On collination groups of finite planes

Supplement to the notes of Arrigo Bonisoli
by Koen Thas

Intensive Course on Finite Geometry and its Applications,
University of Ghent, April 3-14, 2000

Summary

It is our purpose to specify some of the theory of collineation groups of finite projective planes as introduced by A. Bonisoli (although we do not discuss the complete notes here).
We included a lot of relevant exercises of varying degree of difficulty.

1 Notation, definitions and some exercises

Points will always be denoted by small latin letters, lines by capitals (unlike Bonisoli's notations). By (a, b), with $a, b \in \mathbb{N}$, we denote the greatest common divisor of a and b.
Suppose π is a projective plane of order n, and suppose (p, L) is a point-line pair. Then a collineation θ of π is a (p, L)-**collineation** if θ fixes any point on L and every line through p. If (p, L) is a *flag*, then θ is also called a (p, L)-**elation**; if (p, L) is an *anti-flag*, then θ is said to be a (p, L)-**homology**. In both cases, L is called the **axis** and p the **center**. A collineation of a finite projective plane has a center if and only if it has an axis. Collineations with centers are often called **central collineations** or **perspectivities**. A projective plane π of order n is called (p, L)-transitive for a point-line pair (p, L) if the group of (p, L)-collineation is maximal (if (p, L) is a flag, the size

is n, otherwise the size is $n-1$).
A projective plane is said to be (M, L)**-transitive** for not necessarily distinct lines M and L if for every point m on L the plane is (m, L)-transitive [1]. Dually one defines (p, q)-transitivity for points p and q.

Exercise 1.1 *If a projective plane π is (p, L)-transitive and (q, L)-transitive for distinct points p and q on L, show that π is (pq, L)-transitive.*

Exercise 1.2 *Let π be a finite projective plane. If θ is an involutory perpectivity of π and ϕ is a Baer involution, show that $\theta\phi$ has even order.*

Exercise 1.3 *Let π be a finite projective plane. If $\theta \neq \mathbf{1}$ is a (p, L)-collineation and $\phi \neq \mathbf{1}$ is a (q, M)-homology, with $M I p$ and $L \not{I} q$, then show that $\langle \phi, \theta \rangle$ contains a non-identity (p, M)-elation.*

Exercise 1.4 *Let π be a projective plane with a collineation group Γ which has a nonfaithful orbit containing a quadrangle. Show that π must leave some subplane invariant.*

Exercise 1.5 *Suppoae that Γ is an abelian collineation group of a finite projective plane of order n, such that $|\Gamma| \neq n^2+n+1$. Show that $|\Gamma| \leq n^2$.*

Exercise 1.6 *Show that a collineation of a finite projective plane has an equal number of fixed points and lines.*

Exercise 1.7 *Let Γ be a collineation group of a finite projective plane π. Show that Γ has an equal number of point and line orbits.*

2 Ovals, ovoids and a theorem concerning the fixed element structure of a collineation of a finite projective plane

Suppose $\mathcal{O}$ is a k**-cap** of a projective space $\mathsf{PG}(d, q)$, $q > 2$, i.e. a set of k points no three of which are collinear. A line is called **secant, tangent** or **external** to a k-cap if it intersects the cap respectively in $2, 1$ or 0 points.

[1] If $M = L$, then the plane is a **translation plane** with **translation line** L; see further on.

Following Tits [13] in 1962, an **ovoid** is a k-cap of $\mathsf{PG}(d, q)$ such that the union of the tangent lines at a point $p \in \mathcal{O}$ is a hyperplane. Tits counts the number of points of an ovoid, and states a condition on n such that $\mathsf{PG}(d, q)$ admits ovoids.

Note We assume $q > 2$ since $q^2 + 1 = 5$ is not the maximal bound for a k-cap in $\mathsf{PG}(3, 2)$; there do exist 8-caps (as the complements of planes).

Theorem 2.1 (Tits [13], see [3]) *If $\mathsf{PG}(d, q)$ has an ovoid, then $d \leq 3$. Further, an ovoid in $\mathsf{PG}(d, q)$ has $q^{d-1} + 1$ points.*

If $d = 2$ (in this case, the plane doesn't have to be Desarguesian), then an ovoid is also called an **oval**; also, a **hyperoval** of a projective plane of order n is a set of $n + 2$ points no three of which are collinear. Existence of such a hyperoval forces n to be even. A point is said to be **internal**, resp. **external**, to an oval if it is incident with at most one, resp. at least two, tangent lines.

Theorem 2.2 (see e.g. [7]) *Suppose $\mathcal{K}$ is an oval in a $\mathsf{PG}(2, q)$, q even. Then all tangent lines to $\mathcal{K}$ are concurrent with a point p, and this point is called the* **nucleus** *of the oval.*

Corollary 2.3 *For q even, an oval can be uniquely completed to a hyperoval by adding the nucleus.*

Theorem 2.4 (see e.g. [7]) *Suppose $\mathcal{K}$ is an oval in a $\mathsf{PG}(2, q)$, q odd. Then all external points of $\mathcal{K}$ are incident with 2 tangent lines to the cap, and all internal points with 0 tangent lines.*

2.1

Theorem 2.5 *Suppose π is a finite projective plane, and suppose θ is a collineation of the plane. Then there are equally as many fixed points and fixed lines.*

proof:
See e.g. [8]. □

Suppose π is a finite projective plane of order n, and let θ be a nontrivial collineation. If π_θ is the fixed element structure, and π_θ contains an ordinary quadrangle, then π_θ is a projective plane, since every two fixed lines intersect in a fixed point and vice versa. Also, if the order of π_θ is m, then $m \leq \sqrt{n}$ (see section 4). Suppose π_θ contains no quadrangle. If π_θ contains a triangle, then by Theorem 2.5 there is a nonincident point-line pair (q, M) and a natural number r, such that π_θ consists of q, L, r points q_i with $1 \leq i \leq r$ on L and the lines qq_i. Now suppose π_θ doesn't contain a triangle, but still has some fixed elements. Then by Theorem 2.5 there is an $h \in \mathbb{N}$ and a flag (p, L) such that π_θ consists of p, L, h points on L different from p, and h lines through p and different from L. If in both of the last cases, r or h is maximal, then θ is a perspectivity.
We have proved the following theorem (using section 4).

Theorem 2.6 *Suppose π is a finite projective plane of order n, and let θ be a nontrivial collineation. If π_θ is the fixed element structure, then we have the following possibilities:*

1. $\pi_\theta = \emptyset$;

2. *there is a flag (p, L) such that π_θ consists of p, L, k points on L different from p, and k lines through p and different from L, with $k \in \mathbb{N}$;*

3. *there is an anti-flag (q, M) and a natural number k, such that π_θ consists of q, L, k points q_i with $1 \leq i \leq k$ on L and the lines qq_i;*

4. *π_θ is a projective plane of order m, with $m \leq \sqrt{n}$.*

□

3 Translation nets, translation planes, Moufang planes and (p, L)-transitivity

A (finite) **net of order** $k(\geq 2)$ **and degree** $r(\geq 2)$ is an incidence structure $\mathcal{N} = (P, B, I)$ satisfying the following properties:

1. each point is incident with r lines and two distinct points are incident with at most one line;

2. each line is incident with k points and two distinct lines are incident with at most one point;

3. if p is a point and L a line not incident with p, then there is a unique line M incident with p and not concurrent with L.

A net of order k and degree r has k^2 points and kr lines.

3.1 Dual nets and the axiom of Veblen

Now we introduce the *Axiom of Veblen* for dual nets $\mathcal{N}' = (P, B, I)$.

Axiom of Veblen. If $L_1 I p I L_2$, $L_1 \neq L_2$, $M_1 \not{I} p \not{I} M_2$, and if L_i is concurrent with M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2.

The only known dual net which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net H_q^n, $n > 2$, which is constructed in the following way: the points of H_q^n are the points of $\mathsf{PG}(n, q)$ not in a given subspace $\mathsf{PG}(n-2, q) \subset \mathsf{PG}(n, q)$, the lines of H_q^n are the lines of $\mathsf{PG}(n, q)$ which have no point in common with $\mathsf{PG}(n-2, q)$, the incidence in H_q^n is the natural one.
The following theorem characterizes these dual nets H_q^n with the axiom of Veblen.

Theorem 3.1 (Thas and De Clerck [12]) *Let $\mathcal{N}'$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1 > s$. If $\mathcal{N}'$ satisfies the Axiom of Veblen, then $\mathcal{N}' \cong H_q^n$ with $n > 2$ (hence $s = q$ and $t + 1 = q^{n-1}$).*

Suppose $\mathcal{N}$ is a net of order k and degree r; then $\mathcal{N}$ is a **translation net** if there is an automorphism group G of the net which fixes the parallelclasses elementwise, and acting regularly on the points of the net (in this case, the order of G clearly is k^2). In particular, if we put $k + 1 = r$, then $\mathcal{N}$ is an **affine translation plane**. If L is the line at infinity of the affine plane $\mathcal{N}$, then the projective completion of $\mathcal{N}$ is a **(projective) translation plane with translation line** L.
If a projective plane π is a translation plane for every line, then π is said to be a **Moufang plane**. There is a somewhat more natural definition of a

Moufang plane, certainly from the point of view of the theory of the *generalized polygons* (a projective plane is nothing else than a generalized 3-gon); a Moufang plane is just a projective plane which is (p, L)-transitive for every *flag* (p, L) (a flag is an incident point-line pair). From Proposition 2.6 of [2], there follows immediately that this alternative definition directly leads to the fact that every line is a translation line, and the converse is trivial.
There is another reason to see a Moufang plane in the second way; one of the major results in the theory of collineations of finite projective planes (and finite projective planes in general), is the **Lenz-Barlotti classification** [3, 15], which is a classification based on the possible subconfigurations of (not necessarily incident) point-line pairs (p, L) for which the planes are (p, L)-transitive.
The coordinate structure of a Moufang plane is an **alternative division ring**, which is a set with two binary operations — called "addition" and "multiplication" — satisfying the following properties: (a) the additive structure is an abelian group; (b) both distributive laws hold; (c) multiplication has an identity element and each nonzero element has a multiplicative inverse; (d) the identities $x^{-1}(xy) = y = (yx)x^{-1}$ hold for each nonzero element x and any element y, and (e) the alternative laws $x(xy) = (xx)y$ and $(yx)x = y(xx)$ hold for arbitrary elements x and y. A theorem of Artin and Zorn states that in every finite alternative division ring multiplication is associative, and hence such a ring is a finite field by Wedderburn's theorem. Thus any Moufang plane is Desarguesian (for more details, see [8]).

Exercise 3.2 *Two division rings (a* **division ring** *is a quasi-field which satisfies the right distributive law)* D_1 *and* D_2 *are called* **isotopic** *if there is a triple* (P, Q, R) *of non-singular additive mappings from* D_1 *onto* D_2 *such that* $P(x) \odot Q(y) = R(x \oplus y)$ *for all* $x, y \in D_1$ *(where* $\odot$ *is the multplication of* D_2 *and* $\oplus$ *the multplication of* D_1*).*
Show that a division ring D *is isotopic to a commutative division ring if and only if there is a nonzero element* x *in* D *such that* $(xy)z = (xz)y$ *for all* $y, z \in D$*.*

One could also define **Moufang nets** in the same 'classical' (the first) way of the Moufang planes — i.e. a net of which any line is a tranlation line — but here, classifications are not at all obvious; if a net is a translation net in general, then it is not even known whether or not the translation group is abelian, or that the order is a prime-power! The reason of these difficulties

is clear: not every two points are collinear in a general net. Probably, a Moufang net is the dual of a H_q^n for some pair (q, n).
Some classification theorems, quite similar to the theorems of *Ostrom-Wagner* or *Lüneburg* are available.

Theorem 3.3 (Lüneburg) *Let π be a projective plane of order n, with n a prime power, and suppose that π admits a collineation group isomorphic to $\mathsf{PSL}_2(n)$. Then π is Desarguesian.*

Theorem 3.4 (Wagner [14]) *Let $\mathcal{A}$ be a finite affine plane and suppose Γ is a collineation group transitive on the lines of $\mathcal{A}$. Then $\mathcal{A}$ is an affine translation plane and Γ contains the translation group of $\mathcal{A}$.*

Theorem 3.5 (Ostrom-Wagner) *Let π a finite projective plane with a collineation group Γ which is doubly transitive on the points of π. Then π is Desarguesian and Γ contains the little projective group of π.*

proof:
One proves that Γ acts doubly transitive on the lines of π, so that Γ_L is transitive on the affine lines of π^L with L a line of π. By Wagner's theorem, π^L is a translation plane with respect to L and hence, by the transitivity on the lines, π is a Moufang plane. Thus π is Desarguesian. That Γ contains the little projective group follows from the fact that Γ contains all elations with axis L for any L in π (see e.g. [8]). □

A **derivable net $\mathcal{N}$ of degree** r is a net with the property that through each two collinear points there is precisely one affine subplane of $\mathcal{N}$ of order $r - 1$.

Theorem 3.6 (Johnson [9]) *A finite net of order q^2 and degree $q + 1$ is a derivable net if and only if the net admits a collineation group isomorphic to $\mathsf{PSL}(4, q)_L$, where L is a line of the associated 3-dimensional projective space upon which the abstract group acts.*

Theorem 3.7 (Hiramine [5, 6]) *Suppose $\mathcal{N}$ is a net of order q^2 and degree $q + 1$ with q a prime-power that admits a collineation group G with a point-regular normal subgroup T such that $G/T \cong \mathsf{GL}_2(p)$. Then $\mathcal{N}$ must be isomorphic to a regulus net or a twisted cubic net.*

Theorem 3.8 (Hiramine and Johnson [6]) *Suppose $\mathcal{N}$ is a net of order p^2 and degree $p+1$ with p a prime that admits a collineation group G with a point-regular normal subgroup T such that $G/T \cong \mathsf{SL}_2(p)$. Then $\mathcal{N}$ must be a regulus net, a twisted cubic net or one of the three sporadic nets $\aleph_p$ for $p \in \{2,3,5\}$.*

For notions which are not explained in the last two theorems, we refer the reader to [6].

Exercise 3.9 *Show that a finite net of order $q+1$ and degree q^2 is derivable if and only if it is the dual of a H_q^3 for some prime power q. Also, show that if $\mathcal{N}$ is a derivable net of order q^2 and degree $q+1$, then there are* **precisely** *q^3+q^2 affine subplanes of $\mathcal{N}$ of order q and vice versa.*

Exercise 3.10 *Let π be a finite projective plane and let Γ be a collineation group which acts transitively on the points of π. If Γ contains a nontrivial elation, show that π is a Moufang plane.*

Exercise 3.11 *Let π be a projective plane such that the automorphism group fixes no point or line of π. If π is (p,L)-transitive for some flag (p,L), show that π is a Moufang plane.*

Exercise 3.12 *Let π be a finite projective plane of order n and let Γ be a collineation group of π. If every point of π is the center of a nontrivial elation of Γ and if n is not a square, show that one of the following conclusions holds:*

1. *π is Desarguesian and Γ contains its little projective group;*
2. *π is the dual of a translation plane and Γ contains the dual translation group of π.*

Exercise 3.13 *Let π be a finite projective plane of order n and let Γ be a collineation group of π. If every point of π is the center of a nontrivial homology in Γ, show that one of the following conclusions holds:*

1. *π is Desarguesian and Γ contains its little projective group;*
2. *π is the dual of a translation plane and Γ contains the dual translation group of π;*

3. π is a translation plane and Γ contains the translation group of π.

Exercise 3.14 *Let π be a finite projective plane of order n and let Γ be a collineation group of π. If every point of π is the center of a nontrivial perpectivity in Γ and if n is not a square, show that one of the following conclusions holds:*

1. *π is Desarguesian and Γ contains its little projective group;*

2. *π is the dual of a translation plane and Γ contains the dual translation group of π;*

3. *π is a translation plane and Γ contains the translation group of π.*

4 Some easy remarks on the proofs of some propositions

- **Proposition 2.8** About part (3) of the proof: $(\alpha\beta)^p$ equals $\alpha^p\beta^p$ because T is abelian.

- **Proposition 2.9** Suppose $\pi' \subseteq \pi$ is a subplane of order m of a projective plane π of order n. Then the number of lines of π which intersect π' is given by

$$m^2 + m + 1 + (m^2 + m + 1)(n - m) \leq n^2 + n + 1; \qquad (1)$$

  if $m > \sqrt{n}$, then the left-hand side is strictly larger than $n^2 + n + 1$ if $n \neq m$, hence $n = m$ or $m \leq \sqrt{n}$, with the latter equality holding if π' is a Baer subplane of π (with n a square). Moreover, if $m < \sqrt{n}$, then there are lines which don't meet π'.

- **Proposition 3.2** Remark that $T = \cup_{c\in L} T_c = \langle T_c, c \in L\rangle$ by Proposition 2.7, and the fact that $T_a \cap T_b = \mathbf{1}$, for all distinct points a and b on L.

- **Proposition 3.4** Assume that $T^{\alpha\beta} = T$. Then

$$T^{\alpha} = T^{\alpha\beta\beta^{-1}} = T^{\beta^{-1}} = T^{\beta}$$

since β is an involution. Thus,

$$(T^{\alpha})^{\alpha\beta} = T^{\alpha^2\beta} = T^{\beta} = T^{\alpha}$$

hence if $T^{\alpha} \neq T$, then $\alpha\beta$ fixes the two distinct axes T and T^{α}.

5 On Proposition 4.3

The **Klein** 4**-group** is the unique elemetary abelian, non-cyclic group of order 4. A permutation is called **odd**, resp. **even**, if it can be decomposed in an odd, resp. even, number of transpositions.

Lemma 5.1 (Baer) *Let α be an involution of a finite projective plane π of order n. Then either n is a square and α is a Baer involution or α is a central collineation.*

Lemma 5.2 *Let n be an odd square and let π be a finite projective plane of order n with an oval Ω. Let β be a Baer involution of π fixing Ω and let π_0 be the fixed Baer subplane of β. One of the following holds. i) Ω avoids π_0 and β induces an odd permutation on Ω. The lines of π_0 are divided into $(n+1)/2$ secants and $(\sqrt{n}+1)^2/2$ external lines. The points of π_0 are divided into $(n+1)/2$ external points and $(\sqrt{n}+1)^2/2$ internal points. ii) Ω meets π_0 in an oval Ω_0 and β induces an even or an odd permutation on Ω according as $\sqrt{n} \equiv 1$ or $-1 \mod 4$. No point of π_0 is internal to Ω and no line of π_0 is external to Ω.*

Lemma 5.3 *Let n be a square with $\sqrt{n} \equiv 1 \mod 4$ and let π be a finite projective plane of order n with an oval Ω. Assume β_1 and β_2 are distinct commuting Baer involutions fixing Ω and inducing permutations of the same parity on Ω. Then the product $\beta_1\beta_2$ is a homology.*

Lemma 5.4 *The set of all even permutations on a set forms a group. A set of odd permutations on a set, together with the identity, forms a group if and only if the set exists of an involution.*

proof:
Immediate. □

Theorem 5.5 *Let π be a finite projective plane of odd order n with an oval Ω. If G is a Klein 4–group of collineations of π fixing Ω, then G contains at least one involutory homology inducing an even permutation on Ω.*

proof:
First of all, suppose n is not a square. Then G cannot contain Baer involutions, and hence by Lemma 5.1 and Lemma 5.2, every element of G is a homology. Since G is a group of order 4, there follows by Lemma 5.4 that at least one of these homologies induces an even permutation on Ω.
Next, suppose n is a square.
If $\sqrt{n} \equiv -1 \mod 4$, then each Baer involution induces an odd permutation on Ω by Lemma 5.2. Hence, if there is an element of G inducing an even permutation on Ω, then it must be a homology. By Lemma 5.4 there must be at least one such an element.
Finally, suppose $\sqrt{n} \equiv 1 \mod 4$. If every element of G induces an even permutation on Ω, then by Lemma 5.3, G contains at least one homology and we are done. Thus, assume that there is an element inducing an odd permutation. Then by Lemma 5.4, there must be exactly two such elements. We assume that the element inducing an even permutation is a Baer involution, and therefore, by Lemma 5.3, we suppose the other elements to be a Baer involution, say ϕ, and a homology, say θ. There holds, if L is the axis of θ, that L^{ϕ} is the axis of the homology $\phi^{-1}\theta\phi = \theta$ (recall thet G is abelian), and hence $L^{\phi} = L$. As a corollary there holds that $L^{\phi\theta} = L$. The fixed Baer subplane of ϕ doesn't meet Ω by Lemma 5.2, and hence it must be an external line to Ω (it cannot be a secant line since in that case it would follow by Lemma 6.1 (see further on) and by the fact that $n - 1 \equiv 0 \mod 4$ that θ would induce an even permutation on Ω). But, $\phi\theta$ is precisely the unique element of G which induces an even permutation on Ω, and thus by Lemma 5.2, L cannot be an external line to Ω, a contradiction. This proves the theorem.□

Definition

The **generalized quaternion group** $\mathcal{Q}_n$, is defined by

$$\mathcal{Q}_n = \langle x, y \parallel x^{2n} = \mathbf{1}, y^2 = x^n, x^y = x^{-1} \rangle.$$

For $n = 2$, there is another well-known representation:
$\mathcal{Q}_2 = \{\pm\, \mathbf{1},\ \pm i, \pm j, \pm k \parallel i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}$.

The size of $\mathcal{Q}_n$ is $4n$.

Theorem 5.6 *Let π be a finite projective plane of odd order n with an oval Ω. Let E be a 2-group of collineations of π fixing Ω. If E contains no involutory homology then E is cyclic.*

proof:
There is an involution γ in the center of E (this is because any p-group, with p a prime, has a nontrivial center, and if r is a prime which divides the order of a finite group, then this group must contain an element of order r). Any further involution δ in E should commute with γ and thus span with γ a Klein 4-group G. Proposition 5.5 shows that G contains involutory homologies, hence so does E, a contradiction. We conclude that E contains a unique involution (which is a Baer-involution). The finite 2-groups with a unique involution are well characterized: *they are either cyclic or generalized quaternion groups* (note that a generalized quaternion group isn't necessarily a 2-group).
Assume E is a generalized quaternion group $\mathcal{Q}_n$, $n \geq 2$, with Baer involution β whose fixed Baer subplane we denote by π_0. Since $|E| > 2$, we know that β is the square of some collineation in E; if g is an arbitrary nontrivial element of order 4, then $g^2 = \beta$ (since E contains a unique involution). Thus, as a square, β induces an even permutation on Ω. We know then from Lemma 5.2 that $\Omega_0 = \Omega \cap \pi_0$ is an oval of π_0. Since β is in the center of E, we have that π_0 is left (setwise) invariant by the whole of E. This is easy to see; suppose (X, G) is a permutation group, with $N = \langle n \rangle \leq Z(G)$, and let $Y \subseteq X$ the set of elemens of X which are fixed pointwise by N. If $y \in Y$ and $\theta \in G$ are such that $y^\theta \notin Y$, then we have that $y^{\theta n} = y^{n\theta} = y^\theta$, a contradiction since $y^\theta \notin Y$.
We want to show that the kernel of the action of E on π_0 is precisely $\langle \beta \rangle$. Assume the kernel contains a collineation of order 4, say δ. Let L_0 be a line in π_0 which is external to Ω_0. Lemma 5.2 shows that L_0 is a secant to Ω, and hence δ either fixes or interchanges the two points in $L_0 \cap \Omega$, hence $\beta = \delta^2$ necessarily fixes these two points, a contradiction because these two points do not lie in π_0, the fixed subplane of β.
Denote by E_0 the collineation group induced by E on π_0, that is $E_0 = E/\langle \beta \rangle$. Then

$$|E_0| = \frac{|E|}{2} = 2n \geq 4.$$

The group E_0 contains a Klein 4-group G_0; take for instance two collineations of order 4 in E which do not lie in the same cyclic subgroup of order 4: they generate a subgroup inducing a Klein 4-group on π_0.
Suppose that there is an involution α in G_0 which fixes a line M of π_0 which is external to Ω_0. By Lemma 5.2 we again know that M is a secant to Ω. Suppose that α' is an element of E which has the same action on π_0 as α. Since this action is nontrivial, we have that α' is not an involution — since β is the unique involution of E and β acts trivially on π_0 — thus α' either fixes or interchanges the two points of $L \cap (\Omega \setminus \Omega_0)$, and hences α'^2 necessarily fixes these two points. Since α' is not an involution, there follows that the nontrivial collineation α'^2 acts trivially on π_0 but also fixes points outside π_0, a contradiction. We conclude that the Klein 4-group G_0, a collineation group of π_0 fixing the oval Ω_0, does not contain homologies. That contradicts Lemma 5.5. □

Exercise 5.7 *Show that if an abelian group acts transitively on a set, it must act regularly.*

Exercise 5.8 *Let G be a finite group with two distinct elements g, h of order 2. Show that g, h are conjugate in $\langle g, h\rangle$ if and only if gh has odd order. If gh has even order, then show that $\langle g, h\rangle$ contains an element x of order 2 such that x commutes with g and h.*

6 On section 5

If a collineation of a projective plane π fixes an oval pointwise, then it is the identity on the whole plane. A very easy way to see this is the following: if $q = 2$, then the statement is trivial, so suppose $q \neq 2$; then the fixed elements structure is a projective plane π' (see section 2), and this subplane contains an $(n+1)$-arc, which can only be an oval or a hyperoval in π'. Hence $\pi' = \pi$ (see section 4).
Each collineation group fixing an oval or a hyperoval therefore has a faithful permutation representation on the points of the oval or of the hyperoval respectively.

Lemma 6.1 *Let n be odd and let Ω be an oval in π. A nonidentical perspectivity α of π fixing Ω is necessarily an involutory homology and either the*

center is an internal point and the axis is an external line or the center is an external point and the axis is a secant line.
Any two distinct involutory homologies of π fixing Ω have both distinct centers and distinct axes. There cannot exist an elementary abelian group of order 8 *generated by three involutory homologies of π fixing Ω.*

proof:
Let c and L be the center and the axis of α respectively. If c is on Ω and p is any other point on Ω then the line M joining c to p is fixed by α. The points of intersection of M with the oval are precisely c and p. We have $(\Omega \cap M)^\alpha = \Omega^\alpha \cap M^\alpha = \Omega \cap M$, whence $\{c, p\}^\alpha = \{c, p\}$ and since $c^\alpha = c$, we necessarily have also $p^\alpha = p$. So if the center c is on Ω, then α fixes Ω pointwise and so α is the identical collineation of π.
Let N be any line through the center c such that $N \cap \Omega$ is non-empty, hence consists of either one or two points. In either case the collineation α^2 fixes $N \cap \Omega$ pointwise. We conclude that α^2 fixes Ω pointwise and so α is an involution. As an involutory perspectivity in a plane of odd order α must be a homology, because the number of points off N is odd.
Assume c is an external point and let T_1, T_2 be the two tangent lines to Ω through c, meeting Ω at p_1, p_2 respectively. Since α fixes T_1 and Ω, it also fixes their unique common point, that is $p_1^\alpha = p_1$. Similarly, we have $p_2^\alpha = p_2$. As a consequence p_1p_2 is a fixed line not through the center and so it must be the axis. Let, conversely, α have the secant line L as an axis. Set $L \cap \Omega = \{p_1, p_2\}$ and denote by T_1, T_2 the two tangents to Ω at p_1, p_2 respectively. Since α fixes Ω and p_1 it must also fix the unique tangent line to Ω through p_1, that is $T_1^\alpha = T_1$. Similarly, we have $T_2^\alpha = T_2$. The common point c of T_1 and T_2 is therefore also fixed by α, and since c is not on L we have that c is the center of α. Hence α *has an axis which is a secant to* Ω *if and only if its center is an external point.*
If the axis is an external line, then, for each external point p on this line, the homology exchanges the two tangents to Ω through p, hence the center of the homology lies on the line joining the points of contact; since there are at least two external points on the axis, we obtain two distinct lines through the center and so the center is uniquely reconstructed in this case as well. In the converse way, one proves that if the center is an internal point, then the axis is an external line. Hence, *the axis is an external line if and only if the center is an internal point.*
We can conclude that the axis never can be a tangent.

Assume β and δ are distinct commuting involutory homologies fixing Ω. Denote by b and B, resp. d and D, the center and axis of β, resp. δ. The relation $\delta\beta = \beta\delta$ yields $\beta = \delta^{-1}\beta\delta$ and so, since $\delta^{-1}\beta\delta$ is a homology with center b^δ and axis B^δ, we have $b^\delta = b$, showing that b (which is distinct from d) must lie on the axis of δ, that is $b \in D$. Exchanging the roles of β and δ, we obtain $d \in B$.
Let r denote the common point of the axes B and D, and let R be the line joining the centers b and d. Then $\beta\delta$ is an involutory homology with center r and axis R; as the product of two commuting involutions, $\beta\delta$ is itself an involution; furthermore, $\beta\delta$ fixes each one of the points b, d and r, and each one of the lines B, D and R. If $\beta\delta$ were a Baer involution, then it should fix a quadrangle elementwise and so it should fix at least one point p off the triangle formed by these three points and three lines. From the relation $p^{\beta\delta} = p$ we also have $p^\beta = p^\delta$, a contradiction since p^β is collinear with b and p and distinct from p, while p^δ is collinear with d and p and distinct from p. Finally, assume that ϱ is an involutory homology fixing Ω which is distinct from both β and δ and commutes with each one of β and δ. The previous argument shows that the center of ϱ must lie on the axis of β as well as on the axis of δ; furthermore, the center of β must be on the axis of ϱ and also the center of δ must lie on the axis of ϱ. We conclude that ϱ must be an involutory homology with center r and axis R. If, in general, ϕ and θ are involutory homologies with the same center fixing an oval Ω, then, since they both fix each line through the center, their product $\phi\theta$ fixes pointwise the intersection of each such line with the oval, hence $\phi\theta$ fixes the oval pointwise and is thus the identity, yielding $\phi = \theta$. There follows that $\varrho = \beta\delta$, and this completes the proof. □

A finite group G is said to have **p-rank** r (for the given prime p) if p^r is the largest order of an elementary abelian p-subgroup of G (if p divides $|G|$, then the p-rank of G is always strictly positive).

Theorem 6.2 (The 2-rank property) *Suppose π is a finite projective plane of odd order n with an oval Ω, and suppose that G is a collineation group of π fixing Ω. Then the 2-rank of G is at most 3.*

proof:
Let E be an elementary abelian 2-subgroup of G. The involutory homologies

in E together with the identity form a subgroup V of E of order at most 4 by Lemma 6.1. If $V \neq E$, then Lemma 5.5 shows that the product of any two collineations ϕ and θ in $E \setminus V$ — these are Baer involutions — must lie in V, because they generate a Klein 4-group (recall that E is elementary abelian), and hence by Lemma 5.5 we have that $\phi\theta$ is a homology. Thus $|E : V| = 2$ and the assertion follows. □

Remark 6.3 *This property is the basic tool in the detailed analysis required in the proof of the main result of the paper M. Biliotti and G. Korchmáros [1].*

7 Moufang sets and collineations of projective planes

A **Moufang set** $(X, U_{x\|x\in X})$ is a pair which consists of a set X and a family of groups U_x, $x \in X$ and $U_x \in Sym(X)$, for which the following axioms are satisfied:

(MO1) for any $x \in X$, the group U_x fixes x and acts regular on $X \setminus \{x\}$;

(MO2) in the full permutation group of X, the group U_x stabilizes the set $\{U_y \parallel y \in X\}$.

The elements of X are the **points** of the Moufang set, and for any x, the group U_x will be called a **rootgroup**. An element of the group U which is generated by all the rootgroups is a **transvection**, and the group U is the **transvection group** of the Moufang set. If X is a finite set, then the Moufang set also is called **finite**. It is clear that the tranvection group acts 2-transitive on the set of points of the Moufang set.

The following theorem classifies all finite Moufang sets without using the classification of the finite simple groups (see [11, 4]).

Theorem 7.1 (Shult [11]; Hering, Kantor and Seitz [4]) *Suppose $(X, U_{x\|x\in X})$ is a finite Moufang set, and suppose $|X| = s + 1$, with $s < \infty$. Then the transvection group U of the Moufang set must always be one of the following (up to isomorphism):*

1. *a sharply 2-transitive group on X;*

2. $\mathsf{PSL}_2(s)$*;*

3. *the Ree group $R(\sqrt[3]{s})$, with $\sqrt[3]{s}$ an odd power of 3;*

4. *the Suzuki group $Sz(\sqrt{s})$, where $\sqrt{s}$ is an odd power of 2;*

5. *the unitary group $\mathsf{PSU}_3(\sqrt[3]{s^2})$.*

Every root group of course has order s. In the first case, (X, U) is a **Frobenius group**, and it is a known theorem (see any standard work on permutation groups) that $s + 1$ is the power of a prime; in all of the other cases, s is the power of a prime, and we have that $|\mathsf{PSL}_2(s)| = (s + 1)s(s - 1)$ or $(s + 1)s(s - 1)/2$, according as s is even or not, and the group acts (sharply) 3-transitive if and only if s is even. In the other cases, we have that $|R(\sqrt[3]{s})| = (s + 1)s(\sqrt[3]{s} - 1)$, $|Sz(\sqrt{s})| = (s + 1)s(\sqrt{s} - 1)$, and that $|\mathsf{PSU}_3(\sqrt[3]{s^2})| = \frac{(s+1)s(\sqrt[3]{s^2}-1)}{(3,\sqrt[3]{s}+1)}$.

The following theorem is due to J. Cofman (1967).

Theorem 7.2 *Let π be a projective plane of odd order n with an oval Ω. Suppose G is a collineation group of π fixing Ω and acting 2-transitively on the points of Ω, and such that the following property is sattisfied:*

(I) *every involution in G is an involutory homology.*

Then π is Desarguesian, Ω is a conic and $\mathsf{PSL}_2(n) \subseteq G$.

W. M. Kantor improved on this theorem by showing that the same result holds if one replaces Property (I) by the weaker hypothesis "G has **some** involutory homology". Korchmáros proved that one can drop (I) altogether, and Billiotti and Korchmáros [1] improved once more by replacing "2-transitively" by "primitively"(A group G acts **primitively** on a set X if it is transitive and the unique G-invariants are the singletons and X itself (there are no nontrivial G-invariant partitions). It is not hard to see that this equivalent as saying that G acts transitive on X and that for all $x \in X$, G_x is a maximal subgroup of G), with two exceptions (see the last section).

Now suppose π is a projective plane of finite order n, and let Ω be an oval of the plane. Suppose there exists groups G_p, with $p \in \Omega$, such that they fix p

and Ω, and act regularly on $\Omega \setminus \{p\}$, and such that, in the full permutation group of Ω, the group G_p stabilizes the set $\{G_q \parallel q \in \Omega\}$ (hence, assume that $(\Omega, G_{p \parallel p \in \Omega})$ is a finite Moufang set). Then we know that $G = \langle G_p \rangle$ acts as a 2-transitive group on the points of Ω, and that only the following possibilities hold: G is a sharply 2-transitive group on Ω; $G \cong \mathsf{PSL}_2(n)$; $G \cong R(\sqrt[3]{n})$; $G \cong Sz(\sqrt{n})$, or G is isomorphic to the unitary group $\mathsf{PSU}_3(\sqrt[3]{n^2})$. If n is odd, then by the previous remarks we can conclude — since G is always 2-transitive — that π is Desarguesian, that Ω is a conic, and that G is isomorphic to $\mathsf{PSL}_2(n)$. Next, suppose that n is even. First of all one notes that G cannot act as a Ree-group. If $G \cong \mathsf{PSL}_2(n)$, then, because of the fact that G acts faithfully on the oval, the automorphism group of π contains a subgroup isomorphic to $\mathsf{PSL}_2(n)$, and hence by the theorem of Lüneburg, π is Desarguesian.
Now suppose G acts as a $Sz(\sqrt{n})$. Lüneburg proved, in a more general context, that if a group H acts as a $Sz(\sqrt{s})$ on a projective plane of order s, then there are the following three possibilities:

1. G fixes an antiflag (p, L) and acts 2-transitively on the points of L and on the lines incident with p;

2. G fixes an oval Ω and acts 2-transitively on its points;

3. G fixes a line-oval Ω^* and acts 2-transitively on its lines.

All three possibilities occur with the *Lüneburg plane* of order s and its dual, and it is conjectured that these are the only possibilities.

Koen Thas
Ghent University
Department of Pure Mathematics and Computer Algebra
Galglaan 2, B-9000 Ghent
Belgium
kthas@cage.rug.ac.be

APPENDIX

Collineations of finite projective planes and the Petersen graph

With S_n, resp. A_n, we mean the symmetric, resp. alternating, group on n elements (Bonisoli uses the notations $Sym(n)$ and $Alt(n)$).

Theorem 7.3 (Biliotti and Korchmáros [1]) *Let π be a finite projective plane of odd order n, suppose Ω is an oval of π and let G be a collineation group of π fixing Ω. Assume G acts primitively on the points of Ω. Then π is Desarguesian, Ω is a conic and either $n = q$ and G contains a normal subgroup acting on the points of Ω as a $\mathsf{PSL}(2, q)$ in its natural doubly transitive permutation representation, or $n = 9$ and G acts on Ω as an A_5 or an S_5 in their primitive permutation representation of degree* 10.

Note A_5 and S_5 are not isomorphic (as abstract groups).

We give a nice example regarding one of the exceptions of the theorem.
Let X be the set $\{1, 2, 3, 4, 5\}$, and suppose Ω is the set of unordered pairs of distinct elements of X. Then $|\Omega| = 10$. Define a graph $\mathcal{G}$ with vertex set $\mathcal{V}$ as follows: $\mathcal{V}$ is just the set Ω, and two vertices are adjacent if the corresponding pairs are disjoint (as sets). Then this graph is isomorphic to the unique $srg(10, 3, 0, 1)$, the so called **Petersen graph**. Another nice way to construct this graph is by taking as vertices the points of a Desargues configuration, two vertices being adjacent if they are not collinear in the configuration. The automorphism group of a Petersen graph is isomorphic to the symmetric group on five elements S_5. The action of S_5 on the graph is transitive but not 2-transitive on Ω; the Petersen graph is not a complete graph, and hence there are adjacent vertices and nonadjacent ones. The action is primitive though.

References

[1] M. Biliotti and G. Korchmáros; Collineatons groups which are primitive on an oval of a projective plane of odd order; *J. London Math. Soc. (2)* **33** (1986), p.525-534.

[2] A. Bonisoli; Collineations of finite projective planes; *Intensive Course on Finite Geometry and its Applications, University of Ghent, April 3-14* (2000).

[3] H. P. Dembowski; Finite Geometries; *Berlin-Heidelberg-New York, Springer* (1968).

[4] C. Hering, W. M. Kantor and G. M. Seitz; Finite groups with a split BN-pair of rank 1; *J. Algebra* **20** (1972), p.435-475.

[5] Y. Hiramine; On nets of order q^2 and degree $q+1$ admitting $\mathsf{GL}(2,q)$; *Geom. Dedicata* **48** (1993), p.139-189.

[6] Yutaka Hiramine and Norman L. Johnson; Nets of Order p^2 and Degree $p+1$ Admitting $\mathsf{SL}(2,p)$; *Geom. Dedicata* **69** no.1 (1998), p.15-34.

[7] J. W. P. Hirschfeld; Projective Geometries over Finite Fields; *Clarendon Press Oxford* (1979).

[8] Daniel R. Hughes and Fred C. Piper; Projective Planes; *Springer-Verlag, New York Heidelberg Berlin* (1973).

[9] Norman L. Johnson; A group theoretic characterization of finite derivable nets; *J. of Geometry* **40** (1991).

[10] H. Lüneburg; Translation planes; *Springer, Berlin* (1980).

[11] E. E. Shult; On a class of doubly transitive groups; *III. J. Math.* **16** (1972), p.434-455.

[12] J. A. Thas and F. De Clerck; Partial geometries satisfying the axiom of Pasch; *Simon Stevin* **51** (1977), p.123-137.

[13] J. Tits; Ovoïds á translations, *Rend. Mat. e Appl.* **21** (1962), p.37-59.

[14] A. Wagner; On finite affine line transitive planes; *Math. Z.* **87**, p.1-11 (1965).

[15] J. C. D. S. Yaqub; The Lenz-Barlotti classification; *Proc. Proj. Geometry Conference, Univ. of Illinois Chicago* (1967).