Prehistory and History of Polar Spaces and of Generalized Polygons

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Warning

This text is by no means complete. Many items of the bibliography have either not been developed at all in the text itself or they are too briefly mentioned. Some items that have been developed would deserve a more detailed treatment. Some sources may have been forgotten or overlooked. Comments and criticisms are welcome. A period that is fairly well covered is 1956-1974.

Contents

1 Introduction

1.1 In the context of Buildings

The history of polar spaces and of generalized polygons is a major key in order to understand the development of a theory of buildings of spherical type and to understand the emerging new Incidence Geometry. All of this is dominated by the strong personality of Jacques Tits born in 1930. On Tits, see Buekenhout 1990b. Why deal with both subjects in the same study?

Because they are nearly twins, they were born in 1959 and they were a necessary link for the theory of buildings. Actually, the history of polar spaces, generalized polygons and buildings is broadly inseparable.

1.2 The context of Incidence Geometry

A broad context for my study is Incidence Geometry. It can be briefly defined as the kind of geometry that can be performed with a graph as primitive concept together with additional structure. In my opinion this covers potentially any kind of geometry but I shall refrain from futurology and stick to history. Incidence Geometry is surveyed in Buekenhout 1995. It is not yet known under this name by most of those who work in it. In Buekenhout 1998 I describe the rise of Incidence Geometry in the 20th Century during the period ending in 1961, the year where buildings were born. A central role is played by Projective Geometry. A leading and unifying role is played already by Projective Geometry in Klein's Programme of Erlangen (see Klein 1872). During the years 1954-1956 Tits was bringing radically new ideas. He constructed geometries generalizing projective geometries, from any Lie group and over Coxeter-Dynkin diagrams. A projective geometry of dimension n did appear as the particular case where the diagram is of type A_n. The diagram of type $B_n = C_n$ provided geometries that we tend to call classical like projective quadrics but they were not that classical after all. Classically they did appear as objects constructed algebraically inside a projective space. Now they became full spaces on their own, built from smaller objects including projective planes and spaces. Tits own attention went much more to totally new geometries related to the exceptional Lie groups whose geometric understanding had been his most important motivation. The history of these developments has been written by Tits himself. See Tits 1981. It is unknown or misunderstood by many experts. The years 1956-1959 were crucial for our subject. Tits went his path. Since 1953 he had frequent contacts with Hans Freudenthal, the great geometer and expert of Lie groups in Utrecht. This provided room for a lot of osmosis in both directions. Freudenthal's graduate student Ferdinand D. Veldkamp wrote his famous thesis on Polar Geometry and this appeared in print in 1959. One of the obvious characteristics is that the subject stems from the polarities of projective spaces. Veldkamp coins the name *polar geometry.* In December 1991, there was a conference in Utrecht in honor of Veldkamp's 60th birtday with the presence of Tits. I gave a lecture devoted to the history of polar spaces (*A look on Veldkamp's theory of polar spaces and its present role for buildings*). On this occasion, during a private conversation, Tits said that in the 1950s he and Veldkamp did barely meet. But Tits was meeting Freudenthal frequently. Tits said that he had the main ideas for polar spaces in the style of the Veldkamp theory but that he did not work out the details as Veldkamp did. Tits on his side was making a deep, long and difficult study of triality on the seven-dimensional quadrics (hence polar spaces) of Witt index three. The inspiration came from the various kinds of polarities of projective geometry. Tits classified trialities, got the group of each, recognized the groups of type $G_2(F)$ brought to the foreground by Chevalley in 1955, recognized new simple groups now denoted by $^3D_4(F)$ and called twisted Chevalley groups, also discovered independently by Steinberg. Tits produced new geometries consisting of points and lines, from a triality. These were calling for an axiomatic geometric interpretation and became generalized hexagons. At the end, in a short Appendix of two pages he gave formal geometric definitions and used the name: "generalized polygon". We were in 1959. This matters so much because of the broad impact of buildings on various domains of mathematics. The central figure is that of Jacques Tits. Let me recall that buildings were born, in Tits own words, from the meeting of Projective Geometry and of Lie groups. This is not so well known. The role of Lie groups is well understood and recognized. Projective Geometry remains forgotten for reasons that will not be analyzed here.

1.3 In the context of Finite Geometry

Finite Polar Spaces and Generalized Polygons matter a lot in various respects. The history of Finite Geometry deserves a study going beyond existing fragments. As a matter of fact, it existed in the 19th Century. Moreover, the finite fields and the related classical groups were strongly developed in the tracks of Galois and Jordan. In Incidence Geometry and research on its foundations, the necessity of continuity was first assumed then questioned. Hilbert 1899 showed definitely that continuity was not necessary. This was opening a door to finite geometries in a more systematic way around Oswald Veblen especially finite projective planes. A fantastic impulse was given from 1948 on, by Benjamino Segre who founded a great Italian school that has kept growing since then. It suffices here to refer to the book Segre 1961. In many respects all of us, including Tits, were students of Segre. A superb survey of Finite Geometry is given in Dembowski 1968. That book was helping the field to explode far beyond its content. Quite surprisingly in this huge bibliographical survey Veldkamp 1959 does not appear. As to Geometry over a finite field, it has been thoroughly studied in the major treatise of Hirschfeld 1979-1985 and of Hirschfeld-Thas 1991.

1.4 In the context of Linear Algebra

Our subject is truly inseparable from Linear Algebra and from Analytic Geometry. Constructions of our geometric beings rely either on algebra in various guises or on other geometric beings. History of Linear Algebra should be inseparable from our subject too but this is not always the case.

Geometry in the present sense may just be ignored. A very interesting example is Dorier 1997 which is a remarkable study in other respects. A rich source in the direction of geometry is Baer 1952. It provides a deep study of the relationships of Linear Algebra and Projective Geometry. For a classical history of Analytical Geometry I recommend Boyer 1956. Here developments after 1850 are mostly omitted.

1.5 The intersection

A geometry which is both a polar space and a generalized polygon is a Generalized Quadrangle. This has become a big subject on its own growing especially from 1970 on with the work of Thas and Payne in the finite case and many developments related to the school of Ghent. A brilliant synthesis is found in the Red Book Payne-Thas 1984. Another major work is close to appear namely Tits-Weiss 2000 where the Moufang Quadrangles are classified completely after a long progression starting around 1973. This work relies on algebraic developments that will, I believe, matter as much for Algebra than for Geometry. The book does in fact deal with the classification of the Moufang Polygons. The Moufang Quadrangles were curiously enough, the most difficult case.

1.6 In the context of Diagram Geometry

During the years 1954-1958, Tits was developing a theory of geometries over Coxeter diagrams. Actually, he thus dealt with beings that would soon be Buildings and they were amalgamations of beings that would soon be Generalized Polygons. There were no apartments yet. The Coxeter diagrams were generalized by Buekenhout in 1975 as explained in an Oberwolfach lecture. A well-known colleague said to another well-known colleague that my diagrams were too general to ever achieve something with them. The same year, Dan Hughes started to study semi-biplanes on those grounds. Diagrams have now given rise to many developments. A survey is provided by Buekenhout-Pasini 1995. In many respects this subject deals with extensions of, and variations on, Generalized Polygons and Polar spaces.

1.7 In the context of history

The history of mathematics is an exploding science. Too many mathematicians do not care for it and do not realize how important it is both for the public audience of their science and for its development. Some references that we recommend are Kline 1972, Kline 1963, Dieudonné 1978. I recommend two sites:

`http://www-groups.dcs.st-andrews.ac.uk/`

(called also Mac Tutor) and

http://forum.swarthmore.edu/epigone/historia_matematica/

Bouckaert-Buekenhout 1998 provide a handy introductory bibliography of the history of mathematics with many comments fitting the needs of students. I like to mention a new and huge prehistory of geometry developed in Keller 1998. No doubt, geometry has been with mankind for many thousands of years before writing was invented.

1.8 The context of finite distance transitive permutation groups and distance regular graphs

The finite polar spaces and generalized polygons have provided one of the richest sources at the origin of the concept of distance transitive group and distance regular graph. A classification of these permutation groups is close to completion. That matter will not be further investigated here. I refer to the rich and beautiful Brouwer-Cohen-Neumaier 1989 and for a very recent work on distance-transitivity allowing to complete references, see Cohen-Ivanov 2000.

Acknowledgments

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2 Projective spaces

2.1 Complicated

The history of projective spaces is itself rather complicated. We looked at it in Buekenhout 1998 as far as the 20th Century is concerned but a lot was happening in the 19th Century and earlier of course.

2.2 A standard subject

Most texts on the history of mathematics include a chapter on the history of projective geometry especially in relationship to perspective in painting. A major figure getting out of this path is Desargues. In his famous book there is a page that we can consider as an axiomatic of projective three dimensional space.

2.3 The 19th Century

Some of the great characters of this brilliant period are Monge, Poncelet 1822, Möbius 1827 (see Fauvel e.a. 1993), Steiner, Chasles 1837, Cayley, von Staudt (1847), Klein 1872. Hilbert 1899, Veblen and Young 1910

2.4 Hilbert 1899, Veblen and Young 1910

Projective spaces were definitely born and their relationship to division rings and vector spaces was established. We know that projective spaces seen as point-line spaces correspond to division rings and vice-versa. This is a result coming from Incidence Geometry, a consequence of Hilbert 1899 and of Veblen-Young 1910. It is not apparent from Linear Algebra. The right axioms for Linear Algebra as far as Foundations are concerned, are consequences of Projective Geometry. This correspondence has not been made completely explicit until quite recently. At this day, this important fact is often obscured by geometers themselves who like to emphasize coordinates rather than vector spaces.

2.5 Projective spaces as lattices

Around 1935, Menger and Birkhoff got a theory of projective spaces in terms of lattices. Birkhoff got rid of earlier restrictions on the rank. He got rid of the restriction that asks for lines to have at least three points. We refer to Birkhoff 1948 for a detailed account. During the period 1935-1965, texts devoted to Foundations of Projective Geometry consider these as lattices. This is the case for instance in Baer 1952 and Segre 1961. The theory of Polar Geometry of Veldkamp grew in this context. Tits' theory of polar spaces remains close to it.

2.6 Point-line spaces

The Buekenhout-Shult theorem expressed in terms of points and lines only brought the point-line view to the forefront and a number of experts started explaining that Projective Geometry had been point-line since Veblen-Young 1910. The axioms there are definitely point-line. However, there is no general theory of projective spaces. There is a general theory of three-dimensional projective spaces with, in particular, a clear construction of a division ring under a different name. This construction is totally ignored by historians of Algebra. Do experts of Projective Geometry read Veblen and Young? References to it bare various wrong dates such as 1916 and 1918. Volume I appeared in 1910. It is the volume that matters for point-line geometry. Volume II appeared in 1917. Also, there is already a Veblen-Young 1908 paper where the axioms are provided.

2.7 The theorem of Dembowski-Wagner 1960

This is one of the most remarkable theorems on designs and on Finite Projective Spaces. Start with a 2-design. For any two points a, b define the line ab as the intersection of all blocks containing a and b. Assume that every line and every block have a common point. Then the points and the blocks are the points and the hyperplanes of a projective space.

3 Duality and polarity

3.1 Inseparable!

Projective geometry, conics and duality are truly inseparable in history but I decided to separate them here because of their respective importance for the present subject.

3.2 Prehistory

Duality and polarity have had a long prehistory. Duality and polarity arose explicitly around 1825 and were the object of a fierce battle between Gergonne and Poncelet. Their prehistory is quite interesting, long and involved. We refer to Chemla-Pahaut 1988 and Chemla 1989. Their detailed study covers the period from 1780 to 1925 and starts with Euler.

3.3 Polarity and reflexive sesquilinear form

A duality is a projective concept. Its counterpart in vector spaces is the concept of a sesquilinear form. This is clear in Baer 1952 where that theory is fully developed. Baer attributes the first proof to Birkhoff and von Neumann 1936. This paper is certainly not well-known among geometers and algebraists. It is interesting to observe that a 20th century genius such as von Neumann plays a role in the deep understanding of the Foundations of Projective Geometry.

3.4 The Theorem of Parmentier (1974)

This beautiful result shows how to get a projective space from a polarity. Consider a linear space L all of whose lines have at least three points and a symmetric relation S defined on the set of points such that for every point p and every line l, the point p is S-related to either one or all points of l, no point is S-related to all other points and for every line l, the set $S(S(l))$ is l itself. Then L is a projective space and S is a (quasi) polarity. See Buekenhout 1993. In 1998, after a talk given by Bart Van Steirteghem from the VUB, in the Seminar Gent-Bruxelles, I observed that the Theorem of

Parmentier allows to dispense with one of the axioms in the Foundations of Quantum Mechanics.

4 The circle, the round bodies, the conics and the quadrics

4.1 Projective geometry and polarity again

We said already that a basic geometric ingredient for polar spaces is provided by a projective space and a polarity. As said earlier again, the history of projective spaces is rather complicated. The same statement may be made about polarities. It went more or less as follows for a long time: any (non-empty and non-degenerate) quadric of a projective space determines a polarity and conversely, every (suitable) polarity determines a quadric. All of this is strongly dependent on algebraic machinery: quadratic form and symmetric bilinear form. It was using the real and complex numbers, it was a real-complex matter. The simplest case is provided by the conic actually, the circle.

4.2 Circle and conic

The history of the circle (and its rich prehistory) are in the prehistory of polar spaces. In my opinion, this is not a joke. A lot of ideas that matter for the theory of polar spaces take their source in the famous work of Apollonius (-240) called *Conics*. The classical three round bodies: sphere, cone and cylinder are here as well. To give but one example, incidence geometry deals with the concept of a tangent in so many circumstances. In our context, it is a line that hits the object in one point or which is contained in it. It is our duty to make this known and defended in elementary geometry. Too often this view is considered as obsolete in view of the derivative and its generality. If the older view is obsolete then polar spaces and ovoids are likely to be obsolete also. Friends, your mathematics and positions depend on the history of your subject and how it is known.

4.3 Coolidge 1945

A basic reference here is Coolidge 1945. I owe a copy of it to Jean Doyen. It may be worth to list here all chapters and sections.

Ch. I. The Greeks
1. Menaechmus; 2. Aristaeus, Euclid and Archimedes; 3. Pappus of Alexandria.

Ch. II. Apollonius of Perga

Ch.III. Renewed interest in the conics
1. The first awakening; 2. Girard Desargues; 3. Blaise Pascal; 4. Claude Mydorge; 5. St. Vincent; 6. Jan de Witt; 7. Philippe de la Hire.

Ch. IV. The great projective school
1. Newton, MacLaurin and Brianchon; 2. Jean Victor Poncelet; 3. Michel Chasles; 4. Jacob Steiner; 5. Georg von Staudt; 6. Linear Systems

Ch. V. The development of the algebraic treatment
1. Fermat, Descartes and Newton; 2. The immediate successors; 3. Leonhard Euler.

Ch. VI. The introduction of new algebraic techniques
1. Abridged notation and linear dependence; 2. Linear coordinates; 3. The invariants

Ch. VII. Miscellaneous metrical theorems
1. Maxima and minima; 2. Closure problems; 3. Curvature; 4. Parametric representation; 5. Areas and lengths.

Ch. VIII. Systems of conics
1. Linear systems. 2. Chasles' theory of characteristics

Ch. IX Conics in space
1. Algebraic systems; 2. Differential systems.

Ch. X. Mechanical construction of conics

Ch. XI. Quadric surfaces, synthetic treatment
1. The informal period; 2. The formal period.

Ch. XII. Quadric surfaces, algebraic treatment
1. The earlier methods; 2. The more modern techniques.

Ch. XIII Quadric surfaces, higher algebraic treatment.
1. Pairs of quadrics; 2. Linear systems; 3. Focal properties

Ch. XIV. Quadric surfaces, differential properties.

Let me recall that Apollonius is considered unanimously as one of the greatest mathematicians of Antiquity, that he wrote an impressive book on conics and that he proved more than 400 propositions in it.

5 The classical groups Jordan 1870 and Dickson 1901

5.1 Algebra, group theory, geometry

Our subject can hardly be understood if one ignores groups. Let us start again with a little sketch of the relationships. First and most often comes a field or division ring that may be restricted or not in various ways. Then comes a space namely a vector space, an affine space or a projective space over the ground field. This is a major legacy of the analytic geometry whose history does also matter here (Boyer 1956). Then comes a form, quadratic, bilinear, hermitian, etc. Let us summarize these data by speaking of the algebra. From here on, at this time, there are two main trends. In the group theory trend, you define and study a group from the algebra. This is your goal and you ignore the rest except when needed. In the geometry trend, you define a geometric object, a polarity, a polar space and you ignore the rest except when needed. As a matter of fact, algebra, group theory and geometry are deeply intertwined. The reason for the choices made in the algebra is that the groups are so large and nicely acting and that the geometric axioms are so simple and neat. Unfortunately, most of us choose only one trend, tell others to do as they did and most of us understand only one third of what is going on. We may dream of a better understanding. In the present study, the emphasis is on the geometric side. For a short while I want to deal a bit more with the group theory. First I recall Tits' words:

> *Buildings were born from the meeting of Projective Geometry and Lie groups.*

5.2 After the great pioneers of group theory

After the great pioneers of groups such as Ruffini, Lagrange, Galois, Cauchy and Cayley. The first one to understand it all, to transfer groups to euclidean geometry and to build a genuine theory of groups was Jordan. His book, Jordan 1870, of about 690 pages is a fantastic source. He took the field of prime order from Galois and started its group theory. Under names that do not matter here, he got the linear group, the various orthogonal groups and the symplectic groups. He knew the reduction of quadratic forms and so, he kind of knew that there are no empty conics over that field. In geometry it would take the 1950s to make it a standard fact. Many of us and even more colleagues believe that classical groups were first seen over the real-complex numbers. There is no work that I know of and that goes as deep into it as Jordan does over GF(p).

5.3 Dickson 1901

The next great synthesis was due to Dickson as early as 1901. He got the classical groups over any Galois field. He got their simplicity and the isomorphisms. In the stream he created, geometry was not absent as one can see from the literature but there was no systematic geometric development for each family of classical groups. Later on, classical groups were developing with still more generality as to the groundfield in the work of van der Waerden, then Dieudonné 1955. This is a kind of culmination. Coordinates tend to vanish for the sake of structure and simplicity. Projective geometry is carefully wiped out. At some stage, the word "quadric" has escaped Dieudonné's attention. His Géométrie does nevertheless justify its title and I did learn a lot in it. A recent version in a deliberate geometric setting is Taylor 1992. Moreover, classical groups grow further over rings rather than fields.

5.4 Algebras

As mentioned earlier, groups and geometries are related to still another type of structure. I mean algebras. In first instance I think of the great theory of Lie algebras developed by Chevalley 1955. This had a great influence on the work of Tits during that period and later on. There is a lot more especially for an algebra endowed with an involution. The history and present status is beautifully described in the recent book Knus e.a. 1998.

6 Tits 1956

6.1 A major paper

This is a major paper on the road to polar spaces, generalized polygons, buildings and diagram geometry. It has received little attention and when it does it is often misunderstood. Its content and role has been put in a historical context in Tits 1981. That paper was based on ideas that appear already in Tits 1954 in a rather concise form and then with many developments in Tits 1955. Tits was after a geometric interpretation of the five exceptional simple Lie groups.

6.2 Diagrams first

Thanks to Dynkin, also Witt (according to a talk given by Tits in 1992), the simple Lie groups were represented by a simple drawing: the diagram. Therefore Tits thought that the geometric objects he was looking for had to be quite simple also. In the case of a diagram of type A_n, the nodes correspond to points, lines, planes,..., hyperplanes as Tits observed and the diagram displays the duality of projective geometry. In the diagram of type D_4 we can likewise read the principle of triality. Hence, the diagrams are among other things an aide mémoire. Tits got a uniform construction of a geometry for each group. Properties of the geometry could be read from the diagram and more could be proved on that basis. Therefore, Tits started an axiomatic approach that relies on the Coxeter-Dynkin diagram. It assumes that the rank 2 geometries are known. They are prototypes of the future generalized polygons. One of them is the prototype of a generalized hexagon. Tits did not write it but the need for axioms corresponding to these rank 2 geometries must have been there already. The axioms were well known in the case of projective planes.

6.3 Chevalley groups

Tits realized that his axiomatization allowed to find more properties of his new geometries in a way quite similar to the development of projective geometry. Thus he came to the idea that he might produce his geometries over any ground field. He realized that program for E_6 and E_7 and got new finite simple groups of type E_6. Then he became aware of Chevalley's construction in Chevalley 1955. His own project lost its interest as far as the construction of geometries from the simple Lie groups was concerned. On the other hand, his former construction of geometries from the simple Lie groups carried over immediately to the Chevalley groups. In Tits 1981, the author speaks of the algebraization of the Lie-Cartan theory.

6.4 And polar spaces?

As a matter of fact, the important geometries of type C_n were present in all of the geometric work of Tits during this period. They appear in various ways in the geometries related to the exceptional groups. Another important discovery was the geometry of type C_3 whose planes are real Moufang planes which is in Tits 1954b and in Freudenthal 1954-1963. In a private conversation around 1991, Tits told me that the "octave polar space" is due to him. The construction is in Tits 1954b and it is discussed again in Tits 1955 (page 89) a text finished in May 1954. In Tits 1954b, the author says that an axiomatization of this geometry is possible but he does not give the details. As to Freudenthal 1954-1963, the C_3 geometry is there in 1955, constructed and axiomatized. There is no reference to Tits. The axiomatization requires further study for me in the future. It may entail the first piece of Polar Geometry coming before the work of Veldkamp. At any rate, we see that Tits and Freudenthal play an explicit role in the gestation of Polar Geometry that would soon come to birth.

7 Veldkamp 1959

7.1 Birth of polar geometry

We are getting at the crucial moment for our subject: its birth. It is indeed not exagerated to say that the history of polar spaces began in 1959 with the publication of Veldkamp's thesis in a series of five papers for the Dutch Academy. His work was communicated the same year to an important group of geometers through the Colloquium organized in Utrecht by Veldkamp's promotor Hans Freudenthal (1905-1990). See Veldkamp 1959-1960 and Veldkamp 1959. This work is very readable and pleasant. The reasons for the theory and all related subjects are clearly presented. The term polar geometry is coined here. Later on, the name polar space was coined by Tits at least as early as 1968, developed in Tits 1974 and taken over also in Buekenhout-Shult 1974.

7.2 Strictly isotropic subspaces

Let us briefly discuss the content of Veldkamp 1959.

> *We consider systems of strictly isotropic subspaces with respect to a polarity in a projective space; we call such a system when partially ordered by inclusion a polar geometry (which we have introduced as a common noun for orthogonal, unitary and symplectic geometry).*

This is a modest statement: it is indeed daring to unify classically distinct geometries and, moreover, to study them from an intrinsic viewpoint.

Chapter one is devoted to *Strictly isotropic subspaces*. Strictly isotropic had been called totally isotropic in Dieudonné 1955. The source is Baer 1952. I dont follow the terminology of Veldkamp. He starts with a not necessarily commutative division ring K, a left vector space V over K and the projective space $P(V)$. It is implicitly assumed that the rank is finite. This becomes entirely clear when Veldkamp deals with spaces of infinite rank after having discovered and studied Lenz 1954. Projectivity, collineation, linear transformation and semi-linear transformation follow. A duality between projective spaces is defined as an order-reversing transformation. An anti-automorphism of F is introduced as well as a related sesquilinear form. The link with auto-duality is explained. Given an auto-duality d of the space $P(V)$, a strictly isotropic subspace U is a subspace contained in its image under d. Polarities and the corresponding forms are then described. Now the author proves properties of the polar geometry corresponding to a given polarity. These provide the axioms of his geometric theory in chapter III. At this time we may save on these properties in view of the Buekenhout-Shult axioms that are rather immediate to check.

7.3 Chapter II. Polar Geometry

Consider a projective space and a polarity s in it. Then the system of strictly isotropic subspaces together with the partial order provided by inclusion is defined as the polar geometry corresponding to s. Assume that the maximal rank of a strictly isotropic subspace is at least three. In the present language, the polar space is of rank 3 at least. This restriction is motivated by isomorphism properties that fail to hold for the hyperboloid of one sheet whose rank is two. In Tits 1974, we see nevertheless that other rank 2 polar geometries in projective spaces behave as nicely as larger rank ones. what is this behaviour? Roughly, every automorphism of the polar geometry extends to an automorphism of the projective space and the extension is unique provided that the polarity is trace-valued. The latter concept is defined algebraically, for the defining sesquilinear form, but we know that it can be explained geometrically: it means that the set of points of the polar geometry spans the projective space. The algebraic concept is explained in Dieudonné and it is shown that non trace-valued polarities can occur only in characteristic 2. This theory was first obtained by Chow (see Dieudonné 1955). Veldkamp's treatment is somewhat different. This theory is giving faith: if the polar geometry of rank 3 at least is given, then it should be possible to reconstruct the projective space and the polarity around it. This is bringing us to the point, to Veldkamp's programme. At this time, the theory of Chow is not needed anymore as a preliminary. It follows from stronger results established by the theory.

7.4 Chapter III. Axioms for Polar Geometry

Here we go. Inspiration is provided by Birkhoff 1948.

> Axioms I and II say that S, is a partially ordered set.
> Axiom III. Every nonempty set of elements of S has an intersection.
> Axiom IV. For every element a of S, the set of all x such that $x \leq a$ is a projective space of finite rank. Points, lines, etc. are defined as usually.
> Axiom V. Every element of S is contained in some maximal element. All maximal elements have the same (finite) rank which is called the index of S.
> Axiom VI. If x and y are elements of S with intersection 0 then there exist maximal elements a and b in S such that $x \leq a$, $y \leq b$ and the intersection of a and b is 0.
> Axiom VII. Let a and b be maximal and have intersection 0. For every xa there exists one and only one $d(x) \leq b$ ($d(x) = x^{\perp}$) such that the join of x and $d(x)$ exists and is maximal. The mapping d is a duality of the projective space a upon the projective space

b.
Then imaginary lines are defined. Today we would call them hyperbolic lines. Now, a number of properties are deduced. Then a flat subset of S is defined by closure with respect to lines and to imaginary lines.
Axiom VIII. There exists a finite number of points such that S is the only flat subset of S that contains all those points.
Axiom IX. If u and v are maximal elements of S and are considered as projective spaces, then there exists a projectivity (isomorphism) of u upon v.
Axiom X. If a is an element of S, then the projective geometry of all $x \leq a$ is Desarguesian.

Veldkamp observes that the geometry treated by Freudenthal 1954-1955 (the symplectic geometry of the Cayley numbers) satisfies all axioms except axiom X.

7.5 Chapter IV. The embedding of axiomatic polar geometry in a projective space

This is a great moment. We now distinguish two major results in Chapter IV.

A. The functorial embedding of S in a projective space P.

B. The functorial construction of a polarity in P whose polar geometry contains S.

Veldkamp defines subsets of S that we would call *projective hyperplanes* or *geometric hyperplanes* according to authors. He considers the set $V(S)$ of all projective hyperplanes and defines lines on them that Cohen and I decided for the purpose of Buekenhout-Cohen 1982-2000 to call Veldkamp lines. Thus $V(S)$ becomes what Cohen and I call the Veldkamp space of S. This took place during our revision of Veldkamp's proof in 1986-1988. The embedding of S in $V(S)$ is easy. Every point p of S corresponds immediately to its cone, consisting of all points collinear with p and this is obviously a projective hyperplane. Then Veldkamp proceeds to show that $V(S)$ is a projective space. This is a very difficult task. Cohen and I met two arguments that we could not justify shortly. But we could. This is for Problem A.

Problem B is attacked and solved also by Veldkamp in a long and difficult argument. Again, Cohen and I needed a serious reparation at some place.

After Problem B, Veldkamp meets perhaps his main worry. Now S is embedded in a projective space and even in the polar geometry S^* of a polarity and all of the construction is functorial. He wants the equality $S = S^*$. He succeeds in proving this only under the assumption that the characteristic is different from two. In conclusion, Veldkamp has classified the polar geometries satisfying Axioms I to X, except that there remains some doubt in characteristic 2 and that there might still be unknown polar geometries there hidden in classical ones.

7.6 Chapter V

This is an addition with respect to the thesis. It has been often overlooked by the experts of polar spaces. The important result here is that in projective spaces of infinite rank, polarities have to be replaced by the quasi-polarities introduced and studied by Lenz 1954 whose paper appears as a beautiful complement to Baer 1952. In Veldkamp's theory, this means that Axiom VIII is thrown away. All of his results carry over except that now the embedding occurs in a quasi-polarity. In conclusion, the theory of Veldkamp does not restrict to the finite rank.

7.7 One more axiom

In the Yellow Book namely Tits 1974, already in the preprint of 1968, Tits observes that Veldkamp uses implicitly one more axiom:

> (*) for any two maximal subspaces M, M' such that the intersection of M and M' is a hyperplane in M and in M', there exists an isomorphism of projective spaces of M onto M' fixing their intersection pointwise. Tits observes that this condition is always satisfied for a thick polar geometry S.

7.8 Kreuzer 1991

Kreuzer shows that the implicitly used axiom (*) pointed out by Tits can be proved from the other Veldkamp axioms. He is also giving a weakened version of axiom VII.

7.9 New version: Buekenhout-Cohen

For the purpose of the book Buekenhout-Cohen 1982-2000, we undertook a systematic revision of Veldkamp's proofs using the Buekenhout-Shult 1974 first principles. This work was carried out in 1986-1988. We had the feeling to be the first who did really go entirely through the arguments of Veldkamp. I got a conversation with Tits that was very helpful to clarify the basic ideas. We succeeded in giving a totally new solution of Problem A (the embedding of the polar geometry in a projective space) respecting all main ideas of Veldkamp. We were influenced by Teirlinck 1980 who led us to the concept of ample connectedness. The proof was relatively simple for the rank at least 4. We could handle the rank three case also, through a very long proof whose completion got some help by Antonio Pasini. We got a neat text by 1988. On its basis, Cuypers, Johnson and Pasini 1992 got a shorter proof of the case of rank at least 4. Buekenhout-Cohen 1982-2000 does also revise Veldkamp's proof for Problem B but this requires longer explanations that will be postponed. In the meantime, the Buekenhout-Cohen revision has been broadly used in other contexts, sometimes with no reference to it.

7.10

When I lectured on the history of polar spaces at the Veldkamp celebration in 1991, J. J. Seidel drew my attention on a paper which matters here namely J. McLaughlin 1969. Some subgroups of $SL_n(F_2)$. Illinois J. Math. 13, 108-115. I need to read it again sometime.

8 Tits 1959, Feit-Higman 1964

8.1 Birth of Generalized Polygons

Generalized polygons appear in a short and rather dense appendix of only two pages of the long and difficult Tits 1959. What is this paper really about? Diagram geometry as founded by Tits in papers published during the period 1954-1956 had given him a simple view of the pinciple of triality. This went back to Study in 1912. Consider the Coxeter-Dynkin diagram of type D_4. The corresponding geometry is a projective quadric Q of index three in the 7-dimensional projective geometry over a field F. It is called a hyperbolic quadric sometimes because its index is the largest possible in that dimension for a nondegenerate quadric. For a given F, that space is unique and that quadric is unique too up to isomorphism. Its equation is

$$x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 = 0.$$

It is the simplest quadratic equation you can dream of in that dimension. It does not depend on F after all. If you take two points of Q not on a line of Q, the intersection of Q with their tangent hyperplanes is a hyperbolic quadric in dimension 5 which is known under the name of Klein quadric because of a famous isomorphism. Its equation is

$$x_1x_2 + x_3x_4 + x_5x_6 = 0.$$

Repeat the preceding procedure with two noncollinear points of the Klein quadric and you get the threedimensional regulus. Its equation is

$$x_1x_2 + x_3x_4 = 0.$$

Guess what you get if you do everything one more time. The quadric Q contains points, lines, planes and threedimensional projective subspaces. Let inclusion provide the incidence relation between them. The threedimensional subspaces fall in a unique partition, say the reds and the blues such that: every plane in Q is on a unique red and a unique blue, any red and any blue intersect either in a plane or a single point, any two distinct reds (respectively blues) intersect either in a line or the empty set. All of this relies on the two classical sets of lines of a regulus first discovered by the great architect Christopher Wren. Decide that a red and a blue are incident whenever they intersect in a plane. Consider the rank 4 geometry consisting of the points, the lines, the reds and the blues. Consider its automorphisms. It turns out that the points, the reds and the blues are in the same orbit. It is not so easy to prove it but it will become much easier in the future. Beutelspacher-Rosenbaum 1998 have given a coordinate free proof of the Klein correspondence and this will be done one day or the other for the fact just stated in a direct and simple way. That fact is the principle of triality. Actually, there exists some automorphism of order three say a^* such that a^* maps every point on some red, every red on some blue and every blue on some point. Such an automorphism is called a *triality* in Tits 1959. All of these developments follow the pattern of the principle of duality and of the polarities in projective spaces. Polarities are equivalent to reflexive sesquilinear forms. This is one of the great achievements of Birkhoff-von Neumann 1936. Reflexive sesquilinear forms are classified algebraically in Dieudonné 1955. We get orthogonal polarities namely symmetric bilinear forms, symplectic polarities namely alternating forms and unitary polarities namely hermitian forms. This study entails all classical groups. In the case of triality, we dont have an algebraic equivalent. Is there any? In Tits 1959, one of the purposes is nevertheless to classify all trialities in some types and this goal is achieved. How many types? Each type provides also a type of group. Some of these groups, especially ${}^3D_4(F)$, are new.

Consider a triality a^* of some type. The analog of a quadric, let us call it the *Tits triality geometry* or $TTG(a^*)$ consists of: all points p such that $a^*(p)$ is incident to p and all lines invariant under a^*. This is a rank two geometry of points and lines. It may be empty. If it is nonempty, Tits goes into a difficult analysis of its structure. It is a generalized hexagon. This name is not used. The concept is discovered. This is understood and expressed in Freudenthals review for Zentralblatt für Mathematik, provided to me by Frank De Clerck:

> *Les relations d'incidence du système des points autoconjugués et des droites invariantes d'une trialité sont décrites au moyen de la notion d'hexagone généralisé...*

The structure of $TTG(a^*)$ together with other rank two geometries namely projective planes and classical generalized quadrangles brought Tits to his generalized polygons. In a way, it is all a matter of quadrics as they were known to Poncelet around 1825.

Let me make a remark of futuristic type. Given the axiomatic definition of a polarity, it is easy to prove that its quadric of totally isotropic points and lines is a polar space. The parallel proof leading from a triality to a generalized hexagon is difficult and long in Tits 1959. It is made a lot simpler in the book Van Maldeghem 1998 where it is a matter of few pages. Can it still be made simpler? The concept of Generalized Polygon was immediately understood and taken over by Freudenthal in Utrecht who got another PhD student to work with it. It gave us Schellekens 1962 devoted to generalized hexagons. One of the contributions is to study triality and generalized hexagons in the context of a split Cayley algebra. For Buekenhout-Cohen 2000, this appears as the optimal present approach of the subject.

8.2 Generalized octagons over the Ree groups

These beautiful objects were constructed in Tits 1960. They are still not entirely understood from a geometrical viewpoint. See Van Maldeghem 1998 for details, in particular an unpublished Theorem of Tits providing those Generalized octagons from a polarity of a geometry of type F_4, together with the proof. Again, the theme of polarities coming back.

8.3 Feit-Higman 1964

A thick finite generalized n-gon can exist only for $n = 2, 3, 4, 6$ and 8. This relies on purely algebraic methods. The geometry leads to a symmetric matrix whose eigenvalues have necessarily integer multiplicities. From the geometry, these multiplicities are evaluated in terms of n and of the orders of the polygon. This gives an expression involving square roots and when this expression is required to be an integer the result follows. This work has served as a model for many investigations in various directions. Some of those directions are algebraic graph theory, Moufang polygons and topological polygons. I refer to Van Maldeghem 1998 for many details on these matters that I cannot develop further for the time being. A detailed and historical account on the topic of this section is provided in this course by Haemers 2000.

9 Tits 1961-1974

9.1 Birth of buildings

This deals with the official birth of buildings. All experts will tell you that it was happening in the Yellow Book Tits 1974 but that is completely wrong. We can trace the fact to 1961 with the definition and study of polyhedral geometries. In September 1961, at the second meeting of the Groupement de mathématiciens d'expression latine held in Florence and

Bologna, Tits defined his polyhedral geometries and started their theory. The paper appeared in the Proceedings. See Tits 1962. As I said, it is almost universally ignored by those who should have read it. Tits did also introduce here the algebraic counterpart of his geometric objects namely the concept of a group with a BN-pair. It was a major turning point taken in almost complete isolation. Tits was ahead of his time and had been in this situation since 1955. This was the official birth of buildings but actually, there had been a previous somewhat different view of geometries on a Coxeter diagram developed in the years 1954-1959 as I explained earlier. At that time, the Yellow Book was already a project. An important piece is Tits 1961 devoted to Coxeter groups and Geometries, a paper of 26 pages which is very difficult to obtain. It starts by stating that the paragraphs 1 to 3, the essential part of the note, are taken without modification, from the project of a book devoted to the groups satisfying a theorem of Bruhat and to the corresponding geometries. A great deal is taken over in Bourbaki 1968 and it provides the basis for Chapter 2 in Tits 1974. Another trace of the book appears in the Preface to the second edition of Dieudonné 1955 written in October 1962.

> *Several mathematicians following CHEVALLEY himself, have already successfully used these general methods and we have all good reasons to hope that their future development will practically cover all material dealt with in this volume; in the meantime, an exposition of the results obtained at this day must be soon made by J. TITS in the yellow collection.*

In a review of Tits 1962 written for Zentralblatt für Mathematik and provided to me by Frank De Clerck, Veldkamp writes:

> *A complete exposition on polyhedral geometries, with proofs, is to be published in a forthcoming volume of the Lecture Notes in Mathematics, Springer Verlag, Berlin, heidelberg, New York, by the same author*

This is another rather accurate annoucement of Tits 1974 made in the early 1960s. Veldkamp repeats that message in his review of Tits 1964.

9.2 Geometries of type C_n and polar geometries

In October 1963, Tits gave a lecture on the matter at the Simposio internazionale sulle geometrie finite in Rome. This is about Finite Polyhedral Geometries, in other words, finite buildings. See Tits 1964. We get a sketch of the theory of buildings of spherical type. In particular, we get the link of buildings with polar spaces, actually geometries of type C_n and Veldkamp's theory. Tits realizes that the geometries (in his sense) of type C_n are those satisfying the axioms I to VII in Veldkamp provided that the projective spaces of axiom IV do not exclude lines of two points. A geometry satisfying Veldkamp's axiom VIII is called finitely generated. His Theorem 6.1 attributed to Veldkamp, states that every thick polyhedral geometry of type C_n, $n \geq 3$, whose planes are Desarguesian and which is finitely generated is the geometry of totally isotropic subspaces of a polarity in some projective space. According to an ancient but non-published result of his, he states, without proof, Theorem 6.2 saying that the planes of a geometry of type C_3 are either Desarguesian and self-dual or non-Desarguesian and Moufang. Consequences are derived in the finite case. Every finite geometry of type C_n is the geometry of totally isotropic subspaces of some polarity and the latter is orthogonal, symplectic or hermitian. Basic properties of a geometry of type D_n are stated likewise. A footnote explains that for $n = 4$, the 3-dimensional projective residual subspaces are provided with symplectic polarities constructed explicitly from the geometry. This indication was developed and explained in Huybrechts 1993.

9.3 The real-estate terminology

Quite an event for our subject is Bourbaki 1968. This is part of the book on Lie groups and algebras. It contains three chapters. Chapter IV is devoted to Coxeter groups and Tits systems. The name Tits system is a novelty replacing Tits Group with a BN-pair. It has been widely followed. Chapter V is on groups generated by reflections and Chapter VI is on root systems. Bourbaki does not often come so close to Geometry! In Chapter IV, there is even an appendix on graphs. At the end of the Introduction we understand this sudden geometric influence better:

> *Pour la rédaction de ces trois chapitres, de nombreuses conversations avec J. Tits nous ont apporté une aide précieuse. Nous l'en remercions très amicalement.*

The standard joke to make here is that Tits had the privilege to meet Bourbaki but his explanation is that he met J. P. Serre which is a privilege too. In chapter IV, another event occurs. The exercises 15 to 24 of §1 and 3 to 17 of §2 are communicated by J. Tits. They deal with mostly unpublished results. It is an exposition of the theory of buildings with the new terminology and a concise version indeed of some chapters of Tits 1974. We read about chambers, facets, galleries and . . . buildings. Apartments are not far away as well as half apartments, walls and foldings. The historical note is very interesting in view of the role of Tits, often together with A. Borel or F. Bruhat, in the theory of algebraic groups. Once more, this role of Bourbaki is often ignored in the literature mentioning buildings.

9.4 Oberwolfach 1968

In 1968, there was an Oberwolfach conference for which Tits presented an impressive preprint of parts of his future Yellow Book. It included the chapters 7 to 9 on polar spaces that were going to be in Tits 1974. I was fortunate enough to get a copy of the 1968 preprint from Tits as I had obtained most of his papers in the 1960's. I spent always much efforts on the study of these works. Thus I got acquainted with polar spaces. This would matter a lot later on for the Buekenhout-Shult Theory. In this preprint of 1968, the name polar space occurs for the first time. About 20 years later, I was stupid enough to throw that historical preprint away on the grounds that I had two copies of the Yellow Book. I had known Veldkamp's work explicitly at least in 1966. It was an important reference of Tits 1963.

9.5 Ovoïdes à translations

I was much impressed by Tits 1962b, the paper on Translation ovoids. This is broadly ignored in the present work on ovoids. It is too bad. Here Tits starts from the work of Segre and his students on ovals and caps in finite projective spaces, in particular the characterization of conics and oval quadrics in odd characteristic. Tits suggests an alternative definition which has the advantage to work also in infinite spaces. This is how the word ovoid arises. The definition is based on the idea of the tangent hyperplane that would become so fruitful in a variety of circumstances. Tits characterizes the oval quadrics by properties of homogeneity namely the presence of enough automorphisms. His ideas belong to the history of polar spaces of rank one. I was using that paper in my work on ovals developed from 1964 on and published in 1966. In my studies, Tits 1962b and Veldkamp 1959 did interfere (among other things) and I started to think about nondegenerate quadrics in general, in a geometric way, in the style of Tits with ovoids. In 1966, I wrote a long memoir "Espaces conformes et quadriques généralisées". It has 96 pages and was not published. The purpose was an important Belgian award, the Prix Empain. I did not get the award but got several rewards. One of the jury members got in touch in order to encourage me. This kind man was J. Bilo, the promotor of J. A. Thas around that year 1967. It was the first step of a lasting collaboration and friendship with the school of Ghent. I got my ideas concerning inversions in circular spaces which would lead to a major characterization of affine spaces. This is not much related to my present subject. The third reward was to get the concept "quadrique généralisée" and to start its theory. These objects became "Quadratic sets" whose theory appears in Buekenhout 1969. Now, this is clearly part of the theory of polar spaces. I believe that this paper remains very readable but the best place to read about "Quadratic sets" may be Beutelspacher-Rosenbaum 1998.

9.6 Tits' theory of polar spaces

In Tits 1974, chapters 7, 8, 9 and about 100 pages are devoted to Polar Spaces. Thus, it is not exagerated to claim the centrality of these spaces in the Theory of Buildings of Spherical Type. Tits simplifies and completes the theory of Veldkamp to a large extent. Actually, it is hard to recognize the subject. Tits starts with a set S in which some subsets are distinguished and called subspaces. There is an integer $n \geq 1$ called the rank of the polar space S. There are four axioms (P1) to (P4).

(P1) A subspace L, together with the subspaces it contains, is a d-dimensional projective space with $-1 \leq d \leq n-1$.

(P2) The intersection of two subspaces is a subspace.

(P3) Given a subspace L of dimension $n-1$ and a point p in $S - L$, there exists a unique subspace M containing p and such that the intersection of M and L is of dimension $n-2$; it contains all points of L which are collinear with p.

(P4) There exist two disjoint subspaces of dimension $n-1$.

The polar space S is called thick if every line contains at least three points and if every $(n-2)$-dimensional subspace is contained in at least three maximal subspaces. The relationship with buildings of type C_n is established in a rather long and difficult proof. The maximal subspaces of a thick polar space are self-dual projective spaces. The planes of a thick polar space of rank 3 are Moufang. The relationship with buildings of type D_n is established. Chapter 8 is devoted to Projective embeddings of polar spaces. Sesquilinear forms are studied. Then comes a great surprise: pseudo-quadratic forms. This turns out to be the algebraic concept that was missing to Veldkamp in order to finish his classification problem. There were indeed new hidden polar spaces embedded in classical ones. Tits had understood this by 1968. Next, we find a study op polarities and associated polar spaces. These polarities are more general than the classical ones, they are really the quasi-polarities of Lenz 1954. Here Tits touches almost polar spaces of infinite rank but he avoids it explicitly to remain coherent with his definition of a polar space. Now, every pseudo-quadratic form (of finite Witt index) is associated to a polar space. There comes now a deep and difficult theory of embeddings of a polar space in a pair consisting of a projective space and a polarity in which one embedding is given. So, in a way, it studies the embeddings of embedded polar spaces. It includes the automorphisms of embedded polar spaces. It is an important step of the classification: control is achieved over the hidden spaces inside an embedded polar space and that control requires pseudo-quadratic forms, no more. At the end of the chapter, Tits is solving problems (A) and (B) in a totally original way, getting the embedding of any thick polar space of rank ≥ 3 whose maximal subspaces are Desarguesian. Chapter nine is devoted to the non-embeddable polar spaces and classifies them.

9.7 Moufang

There is a lot to say about Moufang polygons, Moufang sets and Moufang buildings. The latter occupy an Appendix of three pages in the Yellow Book. There are many connections to other structures such as Twin Buildings. A concept such as *weakly perspective set* that I used at different places is clearly related (see Buekenhout 1975, Buekenhout-Lefèvre 1976). When did Tits come to his general view on the Moufang condition for Buildings and for Generalized Polygons? Of course, it is in Tits 1974. It is also one of the main purposes of Tits 1976, a paper for a conference held in Rome in 1973. This would have a great influence on work that extends till now. Van Maldeghem 1998 writes that:

> *The Moufang hexagons were explicitly classified in the 1960s by Tits (unpublished).*

More details are wanted. Contrary to some belief, the purpose was not to generalize the Moufang planes but it turned out to be like that. The main motivation was (Tits 1976):

> *the fact that, as follows from the grouptheoretical description... all buildings of spherical type and rank 2 associated ... to classical or algebraical simple groups, to mixed groups of type G_2 or to Ree groups of type $^2F_4$ are Moufang.*

Tits went on:

> *We conjecture that the converse is "practically true". More precisely: CONJECTURE. Every Moufang polygon is associated to one of the groups listed above or is isomorphic to one of the generalized quadrangles constructed in 4.5 below. In particular, there is no Moufang m-gon for m different from 2, 3, 4, 6, 8.*

In the finite case, Tits observed that:

> *the conjecture easily follows from the classification of finite "split BN-pairs" obtained in Fong-Seitz 1973-1974.*

Tits added in proofs (January 1976) that the conjecture is now proved for all m different from 4, 8. The classification of the Moufang octagons was given in Tits 1983. The case $m = 4$ would have a long story. A new class would be detected by Weiss 1996. Weiss 1979, followed by Tits 1979, showed that Moufang m-gons exist only for $m = 3$, 4, 6, 8. Tits 1976c, a preprint that remained unpublished, classified the case $m = 4$ under the restriction that points and lines are regular. In some occasions, he said that he had a full proof of the conjecture for $m = 4$. At some stage in the early 1990s Bernhard Mühlherr wanted to attack the proof and he consulted Tits for permission. Tits said that it was very difficult and that he had really the details in his head so it would be a waste of time to look again for it. His paper Tits 1994 gave a new start to his project. A great surprise came when Weiss 1996 got the new class of examples. The important book Tits-Weiss 2000 classifying all Moufang polygons is coming soon.

9.8 To be or not to be in Mathematical Reviews

My most favorite mathematical journal is by far Mathematical Reviews. I have nothing against other journals with a similar object. Nowadays, colleagues start criticizing young researchers because some of their work is not reviewed. Young colleagues, you are in good company. Tits 1959 was not reviewed. They were missing the birth of Generalized Polygons and the classification of trialities. The discovery of new finite simple groups is little by little denied to Tits. Tits 1961 was not reviewed. This is entirely understandable. It was a modestly printed kind of preprint and had little diffusion. Tits 1962 was not reviewed. They were missing the birth of buildings and of Tits systems and any way, these objects were born under another name. No complain. Mathematical Reviews is my favorite. During the lectures, I mentioned that I did not check Zentralblatt für Mathematik. Frank de Clerck did it and came up with interesting reviews. Here, Tits 1961 is missing quite normally; this was a kind of preprint. Hence, the birth of Generalized polygons and that of Buildings were announced after all. If you are not reviewed in Zentralblatt you may worry.

9.9 All you need

Of course, it is love. Thanks to the Beatles. We also need a publication of the Complete Works of Jacques Tits. And a translation in English of many of his papers. This joke about the Beatles went on during the lectures. Philippe Cara produced a copy of the song that appears as an appendix to this paper. A team of six Pseudo-quadratic Beatles then produced another version of the song that is appended too. The authors are Philippe Cara, Alice Devillers, Michel Sebille, Paul-Olivier Dehaye, Gavin Seal and Ilaria Cardinali.

10 Buekenhout-Shult 1974

10.1 Oberwolfach and Jaap Seidel

In 1972, polar spaces had become rather familiar to me. Other important characters were going to enter the scene. The first is J. J. Seidel called Jaap by almost everybody. We first met at the famous Institute of Oberwolfach to which I was owing already a great debt in view of several invitations made by Reinhold Baer and in view of the fantastic atmosphere he developed. I met Jaap there in 1971, for Gruppen und Geometrien under D. G. Higman and H. Salzmann. Then came the great moment in May 1972, for Gruppen und Geometrien again, where Seidel gave a lecture on E. E. Shult's characterization of symplectic and orthogonal geometries over GF(2) (see Shult 1972). This is a lovely tale about graphs, finite regular graphs, related to symplectic and orthogonal geometry over GF(2). If two points are orthogonal, they are contained in a 3-clique with a remarkable 1-all property. Shult's Theorem characterizes finite regular graphs with the triangle property: any adjacent pair of vertices can be completed to a triangle (a clique) such that each vertex off the triangle is adjacent to one or three vertices of the triangle. Shult gets more. In those triangles I did recognize the lines of a polar space in the sense of Tits. Seidel got an alternative proof of Shult's theorem. I could see a short proof using the Veldkamp-Tits theory of polar spaces but there was clearly something new and strong that was not in Tits. It was the power of triangles actually lines in my view. I got the idea because of my long training and my own great feelings for lines in all circumstances. It is rather fortunate that Shult and Seidel did not know about Tits' theory. Seidel knew Veldkamp's work but the relationship with Shult's ideas was not obvious. Seidel and I got into excited conversations. Quite soon I thought that a generalization was possible. At some point, Seidel showed a sheet of paper with a conjecture about polar graphs if you allow me that expression, written by Shult and expressed over any finite field GF(q). Here we go.

Start with a finite strongly regular graph. For any pair of adjacent vertices x and y you ask for a clique of $q + 1$ vertices containing x and y with the 1-all property. Then, the graph should be a complete graph, a totally disconnected graph, the orthogonality graph of some polarity in a projective space over GF(q) or the collinearity graph of some generalized 4-gon (I was adding the last case). I could not believe this and I said it. If this was true, it would mean an improvement on Tits' theory of polar spaces and according to my faith, it was not possible to watch an improvement on Tits' work. Seidel laid much pressure on me with his unique insight and skill to coach. He insisted that I got in touch with Shult and I did so. Eventually, the story lead to the Buekenhout-Shult 1974 theory. The team was built by Seidel.

10.2 Buekenhout and Shult

This work was built from June to October 1972 on the basis of letters between Brussels and Gainesville, Florida. The starting point was Shult's tantalizing conjecture. Then a substantial letter from me to Shult (June 13). Then Shult came with another substantial letter on September 17. On October 16, I wrote to Shult:

> *In conclusion it seems that our combined work leads to the following fantastic result going far beyond your initial conjecture.*

The reply of Shult on November 15, 1972 included:

> *When you sent your theorem and notes I celebrated by taking my wife to dinner.*

Indeed, at the end of an inspiring exchange of ideas, the graphs were not needed anymore, nor finiteness, nor strong regularity. What remained were the cliques that we called lines and the 1-all property. Also, for one of the two main results, a restriction to the finite rank that is discussed further in another section. Our method was to prove the axioms of Tits as consequences of our axioms.

10.3 Early propaganda

In July 1972, I gave a lecture in Tübingen for quite a large audience with the title: *Characterization of the absolute points adjacency in a polarity of a Desarguesian projective space.* It was the story of Veldkamp, Tits, Shult and what was happening.

10.4 Further developments.

In 1972, I could not realize that the Buekenhout-Shult theory would be so influential on research in nearby fields. One of the first developments was the thesis of B. Cooperstein on geometry of type E_6 in 1975 (see Cooperstein 1977). This would grow to a huge theory of point-line geometries that are locally polar spaces. Buekenhout 1983 seems to have played a role as well as work of Cohen and Cohen-Cooperstein for which I refer to Cohen 1995. Another development was diagram geometry that I started in a new generalized setting in 1975. Polar spaces along the view of Buekenhout-Shult allowed me to recognize easily various geometries for the sporadic groups that were extensions of polar spaces. The most striking are the extended polar spaces for the three sporadic Fischer groups. Still another surprise was the use of polar spaces by M. Aschbacher in an important step of the classification of finite simple groups around 1977.

10.5 The manuscript of Chevalley

In 1980 I had the honor to speak at the Séminaire C. Chevalley at the Ecole Normale in Paris. I dealt with diagram geometry. Claude Chevalley (1909-1984) was present and I felt impressed to speak in front of this great founding member of Bourbaki and discoverer of the Chevalley groups. When he died, Jacques Tits got in touch with me about the latest manuscript written by the great man. It was devoted to the elementary theory of polar spaces along the view of Buekenhout-Shult in (roughly) 25 pages.

10.6 Projective lines

Who cares for polar spaces of rank one? What is more, who cares for polar spaces inside a projective line? I did. Because of the influence of my promotor Paul Libois and the early works of Tits. This is developed in Buekenhout 1975, a paper published in Hamburg. That theory is now integrated and used for the general theory of embedded polar spaces presented in Buekenhout-Cohen 2000. It is related to the Moufang sets of which we heard a lot recently.

10.7 Lines of two points

In Tits 1974, lines of two points were already admitted up to some stage. In projective spaces they had been admitted since the time of Birkhoff around 1935 (see Birkhoff 1948 and Segre 1961). Polar spaces admitting some line of two points are classified in Buekenhout-Sprague 1982 and their polarities are classified by De Clerck-Mazzocca 1988. During recent years I have often advocated a study of projective spaces including a line of less than two points, I mean meager projective spaces and polar spaces. The idea arises from a short sentence on freak cases written by E. Artin in his famous Geometric Algebra of 1957 when he discusses affine planes. More generally, meager buildings deserve attention. They matter for the fixed structure of automorphism groups.

10.8 Restrictriction on rank removed

Polar spaces of infinite rank were necessarily calling for our attention. They were not absent from the work of Veldkamp and Tits but were not emphasized very much. The removal of the condition that the rank is finite in the axioms of Buekenhout-Shult came in three independent papers at about the same time in 1988 and there is no point in looking for priority. These papers are Buekenhout 1990, Johnson 1990 and Percsy 1989. The first and third paper arose in the context of a series of seminars in Brussels given by Percsy on his growing theory of Zara graphs that we cannot describe at this moment (see Percsy 1990). This is an alternative approach to polar spaces based on the collinearity graph and the maximal singular subspaces that are seen as maximal cliques. There were quite a number of exchanges in conversations. A central role was played by Teirlinck 1980. At some stage, I saw the impact on good old Buekenhout-Shult, forgot about Zara graphs and got my result. It was obviously clear too in the mind of Percsy who wrote it up also. Once more, graphs had a great role in the inspiration but they were not necessary anymore in the final result. This is coincidence of course. All of Incidence Geometry is a story on graphs. Johnson 1990 arrived the year after. His proof showing that the Buekenhout-Shult hypothesis of finite rank may be removed is simpler than in Buekenhout and Percsy and probably optimal. He develops the general theory of polar spaces of any rank to a deep extent. The classification in this final form of the axioms works well. The embedding in a projective space was already in the revised Veldkamp theory. The rest is also in Buekenhout-Cohen 1982-2000 in the shape it got in the 1990's and in other recent works such as Johnson 1990 and Johnson 1999. A great deal of Tits 1974 applies immediately. Finally, the main difficulties with polar spaces are not with higher and possibly infinite rank. They are with the small rank 3.

11 To be done

11.1 In this section

In this section I give a short mention of themes belonging to the subject that have not been dealt with.

11.2 The greatest omission

The greatest omission in these lectures where I was trying to cover at least the period 1956-1974 with some detail, is the work of J. A. Thas on Generalized Quadrangles in the early 1970's. Actually, it would not have been entirely stupid for my subject to list all of the 193 papers written by Thas as I could watch them some months ago. His Red Book with Payne (1984) is a good concise source. Let me mention at least Thas 1972, 1972b, 1974, 1974b, 1974-75. His teammate Payne appears on the list in 1975.

11.3 Embeddings

This includes different themes: embedded polar spaces and embeddings of polar spaces. And likewise for generalized polygons. Besides Veldkamp 1959 and Tits 1974, I want to mention Buekenhout-Lefèvre 1974 and 1976. Also Johnson 1990 and 2000 and Buekenhout-Cohen 1982-2000. I may not forget Dienst 1980 and 1980b who plays an important role in the classification of embedded polar spaces of rank 2. There is recent, deep and important work due to Thas-Van Maldeghem 1998 and 1999 with a theme going back to Lefèvre in the 1980's. The best may be to look at the Survey in Thas 1998. Actually, in this course, we find a perfect and actual survey namely Thas-Van Maldeghem 2000 with many historical details.

11.4 Point-line geometries

As mentioned earlier, polar spaces have become important as residues of more complex geometries such as the geometries of type E_6 studied in the 1950's by Tits and then again, as a PhD subject in Cooperstein 1977. See also Buekenhout 1979. For further developments see Buekenhout 1995.

11.5 Ovoids

Ovals, hyperovals and ovoids have been the object of many studies on which extended reports are given in this course by Matthew Brown. Much of it started as said earlier with Segre, Barlotti in the 1950's and with various works of Tits such as Tits 1962b. An excellent historical account appears here in Brown 2000.

11.6 Locally polar spaces and extensions of generalized polygons

Locally polar spaces started as a subject on graphs in which every vertex neighborhood is the orthogonality graph of a polar space. It began with Buekenhout-Hubaut 1977. This was not the first paper on graph extensions but perhaps the first one to achieve significant results. It has been followed by many works for which I refer to Buekenhout 1995.

11.7 Dual polar spaces and Grassmannians

See Cameron 1982. Also Cohen 1995 and a thesis in Brussels by S. Lehman.

11.8 Spreads and ovoids of finite classical polar spaces

See Thas 1992.

11.9 Morphisms of polar spaces

This subject is dealt with in Faure-Seal 2000. It is the object of the thesis that will soon be defended in Brussels by Seal 2000. All of this work is related to the Modern Projective Geometry of Faure-Frölicher 2000.

11.10 Still another approach to polar spaces

Still another approach to those polar spaces that are quadrics is presently written up by one of the participants to the course namely Eva Ferrara Dentice from Caserta.

11.11 Conclusion

When I started this work I thought that I knew it quite well and that it would not be too long. Once more, I forgot the connectedness of mathematics by which you may end up relating any two papers given in advance.

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13 All you need is love

All you need is love

by the Beatles

Love, love, love
Love, love, love
Love, love, love
There's nothing you can do that can't be done
Nothing you can sing that can't be sung
Nothing you can say but you can learn how to play the game
It's easy

There's nothing you can make that can't be made
None you can save that can't be saved
Nothing you can do but you can learn how to be you in time
It's easy

All you need is love
All you need is love
All you need is love, love
Love is all you need

Love, love, love
Love, love, love
Love, love, love

All you need is love
All you need is love
All you need is love, love
Love is all you need

There's nothing you can know that isn't known
Nothing you can see that isn't shown
Nowhere you can be that isn't where you're meant to be
It's easy

All you need is love
All you need is love
All you need is love, love
Love is all you need

All you need is love
All you need is love

All you need is love, love
Love is all you need
Love is all you need
That is all you need
That is all you need
That is all you need
That is all you need

14 All you need is love (bis)

All you need is love

by the pseudo-quadratic Beatles

Love, love, love
Love, love, love
Love, love, love

There is nothing you can do withouuuut points
Nothing you can imagine without lines
Nothing you can understand to Veldkamp's game
It's not easy

There is nothing you can do Tits hasn't done
But you can't find him in Math Reviews
BN pairs, polygons, buildings, ovoids, polar space
It's geometry

All you need is points
Together with lines
All you need is points, lines
and polarities

Tits, Tits, Tits
Shult, Shult, Shult
Buek, Buek, Buek

All you need is points
Together with lines
All you need is points, lines
and quadratic sets

Oberwolfach was the place to be, in the seventies
There were born the new axioms
Polar space is now under control: thanks to one or all

It’s easy

All you need is points
Together with lines
All you need is points, lines
with the good axioms

All you need is love
All you need is love
All you need is love, love
Love is all you need
Love is all you need
That is all you need
That is all you need
That is all you need
That is all you need