A SHORTENED INTRODUCTION TO DIAGRAM GEOMETRY
with an Appendix on
FLAG-TRANSITIVE EXTENSIONS OF FINITE POLAR SPACES

Notes for lectures 1, 2 and 5 of the course
Extensions of Geometries

A. Pasini

I will only collect a number of basic definitions and some general results on diagram geometries, taken from my book [42]. I am not going to give proofs or examples here. You can find them in [42]. Some examples will also be described in my lectures, of course.

As for the subject of my third lecture (lecture 5), I'll only give a sketchy report on a subset of a wider topic. I refer you to Chapter 22 of the Handbook of Incidence Geometry [9] for a glance to the whole of that topic.

In Section 8 of these notes I'll say a very few words on Mőbius, Laguerre and Minkowski planes and on special Laguerre planes. For that topic, I refer you to Section 5 of the chapter by A. Delantsheer [18] in the Handbook of Incidence Geometry. More readings will be also be suggested in Section 8.

1 Definitions

Most of the definitions and results to be stated in this section are taken from [42, Chapter 1]. According to [42],

Definition 1.

- A *geometry of rank* 1 is a set (discrete graph) of size ≥ 2.
- A *geometry of rank* $n > 1$ is a connected graph $\Gamma = (V, *)$ (needless to say, $*$ is the adjacency relation of the graph), equipped with an n-partition Θ such that, for

every vertex x, the neighbourhood of x in Γ, with the $n-1$-partition induced by Θ, is a geometry of rank $n-1$.

(In particular, a geometry of rank 2 is just a connected bi-partite graph where every vertex is adjacent to at least two vertices.) Given a geometry (Γ, Θ), with Γ and Θ as above, we call Γ the *incidence graph* of (Γ, Θ). The reflexive closure of the adjacency relation $*$ is the *incidence relation* of (Γ, Θ). We shall denote it too by $*$.

1.1 Types

The following is proved in [42, Chapter 1]:

Proposition 1 *A graph Γ admits at most one partition Θ such that (Γ, Θ) is a geometry.*

That is, the partition Θ, if it exists, is implicit in the structure of the graph Γ. Accordingly, I'll write Γ for (Γ, Θ), omitting to mention Θ.

Let $\Gamma = (V, *)$ be a geometry of rank n and let Θ be its unique n-partition with the above properties. We call Θ the *type-partition* of Γ. Its classes are the *types* of Γ, the *type* of a vertex x being the class $[x]\Theta$ of Θ containing it. Normally, a set I of indices (names, such as integers, words, or whatelse) is chosen for the types. That is, a bijection is given from V/Θ to I and a function $\tau : V \rightarrow I$ is defined, sending every $x \in V$ to the element $\tau(x)$ of I corresponding to $[x]\Theta$. That function is called the *type-function* of Γ and the above stated terminology is changed as follows: the elements of I are called *types*, $\tau(x)$ being the *type* of x. The geometry Γ is now viewed as a triple $(V, *, \tau)$ (needless to say, $(V, *)$ and τ are the incidence graph and the type function of Γ).

1.2 More terminology (flag, chambers, residues,...)

Elements, flags, chambers and panels. Let $\Gamma = (V, *, \tau)$ be a geometry of rank n, over the set of types I. The elements of V (vertices of the graph $(V, *)$) are the *elements* of Γ.

A *flag* of Γ is a set of pairwise incident elements. (Warning: $\emptyset$ and singletons are also flags.) The *rank* (*corank*) of a flag F is the integer $|F|$ (resp. $n - |F|$). The *type* (*cotype*) of F is $\tau(F)$ (resp. $I \setminus \tau(F)$).

A *chamber* is a flag of maximal size (i.e., of size n). A *panel* is a flag of corank 1. Two distinct chambers C, D are said to be (i-)*adjacent* if $C \cap D$ is a panel (of cotype i). The set of chambers of Γ, equipped with the i-adjacency relations (for all $i \in I$) is a graph with coloured edges (I is the set of colours), called the *chamber system* of Γ.

The following is also proved in [42, Chapter 1].

Proposition 2 *Every maximal flag is a chamber.*

Residues. The *residue* of a non-maximal flag F is the graph induced by $(V,*)$ on the common neighbourhood of the elements of F, with the type-partition induced from the type-partition of Γ. It is clear by Definition 1 that the residue of F is a geometry of rank $n-|F|$ (corank of F).

I'll write $Res_\Gamma(F)$ (also $Res(F)$) to denote the residue of F. (I warn the reader that the notation Γ_F is used in [42] instead of $Res(F)$ or $Res_\Gamma(F)$.) The cotype of F is taken as the set of types of $Res(F)$ and it is called the *type* of $Res(F)$. The type-function of $Res(F)$ is the restriction of τ to $Res(F)$, viewed as a function to $I \setminus \tau(F)$.

1.3 Truncations and Strong Connectedness

Given a geometry $\Gamma=(V,*,\tau)$ of rank $n>1$ over the set of types I and a nonempty proper subset J of I, I denote by $Tr_J(\Gamma)$ the structure induced by Γ on the preimage $\tau^{-1}(J)$ of J and I call it the *J–truncation* of Γ. (This structure is denoted by $Tr_J^+(\Gamma)$ and called J_+-truncation in [42].) The following is also proved in [42, Chapter 1]:

Proposition 3 *The structure $Tr_J(\Gamma)$ is a geometry of rank $|J|$ (over the set of types J).*

The special case of the above with J of size 2 is the so-called *Strong Connectedness Property.*

1.4 Thickness, thinness and orders

Given a geometry Γ over the set of types I and a type $i \in I$, we say that Γ is *thick* (*thin*) at i if all panels of Γ of cotype i are contained in at least three (just two) chambers. If Γ is thick (thin) at all types, then we say that Γ is *thick* (*thin*).

If all panels of Γ of cotype i are contained in exactly $q+1$ chambers (for some cardinal number q), then we say that Γ has *order* q at i. (Needless to say, $q+1=q$ when q is infinite; also, Γ is thin at i iff it has order 1 at i.)

1.5 Automorphisms, correlations and flag-transitivity

As the type-partition of a geometry is uniquely determined by its incidence graph, given a geometry $\Gamma=(V,*,\tau)$ over the set of types I, every automorphism g of the graph $(V,*)$ induces a permutation of the set of fibers of τ. Namely, $\tau g = g_{typ}\tau$ for a unique permutation g_{typ} of I. Following Buekenhout [7], I say that g is an *automorphism* of Γ if g_{typ} is the identity permutation, otherwise I call g a *correlation.* I denote the automorphism group of Γ by $Aut(\Gamma)$.

A subgroup G of $Aut(\Gamma)$ is said to be *flag-transitive* if it is transitive on the set of chambers of Γ (whence, also on the set of flags of type J, for any $J \subseteq I$). If $Aut(\Gamma)$ is flag-transitive, then we say that Γ is *flag-transitive.*

Note. The above terminology is different from that of [42]. All automorphisms of $(V, *)$ are called automorphisms of Γ in [42], the automorphisms in the above sense being called *special* (also *type preserving*) automorphisms. Accordingly, the symbol $Aut_S(\Gamma)$ is used in [42] instead of $Aut(\Gamma)$, where $Aut(\Gamma)$ stands for $Aut((V, *))$ in [42].

Isomorphisms. Given two geometries $\Gamma_1 = (V_1, *, \tau_1)$ and $\Gamma_2 = (V_2, *, \tau_2)$ over sets of types I_1 and I_2 respectively, an *isomorphism* from Γ_1 to Γ_2 is just an isomorphism from the graph $(V_1, *)$ to the graph $(V_2, *)$. As above, given an isomorphism g from Γ_1 to Γ_2, a (unique) bijection g_{typ} from I_1 to I_2 exists such that $\tau_2 g = g_{typ}\tau_1$. (Note that, according to the previous definition of automorphism, an isomorphism from a geometry to itself need not be an automorphism: it might be a correlation as well.)

1.6 Comparison with other definitions of geometries

Definition 1 is stronger than other definitions that can be found in the literature. For example, Buekenhout [7] defines a geometry over a (finite) set of types I as a triple $(V, *, \tau)$ where $(V, *)$ is graph and τ is a surjective mapping from V to I such that:

- the fibers of τ form a partition of the graph $(V, *)$ and
- all maximal fags are chambers (flags and chambers being defined as I have done above).

The geometries meeting the requirements of Definition 1 are called *firm* and *residually connected* in [7].

A definition even weaker than the above is chosen by Tits [57], who only asks the first one of the above two condition for a triple $(V, *, \tau)$ to be a geometry.

2 Geometries of rank 2

Reference: [42, Chapter 1]; also Buekenhout [7] and [9, Section 1]

2.1 Gonality and Diameters

I recall that the girth of graph is the minimal length of a closed path with at least three vertices and no repeated edges, if such a path exists (namely, if the graph is not a tree); trees are said to have girth ∞. Clearly, the girth of a bipartite graph, if it is not ∞, is an even integer ≥ 4. In particular, the girth of a geometry $\Gamma = (V, *)$ of rank 2, is either an even integer ≥ 4 or ∞. The *gonality* of Γ is half of its girth (with the convention that $\infty/2 = \infty$).

Let $0, 1$ be the two types of Γ. For $i = 0, 1$, the maximal distance d_i of an element of type i from the remaining elements of Γ is called the *i-diameter* of Γ. The (*incidence*) *diameter* of Γ is the maximum of d_0 and d_1.

Note that the gonality of Γ is not greater than the diameter of Γ (but it might be much smaller than that). Furthermore, $|d_1 - d_0| \leq 1$.

2.2 The point-line language

Normally, given a geometry Γ of rank 2, the elements of one of the two types of Γ (no matter which) are called *points*, those of the other type being called *lines* (also *blocks*, according to the context). Instead of saying that a point a and a line l are incident, we shall say that a is a point of l, or that it *belongs* to l, also that l *passes through* a, and so on.

Chosen the elements of one type as points and the other ones as lines, we may define a *collinearity relation*, declaring two distinct points to be *collinear* when they are incident with a common line. We write $a \perp b$ to mean that two distinct points a, b are collinear. The graph with the points as vertices and $\perp$ as the adjacency relation is called the *collinearity graph* of Γ.

Given a point a, we denote by $a^{\perp}$ the set of points collinear with a, or equal to a. The *distance* in the collinearity graph of a point a from a line l is the minimal distance in that graph from a to points of l. The *distance* of two lines l, m in the collinearity graph is the minimal distance of a point of l from a point of m.

Needless to say, when the point-line look is chosen, it is understood that words such as "distance", "diameter" etc. refer to the collinearity graph.

2.3 Some classes of rank 2 geometries

Given a pair of types, say 0 and 1, and a geometry Γ over $\{0, 1\}$, let g, d_0 and d_1 be the gonality, the 0-diameter and the 1-diameter of Γ. Then:

Definition 2.

- If $g = d_0 = d_1 = m$ for some integer $m \geq 2$, then Γ is called a *generalized m-gon.*
- If $g = d_0 = 3$ and $d_1 = 4$, then Γ is said to be a *linear space* or a *dual linear space*, according to whether we take the elements of type 0 or those of type 1 as points.
- If $g > 2$, then Γ is said to be a *partial linear space.*

Generalized 2-gons, 3-gons, 4-gons ... are also called generalized *digons, triangles, quadrangles,* and so on. Clearly, ordinary m-gons are just thin generalized m-gons.

Note also that all generalized m-gons with $m > 2$ are partial linear spaces.

The above classes of geometries can also be characterized in the point-line language, as follows. Given Γ as above, assume we have taken the elements of type 0 as points, calling lines those of type 1. According to the above, in the sequel the word "distance" will stand for 'distance in the collinearity graph'.

Proposition 4 *Let $m = 2k+1$. Then Γ is a generalized m-gon if and only if the following hold:*

(i) the collinearity graph has girth m and diameter k;

(ii) every point has distance $\leq k$ from any line and k from some lines;

(iii) any two lines have distance $< k$.

In particular,

Corollary 5 *A generalized triangle is just a (possibly degenerate) projective plane.*

Turning to generalized m-gons with m even, we have the following:

Proposition 6 *Let $m = 2k$, $k > 1$. Then Γ is a generalized m-gon if and only if the following hold:*

(i) no two distinct lines meet in two or more points;

(ii) the collinearity graph has diameter k and every point has distance k from some points;

(iii) given a point a and a line l, there is a unique point of l at minimal distance from a.

Proposition 7 *The geometry Γ is a generalized 2-gon if and only if all its points are incident to all lines.*

Proposition 8 *The geometry Γ is a (dual) linear space if and only if any two points (lines) of Γ are joined by (meet in) a unique line (point).*

Proposition 9 *The geometry Γ is a partial linear space if and only if no two distinct lines of Γ meet in two or more points.*

In particular, the lines of a partial linear space can be viewed just as sets of points.

2.4 More classes of rank 2 geometries

To finish this section, I will describe three more classes of rank 2 geometries which will be mentioned in some of the lectures of this course.

Near n-gons. Given an integer $n > 1$, a partial linear space is called a *near n-gon* if it has diameter n and, given any point a and any line l, just one point of l has minimal distance from l. (Notice that generalized $2n$-gons are near n-gons.)

Nets. A *net* is a partial linear space with an equivalence relation $\|$ (called a *parallelism*) defined on the set of lines such that:

- every point is incident to precisely one line in every class of $\|$;
- two lines meet if and only if they do not correspond in $\|$.

For instance, affine planes are nets. Any geometry obtained by removing one or more classess of parallel lines from a net is still a net, provided we keep at least two such classes. The nets with just two parallel classes are the *grids*, namely the generalized quadrangles with two lines on each point.

Designs. Given positive integers t, v, k, λ with $t \leq k \leq v$, a t-(v, k, λ) *design* is a pair $(S, \mathcal{B})$ where S (called the set of *points*) is a set of size v and $\mathcal{B}$ a family of distinguished subsets of S of size k, called *blocks* and such that any set of k distinct points of S belongs to precisely λ blocks.

Note. The inequalities above assumed on t, k, v are only a few of the condition that a quadruple (t, v, k, λ) should satify for a t-(v, k, λ) design might exist, but I am not going to insist on this here.)

Not every design is a geometry of rank 2 in the meaning of Definition 1. For instance: a 1-$(v, k, 1)$ design is not a geometry. Nevertheless, many designs are geometries. For instance, a 2-$(v, k, 1)$ design with $k < v$ is a linear space. A 1-(v, v, λ) design with $v, \lambda > 1$ is a generalized digon. The point-plane system of $PG(3, q)$ is a 1-(v, k, q^2+q+1) design, a 2-$(v, k, q+1)$ design and also a 3-$(v, k, 1)$ design, for $v = q^3+q^2+q+1$ and $k = q^2+q+1$.

3 Diagrams

Reference: [42, Chapter 3].

3.1 Definitions and generalities

Given a finite set I of size ≥ 2, for any choice of distinct elements i, j of I, let $\mathcal{D}_{i,j}$ be a class of rank 2 geometries with $\{i, j\}$ as the set of types. Assume that, if $\mathcal{D}_{i,j}$ is not the class of generalized digons, then no member of $\mathcal{D}_{i,j}$ is a generalized digon. Then the family

$$\mathcal{D} := \ (\mathcal{D}_{i,j})_{i,j \in I, i \neq j}$$

is called a *diagram over the set of types I*. A geometry Γ over the set of types I is said to *belong to $\mathcal{D}$* if, for any two distinct types $i, j \in I$, all residues of Γ of type $\{i, j\}$ are members of $\mathcal{D}_{i,j}$. We say that a geometry *admits a diagram* if it belongs to some diagram, namely if, for any two distinct types i, j, either no or all residues of Γ of type $\{i, j\}$ are generalized digons.

Definition 3 ('re-definition'). A *diagram geometry* (a *geometry*, for short) is a geometry in the meaning of Definition 1, admitting a diagram.

Given a diagram $\mathcal{D} = (\mathcal{D}_{i,j})_{i,j \in I, i \neq j}$ and a subset J of I of size at least 2, let $\mathcal{D}_J$ be the following subfamily of $\mathcal{D}$:

$$\mathcal{D}_J := \ (\mathcal{D}_{i,j})_{i,j \in J, i \neq j}$$

We call $\mathcal{D}_J$ the *subdiagram induced* by $\mathcal{D}$ on J. The following is obvious:

Proposition 10 *For any geometry Γ belonging to $\mathcal{D}$, all residues of Γ of type J belong to $\mathcal{D}_J$.*

3.2 How to depict diagrams

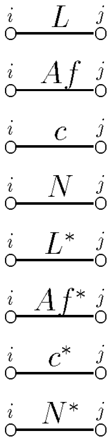
Diagrams are normally drawn as labelled graphs, with the types as vertices. Two types i, j are joined by an edge if and only if $\mathcal{D}_{i,j}$ is not the class of generalized digons, in which case the edge $\{i, j\}$ is given a label denoting the class $\mathcal{D}_{i,j}$.

With this convention, given a diagram $\mathcal{D}$ over the set of types I and a subset J of I of size at least 2, we get the sudiagram $\mathcal{D}_J$ just by deleting the nodes of $I \setminus J$ and the edges of $\mathcal{D}$ that are not contained in J.

Here is a selection of the most popular conventions for edges of diagrams. I recall that 'no edge' stands for 'generalized digons'.

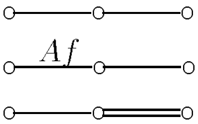
○——○	(no label)	projective planes (possibly degen.)
○══○	(double stroke)	generalized quadrangles
○—m—○	$m \geq 5$	generalized m-gons

In the following pictures I always assume that the elements of type i are taken as points and those of type j as lines. The third picture represents the class of *circular spaces*. These are linear spaces with all lines of size 2. (Namely, a circuar space is a complete graph, where the vertices and the edges are taken as points and lines, respectively.)

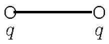


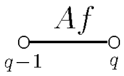
linear spaces
affine planes
circular spaces
nets
dual linear spaces
dual affine planes
dual circular spaces
dual nets

Thus, the following diagrams represent 3-dimensional projective geometries, 3-dimensional affine geometries and polar spaces of rank 3, respectively.

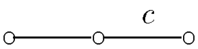


The nodes of a diagram can also be labelled by integers representing orders. When doing so, people normally write types above and orders below the corresponding nodes, if possible. For instance, these are the pictures for $PG(2, q)$ and $AG(2, q)$, respectively:

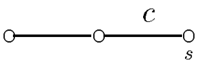


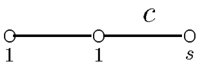


Note that an order is sometimes implicit in the label put above the stroke. In that case, we may omit it. For instance, it is clear from the following diagram that 1, 1 are the orders to put below the first two nodes:



Thus, the following two pictures give us just the same information:





3.3 The basic diagram and the direct sum theorem

Reference: [42, Chapter 4].

The Basic Diagram. Given a diagram $\mathcal{D}$, its *basic diagram* (Buekenhout [7]) is the graph obtained from it by cancelling the labels of the strokes. (That graph is called *diagram graph* in [42].) Accordingly, given a diagram geometry Γ over the set of types I, the *basic diagram* $\Delta(\Gamma)$ of Γ is the graph with I as the set of vertices, two distinct types $i, j \in I$ being joined in $\Delta(\Gamma)$ iff the residues of Γ of type $\{i, j\}$ are not generalized digons. (According to Definition 3, it makes no difference if we read the above either as 'no residue ... of type $\{i, j\}$ is a gen. digon' or as 'some residue ... of type $\{i, j\}$ is not a gen. digon'.)

Direct Sums. Given two geometries $\Gamma_1 = (V_1, *)$ and $\Gamma_2 = (V_2, *)$ with disjoint sets of elements (namely, $V_1 \cap V_2 = \emptyset$) the *direct sum* $\Gamma_1 \oplus \Gamma_2$ of Γ_1 and Γ_2 is the geometry with $V_1 \cup V_2$ as the set of elements and the incidence relation defined as follows:

- all elements of V_1 are incident to all elements of V_2;
- for $i = 1, 2$, two elements of V_i are incident in $\Gamma_1 \oplus \Gamma_2$ iff they are incident in Γ_i.

Theorem 11 (Direct Sum Theorem) *Let Γ be a diagram geometry over the set of types I and let $\{J, K\}$ be a partition of I. Then the following are equivalent* (*see* [42, Chapter 4])*:*

(i) the sets I and J are joins of connected components of the basic diagram $\Delta(\Gamma)$ of Γ;

(ii) we have $\Gamma = Tr_J(\Gamma) \oplus Tr_K(\Gamma)$;

(iii) given any two flags F, G of Γ of types I and J respectively, $\Gamma = Res(F) \oplus Res(G)$.

4 String Diagrams, Points, Lines and Shadows

The material of this section is taken mainly from [42, Chapter 4].

4.1 Definitions

We say that a geometry $\Gamma = (V, *, \tau)$ of rank $n \geq 2$ belongs to a *string diagram* if its basic diagram $\Delta(\Gamma)$ is a string:

$$\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\circ} \;\cdots\; \overset{n-2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{n-1}{\circ}$$

Needless to say, the integers $0, 1, 2, ..., n-1$ are the types. (Note that, with this choice of types, we implicitly give $\Delta(\Gamma)$ an orientation, from left to right.)

We call the elements of Γ of type 0, 1 and 2 *points*, *lines* and *planes*, respectively. The truncation $Tr_{0,1}(\Gamma)$ is called the *point-line system* of Γ. When I say that two points a, b are *collinear* in Γ, I mean that they are collinear in $Tr_{0,1}(\Gamma)$. The collinearity graph of

$Tr_{0,1}(\Gamma)$ is called the *collinearity graph* of Γ. The *diameter* of Γ is the diameter of the collinearity graph of Γ.

For every element x of Γ, the set of points incident to x will be denoted by $\sigma(x)$ and called the *shadow* of x. (Clearly, if x is a point then $\sigma(x) = x$.) The following is an easy consequence of the Direct Sum Theorem:

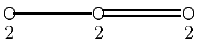
Corollary 12 *If $x * y$ and $\tau(x) \leq \tau(y)$, then $\sigma(x) \subseteq \sigma(y)$.*

Note that the "if and only if" version of the above does not hold in general. Accordingly, the equality $\sigma(x) = \sigma(y)$ does not force $x = y$. For instance, some geometries with string diagram are 'flat', in the following sense:

Definition 4. A geometry Γ of rank $n \geq 3$ is said to be *flat* if all elements of Γ of type $< n-2$ are incident to all elements of type $n-1$. (In particular, when $n = 3$: all points are incident to all planes.)

4.2 A famous flat geometry

It is well known that 30 models of $PG(2,2)$ can be drawn over a given set S of size 7. The symmetric group S_7 transitively permutes those models but the alternating group A_7 doesn't. In fact, A_7 has two orbits of size 15 on that set of 30 models. Let $\mathcal{P}$ be one of them, no matter which. Calling the members of $\mathcal{P}$ *planes*, let us take S as the set of *points* and all 3-subsets of S as *lines*. We declare a point p (a plane π) and a line L to be incident when $p \in L$ (when L is one of the seven lines of π). All points are declared to be incident to all planes. Thus, we get a (flag-transitive) geometry with A_7 as the automorphism group and diagram and orders as follows



Clearly, this geometry is flat.

4.3 The Intersection Property and the LL property

I will only state these properties for geometries with string diagrams, referring to [42, Chapter 6] for the general case.

Definition 5. A geometry $\Gamma = (V, *, \tau)$ with string diagram is said to satisfy the *Intersection Property* (IP, for short), if the following hold:

- for any two elements x, y of Γ, we have $\sigma(x) \subseteq \sigma(y)$ iff $x * y$ and $\tau(x) \leq \tau(y)$;

- for any two elements x, y of Γ, if $\sigma(x) \cap \sigma(y) \neq \emptyset$, then $\sigma(x) \cap \sigma(y) = \sigma(z)$ for some element z (furthermore, z is uniquely determined and we have $x * z * y$ and $\tau(z) \leq \tau(x), \tau(y)$, by the previous condition).

In particular,

Proposition 13 *The geometries of rank 2 other than generalized digons and satisfying IP are precisely the partial linear spaces.*

Proposition 14 *Let Γ be a geometry of rank n with string diagram and assume that Γ satisfies IP. Then IP also holds in residues of points of Γ, in residues of point-line flags of Γ and so on. Furthermore, IP also holds in the dual of Γ (namely, in the geometry with the elements of Γ of type $n-1$, $n-2$, $n-3$ taken as point, lines and planes, respectively).*

(See [42, Chapter 6] for the proof.) The following is easy to prove:

Proposition 15 *Let Γ be a geometry of rank n with string diagram and assume that Γ satisfies IP. Then IP also holds in the $\{0, 1, ..., m-1\}$-truncation of Γ, for any $m = 2, 3, ..., n-1$.*

In, particular, with $m = 2$, the Intersection Property implies that $Tr_{0,1}(\Gamma)$ is a linear space. Namely,

(LL) no two lines of Γ meet in more than one point.

(LL is the name the above properties often gets in the literature.) In the rank 3 case, IP is equivalent to the following four properties

(1) the LL property holds (whence, residues of planes are partial linear spaces);

(2) residues of points are partial linear spaces;

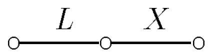
(3) if a line meets a plane in two distinct points, then that line is incident to that plane;

(4) if two planes u, v meet in two distinct points a, b, then $a \perp b$ (and if l is the line joining a with b, unique by (1), then $u * l * v$ by (3), whence $\sigma(l) = \sigma(u) \cap \sigma(v)$, by the above and by (2)).

Needless to say, some geometries might satisfy only some of the above four properties. For instance, (2) and (4) (but neither (1) nor (3)) hold in the previously described flat geometry for A_7 (whereas (1) and (2) but neither (3) nor (4) hold in its dual).

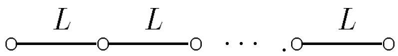
I also notice that in some cases properties weaker than IP (like LL, for instance) are sufficent to force IP to hold. In other cases, IP follows from the diagram. The next two propositions are two such examples (see [42, Chapter 7] for the proofs):

Proposition 16 *The* LL *property is equivalent to* IP *for geometries belonging to diagrams of the following form*

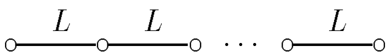


(*where* X *stands for a class of partial planes, no matter which*).

Proposition 17 *The property* IP *holds in any geometry belonging to the following diagram*



Corollary 18 *The geometries of rank* n *belonging to the following diagram*



are precisely the n*-dimensional matroids*

(Buekenhout [6]; also [42, Chapter 7].)

4.4 The T property

A geometry Γ with string diagram is said to satisfy the *Triangular Property* (T-*property*, for short) if any triple of mutually collinear points of Γ is in some plane.

If IP also holds, then the T property can be restated as follows: given any three mutually collinear points not in the same line, there is a unique plane incident to the three of them and that plane is also incident to the three lines joining those points two-by-two.

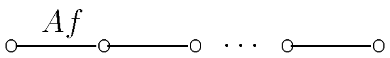
5 Recovering geometries from diagrams

Diagrams are sometimes sufficient to characterize important classes of geometries. Corollary 18 is an example of this kind. The next two results (Buekenhout [6]) are easy consequences of that corollary, via well known characterizations of projective and affine geometries as matroids:

Theorem 19 *All geometries belonging to the following diagram*

are (*possibly degenerate*) *projective geometries.*

Theorem 20 *All geometries belonging to the following diagram*



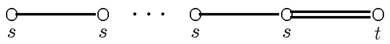
are affine geometries.

In many cases, we need to add some hypotheses to the diagram in order to catch a particular known class of geometries. I mention two results of this kind in the sequel. The first one is due to Tits [57] (see also [42, Chapter 7]). The second one is due to Yoshiara [63].

Theorem 21 *A geometry belonging to the following diagram*

is a (possibly non-thick) polar space if and only if it satisfies the Intersection Property.

Theorem 22 *Let* Γ *be a flag-transitive gemetry with diagram and orders as follows*



where $s, t < \infty$ *and* $s > 1$. *Then* Γ *is either a polar space or the flat geometry of rank 3 for* A_7 *described in Subsection* 4.2.

(More theorems like this will be discussed in Lectures 3-6 of this course.)

The above theorem of Yoshiara ends a long work originated by a result of Aschbacher [1]. The reader may see [42, Chapters 14 and 15] for an account of the intermediate steps of that work (but he will not find the above result in [42], since Yoshiara obtained it one year after the publication of [42].)

6 Coset Geometries and Parabolic Systems

I refer to chapter 10 of [42] for the contents of this section. To make things easier, I will only consider the rank 3 case, adding a very few words on the rank 2 case.

6.1 Minimal and maximal parabolics

Let Γ be a flag-transitive geometry of rank 3 over the set of type $\{0, 1, 2\}$ and let G be a flag-transitive subgroup of $Aut(\Gamma)$.

Stabilizers, kernels and actions. Given an element x (a panel F) of Γ, let G_x (resp. G_F) be its stabilizer in G. We denote by K_x (by K_F) the elementwise stabilizer of $Res(x)$ (of $Res(F)$). Thus, the quotient $\overline{G}_x = G_x/K_x$ (resp. $\overline{G}_F = G_F/K_F$) is the group induced by G_x on $Res(x)$ (by G_F on $Res(F)$). By a little abuse, I call that group the *action* of G_x on $Res(x)$ (of G_F on $Res(F)$), the group K_x (resp. K_F) being the *kernel* of that action.

Parabolic and Borel Subgroup. Given a chamber $C = (c_0, c_1, c_2)$, with c_i of type i, let $B = G_C$ be the stabilizer of C in G. We call it the *Borel subgroup* of G, relative to the choice of C as the *fundamental chamber*.

For $\{i,j,k\} = \{0,1,2\}$ we set $P_i = G_{c_j,c_k}$ (stabilizer of the panel $\{c_j, c_k\}$ of C) and $G_i = G_{c_i}$ (stabilizer of the element c_i of C); also, $K_i = K_{c_i}$.

We call P_i (resp., G_i) the *minimal* (*maximal*) *parabolic subgroup* of G of type i, relative to the choice of C as the fundamental chamber. Clearly:

(1) $G = \langle P_0, P_1, P_2 \rangle$
(2) $B = P_i \cap P_j$ for $0 \leq i < j \leq 2$
(3) $|B : P_i| \geq 2$ for $i = 0, 1, 2$
(4) $G_i = \langle P_j, P_k \rangle$ for $\{i,j,k\} = \{0,1,2\}$
(5) $P_i = G_j \cap G_k$ for $\{i,j,k\} = \{0,1,2\}$
(6) $P_i \cap P_j P_k$ for $\{i,j,k\} = \{0,1,2\}$
(7) $\cap_{g \in G} B^g = 1$

Recovering Γ from B, P_0, P_1, P_2. Given a chamber $X = \{x_0, x_1, x_2\}$ (with x_i of type i), the set of elements of G mapping C onto X is a right coset gB of B, with g any element such that $g(C) = X$. Similarly, for $\{i,j,k\} = \{0,1,2\}$, the set of elements of G sending c_i (resp. $\{c_j, c_k\}$) to x_i (to $\{x_j, x_k\}$) is the right coset gG_i (resp. gP_i) of G_i (of P_i). Thus, the chambers of Γ correspond to the right cosets of B and the elements of Γ of type i correspond to the right cosets of the maximal parabolic G_i. Two cosets gG_i, fG_j (resp. gB, fB) correspond to incident elements (to adjacent chambers) if and only if they are not disjoint (they are contained in the same right coset of a minimal parabolic). Thus, we have constructed a model of Γ by cosets of certain subgroups of G.

6.2 Coset geometries

Conversely, given a group G, let B, P_0, P_1, P_2 be subgroups of G satisfying the properties (1)-(3) and (5)-(7) of the previous subsection, with G_j, G_k in (5) defined as in (4). We call the triple (P_0, P_1, P_2) a *parabolic system* (of *rank* 3) in G. We define a geometry $\Gamma(G, P_0, P_1, P_2)$ of rank 3, as follows:

- for $i = 0, 1, 2$, the right cosets of G_i are the elements of type i;

- for $i, j = 0, 1, 2$ and $f, g \in G$, the elements gG_i and fG_j are declared to be incident if $gG_i \cap fG_j \neq \emptyset$.

I refer the reader to [42, Chapter 10] for the proof that $\Gamma(B, P_0, P_1, P_2)$ is indeed a geometry. I call it the *coset geometry* defined by the parabolic system (B, P_0, P_1, P_2). Needless to say, the chambers (panels) of $\Gamma(G, P_0, P_1, P_2)$ bijectively correspond to the right cosets of B (of the minimal parabolics P_0, P_1, P_2). Furthermore, G acts flag-transitively on $\Gamma(G, P_0, P_1, P_2)$, by left multiplication.

6.3 The rank 2 case

In the rank 2 case, maximal and minimal parabolics are the same and things are easier. More precisely, if $0, 1$ are the type of Γ, then $G_i = P_j$ for $\{i, j\} = \{0, 1\}$ and some of the properties (1)-(7) of Subsection 6.1 trivialize or 'vanish':

(1) becomes:	$G = \langle P_0, P_1 \rangle$	
(2) becomes:	$B = P_0 \cap P_1$	
(3) becomes:	$\lvert B : P_i \rvert \geq 2$	for $i = 0, 1$
(4) becomes:	$G_i = P_j$	for $\{i, j\} = \{0, 1\}$ (as said above)
(5) vanishes		
(6) vanishes		
(7)	$\cap_{g \in G} B^g = 1$	(as in the rank 3 case)

Many properties of Γ (gonality, diameters,...) can be translated in properties of the pair (P_0, P_1) (see [42, Chapter 10]). For instance, the following characterizes generalized digons: $P_0 P_1 = P_1 P_0$.

7 Covers, Quotients and Universal Covers

I refer to [42, Chapters 8 and 11] for this section.

7.1 Covers and quotients

Let Γ_1 and Γ_2 be geometries of rank $n > 1$ over the same set of types I and let m be a positive integer with $1 \leq m < n$. Let $\varphi : \Gamma_1 \to \Gamma_2$ be an incidence- and type-preserving mapping from the set of elements of Γ_1 to the set of elements of Γ_2. We say that φ is an *m-covering* if, for every flag F of Γ_1 of corank m, the morphism φ induces an isomorphism from $Res_{\Gamma_1}(F)$ to $Res_{\Gamma_2}(\varphi(F))$.

Clearly, every m-covering is also an m'-covering for any $m' = 1, 2, ..., m-1$.

It is proved in [42, Chapter 8] that all m-covers are surjective morphism. Actually, more than that is proved, namely the following:

Proposition 23 *If φ is an m-covering from Γ_1 to Γ_2, then, for every subset J of I, φ maps the set of flags of Γ_1 of type J onto the set of flags of Γ_2 of the same type.*

If there is an m-covering from Γ_1 to Γ_2 then we say that Γ_1 (resp. Γ_2) is an *m-cover* of Γ_2 (an *m-quotient* of Γ_1).

An m-covering $\varphi : \Gamma_1 \to \Gamma_2$ is said to be a *t-fold m-cover* (and Γ_2 is said to be a *t-fold m-quotient* of Γ_1) if $|\varphi^{-1}(C)| = t$ for any chamber C of Γ_2. When $t = 2$ or 3, the expressions *double m-covering* and *triple m-covering* are normally used instead of "2-fold m-covering" and "3-fold m-covering".

Let $\varphi : \Gamma_1 \to \Gamma_2$ be an m-covering. The *deck group* $D(\varphi)$ of φ is the group of all automorphisms f of Γ_1 such that $\varphi = \varphi f$. It is easy to see that $D(\varphi)$ acts semiregularly on the set of chambers of Γ_1. Clearly, $D(\varphi)$ is normal in $Aut(\Gamma_1)$. The following is also not difficult to prove:

Proposition 24 *If, for every chamber C of Γ_2, the group $D(\varphi)$ is transitive (whence, regular) on the set of chambers of $\varphi^{-1}(C)$, then, for every flag F of Γ_2, $D(\varphi)$ is also transitive on the set of flags of $\varphi^{-1}(F)$.*

If the hypotheses of the above proposition are satisfied, then we say that the quotient Γ_2 is *defined by* $D(\varphi)$ and we write $\Gamma_2 = \Gamma_1/D(\varphi)$.

7.2 Universal covers

See [42, Chapter 12] for this section (also Tits [57] and Ronan [46]).

An m-covering $\varphi : \widetilde{\Gamma} \to \Gamma$ is said to be *universal* if, for any other m-covering $\psi : \Gamma' \to \Gamma$, there is a unique m-covering $\theta : \widetilde{\Gamma} \to \Gamma'$ such that $\psi\theta = \varphi$.

Clearly, universal m-coverings are uniquely determined up to isomorphisms, when they exist. That, is, if $\varphi : \widetilde{\Gamma} \to \Gamma$ and $\psi : \widehat{\Gamma} \to \Gamma$ are m-coverings of Γ, then a (unique) isomorphism $\theta : \widetilde{\Gamma} \to \widehat{\Gamma}$ exists such that $\psi\theta = \varphi$. Thus, we may speak of *the* universal m-cover of a given geometry Γ, as if it were unique.

A geometry is said to be *m-simply connected* if it is not a proper m-quotient of any other geometry. It is not difficult to see that a geometry is m-simply connected iff it is its own universal m-cover. Clearly, universal m-covers are m-simply connected and an m-covering $\varphi : \widetilde{\Gamma} \to \Gamma$ is universal if and only if $\widetilde{\Gamma}$ is m-simply connected.

The following are proved by Ronan [46] (see also [42, Chapter 12]):

Proposition 25 *Let $\widetilde{\Gamma}$ be m-simply connected and let $\varphi : \widetilde{\Gamma} \to \Gamma$ be an m-covering. Then $\Gamma = \widetilde{\Gamma}/D(\varphi)$.*

Proposition 26 *Let $\varphi : \widetilde{\Gamma} \to \Gamma$ be the universal m-covering of Γ. Then, for every $g \in Aut(\Gamma)$ there is an automorphism $\tilde{g}$ of $\widetilde{\Gamma}$ such that $g\varphi = \varphi\tilde{g}$.*

We call $\tilde{g}$ the *lifting* of g to $\widetilde{\Gamma}$ via φ. Clearly, $\tilde{g}$ is uniquely determined modulo the deck group $D(\varphi)$ of φ.

Conversely, by Proposition 25, if $f \in Aut(\widetilde{\Gamma})$ normalizes $D(\varphi)$, then f induces an automorphism of Γ. Thus, Propositions 25 and 26 together imply the following:

Corollary 27 *Let $\varphi : \widetilde{\Gamma} \to \Gamma$ be the universal m-covering of Γ. Then*

$$Aut(\Gamma) \;=\; N_{Aut(\tilde{\Gamma})}(D(\varphi))/D(\varphi)$$

Warning. Many authors call 2-covers just 'covers' and, accordingly, use the word "simply connected" for '2-simply connected'. According to other authors (including me), the words *covering*, *cover* and *simply connected*, with no prefix m-, stand for $n-1$-covering, $n-1$-cover and $n-1$-simply connected, respectively.

7.3 Do universal covers exist ?

The following is proved by Tits [57] and Ronan [46] (see also [42, Chapter 12]).

Theorem 28 *Every geometry of rank $n > 1$ admits a universal 1-cover and a universal $n-1$-cover.*

Actually, Tits and Ronan prove that the chamber system of any geometry of rank $n > 1$ admits a universal m-cover for any $m = 1, 2, ..., n-1$, but the cover they construct is a chamber system, possibly not a geometry (I refer to [57], [46] and [42, Chapter 12] for the meaning of the above claim). However, when $n = 1$ or $n-1$, then that 'universal' chamber system is indeed the chamber system of a geometry (see [42, Chapter 12]), whence the above theorem holds. When $1 < m < n-1$, nobody has yet ben able to prove that the universal chamber system construct by Tits and by Ronan does come from a geometry.

Unfortunately, 2-coverings are most interesting, but they do not fall under the scope of Theorem 28 when $n > 3$. However, things are not that bad in reality. In many interesting cases it turns out that 2-coverings are in fact $n-1$-coverings. In any case, Theorem 28 applies to 2-coverings in the rank 3 case.

Note. Universal 1-covers are like infinite trees. (Indeed, the chamber system of the universal 1-cover of a geometry is an infinite tree with coloured edges.) They may be interesting in many context, but not so much in diagram geometry. (Note that 1-coverings need not preserve diagrams.)

7.4 How to construct universal covers

The construction given by Tits [57] and Ronan [46] for 'universally covering' chamber systems exploits homotopy classes of paths in chamber systems, but it is not handy at all. However, a more effective construction is available in the flag-transitive case, which sometimes can also be implemented on a computer. I describe that construction in the next theorem. I only state this theorem for universal 2-covers of geometries of rank 3, to make the exposition easier. I refer to [42, Chapter 12] for the precise statement in the general case.

Theorem 29 *Let Γ be a flag-transitive geometry of rank 3 over the set of types $\{0,1,2\}$ and let $\widetilde{\Gamma}$ be its universal 2-cover.*

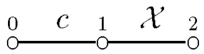
Given a flag-transitive subgroup $G \leq Aut(\Gamma)$, let (B, P_0, P_1, P_2) be the parabolic system in G relative to a given chamber C of Γ and let G_0, G_1, G_2 be the stabilizers of the elements of C, as in Subsection 6.1).

Let $\widetilde{G}$ be the amalgamated product of the groups G_0, G_1, G_2, with amalgamation of the intersection $P_i = G_j \cap G_k$ (for $\{i,j,k\} = \{0,1,2\}$). Then (B, P_0, P_1, P_2) can also be viewed as a parabolic system in $\widetilde{G}$. We have:

$$\widetilde{\Gamma} \ = \ \Gamma(\widetilde{G}, P_0, P_1, P_2)$$

8 Extensions

Let Γ belong to the following diagram, where $\mathcal{X}$ stands for some class of partial planes:



Then we say that Γ is an *extension* of (members of) $\mathcal{X}$.

By Proposition 16, if LL holds in Γ, then IP also holds. Note that the point-line system $Tr_{0,1}(\Gamma)$ of Γ is a (possibly non-simple) graph. Accordingly, LL just says that $Tr_{0,1}(\Gamma)$ is a simple graph (it is just the collinearity graph of Γ).

8.1 Enrichments

Let Γ be as above, but satisfying LL. Then the point-plane system $Tr_{0,2}(\Gamma)$ of Γ uniquely determines the whole of Γ: the lines of Γ are just pairs of points in the same plane and a point a (a plane A) is incident to a line l iff $a \in l$ (resp. $l \subseteq A$).

Conversely, given a geometry Δ of rank 2, with elements called points and blocks, assume the following on Δ:

- every block has at least three points;

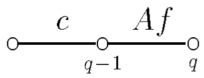
- for every point a, the structure with $a^{\perp} \setminus \{a\}$ as the set of points and $Res(a)$ as the set of blocks, with the incidence relation inherited from Δ, is a member of some given family $\mathcal{X}$ of partial linear spaces.

Then, if we take the points and the blocks of Δ as points and planes and the pairs of points in a same block of Δ as lines, we obtain a geometry Γ with the above diagram, satisfying LL and such that $Tr_{0,2}(\Gamma) = \Delta$. We call Γ the *enrichment of* Δ. Clearly, LL holds in Γ.

I give four examples of this construction in the sequel.

8.2 Inversive, Laguerre and Minkowski planes

Embeddable Finite Inversive Planes Given an ovoid Q in $PG(3,q)$, let Δ be the rank 2 geometry with Q as the set of points, the planes of $PG(3,q)$ intersecting Q in ovals as blocks and the natural incidence relation, inherited from $PG(3,q)$. The geometry Δ admits an enrichment Γ, which is an extended affine plane (of order q):

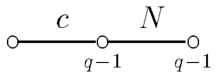


As Γ is a matroid (Corollary 18), both LL and T hold in Γ.

The geometry Δ is called an (*embeddable*) *inversive* (also *Mőbius*) *plane*, a (possibly non embeddable) *inversive plane* being just the point-plane system of a geometry belonging to the above diagram.

If $Q = Q_3^-(q)$, then Δ is said to be *classical* (also Miquelian). If Δ is classical, then all point-residues of Γ are isomorphic to $AG(2,q)$.

Embeddable Finite Laguerre Planes Let C be a cone of $PG(3,q)$ with vertex v and an oval as the basis. Now we take $C \setminus \{v\}$ as the set of points and the planes of $PG(3,q)$ meeting C in ovals as blocks. Let Δ be the rank 2 geometry obtained in this way. It admits an enrichment Γ, with diagram and orders as follows

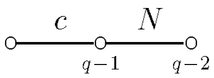


Point-residues are nets obtained by removing one bundle of parallel lines from $AG(2,q)$. Clearly, the LL property holds in Γ (indeed Γ is an enrichment of a rank 2 geometry). It is not difficult to check that the T property also holds.

The geometry Δ is called an (*embeddable*) *Laguerre plane*. If C is a quadratic cone, then Δ is said to be *classical*.

A (possibly non-embeddable) *Laguerre plane* can be defined as the point-plane system of a geometry belonging to the above diagram and satisfying both LL and T. (This is not the usual definition of Laguerre planes, but one can prove that it is equivalent to the usual one; see [41].)

Embeddable Finite Minkowski Planes Given $Q = Q_3^+(q)$, $q \geq 3$, naturally embedded in $PG(3,q)$, let Δ be the rank 2 geometry with Q as the set of points and the planes of $PG(3,q)$ secant for Q (namely, intersecting Q in a non-degenerate conic) as blocks. The geometry Δ admits an enrichment Γ, whith diagram and orders as

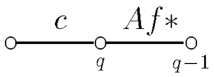


Point residues are nets obtained by removing two bundles of parallel lines from $AG(2,q)$.

The LL property holds in Γ, as Γ is obtained by enrichment. It is not difficult to check that the T property also holds.

The geometry Δ is called an *embeddable* (also, *classical*) *Minkowski plane*. A (possibly non-embeddable) Minkowski plane is a geometry belonging to the above diagram and satisfying both LL and T. (Again, this is not the usual definition of Minkowski planes, but it is equivalent to that; see [41].)

Embeddable Finite Special Laguerre Planes Let C be a cone in $PG(3,q)$ (q even), with a hyperoval as a basis and let v be its vertex. As in the case of Laguerre planes, we take $C \setminus \{v\}$ as the set of points and the planes of $PG(3,q)$ meeting C in hyperovals as blocks. Let Δ be the rank 2 geometry obtained in this way. It admits an enrichment Γ, with diagram and orders as follows



(point-residues are isomorphic to the dual of $AG(2,q)$.) Clearly, the LL property holds in Γ and it is not difficult to check that the T property also holds.

The geometry Δ is called an (*embeddable*) *special Laguerre plane* (also *Laguerre structure*). If the basis of C is a regular hyperoval, then Δ is said to be *classical*. A (possibly non embeddable) special Laguerre plane is a geometry belonging to the above diagram and satisfying both LL and T. (Once again, the usual definition is different from this one, but it is equivalent to it; see Sprague [48], [49].)

Some references. A very accurate survey of Mőbius, Laguerre and Minkowski planes is given by A. Delandtsheer [18, Section 5]; see also Thas [53, 3.3] and Thas [52] for inversive planes and Buekenhout [4] for the axiomatization of Mőbius, Laguerre and Minkowski planes. As for special Laguerre planes, besides the two papers by A. Sprague quoted above, the interested reader might see Willems [58].

8.3 Extensions from graphs

Let Γ be an extension of $\mathcal{X}$, for a given class $\mathcal{X}$ of partial linear spaces. It follows from the diagram of Γ that the (shadows of the) planes of Γ are cliques of the collinearity graph, but they might be not maximal cliques (as when $\mathcal{X}$ is a class of linear spaces; compare Corollary 18). Furthermore, even if the planes of Γ were maximal cliques of the collinearity graph, there might be some maximal cliques in the collinearity graph that have nothing to do with planes.

The following is not difficult to prove:

Proposition 30 *Assume LL and T and assume furthermore that all members of $\mathcal{X}$ have gonality at least 4. Then the planes of Γ are precisely the maximal cliques of the collinearity graph of Γ.*

Thus, under the previous hypotheses, Γ can be entirely recovered from its collinearity graph. In this case, Γ is basically a graph that is *locally $\mathcal{X}$*.

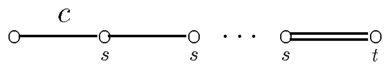
8.4 One point extensions

An extension of a class $\mathcal{X}$ of partial linear spaces is said to be a *one point extension* of (the members of) $\mathcal{X}$ if it satisfies LL and its collinearity graph is a complete graph. Clearly, if $\mathcal{X}$ is a class of linear spaces, then every extension of $\mathcal{X}$ is a one point extension, but one point extensions also exist of partial linear spaces that are not linear spaces. (For instance, a family of one point extensions of generalized quadrangles of order $(q-1, q+1)$ has been constructed by Thas [54]; a one point extension of the dual of $AG(2,4)$ is studied by Sprague [48] and [49].) It is easy to see that T fails to hold in such one point extensions.

9 Flag-transitive extensions of finite classical polar spaces

9.1 The geometries to consider in the sequel

Troughout this section **C** stands for the class of flag-transitive geometries belonging to the following diagram of rank n:



where $s, t < \infty$ and $s > 1$.

When $n > 3$, Theorem 22 tells us that point-residues are either finite polar spaces of rank $n-1$ with all lines of size $s+1 > 1$ (whence, they are classical polar spaces) or the flat geometry for A_7 described in Subsection 4.2. (Actually, results weaker than Theorem 22 are sufficent to obtain this conclusion in the present context, however, as Theorem 22 is at our disposal, let's use it.)

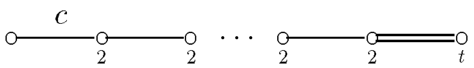
When $n = 3$, we cannot say so much on point-residues. Thus, I make an additional hypothesis in this case, assuming the following:

point-residues are classical thick generalized quadrangles (*whence $t > 1$, too*).

9.2 The examples

Affine polar spaces and their standard quotients. (See [42, Chapters 2 and 8].) Let Δ be a polar space of rank $n \geq 3$ with thick lines (namely, all lines of Δ have at least three points). A *hyperlane* of Δ is a proper subset H of the set of points of Δ such that every line of Δ either is contained in H or it meets H in precisely one point. The *complement* $\Delta \setminus H$ of H in Δ is the geometry having as elements the elements x of Δ with $\sigma(x) \not\subseteq H$. It is called an *affine polar space* (of *order* q if Δ is defined over $GF(q)$). The geometry $\Delta \setminus H$ belongs to the following diagram of rank n

which coincides with



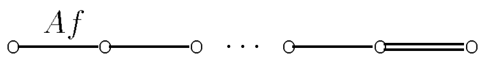
when $\Delta \setminus H$ has order 2. Furthermore, $\Delta \setminus H$ satisfies IP and T.

Let K_H be the subgroup of $Aut(\Delta)$ fixing H elementwise. Then every subgroup $X \leq K_H$ defines an $n-1$-quotient of $\Delta \setminus H$. We call $(\Delta \setminus H)/X$ a *standard quotient* of $\Delta \setminus H$. The quotient $(\Delta \setminus H)/K_H$ is the *minimal standard quotient* of $\Delta \setminus H$.

The geometry $\Delta \setminus H$ and its minimal standard quotient are flag-transitive, with automorphism groups G_H and G_H/K_H, where G_H is the stabilizer of H in $G = Aut(\Delta)$.

All standard quotients of $\Delta \setminus H$ satisfy IP (but not T, except of course when X is the trivial group). Actually, they are the only 2-quotients of $\Delta \setminus H$ that satisfy IP. Indeed (Cuypers and Pasini [17] and Cuypers [14]):

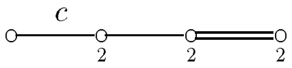
Theorem 31 *Let Γ be a geometry belonging to the following diagram of rank $n \geq 3$:*



and assume that IP holds in Γ (which, in this context, is equivalent to assume that LL holds and that the residues of the points of Γ are polar spaces). Furthermore, when $n = 3$, assume that Γ is finite. Then Γ is an affine polar space or a standard quotient of an affine polar space.

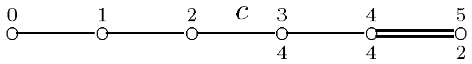
A flat geometry for A_8. Given a set S of size 8, one can draw 30 models of $AG(3,2)$ with the 8 objects of S. The symmetric group transitively permutes those 30 models but the alternating group A_8 does not: it has two orbits of size 15 on that set. Let $\mathcal{S}$ be one of those orbits. Then we can mimick the construction of Subsection 4.2: the elements, the pairs of elements and the 3-subsets of S are the points, the lines and the planes of the geometry we are going to describe; the members of $\mathcal{S}$ are taken as elements of the fourth type, say *spaces*. All points and all lines are declared to be incident to all spaces;

a plane is incident to a space iff it occurs in it; the incidence relation between point, lines and planes is defined in the natural way. A geometry, say Γ, with diagram and orders as follows arises by this construction:



Γ is flat, by definition. The residues of the points of Γ are isomorphic to the flat geometry for A_7 described in Subsection 4.2 and $Aut(\Gamma) = A_8$.

Geometries for the Fischer groups. The Fischer group Fi_{24} is generated by a class D of involutions, called 3-*transpositions*. A graph $\mathcal{F}$ can be defined on D, with 'to commute' as the adjacency relation (see Brouwer, Cohen and Neumaier [3].) The maximal cliques of $\mathcal{F}$ have size 24 and, given a maximal clique X, the stabilizer G_X of X in Fi_{24} acts as M_{24} on X. Thus, the Mathieu design $S(24, 8, 5)$ can be recovered on X from G_X. We can put all those Mathieu designs together, thus forming a rank 6 geometry, say Γ_{24}. More precisely, the elements of Γ_{24} are the vertices, the edges, the 3-cliques and the 4-cliques of $\mathcal{F}$, the 8-cliques of $\mathcal{F}$ that are octads for at least one of the above mentioned Mathieu designs and, finally, the maximal cliques of $\mathcal{F}$. The diagram of Γ_{24} is as follows:

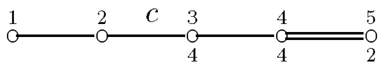


(The integers 0, 1, 2, 3, 4, 5 above the nodes of the diagram are the types and the integers 4, 4, 2 below them are orders; the residues of type $\{3, 4, 5\}$ are isomorphic to the polar space $H_5(4)$ for $U_6(2)$.)

We have $Aut(\Gamma_{24}) = Fi_{24}$ (but the commutator subgroup Fi'_{24} of Fi_{24} is also flag-transitive in Γ_{24}).

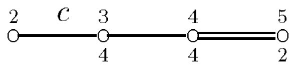
The geometry Γ_{24} is not 5-simply connected but is admits a 2-simply connected triple 5-cover, with $3.Fi_{24}$ as the automorphism group (where the factor 3 is the Schur multiplier of the simple group Fi'_{24}).

Given a 0-element a of Γ_{24}, we set $\Gamma_{23} = Res(a)$. Clearly, Γ_{23} belongs to the following diagram:



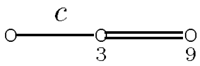
We have $Aut(\Gamma_{23}) = Fi_{23}$. (The stabilizer of a in Fi_{24} is $2 \times Fi_{23}$, but the factor 2 is contributed by the kernel of the action of Fi_{23} on $Res(a)$.) The geometry Γ_{23} is 2-simply connected.

Given a 1-element b of Γ_{23}, we set $\Gamma_{22} = Res(b)$. Clearly, Γ_{22} belongs to the following diagram:



We have $Aut(\Gamma_{22}) = Fi_{22}$ (stabilizer of b in Fi_{23}). The geometry Γ_{22} is 2-simply connected.

A geometry for McL. The MacLaughlin graph (see Brouwer, Cohen and Neumaier [3]) is a strongly regular graph with 275 points, locally $Q_5^-(3)$ and with $Aut(McL) = McL.2$ as the automorphism group. The group McL is transitive on the ordered 3-cliques of that graph. Thus, the vertices, the edges and the maximal cliques of that graph form a flag-transitive geometry satisfying LL and T, with diagram as follows (compare Subsection 8.3),

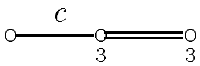


and $McL.2$ as the automorphism group (but McL also acts flag-transitively on it). This geometry is 2-simply connected.

A geometry for $U_5(2)$. Let h_0 be a non-degenerate projective hermitian form in $\Pi_0 = PG(3,4)$. It is well known that the non-isotropic points of Π_0 and the orthogonal bases of non-isotropic points form a model of $W(3)$. Let now h be a non-degenerate projective hermitian form in $\Pi = PG(4,4)$ and define:

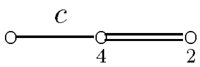
- the points are the points of Π that are not h-isotropic;
- the lines are the orthogonal pairs of non h-isotropic points;
- the planes are the orthogonal bases of non h-isotropic points.

Thus, we get a flag-transitive geometry Γ with diagram and order as follows



and $Aut(\Gamma) = Aut(U_5(2)) = U_5(2).2$ (but $U_5(2)$ is also flag-transitive on Γ). Point-residues are isomorphic to $W(3)$ (with the lines and the planes of Γ as points and lines of $W(3)$, respectively). Both LL and T hold in Γ and Γ is 2-simply connected.

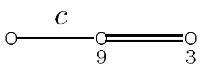
Geometries for $O_6^-(3)$ and $3.O_6^-(3)$. Given a projective non-singular quadratic form q of Witt index 2 in $PG(5,3)$, define a graph as follows: the vertices are the non-singular points of square norm and the adjacency relation is 'being orthogonal'. The vertices, the edges and the maximal cliques of that graph form a flag-transitive geometry Γ for the following diagram:



and $Aut(\Gamma) = O_6^-(3).2^2$ (but $O_6^-(3)$ is also flag-transitive on Γ). Point-residues are isomorphic to $H_3(4)$ and both LL and T hold in Γ.

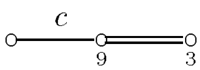
The geometry Γ is not 2-simply connected, but it admits a 2-simply connected triple 2-cover $\widetilde{\Gamma}$ with $Aut(\widetilde{\Gamma}) = 3.O_6^-(3).2^2$ (where the factor 3 is the Schur multiplier of the simple group $O_6^-(3)$). Needless to say, LL and T also hold in $\widetilde{\Gamma}$.

A geometry for Suz. The sporadic Suzuki group Suz acts distance transitively on a graph $\mathcal{G}$ of diameter 4, called the *Patterson graph* (Brouwer, Cohen and Neumaier [3]). The vertices of $\mathcal{G}$ are the centers of the Sylow 3-subgroups of Suz, two vertices being adjacent in that graph when they centralize each other. The graph $\mathcal{G}$ is locally $H_3(9)$. Hence (compare Subsection 8.3) the vertices, the edges and the maximal cliques of $\mathcal{G}$ form a geometry Γ satisfying LL and T and with diagram as follows:



(point residues are isomorphic to $H_3(9)$). The geometry Γ is flag-transitive, with $Aut(\Gamma) = Suz.2$ (but Suz is flag-transitive in it). Furhermore, Γ is 2-simply connected.

A geometry for $Aut(HS)$. The Higman-Sims group HS acts transitively on the paths of length 2 of a strongly regular graph $\mathcal{H}$ with 100 vertices, the so-called Higman-Sims graph (see Brouwer, Cohen and Neumaier [3]). We have $Aut(\mathcal{H}) = Aut(HS) = HS.2$. Yoshiara [60] considered another graph, say $\mathcal{Y}$, defined as follows: the vertices are the edges of $\mathcal{H}$ and the adjacency relation is 'forming a disconnected subgraph of $\mathcal{H}$ of size 4'. The graph $\mathcal{Y}$ has two families of maximal cliques, of size 11 and 10, respectively. If we take the vertices, the edges and the 11-cliques of $\mathcal{Y}$ with the natural incidence relation, then we obtain a geometry Γ with diagram as follows:



(point-residues are isomorphic to $H_3(9)$, as in the geometry for Suz described above).

The geometry Γ is flag-transitive, with $Aut(\Gamma) = Aut(HS)$, it is 2-simply connected and satisfies LL, but not T.

9.3 The classification theorem

The following theorem is an assembling of contributions by Buekenhout and Hubaut [8], Buekenhout [5], Weiss and Yoshiara [59] and Del Fra, Ghinelli, Meixner and Pasini [22]:

Theorem 32 *All geometries belonging to the class* **C** *defined in Subsection* 9.1 *have been mentioned in Subsection* 9.2.

More precisely, besides affine polar spaces of order 2 and their minimal standard quotients, only eight geometries exist in **C**, *namely the geometries for the following groups, described in Subsection* 9.2*:*

$$A_8, Fi_{22}, McL, U_5(2), 3.O_6^-(3), O_6^-(3), Suz \text{ and } Aut(HS)$$

Sketch of the proof. Let Γ be a member of $\mathbf{C}$, with orders $1, s, s, ..., s, t$, as in the diagram of Subsection 9.1, and let G be a flag-transitive subgroup of $Aut(\Gamma)$.

The case of $n \geq 4$. Let $n \geq 4$. By a well known result of Hughes [23], either $s = 2$ or $s = 4$.

When $s = 4$, then Γ is the geometry for Fi_{22}. I refer to Buekenhout and Hubaut [8] for the details of this proof (also Pasechnik [35] and [36]).

The case of $s = 2$. Let $s = 2$. Then point-residues are either polar spaces defined over $GF(2)$ or the flat geometry for A_7 described in Subsection 4.2. It is an easy exercise to check that, in the latter case, Γ is the flat geometry for A_8 described in the second paragraph of Subsection 9.2.

Still with $s = 2$, let point-residues be polar spaces and assume that LL fails to hold in Γ. Then the stabilizer G_a in G of a point a of Γ acts imprimitely on the set of lines incident to a, the lines incident with a an with a given point $b \perp a$ forming an imprimitivity classes of that action. However, the feasible structures of $\overline{G}_a = G_a/K_a$ are known, by a theorem of Seitz [47] (see also Meixner [28] for some corrections to the original statement of [47]). It turns out that $\overline{G}_a$ acts primitively on the set of lines through a. Hence, LL holds. By Theorem 31, Γ is an affine polar space of order 2, or a standard quotient of an affine polar space of order 2 (see also Buekenhout [5] and Buekenhout and Hubaut [8]).

An affine polar space of order 2 admits at most one proper standard quotient: the minimal one, which is a quotient over a group of order 2. Thus, we do really have a complete classification for the case $s = 2$.

The case of $n = 3$. Turning to the case of $n = 3$, if $s = 2$ then the same argument as above shows that Γ is either an affine polar space of rank 3 and order 2 or a standard quotient of such a space. Thus, let $s \geq 3$.

By the above quoted theorem of Seitz, the stabilizer G_a in G of a point a acts as the appropriate classical group on $Res(a)$, except possibly in a few exceptional cases with $s = t = 3$ or $s = 9$ and $t = 3$.

The exceptional cases. Let the group $\overline{G}_a$ induced by G_a on $Res(a)$ be not one of the classical groups appropriate for $Res(a)$. Then an investigation of feasible structures of parabolics shows that $s = 9$, $Res(a) = H_3(9)$ and $\overline{G}_a = L_3(4).2^2$ (Del Fra, Ghinelli, Meixner and Pasini [22].). In this case, a further investigation shows that the parabolic system is precisely as in the Yoshiara example for $Aut(HS)$ (Weiss and Yoshiara [59]). The latter is 2-simply connected (Weiss and Yoshiara [59]). Hence the Yoshiara example is the unique possibility that arises here.

The classical cases. Let $\overline{G}_a$ be classical. Then, given a plane u on a, $\overline{G}_{a,u}$ is between $PSL_2(s)$ and $P\Gamma L_2(s)$ and acts on the set of $s+1$ lines incident to the panel $\{a, u\}$ like on the $s+1$ points of $PG(1, s)$. However, $\overline{G}_{a,u}$ is a subgroup of $\overline{G}_u$ which, being flag-transitive on the circular space $Res(u)$, is a doubly-transitive permutation group of degree $s+2$. By a teorem of Suzuki [50] (also Tits [55]), we have $s = 3$, 4 or 9. There are two ways to finish the investigation of this case.

The graph-theoretic approach. By the same argument used in the case of $s = 2$ one can prove that LL holds. A slightly different argument shows that T also holds. Thus, we are led back to the investigation of graphs that are locally so-and-so (Subsection 8.3). By examining subgroups of $\overline{G}_a$, one can describe the feasible structures of the subgraph of the collinearity graph of Γ induced on the common neighbourhood $a^\perp \cap b^\perp$ of two points at distance 2. Then, one can try to do the same for pairs of points at distance 3 ... eventually, only one of the several possibilities to consider should survive and, thus, the graph will be uniquely determined.

The above is the approach chosen by Buekenhout and Hubaut [8]. However, if we want to work in that way, things will be easier if the diameter d of Γ is small, $d = 2$ or 3, say. The diameter d of Γ indeed admits an upper bound: $d \leq s$ (Cameron, Hughes and Pasini [12]). Unfortunately, that bound is not tight enough as we might wish for the graph-theoretic approach. Actually, the geometries for $3.O_6^-(3)$ and Suz mentioned in Subsection 9.2 have diameter 4. (Accidentaly, Buekenhout and Hubaut missed those two examples in [8].) Eventually, Meixner [30] has been able to apply the above strategy to the case leading to Suz, whereas a graph-theoretic uniqueness proof for the case related to $3.O_6^-(3)$ has been found by Brouwer (quoted in Brouwer, Cohen and Neumaier [3]).

The group-theoretic approach. This is the approach followed by Weiss and Yoshiara [59] for the case of $s = 9$ (both with $\overline{G}_a$ classical and with $\overline{G}_a = L_3(4).2^2$, as in the $Aut(HS)$ example) and by Yoshiara [61] for the cases of $s = 3$ or 4.

According to this way of tackling the problem, we need not distinguish between exceptional and classical cases. We have not even to worry for LL or T. We only need to determine the feasible parabolic systems. Once that is done, we should compute the amalgamated products of the parabolic systems we have got, ruling out those possibilities that collapse, if any. (We may ask a computer to do that job for us, if we are lazy to invent smart tricks.) If it turns out that the surviving possibilities bijectively correspond to the 2-simple connected examples that we knew, then we are done.

This has indeed been the case here.

A different approach. Attempts to get rid of groups (to a large extent if not completely) have been made by Pasechnik [37], [38] and by myself [40].

10 Going on

10.1 Extensions of non-classical generalized quadrangles.

Three cases are to examine here.

Point-residues are thick. Up to duality, only two flag-transitive finite thick non-classical generalized quadrangles are known. They are both of type $T_2^*(O)$, with O a hyperoval of $PG(2,4)$ (hence the generalized quadrangle as orders $(3,5)$) or the Lunelli-Sce hyperoval of $PG(2,16)$ (the orders now are $(15,17)$). Let us call Q_4 and Q_{16} those two generalized quadrangles. Yoshiara [65] has proved that there are precisely two 2-simply connected flag-transitive extensions of Q_4 and just one 2-simply connected flag-transitive extension of the dual of Q_4. I am not aware of any result on flag-transitive extensions of Q_{16} or of its dual.

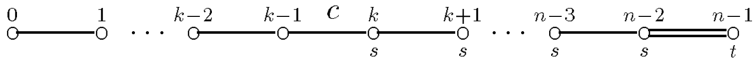
I warn the reader that many non flag-transitive extensions of generalized quadrangles of order $(q-1,q+1)$ and $(q+1,q-1)$ are also known (Thas [51], [54], Yoshiara [64], Del Fra, Pasechnik and Pasini [21], Del Fra and Pasini [20]).

Point-residues are grids. Many flag-transitive examples are known (see Meixner and Pasini [33]) but a complete classification is not yet in reach. I refer to Meixner and Pasini [34] for a classification in the case where the action induced on planes is 'quite large'.

Point-residues are dual grids. (This case includes polar spaces of rank 3 with all lines of size 2.) S. Rees [44] has shown that the extended generalized quadrangles of this class correspond to certain triples of mutually orthogonal latin squares. However, this is not at all a classification. As far as I know, the problem of classifying flag-transitive geometries of this kind is still open.

10.2 Multiple extensions

I shall briefly discuss flag-transitive geometries belonging to diagrams of the following form



with $s,t<\infty$, $s>1$ and $k>1$. Let Γ be such a geometry and let Δ be the isomorphism type of the residues of type $\{k,k+1,...,n-1\}$. Then we say that Γ is a *k-times extension* of Δ, also a *multiple extension* of Δ.

It follows from Hughes [23] that, if $n-k\geq 3$, then $s=4$, $n-k=3$ and $k=2$ or 3. Therefore, by Theorem 32, the following are the cases to consider, where Γ' denotes the isomorphism type of the residues of type $\{k-1,k,...,n-1\}$.

1. We have $\Delta = H_5(4)$ and Γ' is the geometry for Fi_{22} described in the third paragraph of Subsection 9.2. The geometries for Fi_{23}, Fi_{24} and $3.Fi_{24}$ are the only possibilities here (Meixner [29]; also Pasechnik [35], [36]).

2. We have $\Delta = Q_5^-(3)$ and Γ' is the above mentioned geometry for McL. Then $k=2$ and there is a unique 2-simply connected flag-transitive example, for the group $2\times Co_3$ (Meixner [31]). It admits a 2-fold 2-quotient, where Co_3 acts.

3. We have $\Delta = H_3(9)$ and Γ' is the above mentioned geometry for Suz. Then $k = 2$ and there is a unique 2-simply connected flag-transitive example, for the group $2^{1+24}Co_1$ (Ivanov and Shpectorov [27]). It admits at least two flag-transitive 2-quotients, one for $2^{24}Co_1$ and another one for Co_1.

4. We have $\Delta = H_3(9)$ and Γ' is the geometry for $Aut(HS)$. This case is impossible: no flag-transitive extensions exist of the geometry for $Aut(HS)$. (Meixner [31]).

5. We have $\Delta = H_3(4)$ and Γ' is the geometry for $O_6^-(3)$. An infinite series of extensions exists for this geometry (Meixner [29]). I do not know if the same is true if we start with the geometry for $3.O_6^-(3)$.

5. We have $\Delta = W(3)$ and Γ' is the geometry for $U_5(2)$. Again an infinite series of extensions exists (Meixner [29]).

6. We have $\Delta = W(2)$ and Γ' is the affine polar space $Q_6(2) \setminus H$, with $H = Q_5^+(2)$ or $Q_5^-(2)$. For each choice of H an infinite series of extensions exists (Meixner [29]).

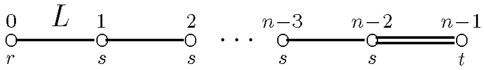
7. Δ is a finite thick but not classical generalized quadrangle. I am not aware of any result on this case.

8. Δ is a grid. I refer to Meixner and Pasini [33] and [34] for this.

9. Δ is a dual grid. Not so much is known on this case, either.

10.3 $L.C_n$-geometries

These are geometries for the following diagram



where $0, 1, ..., n-1$ are the types, $r, s, s, ..., s, t$ are orders, $r, s, t < \infty$ and $s > 1$.

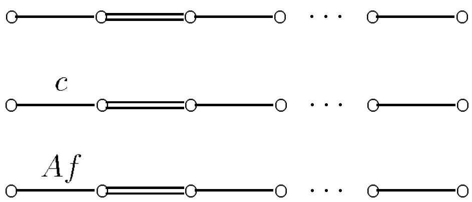
Let Γ be a flag-transitive geometry of rank n with diagram and orders as above and let $G = Aut(\Gamma)$. When $n = 3$, assume furthermore that point-residues are classical. Then we can apply the theorem of Seitz [47] to the group $\overline{G}_a$ induced on $Res(a)$ by the stabilizer in G of a point a. We can also also exploit the classification of finite flag-transitive linear spaces by Buekenhout, Delandtsheer and Doyen [10] to get control over residues of type $\{0, 1\}$. Comparing the information thus obtained on $\overline{G}_a$ and on $\{0, 1\}$-residues, we see that the latter falls in one of the following cases:

- projective spaces;
- circular spaces;
- affine planes.

The first two case are covered by Theorems 22 and 32. In the third case one can prove that LL holds (Del Fra [19]). Hence, by Theorem 31, we have an affine polar space or a standard quotient of an affine polar space.

10.4 Extended dual polar spaces

I only give a few reference for this topic. Three types of extensions can be considered, corresponding to the following three diagrams:



The first diagram describes buildings (of type F_4 when the rank is 4.) I refer to Cuypers [15], [16], Meixner [32] and Yoshiara [62] for results on extensions of the second kind and to Van Bon and Cuypers [2] and Ivanov [25] and [26] for those of the third kind.

References

[1] M. Aschbacher, *Finite geometries of type C_3 with flag-transitive automorphism group*, Geom. Dedicata **16** (1984), 195-200.

(1996), 3-22.

[2] J. van Bon and H. Cuypers, *Affine extensions of near hexagons related to the spin module of type B_3*, in Groups and Geometries (eds. L. Di Martino et al.) Birkhauser, Basel (1998), 215-230.

[3] A. Brouwer, A. Cohen and A. Neumaier, Distance Regular Graphs, Springer, Berlin, 1989.

[4] F. Buekenhout, *Le plans de Benz: une approche unifiée des plans de Mőbius, Laguerre et Minkowski*, J. Geometry **17** (61-68).

[5] F. Buekenhout, *Extensions of polar spaces and the doubly transitive symplectic groups*, Geom. Dedicata **6** (1977), 13-21.

[6] F. Buekenhout, *Diagrams for geometries and groups*, J. Comb. Th. A **27** (1979), 121-171.

[7] F. Buekenhout, *Foundations of incidence geometry*, in Handbook of Incidence Geometry (F. Buekenhout ed.), North Holland (1995), 63-106.

[8] F. Buekenhout and X. Hubaut, *Locally polar spaces and related rank 3 groups*, J. Algebra **45** (1977), 391-434.

[9] F. Buekenhout and A. Pasini, *Finite diagram geometries extending buildings*, in Handbook of Incidence Geometry (F. Buekenhout ed.), North Holland (1995), 1143-1254.

[10] F. Buekenhout, A. Delandtsheer and J. Doyen, *Finite linear spaces with flag-transitive groups*, J. Comb. Th. A **49** (1988), 268-293.

[11] F. Buekenhout, A. Delandtsheer, J. Doyen, P. Kleidman, M. Liebeck and J. Saxl, *Linear spaces with flag-transitive automorphism groups*, Geom. Dedicata **36** (1990), 89-94.

[12] P. Cameron, D. Hughes and A. Pasini, *Exended generalized quadrangles*, Geom. Dedicata **35** (1990), 193-228.

[13] A. Cohen and E. Shult, *Affine polar spaces*, Geom. Dedicata **35** (1990), 73-105.

[14] H. Cuypers, *Finite locally generalized quadrangles with affine planes*, European J. Comb. **13** (1992), 439-453.

[15] H. Cuypers, *A graphical characterization of the group* Co_2, preprint, 1992.

[16] H. Cuypers, *Flag-transitive extensions of buildings of type* G_2 *and* C_3^*, in Groups of Lie Type and their Geometries (eds. L. Di Martino et al.), Cambridge Univ. Press, Cambridge (1995), 97-110.

[17] H. Cuypers and A. Pasini, *Locally polar geometries with affine planes*, European J. Comb. **13** (1992), 39-57.

[18] A. Delandtsheer, *Dimensional linear spaces*, in Handbook of Incidence Geometry (F. Buekenhout ed.), North Holland (1995), 193-294.

[19] A. Del Fra, *Flag-transitive* $Af.C_2$*–geometries*, Forum Math. **7** (1995), 393-403.

[20] A. Del Fra and A. Pasini, *A census of extended generalized quadrangles of order* $(q-1, q+1)$ *and* $(q+1, q-1)$, to appear in Discrete Mathematics.

[21] A. Del Fra, D. Pasechnik and A. Pasini, *A new family of extended generalized quadrangles*, European J. Comb. **18** (1977), 155-169.

[22] A. Del Fra, D. Ghinelli, T. Meixner and A. Pasini, *Flag-transitive extensions of* C_n*–geometries*, Geom. Dedicata **37** (1990), 253-273.

[23] D. Hughes, *Extensions of designs and groups: projective, symplectic and certain affine groups*, Math. Zeit. **89** (1965) 199-205.

[24] A. Ivanov, *On the geometries of the Fischer groups*, European J. Comb. **16** (1995), 163-184.

[25] A. Ivanov, *Exceptional extended dual polar spaces*, European J. Comb. **18** (1997), 859-886.

[26] A. Ivanov, *Affine extended dual polar spaces*, in Groups and Geometries (eds. L. Di Martino et al.), Birkhauser, Basel (1998), 107-122.

[27] A. Ivanov and S. Shpectorov, *A new cover of the 3-local geometry of* Co_1, to appear.

[28] T. Meixner, *Klassische Tits kammersysteme mit einer transitiven automorphismen-gruppe*, Mitt. Math. Sem. Giessen **174** (1986).

[29] T. Meixner, *Some polar towers*, European J. Comb. **12** (1991), 397-415.

[30] T. Meixner, *A computer-free proof of a theorem of Weiss and Yoshiara*, preprint.

[31] T. Meixner, *Two geometries related to the sporadic groups* Co_1 *and* Co_3, preprint.

[32] T. Meixner, *A geometric characterization of the simple group* Co_2, preprint.

[33] T. Meixner and A. Pasini, *A census of known flag-transitive extended grids*, in Finite Geometries and Combinatorics (eds. F. De Clerck et al.), Cambridge Univ. Press, Cambridge (1993), 249-267.

[34] T. Meixner and A. Pasini, *On flag-transitive extended grids*, Atti Sem. Mat. e Fis. Univ. Modena **44** (1996), 67-80.

[35] D. Pasechnik, *Geometric characterization of the sporadic groups* Fi_{22}, Fi_{23} *and* Fi_{24}^*, J. Comb. Th. A **68** (1994), 100-114.

[36] D. Pasechnik, *Extending polar spaces of rank at least* 3, J. Comb. Th. A **72** (1995), 232-242.

[37] D. Pasechnik, *Extensions of* $GQ(3,3)$, Bull. B. M. S. **2** (1995), 509-518.

[38] D. Pasechnik, *The extensions of the generalized quadrangle of order* $(3,9)$, European J. Comb. **17** (1996), 751-756.

[39] A. Pasini, *On the classification of finite* C_n *geometries with thick lines*, Contemp. Math. **111** (1990), 153-170.

[40] A. Pasini, *Remarks on double ovoids in finite classical generalized quadrangles, with an application to extended generalized quadrangles*, Simon Stevin **66** (1992), 41-68.

[41] A. Pasini, *Diagram geometries for sharply n-transitive sets of permutations or of mappings*, Designs, Codes and Cryptography **1** (1992), 275-297.

[42] A. Pasini, Diagram Geometries, Oxford Univ. Press, 1994.

[43] A. Pasini and S. Yoshiara, *Generalized towers of flag-transitive circular extensions of a non-classical C_3-geometry*, J. Comb. Th. A **66** (1994), 313-320.

[44] S. Rees, *Finite C_3 geometries in which all lines are thin*, Math. Zeit. **189** (1985), 67-85.

[45] S. Rees, *A classification of a class of C_3-geometries*, J. Comb. Th. A **44** (1987), 173-181.

[46] M. Ronan, *Coverings and automorphisms of chamber systems*, European J. Comb. **1** (1980), 259-269.

[47] G. Seitz, *Flag-transitive subgroups of Chevalley groups*, Ann. Math. **97** (1973), 27-56.

[48] A. Sprague, *Extended dual affine planes*, Geom. Dedicata **16** (1984), 107-121.

[49] A. Sprague, *A characterization of special Laguerre planes and extended dual affine planes*, Geom. Dedicata **16** (1984), 149-156.

[50] M. Suzuki, *Transitive extensions of a class of doubly transitive groups*, Nagoya Math. J. **27** (1966), 159-169.

[51] J. Thas, *Extensions of finite generalized quadrangles*, Symposia Math. **28** (1985), 127-143.

[52] J. Thas, *The affine plane $AG(2,q)$, q odd, has a unique one point extension*, Invent. Math. **118** (1994), 133-139.

[53] J. Thas, *Projective geometry over a finite field*, in Handbook of Incidence Geometry (F. Buekenhout ed.), North Holland (1995), 295-349.

[54] J. Thas, *Some new classes of extended generalized quadrangles of order* $(q+1, q-1)$, to appear.

[55] J. Tits, *Non-existence de certain extensions transitives I. Groups projectifs a une dimension*, Bull. Soc. Math. Belgique **23** (1971), 481-492.

[56] J. Tits, Buildings of Spherical type and Finite BN-pairs, Springer L.N. **386** (1974).

[57] J. Tits, *A local approach to buildings*, in The Geometric Vein (the Coxeter Festschrift), Springer (1981), 519-547.

[58] M. Willems, *Optimal codes, Laguerre and special Laguerre i-structures*, European J. Comb. **4** (1983), 87-92.

[59] R. Weiss and S. Yoshiara, *A geometric characterization of the groups Suz and HS*, J. Algebra **133** (1990), 182-196.

[60] S. Yoshiara, *A $(c.c_2)$-geometry associated with the groups HS*, European J. Comb. **11** (1990), 81-93.

[61] S. Yoshiara, *A classification of flag-transitive classical cC_2 geometries by means of generators and relations*, European J. Comb. **12** (1991), 159-181

[62] S. Yoshiara, *On some extended dual polar spaces I*, European J. Comb. **15** (1994), 73-86.

[63] S. Yoshiara, *The flag-transitive C_3-geometries of finite order*, J. Algebraic Comb. **5** (1996), 251-284.

[64] S. Yoshiara, *A construction of extended generalized quadrangles using the veronesian*, European J. Comb. **18** (1997), 835-848.

[65] S. Yoshiara, *On some flag-transitive non-classical $c.C_2$-geometries II*, Contributions to Algebra and Geometry, **38** (1997), 357-376.