condition a point has to satisfy in order to extend a normal rational curve to a $(q+2)$-arc is given in the following theorem.

Theorem 1.16 (Storme and Thas [75]) *A point $P=(a_0,\ldots,a_{q-2})$ of $PG(q-2,q)$, q even, extends the normal rational curve $\mathcal{K}=\{(1,t,\ldots,t^{q-2})\,\|\,t\in\mathbf{F}_q^+\}$ to a $(q+2)$-arc if and only if $F(X)=\sum_{i=0}^{q-2}a_{q-2-i}X^{i+1}$ defines a $(q+2)$-arc $\mathcal{K}'=\{(1,t,F(t))\,\|\,t\in\mathbf{F}_q\}\cup\{e_1,e_2\}$ in $PG(2,q)$; in this case, $\mathcal{K}'$ is a C-dual $(q+2)$-arc of $\mathcal{K}\cup\{P\}$.*

This correspondence between the points P and the $(q+2)$-arcs $\mathcal{K}'$ is one-to-one if the condition $F(1)=1$ is added.

The C-duality principle also makes it possible to determine the complete spectrum of n for which a complete n-arc exists in $PG(N,q)$, with N close to q.

Table 1.7 presents the results by Storme and Thas [73] for the values of n for which there exist complete n-arcs in the respective spaces $PG(N,q)$. Note that (a) $N+5=q+1$ when $N=q-4$; (b) $N+4=q+1$ when $N=q-3$; (c) $N+4=q+2$ when $N=q-2$.

q	N	$n=N+4$	$n=N+5$	$n=q+1$	$n=q+2$
$q=p^{2e},\ e\geq 1,$ $p>2$	$q-2\geq N$ $>q-\sqrt{q}/4-39/16$			+	
$q=p^{2e+1},\ e\geq 1,$ $p>2$	$q-2\geq N>$ $q-\sqrt{pq}/4+29p/16-3$			+	
q prime, $q>2$	$q-2\geq N>44q/45-28/9$			+	
$q=p^h,\ p\geq 5$	$q-2\geq N>q-\sqrt{q}/2+1$			+	
$q=p^h,\ p\geq 3,\ q\geq 23^2,$ h even for $p=3,$ $q\neq 5^5,\ 3^6$	$q-2\geq N>q-\sqrt{q}/2-1$			+	
$q=2^h,\ q\geq 32,$ $q\neq 64$	$q-5\geq N$ $>q-\sqrt{q}/2-11/4$	+	+	+	
$q=64$	$N=58$ or $N=59$	+		+	
$q=2^h,\ q\geq 8$	$q-4$	+	+	+	
$q=2^h,\ q\geq 8$	$q-3$	+		+	
$q=2^h,\ q\geq 4$	$q-2$	+			+

Table 1.7: Spectrum of complete arcs

2 n-Caps in $PG(N,q)$

2.1 Definition and first results

An *n-cap in $PG(N,q)$* is a set of n points, no three of which are collinear. It is *complete* when it is not contained in an $(n+1)$-cap of $PG(N,q)$. A point R of $PG(N,q)$ *extends* an n-cap $\mathcal{K}$ to an $(n+1)$-cap if and only if $\mathcal{K} \cup \{R\}$ is an $(n+1)$-cap.

Table 2.1 presents the known values for $m_2(N,q)$, the size of the largest caps in $PG(N,q)$.

An $m_2(3,q)$-cap of $PG(3,q)$ is also called an *ovoid*.

q	N	$m_2(N,q)$	
q odd	3	q^2+1	[3]
q even, $q>2$	3	q^2+1	[63]
$q=2$	N	2^N	[3]
$q=3$	4	20	[58]
$q=3$	5	56	[28]
$q=4$	4	41	[16]

Table 2.1: $m_2(N,q)$

2.2 The known results on $m_2(N,q)$-caps

The following result gives the known characterizations of $m_2(N,q)$-caps in $PG(N,q)$.

Theorem 2.1 (i) (Barlotti [1], Panella [53]) *For q odd or $q=4$, an ovoid in $PG(3,q)$ is an elliptic quadric.*

(ii) (Tits [81], [33, §16.4]) *There exists an ovoid in $PG(3,2^{2e+1})$, $e \geq 1$, which is not an elliptic quadric. This ovoid, the* Tits ovoid, *is projectively equivalent to the set $\mathcal{K} = \{(0,1,0,0)\} \cup \{(1,z,y,x) \,\|\, z = xy + x^{\sigma+2} + y^{\sigma}$, with $(x,y) \in \mathbf{F}_q^2$ and $\sigma = 2^{e+1}\}$.*

(iii) (Fellegara [20], Penttila and Praeger [59]) *In $PG(3,8)$, an ovoid is an elliptic quadric or a Tits ovoid.*

(iv) (O'Keefe and Penttila [47, 50]) *In $PG(3,16)$, an ovoid always is an elliptic quadric.*

(v) (O'Keefe, Penttila and Royle [51]) *In $PG(3,32)$, an ovoid is an elliptic quadric or a Tits ovoid.*

(vi) (Segre [68]) *In $PG(N,2)$, a 2^N-cap is the complement of a hyperplane.*

(vii) (Hill [30]) *There are nine projectively distinct 20-caps in $PG(4,3)$.*

(viii) (Hill [29]) *The 56-cap in $PG(5,3)$ is projectively unique.*

(ix) (Edel and Bierbrauer [16]) *There exist at least two types of 41-caps in $PG(4,4)$.*

2.3 Extendability of caps

As for arcs, there are a number of results on the extendability of caps in $PG(3,q)$ to ovoids and, more generally, on the extendability of caps in $PG(N,q)$ to $m_2(N,q)$-caps.

Of particular interest here is the value $m'_2(3,q)$. This is the cardinality of the second largest complete n-cap in $PG(3,q)$.

Table 2.2 displays the values of n for which an n-cap in $PG(N,q)$ is contained in an $m_2(N,q)$-cap. The table is in two parts. The first part shows upper bounds for q even. The second part of the table displays the upper bounds on $m'_2(3,q)$ for q odd, using the following relation between $m'_2(2,q)$ and $m'_2(3,q)$. The technique used by Nagy and Szőnyi to obtain these formulas is based on the following idea.

Suppose $\mathcal{K}$ is an n-cap in $PG(3,q)$, q odd, having a large cardinality n. Then there exist a large number of planes intersecting $\mathcal{K}$ in more than $m'_2(2,q)$ points. Hence, these planes intersect $\mathcal{K}$ in a subset of a conic (Table 1.3).

The main goal is now to put all these conics onto one elliptic quadric. This then implies immediately that the cap $\mathcal{K}$ is contained in an elliptic quadric.

In Table 2.2(ii), for the corresponding value of $m'_2(2,q)$, see Table 1.3.

Theorem 2.2 (Nagy and Szőnyi [46]) *If, for q odd, $m'_2(2,q) \geq (5q+19)/6$, then*

$$m'_2(3,q) < qm'_2(2,q) + \tfrac{3}{4}(q + \tfrac{10}{3} - m'_2(2,q))^2 - q - 1.$$

N	q	$m'_2(N,q)$	$m_2(N,q)$-cap	
3	q even, $q \geq 8$	$\leq q^2 - q + 5$	ovoid	[9]
N	2	$2^{N-1} + 2^{N-3}$	complement of a hyperplane	[15]

Table 2.2(i): Upper bounds for and values of $m'_2(N,q)$

q odd, $q \geq 17$	$m'_2(3,q)$
$q = p^{2e}$, $p > 2,\ e \geq 1$	$< q^2 - q^{3/2}/4 + 39q/64 + 85q^{1/2}/128 + 4153/3072$
$q = p^{2e+1}$, $p > 2,\ e \geq 1$	$< q^2 - p^{3e+2}/4 + 119p^{2e+2}/64 - 87p^{e+2}/128 + 7p^{e+1}/8 + 2523p^2/1024 - 203p/32 + 37/12$
q prime, $q > 2$	$< 2641q^2/2700 - 4q/135 + 94/27$
$q = p^h,\ p \geq 5$	$< q^2 - q^{3/2}/2 + 67q/16 - 5q^{1/2}/4 + 13/12$
$q = p^h,\ p \geq 3,\ q \geq 23^2,\ q \neq 5^5,\ 3^6,$ h even for $p = 3$	$< q^2 - q^{3/2}/2 + 35q/16 + q^{1/2}/4 - 11/12$

Table 2.2(ii): Upper bounds for $m'_2(3,q)$

Among the known results on caps, two results discuss the spectrum for the values n for which a complete n-cap can exist. The first result gives the precise values n for which a complete n-cap in $PG(N,2)$, with $n \geq 2^{N-1}+1$, exists. The latter result for caps in $PG(3,q)$, q even, gives a number of intervals on n for which it is impossible to have a complete n-cap in $PG(3,q)$.

Theorem 2.3 (i) (Davydov and Tombak [15]) *In* $PG(N,2)$, $N \geq 3$, *a complete* n*-cap* $\mathcal{K}$, *with* $n \geq 2^{N-1}+1$, *has order* $n = 2^{N-1}+2^{N-1-g}$ *for some* $g = 0,2,3,\ldots,N-1$. *For each* $g = 0,2,3,\ldots,N-1$, *there exists a complete* $(2^{N-1}+2^{N-1-g})$*-cap in* $PG(N,2)$.

(ii) (Storme and Szőnyi [72]) *There does not exist a complete* n*-cap in* $PG(3,q)$, q *even,* $q \geq 64$, *for which*

$$n \in \left[q^2-(c-1)q+c\sqrt{q}+2-c+\binom{c}{2},\ q^2-(c-2)q-c^2\sqrt{q}\right]$$

where c *is an integer which satisfies*

$$2 \leq c \leq \frac{-2\sqrt{q}+3+\sqrt{16q\sqrt{q}+12q-44\sqrt{q}-7}}{4\sqrt{q}+2} \approx \sqrt[4]{q}.$$

2.4 Caps in $PG(N,q)$, $N > 3$

Remark 2.4 The classical examples of caps are the *conics in* $PG(2,q)$ and the *elliptic quadrics in* $PG(3,q)$. In other words, they are the quadrics which only have points, and no lines.

In dimension greater than three, every quadric contains lines. Hence, for dimensions at least four, it is impossible to select quadrics as the classical examples of caps.

This leads to the fundamental problem:

In dimension greater than three, what are the classical examples of caps?

This problem is one of the most fascinating problems in finite geometry.

The great difficulty of it can easily be seen in the differences between the orders of the upper bounds on the size of a cap (Section 2.5), and the sizes of the cardinalities of the known examples of caps (Section 2.6).

In Tables 2.3(i) and 2.3(ii), the known upper bounds on $m_2(4,q)$ are presented. The entries in Table 2.3 are a consequence of the results in Theorem 3.5.

The upper bound for q odd is based on the following idea. Let $\mathcal{K}$ be an n-cap in $PG(4,q)$, q odd, of large cardinality n. Then many 3-dimensional subspaces must intersect $\mathcal{K}$ in more than $m_2'(3,q)$ points; hence, they intersect $\mathcal{K}$ in a subset of an elliptic quadric. These subsets of elliptic quadrics are put together onto one quadric, thus showing that $\mathcal{K}$ has a *too large* intersection with a quadric.

This implies a corresponding upper bound on the size of a cap in $PG(4,q)$, q odd.

The upper bound for $m_2(4,q)$, q even, is based on arguments which avoid the classification of ovoids in $PG(3,q)$.

Theorem 2.5 (i) (Storme, Thas and Vereecke [76]) *If* $m_2'(3,q) \geq (41q^2+237q-32)/48$ *for* q *odd, then*

$$m_2(4,q) \leq (q+1)(m_2'(3,q)-m_2'(2,q))+m_2'(2,q).$$

(ii) (Hirschfeld and Thas [38]) *If* $m_2'(3,q) \geq q^2 - q - 1$ *for* q *even and* $q \geq 8$, *then*

$$m_2(4,q) \leq (q-1)m_2'(3,q) + q^2 + 2.$$

q	$m_2(4,q)$	
q even, $q \geq 8$	$\leq q^3 - q^2 + 6q - 3$	[9, 38]

Table 2.3(i): Upper bounds for $m_2(4,q)$, q even

q odd	$m_2(4,q)$
$q = p^{2e},\ p > 2,\ e \geq 1,$ $q \geq 49$	$< q^3 - q^{5/2}/4 + 39q^2/64 + 85q^{3/2}/128 +$ $1225q/3072 + 85q^{1/2}/128 + 4153/3072$
$q = p^{2e+1},$ $p > 2,\ e \geq 1,\ q \geq 41$	$< q^3 - p^{e+1}q^2/4 + 119pq^2/64 - 87p^{e+2}q/128 +$ $7p^{e+1}q/8 + 2523p^2q/1024 - 403pq/64 + 25q/12$ $-87p^{e+2}/128 + 7p^{e+1}/8 + 2523p^2/1024 - 203p/32 + 37/12$
q prime, $q \geq 41$	$< 2641q^3/2700 - 79q^2/2700 + 346q/135 + 94/27$
$q = p^h,\ p \geq 5,\ q \geq 41$	$< q^3 - q^{5/2}/2 + 67q^2/16 - 5q^{3/2}/4 + 13q/48 - 5q^{1/2}/4 + 13/12$
$q = p^h,\ p \geq 3,$ $q \geq 23^2,\ q \neq 5^5,\ 3^6,$ h even for $p = 3$	$< q^3 - q^{5/2}/2 + 35q^2/16 + q^{3/2}/4 - 83q/48 + q^{1/2}/4 - 11/12$

Table 2.3(ii): General upper bounds for $m_2(4,q)$, q odd

2.5 Upper bounds on caps in $PG(N,q)$, $N > 4$

The difficulties for obtaining results on caps are also seen in the problem of finding formulas which link values $m_2(N,q)$ for different values of the dimension N. For upper bounds, the best formula is deduced by Hill and relates $m_2(N-1,q)$ to $m_2(N,q)$.

Theorem 2.6 (Hill [29]) *For* $q > 2$ *and* $N > 4$,

$$\begin{aligned} m_2(N,q) &\leq qm_2(N-1,q) - q - 1, \\ m_2(N,q) &\leq q^{N-4}m_2(4,q) - q^{N-4} - 2(q^{N-5} + \cdots + q) - 1. \end{aligned}$$

Inductive application, for q odd, of this formula starting from the formulas for $m_2(4,q)$ (Table 2.3(ii)) gives the results in Table 2.4(ii).

For q even, an alternative argument, which avoids Hill's formula, was used to find the upper bounds on $m_2(N,q)$ [76].

Comparing these upper bounds to the sizes of the known caps, it is believed that these upper bounds are much too large.

$q,\ N$	$m_2(N,q)$	
$q=3$	$m_2(N,3)\le 2\cdot 3^{N-2}+2$	[29]
$q=4$	$m_2(N,4)\le 4^{N-1}-4^{N-2}-2\cdot 4^{N-3}-2\cdot 4^{N-4}/3+5/3$	
q even, $q\ge 8$	$m_2(5,q)<q^4-q^3+5q^2+2q-2$	[76]
q even, $q\ge 8$	$m_2(6,q)<q^5-q^4+5q^3+q^2+2q-2$	[76]
q even, $q\ge 8$, $N>6$	$m_2(N,q)<q^{N-1}-q^{N-2}+5q^{N-3}+q^{N-4}+$ $(q^{N-5}+\cdots+q^2)+2q-2$	[76]

Table 2.4(i): Upper bounds for $m_2(N,q)$

q odd	$m_2(N,q)$
$q=p^{2e},\ p>2,$ $e\ge 1,\ q\ge 49$	$< q^{N-1}-q^{N-3/2}/4+39q^{N-2}/64+85q^{N-5/2}/128$ $+1225q^{N-3}/3072+85q^{N-7/2}/128$ $+1081q^{N-4}/3072-2\theta_{N-5}+1$
$q=p^{2e+1},$ $p>2,\ e\ge 1,\ q\ge 41$	$< q^{N-1}-p^{e+1}q^{N-2}/4+119pq^{N-2}/64$ $-87p^{e+2}q^{N-3}/128+7p^{e+1}q^{N-3}/8+2523p^2q^{N-3}/1024$ $-403pq^{N-3}/64+25q^{N-3}/12-87p^{e+2}q^{N-4}/128$ $+7p^{e+1}q^{N-4}/8+2523p^2q^{N-4}/1024-203pq^{N-4}/32$ $+25q^{N-4}/12-2\theta_{N-5}+1$
q prime, $q\ge 41$	$< 2641q^{N-1}/2700-79q^{N-2}/2700+346q^{N-3}/135$ $+67q^{N-4}/27-2\theta_{N-5}+1$
$q=p^h,\ p\ge 5,\ q\ge 41$	$< q^{N-1}-q^{N-3/2}/2+67q^{N-2}/16-5q^{N-5/2}/4$ $+13q^{N-3}/48-5q^{N-7/2}/4+q^{N-4}/12-2\theta_{N-5}+1$
$q=p^h,\ p\ge 3,\ q\ge 23^2,$ h even for $p=3,\ q\ne 5^5,\ 3^6$	$< q^{N-1}-q^{N-3/2}/2+35q^{N-2}/16+q^{N-5/2}/4$ $-83q^{N-3}/48+q^{N-7/2}/4-23q^{N-4}/12-2\theta_{N-5}+1$

Table 2.4(ii): General upper bounds for $m_2(N,q),\ q$ odd

2.6 Lower bounds on the sizes of caps

Another way for obtaining new results on caps is to construct caps which are as large as possible.

The most important construction techniques are recursive. They use constructions of caps in low-dimensional spaces to construct new and larger caps in higher-dimensional spaces.

We first describe the technique by Segre which has recently been improved by Storme, Thas and Vereecke.

(1) The Segre method. Let $\mathcal{K}$ be a k-cap in $PG(N,q)$. Consider $PG(N+3,q)$ and identify the last three coordinates of a point of $PG(N+3,q)$ with one element of $\mathbf{F}_{q^2}\times\mathbf{F}_q$. Then, for fixed coordinates y of the points of $\mathcal{K}$,

$$\mathcal{K}'=\{(y,x,x^{q+1})||y\in\mathcal{K},x\in\mathbf{F}_{q^2}\}\cup\{(0,\ldots,0,0,1)\}$$

is a (q^2k+1)-cap of $PG(N+3,q)$.

(2) The Storme-Thas-Vereecke method. Let $\mathcal{K}$ be a cap in $PG(N,q)$. Consider $PG(N+3r,q)$, $r \neq 3$, and identify the last $3r$ coordinates of a point of $PG(N+3r,q)$ with an r-tuple of $\mathbf{F}_{q^2} \times \mathbf{F}_q$. Then, for fixed coordinates y of the points of $\mathcal{K}$, the set of points with coordinates

$$\begin{array}{c}(y, y_1, y_1^{q+1}, y_2, y_2^{q+1}, \ldots, y_r, y_r^{q+1}) \\ (0,0,1,x_2,x_2^{q+1},x_3,x_3^{q+1},\ldots,x_r,x_r^{q+1}) \\ (0,x_1,x_1^{q+1},0,1,x_3,x_3^{q+1},\ldots,x_r,x_r^{q+1}) \\ \ddots \\ (0,x_1,x_1^{q+1},x_2,x_2^{q+1},\ldots,x_{r-1},x_{r-1}^{q+1},0,1)\end{array}$$

with $y_1,\ldots,y_r \in \mathbf{F}_{q^2}$, with $x_1,\ldots,x_r \in \mathbf{F}_{q^2}$ where $x_i^{q+1}+x_j^{q+1} \neq -1$ when $i \neq j$, is a cap of $PG(N+3r,q)$.

These two methods lead to the following formulas which give lower bounds on the sizes of caps in $PG(N,q)$.

q	Conditions	$m_2(N,q)$	
q		$m_2(N+3,q) \geq q^2 m_2(N,q)+1$	[68]
q		$m_2(N+3r,q) \geq q^{2r} m_2(N,q) + q^{2r-2}+\cdots+q^2+1$	[68]
q		$m_2(N+6,q) \geq q^4 m_2(N,q)+2q^2$	[76]
q		$m_2(N+9,q) \geq q^6 m_2(N,q)+2q^4+1$	[76]
q	$4 \leq r \leq \sqrt{2q/3}$	$m_2(N+3r,q) \geq q^{2r} m_2(N,q)$ $+r(q^{2r-2}-(r-1)(r-2)(q^3-q)q^{2r-6}/2)$	[76]
q	$4 \leq r < \sqrt{q}$	$m_2(N+3r,q) \geq q^{2r} m_2(N,q) +$ $rq^2(q^2-(q+1))\cdots(q^2-(r-2)(q+1))$	[76]

Table 2.5: General inequalities

We now present lower bounds on $m_2(N,q)$. The lower bounds for arbitrary values of the dimension N follow from an inductive application of the formulas of Table 2.5. To obtain such a lower bound, in general, we start with the cardinality of a known n-cap in a particular projective space of small dimension. These cardinalities are listed in Table 2.6(i).

q	N	$m_2(N,q) \geq$	
q even, $q > 4$	4	$2q^2 + q + 9$	[18]
$q \equiv 1 \pmod 8$	4	$(5q^2 - 2q - 7)/2$	[19]
$3 < q \equiv 3 \pmod 8$	4	$(5q^2 - 8q - 13)/2$	[19]
$q \equiv 5 \pmod 8$	4	$(5q^2 - 6q - 11)/2$	[19]
$q \equiv 7 \pmod 8$	4	$(5q^2 - 4q - 9)/2$	[19]
q odd	5	$q^3 + q^2 + 3q + 3$	[17]
q even	5	$q^3 + 2q^2 + q + 2$	[45]
q	6	$q^4 + 2q^2$	[17]

Table 2.6(i): Lower bounds on $m_2(N,q)$, $N = 4, 5, 6$

In Tables 2.6(ii), 2.6(iii) and 2.6(iv), we use the general inequalities to obtain lower bounds for $m_2(N,q)$ for arbitrary dimensions N [76]. We first discuss dimensions $N = 8$ to $N = 14$. To obtain a formula for $N = 7, 10, 13$ which is valid for all q, we define the coefficients of q^2, q and 1 of the lower bound $m_2(4,q)$ in Table 2.6(i) as a, b, and c.

q	N	$m_2(N,q) \geq$	Condition
$q > 4$	7	$aq^4 + bq^3 + cq^2 + 1$	$m_2(4,q) \geq aq^2 + bq + c$
q odd	8	$q^5 + q^4 + 3q^3 + 3q^2 + 1$	
q even	8	$q^5 + 2q^4 + 2q^3 + 4q^2$	
q	9	$q^6 + 2q^4 + 1$	
$q > 4$	10	$aq^6 + bq^5 + cq^4 + 2q^2$	$m_2(4,q) \geq aq^2 + bq + c$
q odd	11	$q^7 + q^6 + 3q^5 + 3q^4 + 2q^2$	
q even	11	$q^7 + 2q^6 + 2q^5 + 4q^4 + q + 2$	
$q,\ q < 25$	12	$q^8 + 2q^6 + 2q^2$	
$q,\ q \geq 25$	12	$q^8 + 4q^6 - 12q^5 + 12q^3$	
$q > 4$	13	$aq^8 + bq^7 + cq^6 + 2q^4 + 1$	$m_2(4,q) \geq aq^2 + bq + c$
$q = 8, 16$	14	$q^9 + 2q^8 + q^7 + 2q^6 + 2q^4 + 1$	
$q \geq 32$, q even	14	$q^9 + 2q^8 + 4q^7 - 4q^6$ $-24q^5 + 12q^4 + 24q^3$	
q odd, $q < 25$	14	$q^9 + q^8 + 3q^7 + 3q^6 + 2q^4 + 1$	
q odd, $q \geq 25$	14	$q^9 + q^8 + 4q^7 - 8q^6$ $-12q^5 + 12q^4 + 12q^3$	

Table 2.6(ii): Lower bounds on $m_2(N,q)$, N small

Now we present, in the following two tables, lower bounds on $m_2(d,q)$ for arbitrary dimension d (Remark the change in notation for the dimension). In Table 2.6(iv), $R = \left\lfloor \sqrt{2q/3} \right\rfloor$ is equal to the largest integer smaller than or equal to $\sqrt{2q/3}$. Again, to describe easily the formula for dimension $3N + 4$, we assume $m_2(4,q) \geq aq^2 + bq + c$ (See Table 2.6(i)).

q	d	$m_2(d,q) \geq$	
$q=3$	$3N+2$	$(2 \cdot 3^{2N+3} - 3^{2N+1} - 3^{2N} - 3^{2N-2} - 1)/8$	[52]
$q=3$	$3N+3$	$(2 \cdot 3^{2N+3} - 3^{2N+1} - 3^{2N} - 3^{2N-2} - 1)/4$	[52]
$q=3$	$3N+4$	$(2 \cdot 3^{2N+3} - 3^{2N+1} - 3^{2N} - 3^{2N-2} - 1)/2$	[52]
q	$3N$, $4 \leq N \leq \sqrt{2q/3}$	$q^{2N}+$ $N(q^{2N-2} - (N-1)(N-2)(q^3-q)q^{2N-6}/2)$	[76]
q	$3N+4$, $4 \leq N \leq \sqrt{2q/3}$	$aq^{2N+2} + bq^{2N+1} + cq^{2N} +$ $N(q^{2N-2} - (N-1)(N-2)(q^3-q)q^{2N-6}/2)$	[76]
q odd	$3N+5$	$(q+1)m_2(3N+3,q)$	[45, 76]
q even	$3N+5$	$(q+2)m_2(3N+3,q)$	[45, 76]

Table 2.6(iii): Lower bounds on $m_2(d,q)$

q	d	$m_2(d,q) \geq$	
q	$3N$, $N > \sqrt{2q/3}$	$q^{2N} + Rq^{2N-2} - R(R-1)(R-2)(q^3-q)q^{2N-6}/2$ $+q^{2N-2R-2} + q^{2N-2R-4} + \cdots + q^2 + 1$	[76]
q	$3N+4$, $N > \sqrt{2q/3}$	$aq^{2N+2} + bq^{2N+1} + cq^{2N} +$ $Rq^{2N-2} - R(R-1)(R-2)(q^3-q)q^{2N-6}/2$ $+q^{2N-2R-2} + \cdots + q^2 + 1$	[76]
q odd	$3N+5$, $N > \sqrt{2q/3}$	$(q+1)m_2(3N+3,q)$	[45, 76]
q even	$3N+5$, $N > \sqrt{2q/3}$	$(q+2)m_2(3N+3,q)$	[45, 76]

Table 2.6(iv): Lower bounds on $m_2(d,q)$

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