Unitals

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Intensive Course on Galois Geometry and Generalized Polygons

1 Preliminaries

Throughout these notes, but with the exception of Section 4.5, the unitals we will study are assumed to be embedded in a projective plane of square order q. We will often restrict ourselves to translation planes of square order q and kernel containing $\mathrm{GF}(\sqrt{q})$ or to Desarguesian projective planes of square order q. We will use the notation π_q to denote a projective plane of order q and $\mathrm{PG}(2,q)$ for the Desarguesian projective plane of order q. In the n-dimensional Desarguesian projective space $\mathrm{PG}(n,q)$ a point X has coordinates $X = (x_0, x_1, \ldots, x_n)$ and a hyperplane H has coordinates $H = [u_0, u_1, \ldots, u_n]$. Thus a point X lies in a hyperplane H if and only if $X^T H = XH^T = H^T X = HX^T = 0$.

Recall that a *Baer subplane* of π_q, q square, is a subplane of order $\sqrt{q}$, and any line of π_q meets a Baer subplane in 1 or $\sqrt{q}+1$ points. A set of $\sqrt{q}+1$ points which is the intersection of a line with a Baer subplane is a *Baer subline.*

We will use the following representation of a translation plane π_q, q square, with kernel containing $\mathrm{GF}(\sqrt{q})$ in $\mathrm{PG}(4,\sqrt{q})$, due to André [1954] and rediscovered by Bruck and Bose [1964, 1966] and Segre [1964]. The construction is discussed in Hirschfeld [1985], Section 17.7.

Let $\Sigma_\infty = PG(3,\sqrt{q})$ be embedded as a hyperplane in $\mathrm{PG}(4,\sqrt{q})$. Let $\mathcal{S}$ be a spread of Σ_∞. We define a new incidence structure: the *points* are the points of $\mathrm{PG}(4,\sqrt{q})$ not lying in Σ_∞, the *lines* are the planes of $\mathrm{PG}(4,\sqrt{q})$ which meet Σ_∞ in a line of $\mathcal{S}$ and *incidence* is natural. This incidence structure is an affine plane and can be completed to a projective plane $\pi(\mathcal{S})$ by the addition of an ideal line ℓ_∞ whose points are the elements of the spread $\mathcal{S}$. Conversely, any translation plane π_q of square order q and kernel containing $\mathrm{GF}(\sqrt{q})$ can be represented in this way as $\pi(\mathcal{S})$ for some spread $\mathcal{S}$. The plane $\pi(\mathcal{S})$ is Desarguesian if and only if the spread $\mathcal{S}$ is regular.

Under this representation of $\pi_q = \pi(\mathcal{S})$ in $\mathrm{PG}(4,\sqrt{q})$, a plane of $\mathrm{PG}(4,\sqrt{q})$ which does not lie in Σ_∞ and does not contain any line of $\mathcal{S}$ corresponds to a Baer subplane of π_q with $\sqrt{q}+1$ points on ℓ_∞, see Vincenti [1983], Proposition 2. It follows that a line of $\mathrm{PG}(4,\sqrt{q})$ which does not lie in Σ_∞ corresponds to a Baer subline of π_q with a point on ℓ_∞.

If $\pi_q = \mathrm{PG}(2,q)$ then the converse of each of these results is also true, see Vincenti [1983], Proposition 1. However if π_q is not Desarguesian then there may be counter-examples, such as are known to exist in certain translation planes of fourth-power order (see Vincenti [1983] and Freeman [1980]).

Recent accounts of unitals can be found in the excellent reference work: Hirschfeld [1998]. See also Beutelspacher [1995] and Thas [1995].

2 Hermitian Curves

Let $\mathcal{P}$ denote the set of points and let $\mathcal{L}$ denote the set of lines of π_q. A *polarity* of π_q is a bijective involutory map $\phi : \mathcal{P} \to \mathcal{L}, \mathcal{L} \to \mathcal{P}$ which preserves incidence (that is, for $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$ we have $P \in \ell$ if and only if $\phi(\ell) \in \phi(P)$). A point P is *absolute* if $P \in \phi(P)$ and a line ℓ is *absolute* if $\phi(\ell) \in \ell$.

We suppose for the rest of this section that $\pi_q = \mathrm{PG}(2,q)$. Since a polarity of $\mathrm{PG}(2,q)$ is a collineation from $\mathrm{PG}(2,q)$ to its dual plane, a polarity has equation $P \mapsto AP^\sigma$ and $\ell \mapsto A^{-T}\ell^\sigma$ where $A \in \mathrm{GL}(3,q)$ and $\sigma \in \mathrm{AutGF}(q)$.

There are five different types of polarity, one of which is the *Hermitian polarities* of $\mathrm{PG}(2,q)$, for q a square, each of which has an equation of the form $\phi : P \mapsto AP^\sigma$ and $\ell \mapsto A^{-T}\ell^\sigma$ where $A \in \mathrm{GL}(3,q)$ satisfies $A^\sigma A^{-T} = I$ and $\sigma \in \mathrm{AutGF}(q)$ is the map $\sigma : P \mapsto P^{\sqrt{q}}$. It is straightforward to check that this is indeed a polarity. First, certainly ϕ is a bijective map of $\mathcal{P}$ to $\mathcal{L}$ and $\mathcal{L}$ to $\mathcal{P}$. Next, $\phi(\ell)^T\phi(P) = (A^{-T}\ell^\sigma)^T AP^\sigma = (\ell^T)^\sigma A^{-1}AP^\sigma = (\ell^T P)^\sigma$; so incidence is preserved. Finally, ϕ is an involution since $\phi(\phi(P)) = \phi(AP^\sigma) = A^{-T}(AP^\sigma)^\sigma = P^{\sigma^2} = P$.

The set of absolute points with respect to a Hermitian polarity $\phi : P \mapsto AP^\sigma, \ell \mapsto A^{-T}\ell^\sigma$ is called a *Hermitian variety* and is the set of points $\{X : X^T AX^\sigma = 0\}$. A *Hermitian curve* is a non-singular Hermitian variety.

Example 1 Let $A = I$ be the 3×3 identity matrix and let $\sigma : P \mapsto P^{\sqrt{q}}$. Then $A^\sigma A^{-T} = I$ so $\phi : P \mapsto P^\sigma$ and $\ell \mapsto \ell^\sigma$ is a Hermitian polarity. The set of absolute points is $\{X : X^T X^\sigma = x_0^{\sqrt{q}+1} + x_1^{\sqrt{q}+1} + x_2^{\sqrt{q}+1} = 0\}$. In fact it is straightforward to verify that this is a Hermitian curve.

Theorem 2 (See Hirschfeld [1998, 5.7, 7.18, 7.19]) There is a single orbit of $\mathrm{PGL}(3,q)$ on Hermitian curves in $\mathrm{PG}(2,q)$, q square. Such a Hermitian curve therefore has canonical form $x_0^{\sqrt{q}+1} + x_1^{\sqrt{q}+1} + x_2^{\sqrt{q}+1} = 0$, and has $q\sqrt{q}+1$ points. The stabiliser $\mathrm{PGU}(3,q)$ in $\mathrm{PGL}(3,q)$ of a Hermitian curve has order $q\sqrt{q}(q\sqrt{q}+1)(q-1)$, and the number of Hermitian curves is $q\sqrt{q}(q\sqrt{q}-1)(q+1)$. The group $\mathrm{PGU}(3,q)$ is called the *projective unitary group* and acts 2-transitively on the points of $\mathcal{H}$.

The next theorems collect together several properties of Hermitian curves.

Theorem 3 (See Hirschfeld [1998, 7.20]) Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q)$, q square. There are $q\sqrt{q}+1$ lines which each meet $\mathcal{H}$ in a single point, and the remaining $q(q-\sqrt{q}+1)$ lines each meet $\mathcal{H}$ in $\sqrt{q}+1$ points. The points of $\mathcal{H}$ on a line meeting $\mathcal{H}$ in $\sqrt{q}+1$ points form a Baer subline. Through a point P of $\mathcal{H}$ there passes one line meeting $\mathcal{H}$ in a single point; the remaining q lines on P each meet $\mathcal{H}$ in $\sqrt{q}+1$ points. Through a point Q not on $\mathcal{H}$ there are $\sqrt{q}+1$ lines meeting $\mathcal{H}$ in a single point; the remaining $q-\sqrt{q}$ lines on Q each meet $\mathcal{H}$ in $\sqrt{q}+1$ points.

Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q)$, q square. We call a line a *tangent* or a *secant* of $\mathcal{H}$ according as it meets $\mathcal{H}$ in one or $\sqrt{q}+1$ points.

Corollary 4 (See Hirschfeld [1998, 7.21]) Let ϕ be a Hermitian polarity with associated Hermitian curve $\mathcal{H}$. If $P \in \mathcal{H}$ then $\phi(P)$ is the unique tangent to $\mathcal{H}$ on P, and if $Q \notin \mathcal{H}$ then $\phi(Q)$ is the secant of $\mathcal{H}$ which contains the points of $\mathcal{H}$ on the $\sqrt{q}+1$ secants on Q.

We therefore see that the set of secants of $\mathcal{H}$ is the set of lines of $\mathrm{PG}(2,q)$ which are not absolute with respect to ϕ. Also, the points of $\mathcal{H}$ which lie on the $\sqrt{q}+1$ tangents through a point $Q \notin \mathcal{H}$ are collinear and in fact form a Baer subline.

Theorem 5 (See Hirschfeld [1998, 7.22]) In $\mathrm{PG}(2,q)$, q a square, the intersection of a Hermitian curve and a Baer subplane contains either 1, $\sqrt{q}+1$ or $2\sqrt{q}+1$ points.

3 Unitals

Inspired by the example of the Hermitian curves in $\mathrm{PG}(2,q)$, we define an object in any projective plane of square order which has the combinatorial properties of a Hermitian curve. We then investigate whether in $\mathrm{PG}(2,q)$ it must be a Hermitian curve and whether there exist examples in other planes.

A *unital* $\mathcal{U}$ in π_q, q square, is a set of $q\sqrt{q}+1$ points, such that each line meets it in either 1 or $\sqrt{q}+1$ points. A line is a *tangent* or a *secant* of $\mathcal{U}$ if it contains 1 or $\sqrt{q}+1$ points of $\mathcal{U}$ respectively.

It turns out that even in the case of non-Desarguesian planes examples of unitals can arise from certain polarities, as follows. Baer [1946] showed that a polarity of π_q has at least $q+1$ absolute points, and if equality occurs then the set of absolute points is a line if q is even and an oval if q is odd. Further, he showed that if there are more than $q+1$ absolute points then q is a square and if every non-absolute line has $\sqrt{q}+1$ absolute points then the set of absolute points is a unital. Seib [1970] improved this result as follows:

Theorem 6 (Seib [1970]) A polarity of π_q has at most $q\sqrt{q}+1$ absolute points and if equality occurs then the set of absolute points is a unital.

This observation has been used to construct unitals in the Figueroa planes of order q^3, q square, as outlined in Section 4.3 below.

Lemma 7 (Tallini Scafati [1966]) Let $\mathcal{U}$ be a unital in π_q, q square. Through a point P of $\mathcal{U}$ there pass one tangent and q secants, while through a point Q not on $\mathcal{U}$ there pass $\sqrt{q}+1$ tangents and $q-\sqrt{q}$ secants. It follows that $\mathcal{U}$ has $q\sqrt{q}+1$ tangents and $q(q-\sqrt{q}+1)$ secants.

Proof First, suppose that $P \in \mathcal{U}$ lies on a tangents and b secants. Then $a+b=q+1$ and $b\sqrt{q}+1 = q\sqrt{q}+1$, and the result follows. Similarly, if $Q \notin \mathcal{U}$ lies on a tangents and b secants then $a+b=q+1$ and $a+b(\sqrt{q}+1) = q\sqrt{q}+1$, and the result follows. Now counting ordered pairs (P,ℓ) where $P \in \mathcal{U} \cap \ell$ and ℓ is a tangent of $\mathcal{U}$ shows that $\mathcal{U}$ has $q\sqrt{q}+1$ tangents. Finally, $\mathcal{U}$ has $q^2+q+1-(q\sqrt{q}+1) = q(q-\sqrt{q}+1)$ secants. $\square$

Corollary 8 (Tallini Scafati [1966]) Let $\mathcal{U}$ be a unital in π_q, q square. The set of tangents of $\mathcal{U}$ form the *dual unital* $\mathcal{U}^d$ in the dual plane π_q^d.

Proof By Lemma 7, $\mathcal{U}^d$ has $q\sqrt{q}+1$ points and a line of π_q^d meets $\mathcal{U}^d$ in either 1 or $\sqrt{q}+1$ points. $\square$

Lemma 9 (Grüning [1987]) Let $\mathcal{U}$ be a unital and let $\mathcal{B}$ be a Baer subplane of π_q, q square. If b denotes the number of lines of $\mathcal{B}$ which, when extended to π_q, are tangents of $\mathcal{U}$ and s denotes the number of points in $\mathcal{B} \cap \mathcal{U}$ then $b+s = 2(\sqrt{q}+1)$. In particular, $0 \leq |\mathcal{B} \cap \mathcal{U}| \leq 2(\sqrt{q}+1)$.

Proof We count in two ways the ordered pairs (P,ℓ) where P is a point of $\mathcal{U}$, ℓ is a line of $\mathcal{B}$ (extended to $\mathrm{PG}(2,q)$) and $P \in \ell$. We obtain $s(\sqrt{q}+1) + (q\sqrt{q}+1-s).1 = b.1 + (q+\sqrt{q}+1-b)(\sqrt{q}+1)$ and the result follows. $\square$

Recall that a *blocking set* in π_q is a set of points which is met by every line, and a blocking set is *minimal* if no point can be deleted to obtain a blocking set with fewer points. Since each line of π_q meets a unital, and each point of a unital lies on a tangent, it is immediate that a unital is a minimal blocking set.

Theorem 10 (Bruen and Thas [1977]) Let $\mathcal{B}$ be a minimal blocking set in a projective plane π_q of order q. Then $|\mathcal{B}| \leq q\sqrt{q}+1$ with equality if and only if $\mathcal{B}$ is a unital.

A (k,n)-*arc* $\mathcal{K}$ *of type* (m,n) in π_q is a set of k points such that each line meets the set in either m or n points, where $m<n$. If $\mathcal{K}$ has an m-secant ℓ then counting the points of $\mathcal{K}$ on the lines through a point of $\mathcal{K} \cap \ell$ shows that $k \leq (n-1)q+m$. We see immediately that a unital is a $(q\sqrt{q}+1, \sqrt{q}+1)$-set of type $(1, \sqrt{q}+1)$, which has $k=(n-1)q+m$. The following theorems show that these properties characterise the set of unitals, Baer subplanes and/or their complements. The proofs are quite straightforward, involving counting appropriate sets in two ways and solving the resulting equations.

Theorem 11 (Tallini Scafati [1966]) Let $\mathcal{K}$ be a (k,n)-arc of type (m,n) in π_q, with $m \neq 0$. If $k = (n-1)q + m$ then $\mathcal{K}$ is either a unital or the complement of a Baer subplane.

Corollary 12 (Tallini Scafati [1966]) A (k,n)-arc of type $(1,n)$ in π_q is either a unital or a Baer subplane.

4 The Known Examples of Unitals

We have already seen one infinite family of unitals, namely the set of absolute points of a unitary polarity of PG$(2,q)$, q square. Such a unital is called a *classical unital.*

4.1 The Buekenhout-Metz Unitals

Buekenhout [1976] proved the existence of unitals in every translation plane π_q of square order q with kernel containing $GF(\sqrt{q})$, as follows.

Let $\pi_q = \pi(\mathcal{S})$ for a spread $\mathcal{S}$ of a hyperplane Σ_∞ of PG$(4,\sqrt{q})$.

Let V and W be points on a line t of $\mathcal{S}$, and let $\mathcal{O}$ be an ovoid in some 3-dimensional subspace of PG$(4,\sqrt{q})$ not containing V and such that $\mathcal{O}$ meets Σ_∞ in exactly the point W. Let $\mathcal{U}^*$ be the cone with vertex V projecting $\mathcal{O}$, that is, $\mathcal{U}^*$ is the union of the lines joining V to each point of $\mathcal{O}$. Equivalently, let V be a point on a line t of $\mathcal{S}$ and let $\mathcal{O}$ be an ovoid of the quotient space with respect to V which meets Σ_∞ in exactly t (so $\mathcal{O}$ is a set of $q+1$ lines through V, no three of which lie in a plane through V, and $\mathcal{O}$ contains t). In either case, let $\mathcal{U}^*$ be the union of the lines so constructed. Then the set $\mathcal{U}$ of points of π_q corresponding to $\mathcal{U}^*$ is a unital, with ℓ_∞ tangent at a point T.

We verify directly that $\mathcal{U}$ is a unital in π_q, using the representation in PG$(4,\sqrt{q})$. A plane of PG$(4,\sqrt{q})$ on t and not contained in Σ_∞ contains exactly one line of $\mathcal{U}^*$, hence a line of π_q on T contains exactly $\sqrt{q}+1$ points of $\mathcal{U}$. A plane of PG$(4,\sqrt{q})$ not contained in Σ_∞ and not on t meets $\mathcal{U}^*$ in either 1 point or the $\sqrt{q}+1$ points of an oval, hence a line in π_q not on T contains 1 or $\sqrt{q}+1$ points of $\mathcal{U}$.

We recall that all known ovoids in PG$(3,\sqrt{q})$ are either elliptic quadrics (which exist for all q) or Tits ovoids (which exist for $q > 2$ even and not a square) (see Hirschfeld [1985], 16.4). However the classification of ovoids in PG$(3,\sqrt{q})$ is complete only for all $\sqrt{q}$ odd and for $\sqrt{q}$ even with $\sqrt{q} \leq 32$ (Barlotti [1955], Panella [1955], Fellegara [1962], O'Keefe and Penttila [1990, 1992], O'Keefe, Penttila and Royle [1994] and Penttila and Praeger [1998]; see Hirschfeld [1985]).

Buekenhout [1976] showed that the classical unital in PG$(2,q)$ arises by this construction with $\mathcal{S}$ a regular spread and $\mathcal{O}$ an elliptic quadric. Thus, choosing $\mathcal{S}$ to be regular and $\mathcal{O}$ to be a Tits ovoid gives rise to a non-classical unital in PG$(2,q)$ for $\sqrt{q} > 2$ even and not a square. Metz [1979] later observed that for any $\sqrt{q} > 2$ it is possible to choose a regular spread $\mathcal{S}$ and an elliptic quadric $\mathcal{O}$ such that Buekenhout's construction gives a non-classical unital in $\mathrm{PG}(2,q)$.

A *Buekenhout-Metz unital* in a translation plane π_q of square order q with kernel containing $\mathrm{GF}(\sqrt{q})$ is a unital which arises by this construction for any choice of spread $\mathcal{S}$ and ovoid $\mathcal{O}$. Since the classical unital in $\mathrm{PG}(2,q)$ can be constructed in this way, it is included in the class of Buekenhout-Metz unitals.

Baker and Ebert [1992] and Ebert [1992, 1997a], extending results of Hirschfeld and Szőnyi [1991], gave coordinates for the Buekenhout-Metz unitals in $\mathrm{PG}(2,q)$ with underlying ovoid an elliptic quadric or a Tits ovoid. These representations also allowed the calculation of the number of inequivalent such unitals and their collineation groups.

Theorem 13 (Baker and Ebert [1992], Ebert [1992], Hirschfeld and Szőnyi [1991]) Let q be a square and for $a,b \in \mathrm{GF}(q)$ let $U_{ab} = \{(1,y,ay^2+by^{\sqrt{q}+1}+r) : y \in \mathrm{GF}(q), r \in \mathrm{GF}(\sqrt{q})\} \cup \{(0,0,1)\}$. Then $\mathcal{U}_{ab}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$ if and only if:
(i) q is odd and $4a^{\sqrt{q}+1}+(b^{\sqrt{q}}-b)^2$ is a non-square in $\mathrm{GF}(\sqrt{q})$
(ii) q is even, $b \notin \mathrm{GF}(\sqrt{q})$ and $a^{\sqrt{q}+1}/(b^{\sqrt{q}}+b)^2$ has trace zero in $\mathrm{GF}(q)$.
Further, every Buekenhout-Metz unital in $\mathrm{PG}(2,q)$ constructed from an elliptic quadric can be expressed as U_{ab} for some a,b. In particular, U_{ab} contains an oval if and only if q is odd and $b \in \mathrm{GF}(q)$. In this case U_{ab} is the union of a partial pencil of q conics.

Theorem 14 (Ebert [1997a]) Let q be an even square such that $\sqrt{q} > 8$ is a non-square, and let $\sigma \in \mathrm{AutGF}(\sqrt{q})$ be such that $\sigma^2 \equiv 2 \pmod{\sqrt{q}}$. Let β be a primitive element of $\mathrm{GF}(q)$ such that $\beta^{\sqrt{q}+1}$ is a primitive element of $\mathrm{GF}(q)$. Let $\epsilon \in \mathrm{GF}(q) \setminus \mathrm{GF}(\sqrt{q})$ be such that $\epsilon^{\sqrt{q}} = 1+\epsilon$ and $\epsilon^2 = \lambda+\epsilon$ for some $\lambda \neq 1$ with $\mathrm{trace}(\lambda)=1$; so $\{1,\epsilon\}$ is a basis for $\mathrm{GF}(q)$ over $\mathrm{GF}(\sqrt{q})$. The set of points $\mathcal{U}_T = \{(1, s+t\epsilon, r+(s^{\sigma+2}+t^{\sigma}+st)\epsilon) : r,s,t \in \mathrm{GF}(\sqrt{q})\} \cup \{(0,0,1)\}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$ constructed from a Tits ovoid.

For completeness, we give the following theorem which should be compared with Lemma 9 and Theorem 5.

Theorem 15 (Quinn [1995]) Let $\mathcal{U}$ be a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$, $q > 169$ a square, arising from an elliptic quadric. If $\mathcal{B}$ is a Baer subplane of $\mathrm{PG}(2,q)$ then $1 \leq |\mathcal{U} \cap \mathcal{B}| \leq 2\sqrt{q}+1$.

It is interesting to note that computer programs have shown that for small values of q the intersection numbers 0 and $2(\sqrt{q}+1)$ also occur.

4.2 The Buekenhout Unitals

Let π_q be a translation plane of square order q whose kernel contains $\mathrm{GF}(\sqrt{q})$. Suppose that $\pi_q = \pi(\mathcal{S})$ for some spread $\mathcal{S}$ in the hyperplane $\Sigma_\infty = PG(3,\sqrt{q})$ of $\mathrm{PG}(4,\sqrt{q})$. We denote the line at infinity of π_q by ℓ_∞. Suppose further that $\mathcal{S}$ contains a regulus $\mathcal{R}$. We remark that the assumption that $\mathcal{S}$ contain a regulus is equivalent to assuming that the plane $\pi(\mathcal{S})$ is *derivable,* see Section 4.4 below.

Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(4, \sqrt{q})$ which meets Σ_∞ in a non-singular hyperbolic quadric, one of whose sets of generators is the regulus $\mathcal{R}$. Then the set $\mathcal{U}$ of points of π_q corresponding to $\mathcal{Q}$ is a unital, with ℓ_∞ a secant line.

We verify directly that $\mathcal{U}$ is a unital in π_q, using the representation in $\mathrm{PG}(4, \sqrt{q})$. First, a plane of $\mathrm{PG}(4, \sqrt{q})$ containing a line of $\mathcal{Q}$ meets $\mathcal{Q}$ in either just the points of that line or in the points on a pair of lines. Also, a plane of $\mathrm{PG}(4, \sqrt{q})$ containing a line disjoint from $\mathcal{Q}$ meets $\mathcal{Q}$ in either a single point or a conic. Therefore a plane of $\mathrm{PG}(4, \sqrt{q})$ on a line t of $\mathcal{R}$ and not contained in Σ_∞ meets $\mathcal{Q}$ either just in t or in t together with a further line; hence a line in π_q on a point of $\mathcal{U} \cap \ell_\infty$ contains either 1 or $\sqrt{q}+1$ points of $\mathcal{U}$. A plane of $\mathrm{PG}(4, \sqrt{q})$ not contained in Σ_∞ on a line t of $\mathcal{S} \setminus \mathcal{R}$ meets $\mathcal{Q}$ in either a single point or a conic disjoint from Σ_∞; hence a line in π_q on a point of $\ell_\infty \setminus \mathcal{U}$ contains either 1 or $\sqrt{q}+1$ points of $\mathcal{U}$.

A unital constructed in this way in a derivable translation plane π_q of square order q and kernel containing $\mathrm{GF}(\sqrt{q})$ is called a *Buekenhout unital.*

4.3 The Hamilton-de Resmini Unitals

Hamilton and de Resmini [1998] construct a polarity in each Figueroa plane $\mathrm{Fig}(q^3)$ of order q^3, $q > 4$ square, and show that it has $q^4\sqrt{q}+1$ absolute points. Theorem 6 then guarantees that the set of absolute points is a unital. The method is essentially the same as the one used by Julia Brown to construct ovals via appropriate polarities in the Figueroa planes.

The (finite) Figueroa planes form an infinite family of projective planes of order q^3, where $q > 2$ is a prime power. The original construction due to Figueroa [1982] had a further restriction on the value of q, but this was removed by Hering and Schaeffer [1982]. Grundhöfer [1986] later gave a synthetic construction, which we now describe.

Let q be any prime power and let α be a planar collineation of $\mathrm{PG}(2, q^3)$ of order 3, that is, the set of fixed points form a Desarguesian subplane of order q. With respect to α, the points and lines of $\mathrm{PG}(2, q^3)$ each fall into three disjoint classes according to the structure of their orbit under powers of α. A point P has type I if $\alpha(P) = P$; type II if $P, \alpha(P), \alpha^2(P)$ are distinct and collinear and type III if $P, \alpha(P), \alpha^2(P)$ are distinct and not collinear. The line types are defined dually.

If P is a point of type III then let $\mu(P) = \alpha(P)\alpha^2(P)$ and if ℓ is a line of type III then let $\mu(\ell) = \alpha(\ell) \cap \alpha^2(\ell)$. It is straightforward to verify that μ is an involutory bijection mapping the points of type III to the lines of type III and conversely. Let $\mathcal{P}$ and $\mathcal{L}$ denote the sets of points and lines of $\mathrm{PG}(2, q^3)$, respectively. Define a new incidence structure $\mathrm{Fig}(q^3)$ with *points* $\mathcal{P}$, *lines* $\mathcal{L}$ and *incidence* as follows: a point P and a line ℓ are incident

in Fig(q^3) if:
(i) $\mu(\ell) \in \mu(P)$ in the case that P and ℓ are both of type III
(ii) $P \in \ell$ otherwise.
Then Fig(q^3) is a projective plane of order q^3, the *Figueroa* plane, and is non-Desarguesian if $q > 2$.

Theorem 16 (Hamilton and de Resmini [1998]) Let q be a square and let Fig(q^3) be the Figueroa plane constructed using the planar collineation $P \mapsto P^q$ of PG($2, q^3$) as above. The map ϕ which maps the point with coordinates X to the line with coordinates $X^{q\sqrt{q}}$ and the line with coordinates Y to the point with coordinates $Y^{q\sqrt{q}}$ is a polarity of Fig(q^3) with $q^4\sqrt{q} + 1$ absolute points, which therefore form a unital in Fig(q^3).

We remark that, as in Section 2, ϕ is a Hermitian polarity of PG($2, q^3$) and hence its absolute points and absolute lines are the points and tangents of a classical unital $\mathcal{U}$ in PG($2, q^3$).

Question: Does the Hamilton-de Resmini Unital have the same points as this classical unital? If so, then it is really an inherited unital in the sense of Section 4.4.

4.4 Inherited Unitals

A unital in a given projective plane often gives rise to a unital in a closely-related plane. In particular, if two planes are defined on almost the same point sets then sometimes most of the points of a unital in one plane are most of the points of a unital in the second plane. We call these unitals *inherited,* and in this section we discuss such unitals.

Hall planes

The first unitals constructed in Hall planes arise from the original construction of Buekenhout [1976]. Further examples were found by Brouwer [1981].

Now, we investigate unitals arising in projective planes which are obtained from PG($2, q$) by derivation as follows. Let π_q be a projective plane of square order q, let ℓ_∞ be a line of π_q and let $\mathcal{A}$ be the affine plane $\pi_q \setminus \ell_\infty$. A set $\mathcal{D}$ of $\sqrt{q} + 1$ points of ℓ_∞ is called a *derivation set* if for each pair of points $X, Y \in \mathcal{A}$ such that the line XY meets ℓ_∞ in a point of $\mathcal{D}$, there is a Baer subplane of π_q containing X, Y and $\mathcal{D}$. (This Baer subplane is necessarily unique.) Given a derivation set $\mathcal{D}$ for π_q, there is an affine plane $\mathcal{A}^{\mathcal{D}}$ of order q with *points* the points of $\mathcal{A}$ and *lines* each set of points of $\mathcal{A}$ on a line of π_q meeting $\ell_\infty \setminus \mathcal{D}$ together with each set of points of $\mathcal{A}$ on a Baer subplane of π_q containing $\mathcal{D}$. The unique completion of this affine plane to a projective plane by the addition of the points on a line $\ell_\infty^{\mathcal{D}}$ is the plane $\pi_q^{\mathcal{D}}$ *derived* from π_q with respect to $\mathcal{D}$. In the following we will regard π_q and $\pi_q^{\mathcal{D}}$ as having the same affine points and we will also identify the points of $\ell_\infty \setminus \mathcal{D}$ in π_q with the corresponding points of $\ell_\infty^{\mathcal{D}}$ in $\pi_q^{\mathcal{D}}$. (For a full discussion of derivation see Hughes and Piper [1973].)

It is easily seen that a projective plane π_q represented as $\pi(\mathcal{S})$ where $\mathcal{S}$ contains a regulus $\mathcal{R}$ is derivable. In fact $\mathcal{R}$ is a derivation set, and in the process of derivation the lines of π_q which are planes on lines of $\mathcal{R}$ are replaced by the planes on lines of the opposite regulus $\mathcal{R}^{opp}$, which were the Baer subplanes in π. It is clear then that certain Buekenhout-Metz and Buekenhout unitals in $\mathrm{PG}(2,q)$ should inherit to unitals in the Hall plane $\mathrm{PG}(2,q)^d = \mathrm{H}(q)$.

Several authors have verifed that indeed this is the case, for various unitals and various choices of derivation set see Barlotti and Lunardon [1979], Barwick [1996, 1997], Buekenhout [1976] and Grüning [1987]. Some of these constructions were repeated independently by Dover [1997] and Rinaldi [1995b, 1995a].

A first general approach to constructing unitals in derived planes, which includes all these previous examples, relies on the characterisation of unitals among blocking sets (Theorem 10).

Theorem 17 (Blokhuis and O'Keefe [1998]) Let π_q be a projective plane of square order q and let $\mathcal{D}$ be a derivation set in π_q.
(i) If there is a unital $\mathcal{U}$ in π_q such that $\mathcal{U} \cap \mathcal{D} = \emptyset$ and each Baer subplane of π_q containing $\mathcal{D}$ meets $\mathcal{U}$ in 1 or at least $\sqrt{q}+1$ points, then there exists a unital in the derived plane $\pi_q^{\mathcal{D}}$ with the same points as $\mathcal{U}$.
(ii) If there is a unital $\mathcal{U}$ in π_q such that $\mathcal{U} \cap \mathcal{D} = \mathcal{D}$ and each Baer subplane of π_q containing $\mathcal{D}$ meets $\mathcal{U}$ in $\sqrt{q}+1$ or at least $2\sqrt{q}+1$ points, then there exists a unital in the derived plane $\pi_q^{\mathcal{D}}$ with the same affine points as $\mathcal{U}$.

Proof Let $\mathcal{A} = \pi_q \setminus \ell_\infty$, where ℓ_∞ is the line containing $\mathcal{D}$, and let $\mathcal{U}' = \mathcal{U} \setminus \ell_\infty$.
(i) Let $\mathcal{V}'$ be the set of points $P' \in \ell_\infty^{\mathcal{D}}$ which correspond to the points $P \in \ell_\infty \cap \mathcal{U}$ (so $|\mathcal{V}'| = 1$ or $\sqrt{q}+1$) and let $\mathcal{U}^{\mathcal{D}} = \mathcal{U}' \cup \mathcal{V}'$. Then $\mathcal{U}^{\mathcal{D}}$ is a set of $q\sqrt{q}+1$ points in $\pi_q^{\mathcal{D}}$ which meets every line of $\pi_q^{\mathcal{D}}$ in 1 or at least $\sqrt{q}+1$ points. Thus $\mathcal{U}^{\mathcal{D}}$ is a blocking set. It is minimal, for otherwise there is a point $P \in \mathcal{U}^{\mathcal{D}}$ on no tangent, implying $|\mathcal{U}^{\mathcal{D}}| \geq (q+1)\sqrt{q}+1$. By Lemma 10, $\mathcal{U}^{\mathcal{D}}$ is a unital in $\pi_q^{\mathcal{D}}$.
(ii) By assumption $\mathcal{U}'$ has 1 or $\sqrt{q}+1$ points on each line of π_q, other than ℓ_∞, meeting $\ell_\infty \setminus \mathcal{D}$ and 0 or at least $\sqrt{q}$ points on each Baer subplane of π_q on $\mathcal{D}$. The set of Baer subplanes of π_q containing $\mathcal{D}$ is a set of $\sqrt{q}+1$ parallel classes of lines in the affine plane $\mathcal{A}^{\mathcal{D}}$ derived from $\mathcal{A}$; so let $\mathcal{D}'$ be the set of $\sqrt{q}+1$ points of $\ell_\infty^{\mathcal{D}}$ which lie on these lines in the projective completion $\pi_q^{\mathcal{D}}$. It follows that $\mathcal{U}^{\mathcal{D}} = \mathcal{U}' \cup \mathcal{D}'$ is a blocking set of $q\sqrt{q}+1$ points in $\pi_q^{\mathcal{D}}$ which meets every line in 1 or at least $\sqrt{q}+1$ points. As in the proof of part (1), $\mathcal{U}^{\mathcal{D}}$ is a unital. □

An alternative general approach to constructing unitals in derived planes verifies the properties of a unital directly.

Theorem 18 (Blokhuis and O'Keefe [1998]) Let π_q be a projective plane of square order q and let $\mathcal{D}$ be a derivation set on a line ℓ_∞ in π_q. Let $\mathcal{U}$ be a unital in π_q such that $\mathcal{U} \cap \mathcal{D} = \emptyset$,

each Baer subplane of π_q containing $\mathcal{D}$ meets $\mathcal{U}$ in at least 1 point and
(i) if ℓ_∞ is a secant of $\mathcal{U}$ then at least $(\sqrt{q}+1)^2$ Baer subplanes on $\mathcal{D}$ meet $\mathcal{U}$ in exactly one point,
(ii) if ℓ_∞ is a tangent of $\mathcal{U}$ then at least $\sqrt{q}(\sqrt{q}+1)$ Baer subplanes on $\mathcal{D}$ meet $\mathcal{U}$ in exactly one point.
Then there exists a unital in the derived plane $\pi_q^{\mathcal{D}}$ with the same points as $\mathcal{U}$.

Proof Let $\mathcal{U}^{\mathcal{D}}$ be defined as in the proof of Theorem 17. In order to prove that $\mathcal{U}^{\mathcal{D}}$ is a unital in $\pi_q^{\mathcal{D}}$, it suffices to show that each Baer subplane on $\mathcal{D}$ meets $\mathcal{U}$ in either 1 or $\sqrt{q}+1$ points.
Let $\mathcal{S} = \{\mathcal{B}_1, \ldots, \mathcal{B}_s\}$, where $s = q(\sqrt{q}+1)$ be the Baer subplanes on $\mathcal{D}$ and let $t_i = |\mathcal{B}_i \cap \mathcal{U}| \geq 1$. (i) Counting the ordered pairs $(P, \mathcal{B})$ where $P \in \mathcal{U}, \mathcal{B} \in \mathcal{S}$ and $P \in \mathcal{B}$ yields:

$$\sum_{i=1}^{s} t_i = (q\sqrt{q} - \sqrt{q})(\sqrt{q}+1)$$

and counting the ordered triples $(P, Q, \mathcal{B})$ where $P, Q \in \mathcal{U}, \mathcal{B} \in \mathcal{S}$ and $P, Q \in \mathcal{B}$ yields:

$$\sum_{i=1}^{s} t_i(t_i - 1) = (\sqrt{q}+1)(q - \sqrt{q} - 1)(\sqrt{q}+1)\sqrt{q}$$

(this is the number of secants distinct from ℓ_∞ on a point of $\mathcal{D}$ multiplied by the number of ordered pairs of points of $\mathcal{U}$ on such a secant). Thus we obtain:

$$\sum_{i=1}^{s}(t_i - (\sqrt{q}+1))^2 = q(\sqrt{q}+1)^2.$$

Now there are at least $(\sqrt{q}+1)^2$ Baer subplanes $\mathcal{B}_i \in \mathcal{S}$ with $t_i = 1$, and $(\sqrt{q}+1)^2$ of them contribute $q(\sqrt{q}+1)^2$ to the sum. The remaining Baer subplanes in $\mathcal{S}$ therefore contribute zero to the sum, and hence intersect $\mathcal{U}$ in exactly $\sqrt{q}+1$ points.
(ii) The argument in this case is analogous, with

$$\sum_{i=1}^{s} t_i = q\sqrt{q}(\sqrt{q}+1) \quad \text{and} \quad \sum_{i=1}^{s} t_i(t_i - 1) = (\sqrt{q}+1)(q - \sqrt{q})(\sqrt{q}+1)\sqrt{q}.$$

□

Theorem 19 (Blokhuis and O'Keefe [1998]) Let π_q be a projective plane of square order q and let $\mathcal{D}$ be a derivation set in π_q. If there is a unital $\mathcal{U}$ in π_q such that $\mathcal{U} \cap \mathcal{D} = \mathcal{D}$, each Baer subplane of π_q containing $\mathcal{D}$ meets $\mathcal{U}$ in at least $\sqrt{q}+1$ points and there are at least $\sqrt{q}+1$ Baer subplanes on $\mathcal{D}$ that meet $\mathcal{U}$ in exactly $\sqrt{q}+1$ points, then there exists a unital in the derived plane $\pi_q^{\mathcal{D}}$ with the same affine points as $\mathcal{U}$.

Proof Let $\mathcal{U}^{\mathcal{D}}$ be as in the proof of Theorem 17. It suffices to show that each Baer subplane on $\mathcal{D}$ meets $\mathcal{U} \setminus \ell_\infty$ in 0 or $\sqrt{q}$ points of $\pi_q \setminus \ell_\infty$. Our argument is again analogous to that given in Theorem 18(i), but with $t_i = |\mathcal{B}_i \cap (\mathcal{U} \setminus \ell_\infty)| \geq 0$. We find that

$$\sum_{i=1}^{s} t_i = (q\sqrt{q} - \sqrt{q})(\sqrt{q}+1) \quad \text{and} \quad \sum_{i=1}^{s} t_i(t_i - 1) = (\sqrt{q}+1)(q-1)\sqrt{q}(\sqrt{q}-1).$$

$\square$

These theorems can be used to give short proofs of some known constructions of unitals in Hall planes mentioned above, as follow.

Theorem 20 (Buekenhout [1976], Grüning [1987]) Let $\mathcal{U}$ be a classical unital in a Desarguesian plane $\pi_q = \mathrm{PG}(2,q)$. Let $\mathcal{D}$ be a Baer subline comprising the points of $\mathcal{U}$ on a secant line to $\mathcal{U}$. Theorem 17 or 19 guarantees the existence of a unital in the Hall plane $\mathrm{PG}(2,q)^{\mathcal{D}}$, with the translation line a secant and with the same affine points as $\mathcal{U}$.

Theorem 21 (Barwick [1996], see also Rinaldi [1995a]) Let $\mathcal{U}$ be a classical unital in a Desarguesian plane $\pi_q = \mathrm{PG}(2,q)$. Let $\mathcal{D}$ be a Baer subline comprising points on a secant line to $\mathcal{U}$, disjoint from $\mathcal{U}$. Theorem 17 or 18 guarantees the existence of a unital in the Hall plane $\mathrm{PG}(2,q)^{\mathcal{D}}$, with the translation line a secant and with the same affine points as $\mathcal{U}$.

Theorem 22 (Barwick [1997], see also Buekenhout [1976], Rinaldi [1995b]) Let $\mathcal{U}$ be a Buekenhout-Metz unital (that is, a unital constructed as in [1976, Remark 4]) in a Desarguesian plane $\pi_q = \mathrm{PG}(2,q)$, with respect to the tangent ℓ_∞, and let $\mathcal{D}$ be a Baer subline on ℓ_∞, disjoint from $\mathcal{U}$. Theorem 17 or 18 guarantees the existence of a unital in the Hall plane $\mathrm{PG}(2,q)^{\mathcal{D}}$, with the translation line a tangent and with the same affine points as $\mathcal{U}$.

Theorem 23 (Barlotti and Lunardon [1979], see also Dover [1997]) Let $\mathcal{U}$ be a Buekenhout-Metz unital in a Desarguesian plane $\pi_q = \mathrm{PG}(2,q)$, with respect to the tangent ℓ_∞. Let $T = \ell_\infty \cap \mathcal{U}$ and let $\mathcal{D}$ be a Baer subline comprising the points of $\mathcal{U}$ on a secant line through T. Theorem 17 or 18 guarantees the existence of a unital in the Hall plane $\mathrm{PG}(2,q)^{\mathcal{D}}$, with the translation line a tangent and with the same affine points as $\mathcal{U}$.

Hall and dual Hall planes

Brouwer [1981] constructed a unital in the Hall plane of order 9 and its dual. This example was generalized to any Hall plane and its dual by Grüning [1987].

Derived dual translation planes

Barlotti and Lunardon [1979] showed that inherited unitals exist in derived dual translation planes.

Hughes planes and derived Hughes planes

Let q be a square odd prime power and let $\mathrm{GF}(\sqrt{q})$ be a subfield of $\mathrm{GF}(q)$. Let ω be a primitive element of $\mathrm{GF}(q)$, and let an element $x = \omega^r \in \mathrm{GF}(q) \setminus \{0\}$ be *even* if r is even and *odd* if r is odd. Define a new multiplication $\cdot$ on $\mathrm{GF}(q)$ as follows:

$$\begin{aligned} a \cdot b &= ab \quad \text{if } a \text{ is even} \\ &= ab^{\sqrt{q}} \quad \text{if } a \text{ is odd} \\ 0 \cdot b &= a \cdot 0 = 0. \end{aligned}$$

On the set $\mathcal{P} = \mathrm{GF}(q)^3 \setminus \{(0,0,0)\}$ we define an equivalence relation by: $(x,y,z) \equiv (x',y',z')$ if $(x,y,z) = k \cdot (x',y',z')$ for some $k \in \mathrm{GF}(q)$. Consider the incidence structure $\mathcal{H}(q)$ with points $\mathcal{P}$ and lines $\mathcal{P}$, and such that the point $P = (x,y,z) \in \mathcal{P}$ is incident with the line $\ell = [u,v,w] \in \mathcal{P}$ if and only if $xu_1 + yv_1 + zw_1 + (xu_2 + yv_2 + zw_2) = 0$ where $u = u_1 + tu_2, v = v_1 + tv_2$ and $w = w_1 + tw_2$ for some $t \in \mathrm{GF}(q) \setminus \mathrm{GF}(\sqrt{q})$. It is straightforward to verify that $\mathcal{H}(q)$ is a projective plane; in fact it is the Hughes plane of order q and is non-Desarguesian.

Brouwer [1981, 1989] constructed 4 non-isomorphic unitals in the Hughes plane of order 9.

Rosati [1988] later constructed an infinite family of unitals in the Hughes planes, and Brouwer showed that it was different from the four known unitals in the Hughes plane of order 9.

Theorem 24 Rosati [1988] The set of points $\{(x,y,z) : x^{\sqrt{q}+1} + y^{\sqrt{q}+1} + z^{\sqrt{q}+1} = 0\}$ is a unital in the Hughes plane of square odd order q.

Wantz [1996] developed a new coordinatisation for the Buekenhout-Metz unitals in $\mathrm{PG}(2,q)$, q odd, constructed from an elliptic quadric. The advantage of this new representation is that it is easy to prove that the same set of points is a unital in the associated Hughes plane of the same order. Also, the collineation groups and projective equivalence classes were calculated.

Theorem 25 (Wantz [1996]) Let q be a square and let $a \in \mathrm{GF}(\sqrt{q})$ and $b \in \mathrm{GF}(q)$ be such that $b^2 - a^2$ is a square in $\mathrm{GF}(\sqrt{q})$. Let β be a primitive element of $\mathrm{GF}(q)$ and let $\epsilon = \beta^{(\sqrt{q}+1)/2}$; so $\{1, \epsilon\}$ is a basis for $\mathrm{GF}(q)$ over $\mathrm{GF}(\sqrt{q})$. The set of points $U_{(a,b)} =$

$\{(1, y, ay^2 + by^{\sqrt{q}+1} + t\epsilon) : y \in \mathrm{GF}(q), t \in \mathrm{GF}(\sqrt{q})\} \cup \{(0,0,1)\}$ is a unital in the Hughes plane of order q, which is projectively equivalent to a Rosati unital if and only if $a = 0$.

Abatangelo, Larato and Rosati [1990] showed the existence of unitals in derived Hughes planes.

4.5 Other (non-embedded) Unitals

Just for this section, a *unital* with parameter u is any 2–$(u^3+1, u+1, 1)$ design. We will call the blocks of the unital *lines.* It is straightforward to verify that a unital with parameter u has a total of $u^2(u^2-u+1)$ lines and on each point there are u^2 lines.

Ree Unitals

For any integer $e \geq 0$ and $\sqrt{q} = 3^{2e+1}$, there exists a Ree group ${}^2G_2(\sqrt{q})$ Ree [1961] with a 2-transitive action on $q\sqrt{q}+1$ points. Any pair of points is fixed by a unique involution and such an involution fixes exactly $\sqrt{q}+1$ points, which we call a *block*. Lüneburg [1966] showed that the set of blocks on the $q\sqrt{q}+1$ points is a unital, which is not embeddable into a projective plane of order q such that the automorphisms of the unital are induced by collineations of the plane.

Mathon Unitals

Mathon [1987] gives a construction for a family of designs, one instance of which provides a unital of parameter 6. The base blocks which generate such a design are listed in Mathon [1987] in the remarks following Corollary 4.

Bagchi and Bagchi Unitals

Let u be an integer such that $P = u+1$ and $Q = u^2-u+1$ are prime powers. Let G_Q be the multiplicative group of $\mathrm{GF}(Q)$ and let $G_Q(M)$ (where $M \mid Q-1$) be the unique subgroup of G_Q of order M. Let $f\colon G_Q(P-1) \to G_P\left(\frac{P-1}{2}\right)$ be an epimorphism. Let $X = \mathrm{GF}(P) \times \mathrm{GF}(Q)$ and define

$$\begin{aligned} A_0 &= \{(f(x), x) : x \in G_Q(P-1)\} \cup \{(0,0)\} \\ A_j &= (1, \gamma^j)A_0 \qquad 0 \leq j < u-1 \qquad \gamma = \text{generator of } G_Q(P-2) \\ A_\infty &= \mathrm{GF}(P) \times \{0\}. \end{aligned}$$

Define the incidence structure D with *points* the elements of X, *blocks* the distinct additive translates of A_∞ and A_j for $0 \leq j < u-1$. It is straightforward to verify that each block has P points and D has $(u-1)PQ + 1.Q = u^4 - u^3 + u^2$ blocks. In fact, D is a 1-design with $P(u-1)+1 = u^2$ blocks on a point. Further, Bagchi and Bagchi [1989] have given conditions under which D is a 2-design; in that case D is a unital $\mathcal{U}(u)$ of parameter u. They further showed that designs $\mathcal{U}(u)$ with parameters $u = 2, 4$ and 6 exist.

The unital $\mathcal{U}(2)$ is the classical unital with parameter 2 and is embeddable in $\mathrm{PG}(2, 2^2)$. Jackson, O'Keefe and Penttila [1990] showed that $\mathcal{U}(4)$ is not embeddable in any projective plane of order $4^2 = 16$, while Jackson, Piper and Wild [1991] showed that $\mathcal{U}(6)$ cannot be embedded in any projective plane of order 36 such that the automorphisms of the design are induced by collineations of the plane.

Brouwer's Unitals

Brouwer [1981] used a computer search to construct 138 pairwise non-isomorphic unitals with parameter 3. Only 11 of these were embeddable in a projective plane of order 9; two in the Desarguesian plane, four in the Hughes plane, two in the Hall plane, two in the dual Hall plane and one embeddable in both the Hall plane and its dual. (The unitals in $\mathrm{PG}(2,9)$ are the classical unital and the non-classical Buekenhout-Metz unital which is a union of conics discovered by Hirshfeld and Szőnyi [1991]. The unital embeddable in both the Hall plane and its dual was generalized to an infinite family by Grüning [1987]. Brouwer showed that Rosati's unital in the Hughes plane was not among his four examples and gave an algebraic description for one of his unitals in Brouwer [1989].)

5 Characterisation Results

We note that every unital in $\mathrm{PG}(2,2^2)$ is classical and hence is a Buekenhout-Metz unital. Consequently, in the following we will consider unitals in $\mathrm{PG}(2,q)$ where $\sqrt{q} > 2$. A recent computer search by Penttila and Royle [1995] concludes that every unital in $\mathrm{PG}(2,3^2)$ is a Buekenhout-Metz unital.

There are very few known constructions of unitals in $\mathrm{PG}(2,q)$. In fact, the only unitals in $\mathrm{PG}(2,q)$ known to the author are the classical, Buekenhout-Metz and Buekenhout unitals, and the duals of these unitals. We note, in particular, that the unitals investigated by Baker and Ebert [1990, 1992] Ebert [1992, 1997a] and by Hirschfeld and Szőnyi [1991] are shown, in the respective papers, to be Buekenhout-Metz unitals.

It is well-known that the dual of a classical unital in $\mathrm{PG}(2,q)$ is classical. Further, Barwick [1994] has shown that a Buekenhout unital in $\mathrm{PG}(2,q)$ is classical. Further:

Theorem 26 (Baker and Ebert [1992], Ebert [1992] and Casse, O'Keefe and Penttila [1996]) The dual of a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$ is a Buekenhout-Metz unital.

Baker and Ebert [1992] showed that each Buekenhout-Metz unital in $\mathrm{PG}(2,q)$, q odd, is isomorphic to its dual. Ebert [1992] proved the same result for those Buekenhout-Metz unitals in $\mathrm{PG}(2,q)$, q even, for which the underlying ovoid is an elliptic quadric. Casse, O'Keefe and Penttila [1996] provide a proof which is valid for any q and any underlying ovoid.

Putting together all these results shows that in $\mathrm{PG}(2,q)$ every known unital is a Buekenhout-Metz unital.

5.1 Characterisations of the Classical Unital

Theorem 27 (Tallini Scafati [1966]) Let $\mathcal{U}$ be a unital in π_q, q square. For $P_i \in \mathcal{U}$, denote the tangent to $\mathcal{U}$ at P_i by ℓ_i. Suppose that $\mathcal{U}$ is *reciprocal,* that is, for every $P_1, P_2, P_3, P_4 \in \mathcal{U}$ with no three collinear and $\ell_1 \cap \ell_2 \in P_3P_4$ we have $\ell_3 \cap \ell_4 \in P_1P_2$. Then $\mathcal{U}$ is the set of absolute points of a polarity. Thus if $\pi_q = \mathrm{PG}(2, q)$ then $\mathcal{U}$ is classical.

Theorem 28 (Faina and Korchmáros [1983], Lefèvre-Percsy [1982]) In $\mathrm{PG}(2, q)$, $q > 4$ square, a unital $\mathcal{U}$ such that each secant meets it in a Baer subline is classical.

A very recent result improves this characterisation in the case of $q = p^2$, p prime, by showing that assuming fewer than all $q^2 - q\sqrt{q} + q$ secants are Baer sublines suffices.

Theorem 29 (Ball, Blokhuis and O'Keefe [1998]) In $\mathrm{PG}(2, q)$, where $q = p^2$ and p is a prime, a unital $\mathcal{U}$ such that each of $q\sqrt{q} = p^3$ secants meets it in a Baer subline is classical.

The next result shows that the property that the points of a classical unital on the tangents through a point off it are collinear characterises the classical unital. Since the dual of a classical unital is classical, this statement is equivalent to: the tangents at collinear points of a classical unital are concurrent.

Theorem 30 (Thas [1992]) In $\mathrm{PG}(2, q)$, $q > 4$ square, a unital $\mathcal{U}$ such that the tangents at collinear points are concurrent is classical.

Theorem 31 (Barwick [1994]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q)$, q square, and let ℓ be a secant of $\mathcal{U}$. If every Baer subline with a point on ℓ meets $\mathcal{U}$ in either $0, 1, 2$ or $\sqrt{q} + 1$ points then $\mathcal{U}$ is classical.

The next characterisation is a corollary of Theorem 38 below. It is the converse of Theorem 5.

Corollary 32 (Barwick, O'Keefe and Storme [1997]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q)$, q square. Then $\mathcal{U}$ is classical if and only if every Baer subplane meets $\mathcal{U}$ in either $1, \sqrt{q} + 1$ or $2\sqrt{q} + 1$ points.

O'Nan [1972] showed that the classical unital does not contain a configuration of four distinct lines which meet in six points (an *O'Nan configuration*). Piper [1972] conjectured that this property characterises the classical unital, and Wilbrink [1983] proved that this is true if q is even and the unital enjoys a further property:

Theorem 33 Wilbrink [1983] Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q)$, where q is an even square. Suppose that:
(i) there is no O'Nan configuration with points points of $\mathcal{U}$
(ii) for each secant ℓ of $\mathcal{U}$, point $X \in \mathcal{U} \setminus \ell$ and secant m through X and meeting ℓ in a point of $\mathcal{U}$, if Y is a point of $(m \cap \mathcal{U}) \setminus \{X\}$ then there exists a secant ℓ' distinct from m on Y which meets each secant on X that meets ℓ in a point of $\mathcal{U}$.
Then $\mathcal{U}$ is classical.

Baker and Ebert [1990] show that in $\mathrm{PG}(2,q)$, $q = p^h$ square, each Buekenhout-Metz unital constructed from an elliptic quadric intersects a classical unital in 1 (mod p) points, and mention a conjecture of Assmus and Key that every unital has this property with respect to the classical unital. The conjecture was proved by Blokhuis, Brouwer and Wilbrink [1991], since it is an easy corollary of Theorem 34 below, also conjectured by Assmus and Key. Before stating the theorem, we describe the context within which the result resides.

In $\mathrm{PG}(2,q)$, give the points an order $(P_1, P_2, \ldots, P_{q^2+q+1})$ and let $\mathcal{L}$ denote the set of lines of $\mathrm{PG}(2,q)$. For a set $\mathcal{S}$ of points, let the *characteristic vector* be

$$\chi(\mathcal{S}) = (x_1, x_2, \ldots, x_{q^2+q+1}),$$

where $x_i = 1$ if $P_i \in \mathcal{S}$ and $x_i = 0$ otherwise. Note that for two sets $\mathcal{S}_1, \mathcal{S}_2$ of points we have: $|\mathcal{S}_1 \cap \mathcal{S}_2| = \chi(\mathcal{S}_1) \cdot \chi(\mathcal{S}_2)$. Now consider the subspace of $\mathrm{GF}(p)^{q^2+q+1}$ spanned by $\{\chi(\ell) : \ell \in \mathcal{L}\}$ over $\mathrm{GF}(p)$. It is a linear code, called the *code of the plane* $\mathrm{PG}(2,q)$.

Now let $\mathcal{S}$ be a set of points such that $\chi(\mathcal{S})$ is a codeword; so

$$\chi(\mathcal{S}) = \sum_{\ell \in \mathcal{L}} \alpha_\ell \chi(\ell)$$

for some $\alpha_\ell \in \mathrm{GF}(p)$. If m is any line of $\mathrm{PG}(2,q)$, then

$$|m \cap \mathcal{S}| = \chi(m) \cdot \chi(\mathcal{S}) = \sum_{\ell \in \mathcal{L}} \alpha_\ell \chi(m) \cdot \chi(\ell) = \sum_{\ell \in \mathcal{L}} \alpha_\ell$$

modulo p (since $|m \cap \ell| = 1$ or $q+1$), which is therefore independent of m. In other words, a line meets $\mathcal{S}$ in a number of points which is a constant modulo p.

Now lines, Baer subplanes and unitals are examples of sets of points which are met by each line in 1 modulo p points, so we can ask whether the corresponding characteristic vectors are in the code of $\mathrm{PG}(2,q)$. By definition the code contains the characteristic vector of each line.

We now argue that if $\mathcal{B}$ is a Baer subplane of $\mathrm{PG}(2,q)$, q square, then $\chi(\mathcal{B})$ is not in the code of $\mathrm{PG}(2,q)$. Suppose, aiming for a contradiction, that $\chi(\mathcal{B}) = \sum_{\ell \in \mathcal{L}} \alpha_\ell \chi(\ell)$. Let $\mathcal{B}'$ be any Baer subplane of $\mathrm{PG}(2,q)$. Since $|\ell \cap \mathcal{B}'| = 1 \pmod p$ for all $\ell \in \mathcal{L}$, we have $\chi(\ell) \cdot \chi(\mathcal{B}) = 1 \pmod p$ for all $\ell \in \mathcal{L}$. It follows immediately that

$$|\mathcal{B} \cap \mathcal{B}'| = \left(\sum_{\ell \in \mathcal{L}} \alpha_\ell \chi(\ell) \right) \cdot \chi(\mathcal{B}') = \sum_{\ell \in \mathcal{L}} \alpha_\ell$$

which is a constant modulo p. Thus if $\chi(\mathcal{B})$ is in the code, then every Baer subplane $\mathcal{B}'$ of $\mathrm{PG}(2,q)$ meets $\mathcal{B}$ in a number of points which is constant modulo p. Now $\mathcal{B}$ meets $\mathcal{B}$ in q^2+q+1 points, which is 1 modulo p. On the other hand, there exists a Baer subplane disjoint from $\mathcal{B}$ which therefore meets $\mathcal{B}$ in 0 modulo p points. The contradiction proves that $\chi(\mathcal{B})$ is not in the code of $\mathrm{PG}(2,q)$.

In the case of unitals, in fact only the classical unitals are codewords in $\mathrm{PG}(2,q)$, as follows.

Theorem 34 (Blokhuis, Brouwer and Wilbrink [1991]) Let $q=p^h$ be a square. A unital in $\mathrm{PG}(2,q)$ is classical if and only if it is a codeword in the code of $\mathrm{PG}(2,q)$ spanned by the lines over $\mathrm{GF}(p)$.

For other characterisations of Hermitian curves among unitals see Abatangelo [1984] and Faina [1991].

5.2 Characterisations of Buekenhout-Metz Unitals

A non-classical Buekenhout-Metz unital has a special point, and in some sense a classical unital can be considered as a Buekenhout-Metz unital in which all points are special. Many of the characterisations of Buekenhout-Metz unitals are inspired by the observation that such a unital "looks like" a classical unital if local properties of this point are under consideration.

Theorem 35 (Lefèvre-Percsy [1981]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q)$, q square and $\sqrt{q}>2$, and let ℓ be some tangent line to $\mathcal{U}$. If all Baer sublines having a point on ℓ intersect $\mathcal{U}$ in $0,1,2$ or $\sqrt{q}+1$ points, then $\mathcal{U}$ is a Buekenhout-Metz unital.

In fact Theorem 35 is a necessary and sufficient condition, and also holds for unitals in any translation plane of order q whose kernel contains $GF(\sqrt{q})$ which are such that the translation line is a tangent.

This characterisation is equivalent to the following. Let T be the point of $\mathcal{U}$ on the tangent ℓ. If the points of $\mathcal{U}$ on lines through T form Baer sublines and if each Baer subline meeting $\ell\backslash\{T\}$ intersects $\mathcal{U}$ in at most 2 points, then $\mathcal{U}$ is a Buekenhout-Metz unital. (This follows from the fact that two Baer sublines of a given line in $\mathrm{PG}(2,q)$ meet in $0,1,2$ or $\sqrt{q}+1$ points, see Dembowski [1968].) A related result is:

Theorem 36 (Larato [1983]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q)$, q square and odd, and let ℓ be tangent to $\mathcal{U}$. Then $\mathcal{U}$ is a Buekenhout-Metz unital if and only if for any two lines ℓ_1,ℓ_2 such that $\ell_1\cap\ell_2=\ell\cap\mathcal{U}$, there is a Baer subplane $\mathcal{B}$ containing ℓ satisfying $\mathcal{B}\cap\mathcal{U}=(\ell_1\cap\mathcal{U})\cup(\ell_2\cap\mathcal{U})$.

Theorems 35 and 36 have recently been improved:

Theorem 37 (Casse, O'Keefe and Penttila [1996], Quinn and Casse [1995]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q)$, where q is square and $\sqrt{q} > 2$. Then $\mathcal{U}$ is a Buekenhout-Metz unital if and only if there exists a point T of $\mathcal{U}$ such that the points of $\mathcal{U}$ on each of the q secant lines to $\mathcal{U}$ through T form a Baer subline.

For the purposes of this next result, we call a Baer subplane of π_q *linear* if it corresponds to a plane under the representation of π_q in $\mathrm{PG}(4,\sqrt{q})$ due to André [1954] and Bruck and Bose [1964, 1966]. Note that a linear Baer subplane of π_q contains the translation line of π_q.

Recall that if $\mathcal{U}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$ with an associated elliptic quadric base and $\mathcal{B}$ is a Baer subplane, then Quinn [1995] has shown that $1 \leq |\,\mathcal{U} \cap \mathcal{B}| \leq 2\sqrt{q}+1$ when $\sqrt{q} > 13$.

Theorem 38 (Barwick, O'Keefe and Storme [1997]) Let $\mathcal{U}$ be a unital in a translation plane π_q of square order q with kernel containing $\mathrm{GF}(\sqrt{q})$, such that the translation line is a tangent. Then $\mathcal{U}$ is a Buekenhout-Metz unital if and only if every linear Baer subplane of π_q meets $\mathcal{U}$ in 1 modulo $\sqrt{q}$ points.

This theorem admits some immediate corollaries:

Corollary 39 (Barwick, O'Keefe and Storme [1997]) Let $\mathcal{U}$ be a unital in a translation plane π_q of square order q with kernel containing $\mathrm{GF}(\sqrt{q})$, such that the translation line is a tangent. If every Baer subplane of π_q which contains the translation line of π_q meets $\mathcal{U}$ in 1 modulo $\sqrt{q}$ points then $\mathcal{U}$ is a Buekenhout-Metz unital.

Corollary 40 (Barwick, O'Keefe and Storme [1997]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q)$, q square, and let ℓ be a tangent. If every Baer subplane which contains ℓ meets $\mathcal{U}$ in 1 modulo $\sqrt{q}$ points then $\mathcal{U}$ is a Buekenhout-Metz unital.

Now we illustrate some characterisations of a more group-theoretical nature.

Theorem 41 (Abatangelo and Larato [1991], Abatangelo [1992]) In $\mathrm{PG}(2,q)$, q square, if a unital $\mathcal{U}$ is fixed by a subgroup Γ of $\mathrm{PGL}(3,q)$ such that:
(i) there is a fixed point $T \in \mathcal{U}$
(ii) Γ has a normal subgroup acting transitively on $\mathcal{U} \setminus \{T\}$
(iii) the stabiliser in Γ of a point $Q \in \mathcal{U} \setminus \{T\}$ has a cyclic subgroup of order $\sqrt{q}-1$
then $\mathcal{U}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$ arising from an elliptic quadric.

Theorem 42 (Abatangelo and Larato [1996]) Let $\mathcal{U}$ be a Buekenhout-Metz unital in $\mathrm{PG}(2,q)$, q square. If there is a point $T \in \mathcal{U}$ such that the stabiliser of $\mathcal{U}$ in $\mathrm{PGL}(3,q)$ has a subgroup which acts sharply-transitively on $\mathcal{U} \setminus \{T\}$ then $\mathcal{U}$ arises from an elliptic quadric.

Abatangelo and Larato [1996] further showed that the stabiliser Γ in $\mathrm{PGL}(3,q)$ of a Buekenhout-Metz unital $\mathcal{U}$ arising from a Tits ovoid has no normal subgroup acting sharply transitively on $\mathcal{U}\setminus\{T\}$, and the stabiliser in Γ of a point $Q \in \mathcal{U}\setminus\{T\}$ has no cyclic subgroup of order $\sqrt{q}-1$.

Theorem 43 (Ebert and Wantz [1996]) Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q)$, q square, which is fixed by a subgroup of $\mathrm{PGL}(3,q)$ which is a semidirect product of a subgroup of order q by a subgroup of order $\sqrt{q}-1$. Then $\mathcal{U}$ is a Buekenhout-Metz unital with associated ovoid $\mathcal{O}$ an ellliptic quadric.

6 Notes and Nomenclature

Some of the objects studied in this note have different names in the literature.

For example, in some references unitals are called *Hermitian arcs.* We have avoided the use of the terms *parabolic* and *hyperbolic,* since these are used with different scopes in various parts of the literature. Most often they seem to be used to distinguish between Buekenhout's two construction methods for unitals.

Perhaps the most serious difficulty is with regard to the term *Buekenhout-Metz unital.* We include in the class of Buekenhout-Metz unitals the unitals constructed using *any* spread $\mathcal{S}$ and *any* ovoid $\mathcal{O}$ in Buekenhout's construction of Section 4.1. However some authors (notably Baker, Dover, Ebert and Wantz) use the term Buekenhout-Metz unital for a subclass of these ovoids (namely, those with $\mathcal{S}$ regular and $\mathcal{O}$ an elliptic quadric), introducing other terms such as Buekenhout-Tits unital and Buekenhout unital to cover other cases.

References to a number of related works can be found in Hirschfeld [1998, 12.9].

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