On collineation groups of finite planes

Arrigo BONISOLI
Dipartimento di Matematica
Università della Basilicata
via N.Sauro 85
85100 Potenza (Italy)

Socrates Intensive Programme
Finite Geometries and Their Applications
Gent, April of the WMY

Please send all remarks to the author's e–mail address:
`bonisoli@unibas.it`

1 Introduction

From the **Introduction** to P. Dembowski's *Finite Geometries*, Springer, Berlin 1968:

" . . . An alternative approach to the study of projective planes began with a paper by BAER 1942 in which the close relationship between Desargues' theorem and the existence of central collineations was pointed out. Baer's notion of (p, L)–transitivity, corresponding to this relationship, proved to be extremely fruitful. On the one hand, it provided a better understanding of coordinate structures (here SCHWAN 1919 was a forerunner); on the other hand it led eventually to the only coordinate–free, and hence geometrically satisfactory, classification of projective planes existing today, namely that by LENZ 1954 and BARLOTTI 1957. Due to deep discoveries in finite group theory the analysis of this classification has been particularly penetrating for finite planes in recent years. In fact, finite groups were also applied with great success to problems not connected with (p, L)–transitivity.

... The field is influenced increasingly by problems, methods, and results in the theory of finite groups, mainly for the well known reason that the study of automorphisms "has always yielded the most powerful results" (E. Artin, *Geometric Algebra*, Interscience, New York 1957, p. 54). Finite–geometrical arguments can serve to prove group theoretical results, too, and it seems that the fruitful interplay between finite geometries and finite groups will become even closer in the future. ... "

Dembowski's comments may appear somewhat prophetic if one considers the subsequent developments, in particular those that eventually led to the classification of finite simple groups.

It is the purpose of these lectures to point out first of all that Dembowki's observations are still valid today, twenty years after the classification of finite simple groups. In the second place I would like to underline that the difficulty of some problems related to collineation groups of finite planes remains unaltered even after the classification of finite simple groups. Finally I would like to illustrate some of the recent results obtained in the study of collineation groups fixing an oval or a hyperoval of a finite projective plane.

2 The basics

2.1 Notation

I shall usually denote by π a finite projective plane of order n, that is a 2–$(n^2+n+1, n+1, 1)$ design for some integer $n \geq 2$. Points of π will usually be denoted by latin capitals such as P, Q, R. Lines of π will usually be denoted by small latin letters such as a, b, c, ℓ.

Some standard facts from the theory of (finite) permutation groups will be used throughout. The monograph by H. Wielandt, *Finite Permutation Groups*, Academic Press, New York 1964, is a classic. Many good textbooks on group theory or algebra cover now the subject of permutation groups to a fair extent.

A *collineation* of π is simply an automorphism of π. The action of a collineation is faithful on the point–set of π (it is also faithful on the line–set of π) and so I shall usually identify a collineation with the permutation it induces on the point–set of π.

Collineations will usually be denoted by small Greek letters such as α, β, γ. The image of the point P under the collineation α is denoted by P^α. The image of the line ℓ under the collineation α is denoted by ℓ^α. A collineation of order two is called an *involution*.

An *oval* in π is an $(n+1)$–arc. An oval will usually be denoted by Ω. The combinatorics of ovals in finite projective planes is assumed. In particular, in case n is even, I usually denote by Ω' the union of an oval Ω with its nucleus. That is an $(n+2)$–arc, a so called *hyperoval* and, conversely, each point of a hyperoval is the nucleus of the oval which remains after the deletion of the point.

2.2 Central collineations

A quick review of some elementary but important properties of collineations.

Proposition 2.1 *A collineation in a (not necessarily finite) projective plane π fixing every point on each of two distinct lines is the identical collineation.*

Proof. Let ℓ_1 and ℓ_2 be the pointwise fixed lines and let Q be their common point. Let P be a point off ℓ_1 and ℓ_2. Consider two distinct points A_1 and B_1 on ℓ_1 other than Q. Let the line PA_1 meet ℓ_2 in A_2. Let the line PB_1 meet ℓ_2 in B_2. Since A_1 and A_2 are distinct fixed points on the line PA_1, we have that this line is fixed and, similarly, the line PB_1 is fixed. The point P is the common point of two distinct fixed lines and so it is itself a fixed point. □

Proposition 2.2 *A collineation in a (not necessarily finite) projective plane π fixing every point on one line and two further points off the line is the identical collineation.*

Proof. The same argument of the previous proof shows that each point off the pointwise fixed line and off the line joining the two extra fixed points is itself a fixed point. So there must be a further line which is pointwise fixed and we are back to the case of the previous proposition. □

An *axial* collineation is one fixing each point of a line ℓ, called the *axis*. A *central* collineation is one fixing each line through a point C, called the *center*.

Proposition 2.3 *Each axial collineation is central. Each central collineation is axial. The fixed points of a non–identical central collineation are the center itself and all points on the axis, while the fixed lines are the axis and all lines through the center. A central collineation α is completely determined by its center C, its axis ℓ and the mapping $P \mapsto P^\alpha$ of any point P not on ℓ and different from C.*

Proof. Let α be a non–identical axial collineation with axis ℓ. If α fixes a point C off the axis ℓ, then each line though C is fixed, since it contains two distinct fixed points, namely C and the point of intersection with ℓ. Hence α is a C–ℓ homology in this case.

Assume α fixes no point off the line ℓ. Consider an arbitrary point P off the line ℓ. The point P^α is distinct from P. The line a joining P to P^α meets ℓ at a fixed point A and we have thus $a = AP = AP^\alpha = A^\alpha P^\alpha = a^\alpha$. Hence every point P off the axis lies on a fixed line. Should two such fixed lines meet off the axis ℓ, then their common point would be a fixed point off the axis, contradicting our assumption. Hence any two fixed lines meet in one and the same point of the axis which is thus the center and α is thus an elation.

We have proved that a collineation with an axis must also have a center. The dual argument shows that each central collineation is axial. The statement on fixed points and fixed lines is an immediate consequence of the two previous propositions.

It is immediately seen that if two central collineations have distinct centers then they have distinct actions on at least one point off the axes and off the line joining the centers. Similarly, two axial collineations with distinct axes have distinct actions on at least one line.

Assume α and β are central collineations with the same center C and the same axis ℓ. If there exists a point P distinct from the center and off the axis with $P^\alpha = P^\beta$, then the collineation $\alpha^{-1}\beta$ fixes each point on ℓ, each line through C and the point P, and so $\alpha^{-1}\beta$ is the identity. □

A central collineation is often called a *perspectivity*. We shall speak for short of a C–ℓ perspectivity to mean a perspectivity with center C and axis ℓ. We distinguish further between a *homology* when the center lies off the axis and an *elation* when the center is on the axis.

The fixed points of a non–identical central collineation are the center itself and all points on the axis; the fixed lines are the axis and all lines through the center. A perspectivity acts thus semiregularly on the points of each line through the center other than the center itself and the point of intersection of the line with the axes (dually: on the lines of each pencil through a point of the axis other than the axis itself and the line of the pencil through the center). The case of a perspectivity of order 2 will be of special interest and we record it as a separate statement.

Proposition 2.4 *A perspectivity of order* 2 *of a finite projective plane is an elation or a homology according as the order of the plane is even or odd respectively.* □

Given a point C and a line ℓ we say that the plane π is C–ℓ *transitive* if for any pair of distinct points P and Q which are collinear with C and not on ℓ there exists a C–ℓ perspectivity mapping P to Q.

It is known that the existence of central collineations is related to the validity of special instances of Desargues' theorem.

Proposition 2.5 *A plane π is C–ℓ transitive if and only if the Theorem of Desargues holds for all triangles which are perspective with respect to C and having two pairs of corresponding sides intersect on ℓ, whence the third pair also intersect on ℓ.*

Proof. A thorough discussion can be found in §20.2 of M. Hall, *The Theory of Groups*, Macmillan, New York 1959. □

The detailed analysis of the "configuration" formed by the point–line pairs (C, ℓ) for which the plane π is C–ℓ transitive is the essence of the so called Lenz–Barlotti classification that was mentioned in the Introduction.

A line ℓ of π such that for each point P on ℓ the plane π is P–ℓ transitive is said to be a *translation* line and π is said to be a *translation plane* with respect to ℓ.

Proposition 2.6 *A sufficient condition for a line ℓ to be a translation line for the plane π is that π be A–ℓ transitive and B–ℓ transitive for two distinct points A, B on ℓ.*

Proof. Let P, Q be distinct points off the line ℓ such that the line PQ meets ℓ at a point C which is different from both A and B. Let the lines AP and BQ meet at a point R. Let α be the A–ℓ elation mapping P to R; let β be the B–ℓ elation mapping R to Q. Then $\alpha\beta$ is an elation with axis ℓ such that $P^{\alpha\beta} = R^\beta = Q$ holds, and so the center of $\alpha\beta$ is C. We have proved that π is C–ℓ transitive. □

Translation planes form a chapter of their own in the theory of finite planes. They can be studied from different points of view. The most famous textbook on this subject is perhaps H. Lüneburg, *Translation planes*, Springer, Berlin, 1980. I would like to mention the very recent treatment by V. Jha, N.L. Johnson, *The Bella–Muro lectures on translation planes*, given at the 1997 Summer School on Finite Geometries organized by the Università della Basilicata. The final version appears as a special volume of the Department of Mathematics, Università di Lecce.

The set of all collineations of π with given center or with given axis or with given center and axis clearly forms a group. More generally, for a given collineation group G of π one can consider the subgroup of G consisting of all perspectivities with given center C or with given axis ℓ or with center C and axis ℓ: these will be denoted by $G(C)$, $G(\ell)$ and $G(C, \ell)$ respectively.

Proposition 2.7 *Let A and B be distinct points on a line ℓ. Let α be a non–identical A–ℓ elation. Let β be a non–identical B–ℓ elation. The product $\alpha\beta$ is an elation with axis ℓ whose center C is different from both A and B.*

Proof. Clearly $\alpha\beta$ is an axial collineation with axis ℓ. In order to see that $\alpha\beta$ is an elation we must show it does not fix any point off the axis ℓ. Let P be one such point and assume $P^{\alpha\beta} = P$. Then $P^\alpha = P^{\beta^{-1}}$, the points A, P, P^α are collinear and so are the points B, P, $P^{\beta^{-1}}$. Since $P \neq P^\alpha$ and $P \neq P^{\beta^{-1}}$ the relation $P^\alpha = P^{\beta^{-1}}$ forces $A = B$, a contradiction. Hence $\gamma = \alpha\beta$ is an elation with axis ℓ.

Let C denote the center of γ. If $C = A$ then $\beta = \alpha^{-1}\gamma$ should also be an elation with center A, a contradiction. Similarly we cannot have $C = B$ and the assertion is proved. □

Proposition 2.8 (Baer). *Let G be a collineation group of π. If for two distinct centers A and B on ℓ the groups $G(A, \ell)$ and $G(B, \ell)$ are non–trivial, then the subgroup T consisting of all elations with axis ℓ in G is elementary abelian.*

Proof. Take non–identical elations $\alpha \in G(A, \ell)$ and $\beta \in G(B, \ell)$ and let P be a point not on ℓ. The points A, P, P^α are on a line a, the points B, P, P^β are on a line b. The points A, P^β, $P^{\beta\alpha}$ are also on a line a' and the points B, P^α, $P^{\alpha\beta}$ are on a line b'. We have $a' = a^\beta$ and $b' = b^\alpha$, hence the common point of a' and b' must be simultaneously equal to $P^{\beta\alpha}$ and to $P^{\alpha\beta}$. We conclude that $P^{\beta\alpha} = P^{\alpha\beta}$ holds for each point P off the line ℓ. Since the relation also holds for all points on ℓ, we have $\beta\alpha = \alpha\beta$.

We have proved that α commutes elementwise with each group $G(C, \ell)$ whenever C is a point on ℓ different from A. Let α' be a non–identical collineation in $G(A, \ell)$ with $\alpha' \neq \alpha$. We know from the previous Proposition that $\alpha'\beta$ is a C–ℓ elation for some center C which is different from both A and B. As before we must have $\alpha(\alpha'\beta) = (\alpha'\beta)\alpha$, whence also $(\alpha\alpha')\beta = \alpha(\alpha'\beta) = (\alpha'\beta)\alpha = \alpha'(\beta\alpha) = \alpha'(\alpha\beta) = (\alpha'\alpha)\beta$, that is $(\alpha\alpha')\beta = (\alpha'\alpha)\beta$, yielding $\alpha\alpha' = \alpha'\alpha$. We have proved that any two elations with axis ℓ in G commute and so T is abelian.

As a non–trivial finite group T contains some element α of prime order p. Assume α is an A–ℓ elation. Let β be a non–identical B–ℓ elation in G with $B \neq A$. Then $\alpha\beta$ is a C–ℓ elation in G with $C \neq A, B$. We have $(\alpha\beta)^p = \alpha^p\beta^p = \beta^p$. Since $(\alpha\beta)^p \in G(C, \ell)$, $\beta^p \in G(B, \ell)$ and these subgroups have only the identity in common, we see that β^p is the identity.

Hence the fact that α has order p implies that every non–trivial elation in G with axis ℓ and center different from A has order p. Similarly, the fact that β has order p implies that every non–trivial elation in $G(A, \ell)$ has order p.

We have proved that T is an abelian group in which every non–trivial element has order p, that means T is an elementary abelian p–group. □

2.3 Involutions and Baer collineations

Consider a plane π of square order n admitting a Baer subplane. A *Baer collineation* α of π is a collineation fixing a Baer subplane π_0 elementwise (pointwise and linewise). An involution which is a Baer collineation is called a *Baer involution*.

Proposition 2.9 (Baer). *Let α be an involution of a finite projective plane π of order n. Then either n is a square and α is a Baer involution or α is a central collineation.*

Proof. Assume the point P is not fixed by α. Then P and P^α are distinct points which are exchanged by α, and so the line joining them is a line through P which is fixed by α.

Assume the point P is fixed by α. Let Q be another point and assume the line $\ell = PQ$ is not fixed by α. Then the point Q^α is not on ℓ and we have $\ell^\alpha = PQ^\alpha$. Take a point R on ℓ other than P, Q. The point R^α is on ℓ^α and is distinct from P and Q^α. The lines RQ^α and QR^α are exchanged by α and their common point S is distinct from P and is fixed from α. The line joining S to P is a line through P which is fixed by α.

We have proved that each point lies on a fixed line. Dually, each line contains a fixed point. Assume there exists a quadrangle of fixed elements; then the fixed elements of α form a proper subplane π_0 of π the order of which we denote by m. The counting argument involved in the combinatorics of Baer subplanes yields $n \geq m^2$ and it also shows that if $n > m^2$ then there is a line of π missing π_0. The latter possibility is excluded by the previous observation that each line of π must contain a fixed point. We conclude that π_0 is a Baer subplane and α is a Baer involution in this case.

Assume no quadrangle of fixed elements exists. We prove first of all that there is a line containing three fixed points. Pick a line ℓ_1 and a fixed point P_1 on ℓ_1. Choose a second line ℓ_2 not through P_1 and let P_2 be a fixed point on ℓ_2. The line P_1P_2 is fixed. Choose a third point Q on P_1P_2. If Q is fixed then the line P_1P_2 has the required property. If not, then a line ℓ_3 through Q other than P_1P_2 contains a fixed point P_3 not on P_1P_2. Take a line ℓ_4 not through any one of the points P_1, P_2, P_3, and let P_4 be a fixed point on ℓ_4. Since we are assuming that no quadrangle of fixed elements exists, we see that P_4 must lie on one of the sides of the triangle $P_1P_2P_3$, and this side is the line with the required property.

If ℓ is a line with three fixed points then there is at most one fixed point P off ℓ, because otherwise a quadrangle of fixed elements should exist. Take a point Q on ℓ. Choose a line through Q other that ℓ itself and (possibly) PQ. This line must contain a fixed point which, by our choice, must lie on ℓ, hence it must be Q. We conclude that ℓ is pointwise fixed by α and the assertion follows. $\square$

The following powerful result was proved by D.R. Hughes in *Generalized incidence matrices over group algebras* Ill. J. Math. 1 (1957) 545–551.

Proposition 2.10 *A finite projective plane* π *of order* $n > 2$ *where* $n \equiv 2 \mod 4$ *cannot admit a collineation of order* 2.

Proof. A full proof can be found in D.R. Hughes, F.C. Piper, *Projective Planes*, Springer, Berlin 1971. $\square$

2.4 Perspectivities fixing ovals or hyperovals

It is easily seen that if a collineation of π fixes an oval pointwise then it is the identity on the whole plane. Hence each collineation group fixing an oval or a hyperoval has a faithful permutation representation on the points of the oval or of the hyperoval respectively.

Proposition 2.11 *Let n be odd and let Ω be an oval in π. A non–identical perspectivity α of π fixing Ω is necessarily an involutory homology and either the center is an internal point and the axis is an external line or the center is an external point and the axis is a secant line. Any two distinct involutory homologies of π fixing Ω have both distinct centers and distinct axes. There cannot exist an elementary abelian group of order 8 generated by three involutory homologies of π fixing Ω.*

Proof. Let C and ℓ be the center and the axis of α respectively.

If C is on Ω and P is any other point on Ω then the line s joining C to P is fixed by α. The points of intersection of s with the oval are precisely C and P. We have $(\Omega \cap s)^\alpha = \Omega^\alpha \cap s^\alpha = \Omega \cap s$, whence $\{C, P\}^\alpha = \{C, P\}$ and since $C^\alpha = C$ we necessarily have also $P^\alpha = P$. We have proved that if the center C is on Ω, then α fixes Ω pointwise and so α is the identical collineation of π.

Let a be any line through the center C such that $a \cap \Omega$ is non–empty, hence consists of either one or two points. In either case the collineation α^2 fixes $a \cap \Omega$ pointwise. We conclude that α^2 fixes Ω pointwise and so α is an involution. As an involutory perspectivity in a plane of odd order α must be a homology.

Assume C is an external point and let t_1, t_2 be the two tangent lines to Ω through C, meeting Ω at P_1, P_2 respectively. Since α fixes t_1 and Ω, it also fixes their unique common point, that is $P_1^\alpha = P_1$. Similarly, we have $P_2^\alpha = P_2$. As a consequence P_1P_2 is a fixed line not through the center and so it must be the axis.

Let conversely α have the secant line ℓ as an axis. Set $\ell \cap \Omega = \{P_1, P_2\}$ and denote by t_1, t_2 the two tangents to Ω at P_1, P_2 respectively. Since α fixes Ω and P_1 it must also fix the unique tangent line to Ω through P_1, that is $t_1^\alpha = t_1$. Similarly, we have $t_2^\alpha = t_2$. The common point C of t_1 and t_2 is therefore also fixed by α, and since C is not on ℓ we have that C is the center and α is a homology.

If α and β are involutory homologies fixing Ω with the same center, then, since they both fix each line through the center, their product $\alpha\beta$ fix pointwise the intersection of each such line with the oval, hence $\alpha\beta$ fixes the oval pointwise and is thus the identity, yielding $\alpha = \beta$.

If and involutory homology fixing Ω has a secant line as axis then its center cen be uniquely reconstructed as the intersection of the two tangents at the points where the axis meets the oval. If the axis is an external line, then, for each external point P on this line, the homology exhanges the two tangents to Ω through P, hence the center of the homology lies on the line joining the points of contact; since there are at least two external points on the axis, we obtain two distinct lines through the center and so the center is uniquely reconstructed in this case as well.

Assume β, δ are distinct commuting involutory homologies fixing Ω. Denote by B and b resp. D and d the center and axis of β resp. δ. The relation $\delta\beta = \beta\delta$ yields $\beta = \delta^{-1}\beta\delta$ and so, since $\delta^{-1}\beta\delta$ is a homology with center B^δ and axis b^δ we have $B^\delta = B$ showing that B (which is distinct from D) must lie on the axis of δ, that is $B \in d$. Exchanging the roles of β and δ we obtain $D \in b$.

Let R denote the common point of the axes b and d and let r be the line joining the centers B and D. I claim that $\beta\delta$ is an involutory homology with center R and axis r. As the product of two commuting involutions $\beta\delta$ is itself an involution. Furthermore, $\beta\delta$ fixes each one of the points B, D, R and each one of the lines b, d, r. If $\beta\delta$ were a Baer involution then it should fix a quadrangle elementwise and so it should fix at least one point P off the triangle formed by these three points and three lines. From the relation $P^{\beta\delta} = P$ we also have $P^\beta = P^\delta$, a contradiction since P^β is collinear with B and P and distinct from P while P^δ is collinear with D and P and distinct from P.

Assume that ϱ is an involutory homology fixing Ω which is distinct from both β and δ and commutes with each one of β and δ. The previous argument shows that the center of β must lie on the axis of β as well as on the axis of δ; furthermore, the center of β must be on the axis of ϱ and the center of δ must lie on the axis of ϱ. We conclude that ϱ must be an involutory homology with center R and axis r, hence $\varrho = \beta\delta$. □

Proposition 2.12 *Let n be even. Let Ω be a oval in π and let Ω' be the hyperoval obtained from Ω by adding its nucleus. If a collineation of π fixes Ω it also fixes its nucleus. A perspectivity of π fixing Ω is necessarily an* *involutory elation, whose center does not lie on Ω'. If $n > 4$ the axis is a secant of Ω' and the permutation induced on the points of Ω' is even.*

Proof. The proof of the first part goes much like in the odd order case. For the last statement see Lemma 3.2 in T. Penttila, G.F. Royle, *On hyperovals in small projective planes*, J. of Geometry 54 (1995) 91–104. □

3 Some classics

A *Moufang plane* is a projective plane in which every line is a translation line. The coordinate structure of a Moufang plane is an alternative division ring, that is a set with two binary operations (addition and multiplication) satisfying the following properties: i) the additive structure is an abelian group; ii) both distributive laws hold; iii) multiplication has an identity element and each non–zero element has a multiplicative inverse; iv) the identities $a^{-1}(ab) = b = (ba)a^{-1}$ hold for each non–zero element a and arbitrary element b; v) the alternative laws $a(ab) = (aa)b$, $(ba)a = b(aa)$ hold for arbitrary elements a, b. The Artin–Zorn theorem states that in every finite alternative division ring multiplication is associative and consequently each such ring is actually a finite field by the theorem of Wedderburn. Each finite Moufang plane is therefore desarguesian.

Proposition 3.1 (a lemma from permutation groups). *Let H be a permutation group on a finite set X and assume that there exists a prime p such that for each $x \in X$ there is an element of order p in H fixing x and no other element of X. Then H is transitive on X.*

Proof. Assume Y and Z are distinct orbits of H on X. Pick an element $y \in Y$: since there exists a permutation in H fixing y and permuting all other elements of X into cycles of length p, we see that p divides both $|Y| - 1$ and $|Z|$. Taking an element $z \in Z$ instead and repeating the same argument, we see that p divides $|Y|$ and $|Z| - 1$, a contradiction. There is thus just one H–orbit,that is X itself. □

Proposition 3.2 (another lemma). *Suppose that there exists a line ℓ of π such that for all points C on ℓ the groups of all C–ℓ elations have the same order $h > 1$. Then ℓ is a translation line for π.*

Proof. For each point C on ℓ we define T_C to be the group of all C–ℓ elations; we denote by T the group of all elations with axis ℓ, that is $T = \cup_{C\in\ell}T_C$ (this is is sometimes referred to as the *translation group* of π, or better, of the affine plane obtained from π when ℓ is viewed as a line at infinity). If C_1, C_2 are distinct points on ℓ then the subgroups T_{C_1}, T_{C_2} have only the identity in common. We have thus $|T| = (n+1)(h-1)+1$. The group T acts semiregularly off the line ℓ (in fact a non–identical elation fixes precisely the points of its axis). As a consequence each T–orbit of points off the axis ℓ has length $|T|$, which is thus a divisor of n^2, the number of "affine" points: say $n^2 = [(n+1)(h-1)+1]m$ for some positive integer m. Since $h-1 > 0$ we have $m < n$. We also have $n^2 \equiv 1 \mod n+1$ and if we interpret the equation $n^2 = [(n+1)(h-1)+1]m$ modulo $n+1$ we obtain $n^2 \equiv m \mod n+1$. The relation $m \equiv 1 \mod n+1$ with $m < n$ yields $m = 1$, whence also $|T| = n^2$, in other words T permutes the n^2 "affine" points in a single orbit and the assertion follows. □

Proposition 3.3 (Gleason's theorem). *If for any incident point–line pair (P, ℓ) of π there exists a non–trivial P–ℓ elation, then π is desarguesian.*

Proof. By a previous result if the line ℓ admits non–trivial elations for two distinct centers on ℓ, then all elations with axis ℓ form an elementary abelian p–group for some prime p. By the dual of this statement if the point P is the center of non–trivial elations for two distinct lines through P, then the elations with center P form an elementary abelian p–group (where p is the same prime as before). For any incident point–line pair (P, ℓ) of π the group of all P–ℓ elations is an elementary abelian p–group.

Take a given line a of π and let A be an arbitrary point on A. Choose another line b through A and consider a non–trivial A–b elation. This elation has order p and fixes the line a through its center: since A is the unique fixed point on a, all other orbits have length p. By the Lemma above the group of all collineations of π fixing a is transitive on the points of a. In particular, the groups of all C–a elations as C varies on a are all conjugate in this group and have thus the same size $h > 1$. The other lemma shows that a is a translation line. Each line of π is thus a translation line for π, hence π is a finite Moufang plane, therefore also a desarguesian plane. □

Proposition 3.4 (the Ostrom–Wagner theorem). *If π admits a collineation group G acting doubly transitively on points, then π is desarguesian and G*

contains all elations of π*, whence also* $PSL(3, n)$ (*the subgroup generated by all elations*).

Proof. We only prove the first part of the Theorem under the further assumption that n is not a square. Since every 2–transitive group has even order, we see that G contains an involution α. As the order of the plane is not a square, we see that α cannot be a Baer involution and so it is an elation or a homology according as n is even or odd.

We want to prove that if n is odd then G still contains elations. Let the involutory homology α have center C and axis d; choose a point D on d and a point B off d, $B \neq C$. By 2–transitivity there exists a collineation $\gamma \in G$ with $C^\gamma = C$, $D^\gamma = B$. The involutory homology $\beta = \gamma^{-1}\alpha\gamma$ has center C and axis d^γ, a line through B hence different from d. The collineation $\alpha\beta$ fixes each line through C and so it is a central collineation with center C. Assume $\alpha\beta$ fixes a line t not through C. If $t^\alpha \neq t$ then $\alpha\beta$ fixes the two distinct lines t and t^α not through C, hence $\alpha\beta$ is the identity, yielding $\alpha = \beta$, a contradiction since α and β have distinct axes. Hence $t^\alpha = t$, showing that t is the axis of both α and β, again a contradiction. We conclude that the central collineation $\alpha\beta$ fixes no line off the center and so its axis is incident with the center, that is $\alpha\beta$ is an elation.

Consider an elation in G with center C and axis ℓ and let A be another point on ℓ. By 2–transitivity G contains a collineation σ exchanging A and C. Then σ fixes ℓ. As a consequence the stabilizer of ℓ in G acts transitively on the points of ℓ, yielding in particular that the subgroups of G consisting of all elations with axis ℓ and center in a given point P of ℓ have the same order $h > 1$. By one of the lemmas we proved above ℓ is a translation line forπ. Since G is 2–transitive on points, G can map a given pair of points on ℓ onto any other pair, hence can map ℓ onto any other line. Hence every line is a translation line and π is a Moufang plane. A finite Moufang plane is desarguesian as we already observed. □

Proposition 3.5 (Lüneburg's theorem). *Let* n *be a prime power, say* $n = p^e$ *for some prime* p*, and assume* π *admits a collineation group isomorphic to* $PSL(2, n)$*. Then* π *is desarguesian.*

Proof. This is one of the results contained in H. Lüneburg, *Charakterisierungen der endlichen projektiven Ebenen*, Math. Z. 85 (1964) 419–450. □

Observe that in the previous statement it is assumed that the order n of the plane is precisely p^e. Also note that the proof of the result requires some deeper group theory, in particular the classification of the subgroups of $PSL(2, p^e)$, which can be found in the classical exposition by L.E. Dickson, *Linear groups, with an exposition of the Galois field theory*, Teubner, Leipzig 1901 (reprint Dover, New York 1958).

Finally observe that $PSL(2, p^e)$ in its natural 2–transitive permutation representation can be realized in the plane as a group of collineations fixing a non–singular conic. In order to obtain an explicit representation, let more generally $\mathbf{K}$ be an arbitrary field and let $\mathcal{C}$ be the irreducible conic of $PG(2, \mathbf{K})$ defined by the homogeneous equation $X_0X_2 = X_1^2$. We have $\mathcal{C} = \{(1, t, t^2);\ t \in \mathbf{K}\} \cup \{(0, 0, 1)\}$. For a, b, c, d in $\mathbf{K}$ we define the matrix

$$M = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

The relation $\det(M) = (ad - bc)^3$ shows that if $ad - bc \neq 0$ then M induces a linear collineation φ of $PG(2, \mathbf{K})$. We have

$(1, t, t^2)^\varphi = ((a + bt)^2, (a + bt)(c + dt), (c + dt)^2)),$

$(0, 0, 1)^\varphi = (b^2, bd, d^2).$

and so φ fixes $\mathcal{C}$ inducing on it the fractional linear transformation

$t \mapsto (a + bt)/(c + dt).$

Hence if we let a, b, c, d vary on $\mathbf{K}$ subject to $ad - bc \neq 0$, we obtain a collineation group of $PG(2, \mathbf{K})$ which is clearly isomorphic to $PGL(2, \mathbf{K})$. Since the quadratic character of $(ad - bc)^3$ is the same as that of $ad - bc$ we have that the linear collineations induced by the matrices in which $ad - bc$ is a square clearly form a subgroup isomorphic to $PSL(2, \mathbf{K})$. The conic $\mathcal{C}$ is also clearly invariant under the collineation $(X_0, X_1, X_2) \mapsto (X_0^\sigma, X_1^\sigma, X_2^\sigma)$ induced by an arbitrary automorphism σ of the field $\mathbf{K}$. If we add all these collineations as σ varies in $\mathrm{Aut}(\mathbf{K})$, we obtain a representation of $P\Gamma L(2, \mathbf{K})$ fixing the conic $\mathcal{C}$.

4 Machinery

Generally speaking the best results in the study of collineation groups of finite planes are based on the possibility of proving at some stage that the group under consideration contains non–trivial perspectivities. Another rule of thumb which usually applies both in purely group–theoretical considerations and in geometric arguments is that the control of involutions is usually crucial in trying to get better results.

Both observations certainly apply to collineation groups fixing an oval or a hyperoval. It is the purpose of this section to develop some further properties of perspectivities and Baer involutions fixing an oval in a finite projective plane of odd order.

Proposition 4.1 *Let n be an odd square and let π be a finite projective plane of order n with an oval Ω. Let β be a Baer involution of π fixing Ω and let π_0 be the fixed Baer subplane of β. One of the following holds.* i) *Ω avoids π_0 and β induces an odd permutation on Ω. The lines of π_0 are divided into $(n+1)/2$ secants and $(\sqrt{n}+1)^2/2$ external lines. The points of π_0 are divided into $(n+1)/2$ external points and $(\sqrt{n}+1)^2/2$ internal points.* ii) *Ω meets π_0 in an oval Ω_0 and β induces an even or an odd permutation on Ω according as $\sqrt{n} \equiv 1$ or $-1 \mod 4$. No point of π_0 is internal to Ω and no line of π_0 is external to Ω.*

Proof. First of all, if π admits a Baer involution then n is a square and we have $n \equiv 1 \mod 4$, whence $n+1 \equiv 2 \mod 4$, i.e. $(n+1)/2$ is odd.

If $\Omega \cap \pi_0$ is empty, then β fixes no point of Ω and so β induces a permutation on Ω which is the product of $(n+1)/2$ transpositions, hence an odd permutation. If P is a point of Ω, then $P \notin \pi_0$ and so there is exactly one line of π_0 through P; this line cannot be a tangent to Ω, since P is not fixed by β, and so it is a secant. We obtain in this manner $(n+1)/2$ secants in the Baer subplane π_0. The number of remaining lines in π_0 is $n+\sqrt{n}+1-(n+1)/2 = (\sqrt{n}+1)^2/2$. The dual argument yields the behavior of the points of π_0.

Assume $\Omega_0 = \Omega \cap \pi_0$ is non–empty and let P_0 be an arbitrary point in Ω_0. The tangent to Ω through P_0 is fixed by β, hence lies in π_0. Besides this line, there are $\sqrt{n}$ further lines of π_0 through P_0. Each one of them meets Ω at another point which is also fixed by β, hence in Ω_0. Conversely, the line joining a point in $\Omega_0 \setminus \{P_0\}$ to P_0 lies in π_0. We have thus $|\Omega_0| = \sqrt{n}+1$ and so Ω_0 is an oval in π_0. The permutation induced by β on Ω_0 fixes Ω_0 pointwise and exchanges the remaining $(n+1)-(\sqrt{n}+1)$ points of Ω into pairs: the number of transpositions induced by β on Ω is thus $\sqrt{n}(\sqrt{n}-1)/2$, which is even or odd according as $\sqrt{n} \equiv 1$ or $-1 \mod 4$ respectively. Let P be a point in $\Omega \setminus \Omega_0$. The unique line ℓ_0 of π_0 through P cannot be a tangent to Ω, because otherwise P should be fixed by β, hence lie in π_0. The further point of intersection of ℓ_0 with Ω cannot lie in Ω_0, otherwise β should again fix P. We have thus detected $|\Omega \setminus \Omega_0|/2 = (n - \sqrt{n})/2$ lines of π_0 meeting Ω at two points off π_0. Since this is also the total number of lines of π_0 which are external to Ω_0, we have that each line of π_0 which is external to Ω_0 does meet Ω. The statement on points is obtained by the dual argument. □

Proposition 4.2 *Let n be a square with $\sqrt{n} \equiv 1 \mod 4$ and let π be a finite projective plane of order n with an oval Ω. Assume β_1, β_2 are distinct commuting Baer involutions fixing Ω and inducing permutations of the same parity on Ω (that is both even or both odd). Then the product $\beta_1\beta_2$ is a homology.*

Proof. See statements (I) and (II) in the proof of Proposition 2.3 in M. Biliotti, G. Korchmáros, *Collineation groups which are primitive on an oval of a projective plane of odd order*, J. London. Math. Soc. (2) 33 (1986) 525–534. □

Proposition 4.3 *Let π be a finite projective plane of odd order n with an oval Ω. If V is a Klein 4–group of collineations of π fixing Ω, then V contains at least one involutory homology inducing an even permutation on Ω.*

Proof. A subgroup of V of index at most 2 must induce even permutations on Ω.

If n is a non–square then V contains no Baer involutions and the assertion is clear.

Assume n is a square with $\sqrt{n} \equiv -1 \mod 4$. According to Proposition 4.1 each Baer involution induces an odd permutation on Ω in this case and so there must exist a homology in V.

Assume n is a square with $\sqrt{n} \equiv 1 \mod 4$. If all collineations in V induce even permutations on Ω, then Proposition 4.2 shows that the three involutions in V cannot simultaneously be Baer involutions. Assume the collineations in V inducing even permutations on Ω form a subgroup W of index 2 in V. If both involutions in $V \setminus W$ are Baer involutions then Proposition 4.2 shows that their product is a homology inducing an even permutation on Ω. Assume the involution β in W is a Baer involution, one of the involutions in $V \setminus W$, say α, is a homology and the other one is a Baer involution. Since $n \equiv 1 \mod 4$ and α induces an odd permutation on Ω, we see that the axis a of α must be disjoint from Ω. The homology $\beta^{-1}\alpha\beta$ has axis a^β, but the relation $\beta^{-1}\alpha\beta = \alpha$ yields $a^\beta = a$, and so a is a line of the fixed Baer subplane of β missing Ω. Since case ii) in Proposition 4.1 applies here we have a contradiction. □

Proposition 4.4 *Let π be a finite projective plane of odd order n with an oval Ω. Let E be a 2–group of collineations of π fixing Ω. If E contains no involutory homology then E is cyclic.*

Proof. There is an involution γ in the center of E. Any further involution δ in E should commute with γ and thus span with γ a Klein 4–group V. Proposition 4.3 shows that V contains involutory elations, hence so does E, a contradiction. We conclude that E contains a unique involution. The finite 2–groups with a unique involution are well characterized: they are either cyclic or generalized quaternion groups.

Assume E is a generalized quaternion group with Baer involution β whose fixed Baer subplane we denote by π_0. Since $|E| > 2$ we know that β is the square of some collineation in E, hence β induces an even permutation on Ω. We know then from Proposition 4.1 that $\Omega_0 = \Omega \cap \pi_0$ is an oval of π_0. Since β is in the center of E, we have that π_0 is left (setwise) invariant by the whole of E.

We want to show that the kernel of the action of E on π_0 is precisely $\langle\beta\rangle$. Assume the kernel contains a collineation of order 4, say δ. Let ℓ_0 be a line in π_0 which is external to Ω_0. Proposition 4.1 shows that ℓ is a secant to Ω. Hnece δ either fixes or interchanges the two points in $\ell \cap \Omega$, hence $\beta = \delta^2$ necessarily fixes these two points. That is a contradiction because these two points do not lie in π_0, the fixed subplane of π_0.

Denote by E_0 the collineation group induced by E on π_0, that is $E_0 = E/\langle\beta\rangle$. The group E_0 contains a Klein 4–group V_0 (take for instance two collineations of order 4 in E which do not lie in the same cyclic subgroup of order 4: they generate a subgroup inducing a Klein 4–group on π_0). The same argument as before shows that no involution in V_0 fixes a line of π_0 which is external to Ω_0. We conclude that the Klein 4–group V_0, a collineation group of π_0 fixing the oval Ω_0, does not contain homologies. That contradicts Proposition 4.3. □

5 Primitive ovals in projective planes of odd order

A finite group G is said to have p–rank r (for the given prime p) if p^r is the largest order of an elementary abelian p–subgroup of G.

In this section π will denote a finite projective plane of odd order n with an oval Ω and G will denote a collineation group of π fixing Ω.

Proposition 5.1 (the 2–rank property). *The 2–rank of G is at most 3.*

Proof. Let E be an elementary abelian 2–subgroup of G. The involutory homologies in E together with the identity form a subgroup V of E of order at most 4. If $V < E$ then Proposition 4.3 shows that the product of any two collineations in $E \setminus V$ (these are Baer involutions) must lie in V and so $|E : V| = 2$ and the assertion follows. □

The above property is the basic tool in the detailed analysis required in the proof of the main result of the paper M. Biliotti, G. Korchmáros, *Collineation groups which are primitive on an oval of a projective plane of odd order*, J. London. Math. Soc. (2) 33 (1986) 525–534.

Proposition 5.2 *Assume G acts primitively on the points of Ω. Then π is desarguesian, Ω is a conic and either G contains a normal subgroup acting on the points of Ω as $PSL(2, n)$ in its natural doubly transitive permutation representation, or $n = 9$ and G acts on Ω as* Alt(5) *or* Sym(5) *in the primitive permutation representation of degree* 10.

The key idea in the proof of the previous statement is based on the consideration of a *minimal normal subgroup* of the group G under consideration. That is a non–trivial normal subgroup M of G which does not contain properly any non–trivial normal subgroup K of G. Minimal normal subgroups have the important property of being *characteristically simple*. A characteristically simple group is one in which the unique characteristic subgroups are the trivial subgroup and the entire group. A finite characteristically simple group can be represented as the direct product of finitely many pairwise isomorphic finite simple groups, hence either cyclic of prime order or non–abelian simple.

Let M be a minimal normal subgroup of the group G in the statement of the previous Proposition. As a non–trivial normal subgroup of a primitive group, M must be transitive on the oval Ω. If M is the direct product of cyclic groups of the same prime order, in other words if M is elementary abelian, then as a transitive abelian permutation group on Ω, the group M must be regular on Ω. Since the cardinality of Ω is $n+1$, an even number, we have that M is an elementary abelian 2–group. By the 2–rank property the size of M must be 4 or 8, hence n must be 3 or 5, in either case the plane π must be desarguesian and a direct verification is possible. If M is the direct product of pairwise isomorphic non–abelian simple groups, then since each non–abelian finite simple group contains at least two commuting involutions, hence a Klein 4–group, we have that if the number of pairwise isomorphic factors in the direct product is greater than one, then the group M has 2–rank at least 4, which is impossible. We conclude that the number of factors is just one, that is M is a non–abelian finite simple group. Since M itself leaves the oval Ω invariant we also see that the 2–rank of M is at most 3.

The relevant fact is that non–abelian finite simple groups of 2–rank not exceeding 3 are classified by the work of G. Stroth, *Über Gruppen mit 2–Sylow Durchschnitten vom Rang* ≤ 3, J. Algebra 43 (1976) 398–456.

Proposition 5.3 *Let M be a non–abelian simple group of 2–rank at most 3. Two possibilities arise.*

i) *The 2–rank of M is 3 and M is isomorphic to one of the following groups: $G_2(q)$, ${}^3D_4(q)$, where q is odd in both cases, ${}^2G_2(3^n)$ with n odd, $n > 1$, $PSL(2,8)$, $PSU(3,8^2)$, Sz(8), M_{12}, J_1, ON.*

ii) *The 2–rank of M is 2 and M is isomorphic to one of the following groups: $PSL(2,q)$, $q \geq 5$, $PSL(3,q)$, $PSU(3,q^2)$, with q odd in all three cases, $PSU(3,4^2)$, A_7, M_{11}.*

All of the candidates under case i), with the only exception of M_{12}, must be discarded because they contain a unique class of involutions, while we have seen that an elementary abelian group of order 8 fixing an oval must contain involutory elations and Baer involutions simultaneously. The analysis of most of the other cases requires “ad hoc” arguments. The final result is Theorem A in M. Biliotti, G. Korchmáros, *Collineation groups preserving an oval in a projective plane of odd order*, J. Austral. Math. Soc. Ser. A 48 (1990) 156–170.

Proposition 5.4 *If π is a finite projective plane of odd order n with an oval Ω which is left invariant by a simple group M, then M must be isomorphic to $PSL(2, q)$ with q odd ≥ 5.*

6 An excursion into graphs

If a graph on v vertices admits a *one–factor* (or *perfect matching*) then v is necessarily even. The complete graph on v vertices is usually denoted by K_v. If v is even, say $v = 2m$ not only does K_v admit a one–factor but it also admits a *one–factorization*, that is a partition of the set of edges into one–factors. An equivalent concept is that of an edge–coloring of K_{2m} in which precisely $2m-1$ colors are used. Another equivalent concept is that of a sharply transitive permutation set of degree $2m$ consisting of the identity and $2m - 1$ involutory permutations, see for instance E. Ihrig, *Symmetry Groups Related to the Construction of Perfect One Factorizations of K_{2n}*, J. Comb. Theory, Ser. B 40 (1986) 121–151. Incidentally, if we assume S to be a permutation set on $2m$ elements consisting of the identity and of $2m - 1$ involutions, then S will be sharply transitive if and only if each involution in S as well as the product of any two involutions in S is fixed–point–free.

One–factorizations can of course be defined for arbitrary graphs, but the case of a complete graph is sufficiently complicated to require special care. A recent monograph: W.D. Wallis, *One–factorizations*, Kluwer 1997. My interest in one–factorizations of complete graphs in the context of finite planes arises from an interesting link to ovals and hyperovals, as we shall see in this section.

6.1 Motivation

If a finite simple group acts as a collineation group on a finite projective plane, what can be said on the group and on the plane? I believe it was Theodore Ostrom in the fifties who raised this question explicitely in this general form in one of his papers. If I remember correctly the question was immediately followed by the answer "Not much of anything!" or the like, although it was Ostrom himself who studied the problem intensively and gave many important contributions.

One important family of non–abelian finite simple groups was discovered by Michio Suzuki at the beginning of the sixties. The fact that the Suzuki group $\mathrm{Sz}(q)$ does act on a translation plane of order q^2 as a collineation group follows from the constructions of Jacques Tits. The paper J. Tits, *Ovoïdes et groupes de Suzuki*, Arch. Math. (Basel) 13 (1962) 187–198, besides containing the construction of the ovoid of $PG(3, q)$ on which $\mathrm{Sz}(q)$ acts as a linear collineation group, shows that $\mathrm{Sz}(q)$ also leaves invariant a spread of lines of $PG(3, q)$ and therefore acts on the corresponding translation plane of order q^2.

Heinz Lüneburg in his paper *Über projective Ebenen in denen jede Fahne von einer nichttrivialen Elation invariant gelassen wird*, Abh. Math. Sem. Univ. Hamburg 29 (1965) 37–76, described the coordinate structure of this translation plane, a quasifield of order q^2 with kernel $GF(q)$. He also proved that the action of $G \cong \mathrm{Sz}(q)$ as a collineation group of an arbitrary projective plane of order q^2 is of three possible types (see Theorem 28.11 in H. Lüneburg, "Translation planes," Springer, Berlin, 1980):

(a) G fixes an antiflag (P, ℓ) and acts 2–transitively on the points of ℓ and on the lines through P;

(b) G fixes an oval Ω and acts 2–transitively on its points;

(c) G fixes a line–oval Ω^* and acts 2–transitively on its lines.

(the 2–transitive action is the natural one everywhere; the sizes of point and line orbits in the whole plane are also specified).

The fact that (a) occurs in the Lüneburg plane was pointed out by Lüneburg himself. Possibility (c) for the Lüneburg plane was first obtained as a consequence of results of H. Pollatsek in *First cohomology groups of some linear groups over fields of characteristic two*, Ill. J. Math. 15 (1971) 393–417. Later on W.M. Kantor in *Symplectic Groups, Symmetric Designs and Line Ovals*, J. Algebra 33 (1975) 43–58, and G. Korchmáros in *Le ovali di linea del piano di Lüneburg d'ordine* 2^{2r} *che possono venir mutate in sé da un gruppo di collineazioni isomorfo al gruppo semplice* $\mathrm{Sz}(2^r)$ *di Suzuki*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Mem. (8) Mat. Appl. 15 (1979) 295–315, showed that possibility (b) occurs in the dual Lüneburg plane, thus proving possibility (c) again for the Lüneburg plane by duality.

It seems quite natural to ask whether the Lüneburg plane of order q^2 and its dual plane are the unique planes on which $\mathrm{Sz}(q)$ can act as a collineation group.

There are some affirmative answer under additional assumptions on the plane, see R.A. Liebler, *A characterization of the Lüneburg planes*, Math. Z.

126 (1972) 82–90; W. Büttner, *On translation planes of order q^2 containing* Sz(q) *in their translational complement*, Arch. Math. (Basel) 33 (1979/80) 216–221, and *Eine Charakterisierung der Lüneburgebenen*, Abh. Math. Sem. Univ. Hamburg 54 (1984) 15–20.

The question addressed in the paper A. Bonisoli, G. Korchmáros, *Suzuki groups, one–factorizations and Lüneburg planes*, Discrete Math. 161 (1996) 13–24, is whether occurrence (b) characterizes the dual Lüneburg plane of order q^2 (equivalently: whether occurrence (c) characterizes the Lüneburg plane of order q^2).

The approach developed there is based on the possibility of describing a projective plane π of even order possessing an oval Ω by means of the one–factorizations of certain complete graphs arising from the lines of π, see the next subsection: we were able to determine all one–factors which may occur in such one–factorizations, obtaining in particular all one–factorizations of the complete graph on q^2 vertices admitting the one–point–stabilizer of Sz(q) as an automorphism group and having $q-1$ prescribed one–factors, namely those arising from the involutions in the group.

This construction is also interesting from a graph–theoretical point of view: there are not too many infinite families of one–factorizations of complete graphs for which a non–trivial automorphism group is explicitely known. Incidentally, E. Mendelsohn and A. Rosa on page 49 of their paper *One–Factorizations of the Complete Graph: a Survey*, J. Graph Theory 9 (1985) 43–65 quote an unpublished result of P. Cameron stating that the probability that a one–factorization of K_{2m} is *rigid* (that is its full automorphism group is trivial) tends to 1 as m goes to infinity. Hence symmetry is rare for general one–factorizations, those arising from our geometric context seem to have a better chance.

In general, the problem of determining when two of the above one–factorizations may arise from distinct lines in the same plane remains open. Nevertheless, the method seems adequate for computer calculations, which we have actually performed in the smallest case $q = 8$: the dual Lüneburg plane is indeed the only plane of order 64 for which possibility (b) occurs.

6.2 Ovals and one–factorizations

Assume that π is a projective plane of even order n containing an oval Ω and let Ω' denote the hyperoval arising from Ω. Each line ℓ of π leads to a one–factorization of the complete graph whose vertices are the points of Ω'

not lying on ℓ. In fact, the lines through a point P outside Ω' partition Ω' into 2–subsets. Now, if ℓ is an external line of Ω, the set of such partitions, as P varies on ℓ, is a one–factorization of the complete graph on Ω'. If ℓ is a tangent or a secant of Ω (i.e. ℓ meets Ω') the partition induced by $P \in \ell \setminus \Omega'$ contains $\Omega' \cap \ell$ as one part; the set of such partitions as P varies in $\ell \setminus \Omega'$ yields thus a one–factorization of the complete graph on $\Omega' \setminus \ell$. Clearly, any collineation of π fixing Ω and ℓ induces an automorphism of the associated one–factorization; in particular an involutory elation fixing Ω and ℓ yields a so called *one–factor symmetry*.

We now want to consider the above construction in a special situation. Let π be a finite projective plane of order q^2 and let G be a collineation group of π which is isomorphic to $\mathrm{Sz}(q)$. Assume G fixes an oval Ω and acts 2–transitively on its points (case (b) of Theorem 28.11 in Lüneburg's book on translation planes).

As we have seen, if ℓ_X is the tangent line of Ω at the point X, then ℓ_X defines a one–factorization of the complete graph on $\Omega \setminus \{X\}$ as follows. Each point $P \in \ell_X$, $P \neq X$, defines a one–factor by considering the lines through P which are secant to Ω. This observation was stated in the language of minimal edge colorings by P.J. Cameron in section 6 of his paper *Minimal edge–colourings of complete graphs*, J. London Math. Soc. (2), 11 (1975) 337–346.

Equivalently, we may consider the involutory permutation j_P on $\Omega \setminus \{X\}$ mapping each point Q to the further point of intersection of the line PQ with Ω; the permutation set $J_X = \{j_P;\ P \in \ell_X, P \neq X\} \cup \{\mathrm{id}\}$ is sharply transitive on $\Omega \setminus \{X\}$. Whenever needed, we extend the action of J_X to the whole of Ω by agreeing that J_X fixes X. This is the approach introduced by F. Buekenhout in his paper *Etude intrinséque des ovales*, Rend. Mat. Applic. (5), 25 (1966) 333–393: the union of the permutation sets J_X as X varies on the oval Ω is a so called *abstract oval*, also called a *Buekenhout oval* or simply a *B–oval*.

Since G_X fixes X and ℓ_X we see that J_X is invariant under conjugation by G_X. Consider the characteristic subgroup V_X of G_X consisting of the identity and of the $q-1$ involutions in the unique Sylow 2–subgroup of G_X. These involutions, when considered as collineations of π, are necessarily elations, as proved in P. Dembowski, *Zur Geometrie der Suzukigruppen*, Math. Z. 94 (1966) 106–109. Their centers lie on ℓ_X, which means $V_X \subseteq J_X$. Note that V_X commutes with J_X elementwise.

The action of G_X on ℓ_X yields three point–orbits, namely $\{X\}$, an orbit

Δ of $q-1$ points which are precisely the centers of the elations in V_X and an orbit Λ consisting of the remaining q^2-q points, see §28 in Lüneburg's book. It follows that for each $P \in \Lambda$ the involution j_P has q^2-q conjugates under the action of G_X and they all lie in J_X; the centralizer of j_P in G_X is thus precisely V_X and we have $J_X = V_X \cup \{kj_Pk^{-1};\ k \in G_X\}$. The relevant fact here is that all these permutation sets J_X can be described explicitely, see section 3 in A. Bonisoli, G. Korchmáros, *Suzuki groups, one–factorizations and Lüneburg planes*, Discrete Math. 161 (1996) 13–24.

Let Y, ℓ_Y and J_Y be a point of Ω different from X, the tangent to Ω at Y and the corresponding permutation set on Ω respectively. If P, Q are points with $P \in \ell_X$, $P \neq X$, $Q \in \ell_Y$, $Q \neq Y$ and j_P, j_Q denote the corresponding involutory permutations on Ω, then since the line PQ meets Ω in at most two points, we have $|\mathrm{Fix}(j_Pj_Q)| \leq 2$. In particular in our situation there exists an involution $k \in G$ exchanging X and Y, yielding $J_Y = kJ_Xk^{-1}$.

We may now reverse our point of view and assume that a candidate for J_X is available, namely a permutation set on Ω with all the properties we have just described. The previous observation may be used to test whether the given permutation set can actually arise from the tangent to an oval: if a conjugate J' of J_X by an involution in G (not fixing X) is such that $|\mathrm{Fix}(jj')| \geq 3$ holds for at least one pair of involutions $j \in J_X$ and $j' \in J'$ then the candidate for J_X must be rejected.

Once our candidate for J_X has positively passed this test, we have in principle reconstructed one piece of our plane, namely all points (the elements of Ω and all permutations in some conjugate of J_X, where the identity plays the role of the nucleus of the oval), all the tangent lines of Ω (the conjugates of J_X under G with their respective fixed point) and all the secant lines of Ω (to each pair of distinct points of Ω add all permutations in some conjugate of J_X which exchange these points). With the terminology of Buekenhout ovals we have reconstructed the so called *ambient*.

In order to get the whole of the plane, we must be able to define external lines. Assume that the plane π, the oval Ω and an external line ℓ are given. If P, Q are distinct points on ℓ and j_P, j_Q denote the corresponding involutory permutations on Ω then the product j_Pj_Q is fixed–point–free on Ω. By Theorem 28.11 in Lüneburg's book there are two orbits $\mathcal{E}_1$, $\mathcal{E}_2$ of external lines under the action of G, the lines in $\mathcal{E}_1$ resp. in $\mathcal{E}_2$ admitting a dihedral group of order $2(2^{2e+1}+2^{e+1}+1)$ resp. $2(2^{2e+1}-2^{e+1}+1)$ in their stabilizers. Each external line of $\mathcal{E}_1$ resp. $\mathcal{E}_2$ contains $\nu_1 = 2^{2e+1}+2^{e+1}+1$ resp. $\nu_2 = 2^{2e+1}-2^{e+1}+1$ points which are centers of elations in G and are thus obtained as conjugates in G of an involution in V_X; all other points of the line are obtained as suitable conjugates in G of an arbitrary involution $j \in J_X \setminus V_X$.

We reverse our point of view one more time and assume that we have selected ν_1 resp. ν_2 involutions in G which should correspond to elation centers on an external line of $\mathcal{E}_1$ resp. $\mathcal{E}_2$: these involutions must be found in a dihedral subgroup G_1 resp. G_2 of G of order $2\nu_1$ resp. $2\nu_2$.

We fix an involution $j \in J_X \setminus V_X$ and test whether the products of the ν_1 resp. ν_2 previous involutions with a given G–conjugate j' of j are fixed–point–free on Ω. If j' passes the test, then further conjugates passing the test are the G_1–conjugates resp. G_2–conjugates of j'. The set of points of our external line which are not elation centers in G must be partitioned into G_1–orbits resp. G_2–orbits of the type just described: once two such orbits are selected, we must further test whether the product of a given permutation in one orbit with an arbitrary permutation in the other orbit is fixed–point–free.

6.3 The uniqueness of the dual Lüneburg plane of order 64

Skipping the explicit description of the permutation set J_X (it requires some non–trivial notation, see the quoted paper) and for which we essentially have uniqueness, let us see how that information can be used.

Proposition 6.1 *The dual Lüneburg plane is the only plane of order* 64 *admitting* Sz(8) *as a collineation group fixing an oval.*

Proof. The previous observations and some computational work (by hand and by computer) lead to the fact that we can fix the choice of J_X and of an involution $j \in J_X \setminus V_X$ as we wish. The above discussion shows that our assertion will hold as soon as we can prove that external lines in the two orbits $\mathcal{E}_1$ and $\mathcal{E}_2$ can be reconstructed uniquely. To this purpose let G_1 resp. G_2 be a dihedral subgroup of order $2\nu_1$ resp. $2\nu_2$ of G with $\nu_1 = 13$, $\nu_2 = 5$.

We checked by computer that there are precisely two G_1–orbits of conjugates of j (of size 26 each) with the property that the product of a permutation in the orbit with an involution in G_1 is fixed–point–free; it follows from $65 = 2 \cdot 26 + \nu_1$ that an external line in $\mathcal{E}_1$ can be reconstructed uniquely.

We also verified by computer that there are precisely twelve G_2–orbits of conjugates of j (of size 10 each) with the property that the product of a permutation in the orbit with an involution in G_2 is fixed–point–free; the further test to see when all products of a given permutation in one orbit with each permutation in another orbit are fixed–point–free leaves a unique set of six pairwise "compatible" orbits. Consequently an external line in $\mathcal{E}_2$ can also be reconstructed uniquely. □

The computer programs that I have mentioned above were written in CAYLEY and ran on a VAX 6510, it must have been around 1994. I have no idea whether the performances of the current machines would allow the testing of the next case, which would be interesting: the order of the plane is $32^2 = 1024$.

Let me add as a final comment on this subject that if one gets rid of the assumption that the order of the plane on which $\mathrm{Sz}(q)$ acts as a collineation group is *precisely* q^2, practically nothing is known, at least as far as I am concerned.

7 Recent results, open problems

As we have seen, if the action of a collineation group fixing an oval in a projective plane of odd order is assumed to be primitive we have a full classification. Much is known even if the action is only assumed to be transitive. what if we allow more than one orbit? An interesting open case occurs when the group fixes one point and acts primitively (or even 2–transitively) on the oval.

7.1 An example in a commutative twisted field plane

Twisted fields and translation planes coordinatized over twisted fields were introduced by A.A. Albert in the papers *On nonassociative division algebras*, Trans. Amer. Math. Soc. 72 (1952) 296-309 and *On the collineation groups associated with twisted fields*, Calcutta Math. Soc. Golden Jubilee Commemoration, Calcutta Math. Soc. 1958/59, Part II (1963) 485-479. See also Dembowski's book p. 242 and the recent survey article M. Cordero, R.F. Figueroa, *Transitive autotopism groups and the generalized twisted planes*, in *Mostly finite geometries* (ed. N.L. Johnson) Lecture notes in pure and applied mathematics, vol. 190 (1997) 191-196.

For our purpose we will adopt the model of a commutative twisted field plane introduced in V. Abatangelo, M.R. Enea, G. Korchmáros, B. Larato, *Ovals and unitals in commutative twisted field planes*, Discrete Math. (to appear).

Let $GF(q^2)$ be a Galois field of order $q^2 = p^{2m}$, p odd prime, containing a subfield $GF(d)$ such that -1 is not a $(d-1)$–th power in $GF(q^2)$. Set $d = p^s$ and $r = 2m/s$. Then $r \geq 3$ is odd, and every element $x \in GF(q^2)$ can be uniquely expressed in the form $x = a^d + a$ for some element $a \in GF(q^2)$. The commutative twisted field $Q(+, \star)$ of order q^2 (also called commutative Albert twisted field) may be obtained from $GF(q^2)$ by replacing the multiplication in $GF(q^2)$ with a new one defined by the rule $(a^d + a)(b^d + b) = a^d b + ab^d$. Correspondingly, the affine plane π coordinatized by $Q(+, \star)$ is obtained from the desarguesian plane $AG(2, q^2)$ over $GF(q^2)$ by replacing lines not through (∞) (i.e. non–vertical lines) with graphs of functions over $GF(q^2)$. These functions are $y = m^d x + mx^d + w$ where m, w range over $GF(q^2)$. Translations of $AG(2, q^2)$ are also translations of π. Moreover, for every $a, b, c \in GF(q^2)$, we have a collineation $x' = x + a, y' = c^d x + cx^d + y + b$ which is either a translation or a shear with special point (∞) according as $c = 0$ or $c \neq 0$, or a product of them. The group Π consisting of these collineations is a metabelian group of order q^6 and it is a normal subgroup of the full collineation group Γ of π.

Since $Q(+, \star)$ is commutative and distributive, it is easy to check that for every element $a \in GF(q^2)$ the points $P(x, y)$ with $y = x \star x = x^{d+1}$ together with (∞) form a parabolic oval in π. We will denote it by Ω_a.

Incidentally V. Abatangelo, G. Korchmáros and B. Larato in their recent paper *Transitive parabolic unitals in translation planes of odd order* that it is possible to select q ovals Ω_a in such a way that their union is a unital in π.