Reader for the lectures

# Matrix techniques for strongly regular graphs and related geometries

presented by

Willem H. Haemers

at the Intensive Course on Finite Geometry and Applications,
University of Ghent, April 3-14, 2000.

## 1 Strongly regular graphs

A graph (simple, undirected and loopless) of order $v$ is *strongly regular* with parameters $v$, $k$, $\lambda$, $\mu$ whenever it is not complete or edgeless and

(i) each vertex is adjacent to $k$ vertices,

(ii) for each pair of adjacent vertices there are $\lambda$ vertices adjacent to both,

(iii) for each pair of non-adjacent vertices there are $\mu$ vertices adjacent to both.

For example, the pentagon is strongly regular with parameters $(v, k, \lambda, \mu) = (5, 2, 0, 1)$. One easily verifies that a graph $G$ is strongly regular with parameters $(v, k, \lambda, \mu)$ if and only if its complement $\overline{G}$ is strongly regular with parameters $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$. The line graph of the complete graph of order $m$, known as the *triangular graph* $T(m)$, is strongly regular with parameters $(\frac{1}{2}m(m-1), 2(m-2), m-2, 4)$. The complement of $T(5)$ has parameters $(10, 3, 0, 1)$. This is the Petersen graph.

A graph $G$ satisfying condition (i) is called $k$-regular. It is well-known and easily seen that the adjacency matrix of a $k$-regular graph has an eigenvalue $k$ with eigenvector $\mathbf{1}$ (the all-one vector), and that every other eigenvalue $\rho$ satisfies $|\rho| \leq k$ (see Biggs [4]). For convenience we call an eigenvalue *restricted* if it has an eigenvector perpendicular to $\mathbf{1}$. We let $I$ and $J$ denote the identity and all-one matrices, respectively.

**Theorem 1.1** *For a simple graph $G$ of order $v$, not complete or empty, with adjacency matrix $A$, the following are equivalent:*

*(i) G is strongly regular with parameters $(v, k, \lambda, \mu)$ for certain integers $k$, $\lambda$, $\mu$,*

*(ii) $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$ for certain reals $k$, $\lambda$, $\mu$,*

*(iii) A has precisely two distinct restricted eigenvalues.*

**Proof.** The equation in (ii) can be rewritten as

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Now (i) $\Leftrightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii): Let $\rho$ be a restricted eigenvalue, and $u$ a corresponding eigenvector perpendicular to $\mathbf{1}$. Then $Ju = 0$. Multiplying the equation in (ii) on the right by $u$ yields $\rho^2 = (\lambda - \mu)\rho + (k - \mu)$. This quadratic equation in $\rho$ has two distinct solutions. (Indeed, $(\lambda - \mu)^2 = 4(\mu - k)$ is impossible since $\mu \leq k$ and $\lambda \leq k - 1$.)
(iii) $\Rightarrow$ (ii): Let $r$ and $s$ be the restricted eigenvalues. Then $(A - rI)(A - sI) = \alpha J$ for some real number $\alpha$. So $A^2$ is a linear combination of $A$, $I$ and $J$. $\square$

As an application, we show that quasi-symmetric block designs give rise to strongly regular graphs. A quasisymmetric design is a 2-$(v, k, \lambda)$ design such that any two blocks meet in either $x$ or $y$ points, for certain fixed $x$, $y$. Given this situation, we may define a graph $G$ on the set of blocks, and call two blocks adjacent when they meet in $x$ points. Then there exist coefficients $\alpha_1, \dots, \alpha_7$ such that $NN^\top = \alpha_1 I + \alpha_2 J$, $NJ = \alpha_3 J$, $JN = \alpha_4 J$, $A = \alpha_5 N^\top N + \alpha_6 I + \alpha_7 J$, where $A$ is the adjacency matrix of the graph $G$. (The $\alpha_i$ can be readily expressed in terms of $v$, $k$, $\lambda$, $x$, $y$.) Then $G$ is strongly regular by (ii) of the previous theorem. (Indeed, from the equations just given it follows straightforwardly that $A^2$ can be expressed as a linear combination of $A$, $I$ and $J$.) A large class of quasisymmetric block designs is provided by the 2-$(v, k, \lambda)$ designs with $\lambda = 1$ (also known as Steiner systems $S(2, k, v)$) - such designs have only two intersection numbers since no two blocks can meet in more than one point. This leads to a substantial family of strongly regular graphs, including the triangular graphs $T(m)$ (derived from the trivial design consisting of all pairs out of an $m$-set).

Another connection between strongly regular graphs and designs is found as follows: Let $A$ be the adjacency matrix of a strongly regular graph with parameters $(v, k, \lambda, \lambda)$ (i.e., with $\lambda = \mu$; such a graph is sometimes called a $(v, k, \lambda)$ graph). Then, by 2.1(ii)

$$AA^\top = A^2 = (k - \lambda)I + \lambda J,$$

which reflects that $A$ is the incidence matrix of a square ('symmetric') 2-$(v, k, \lambda)$ design. (And in this way one obtains precisely all square 2-designs possessing a polarity without absolute points.) For instance, the triangular graph $T(6)$ provides a square 2-$(15, 8, 4)$ design, the complementary design of the design of points and planes in the projective space $PG(3, 2)$. Similarly, if $A$ is the adjacency matrix of a strongly regular graph with parameters $(v, k, \lambda, \lambda + 2)$, then $A + I$ is the incidence matrix of a square 2-$(v, k, \lambda)$ design (and in this way one obtains precisely all square 2-designs possessing a polarity with all points absolute).

**Theorem 1.2** *Let $G$ be a strongly regular graph with adjacency matrix $A$ and parameters $(v, k, \lambda, \mu)$. Let $r$ and $s$ $(r > s)$ be the restricted eigenvalues of $A$ and let $f$, $g$ be their respective multiplicities. Then*

*(i)* $k(k - 1 - \lambda) = \mu(v - k - 1)$,

*(ii)* $rs = \mu - k, \; r + s = \lambda - \mu$,

*(iii)* $f, \; g = \frac{1}{2}(v - 1 \mp \frac{(r+s)(v-1)+2k}{r-s})$.

*(iv)* *$r$ and $s$ are integers, except perhaps when $f = g$, $(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$ for some integer $t$.*

**Proof.**(i) Fix a vertex $x$ of $G$. Let $\Gamma(x)$ and $\Delta(x)$ be the sets of vertices adjacent and non-adjacent to $x$, respectively. Counting in two ways the number of edges between $\Gamma(x)$ and $\Delta(x)$ yields (i). The equations (ii) are direct consequences of 1.1(ii), as we saw in the proof. Formula (iii) follows from $f + g = v - 1$ and $0 = \text{trace } A = k + fr + gs = k + \frac{1}{2}(r + s)(f + g) + \frac{1}{2}(r - s)(f - g)$. Finally, when $f \neq g$ then one can solve for $r$ and $s$ in (iii) (using (ii)) and find that $r$ and $s$ are rational, and hence integral. But $f = g$ implies $(\mu - \lambda)(v - 1) = 2k$, which is possible only for $\mu - \lambda = 1$, $v = 2k + 1$. □

These relations imply restrictions for the possible values of the parameters. Clearly, the right hand sides of (iii) must be positive integers. These are the so-called *rationality conditions*. As an example of the application of the rationality conditions we can derive the following result due to Hoffman & Singleton [26]

**Theorem 1.3** *Suppose $(v, k, 0, 1)$ is the parameter set of a strongly regular graph. Then $(v, k) = (5, 2)$, $(10, 3)$, $(50, 7)$ or $(3250, 57)$.*

**Proof.** The rationality conditions imply that either $f = g$, which leads to $(v, k) = (5, 2)$, or $r - s$ is an integer dividing $(r + s)(v - 1) + 2k$. By use of 1.2(i)-(ii) we have

$$s = -r - 1, \; k = r^2 + r + 1, \; v = r^4 + 2r^3 + 3r^2 + 2r + 2,$$

and thus we obtain $r = 1$, 2 or 7. $\square$

The first three possibilities are uniquely realized by the pentagon, the Petersen graph and the Hoffman-Singleton graph. For the last case existence is unknown (but see Aschbacher [1]).

Except for the rationality conditions, a few other restrictions on the parameters are known. We mention two of them. The Krein conditions, due to Scott [34], can be stated as follows:

$$(r+1)(k+r+2rs) \leq (k+r)(s+1)^2,$$
$$(s+1)(k+s+2rs) \leq (k+s)(r+1)^2.$$

Seidel's absolute bound (see Delsarte, Goethals & Seidel [17]) reads,

$$v \leq f(f+3)/2, \; v \leq g(g+3)/2.$$

The Krein conditions and the absolute bound are special cases of general inequalities for association schemes - we'll meet them again in the next section. In Brouwer & Van Lint [10] one may find a list of all known restrictions; this paper gives a survey of the recent results on strongly regular graphs. It is a sequel to Hubaut [27] earlier survey of constructions. Seidel [35] gives a good treatment of the theory.

# 2 Association schemes

## 2.1 definition

An *association scheme with $d$ classes* is a finite set $X$ together with $d+1$ relations $R_i$ on $X$ such that

(i) $\{R_0, R_1, \ldots, R_d\}$ is a partition of $X \times X$;

(ii) $R_0 = \{(x,x) | x \in X\}$;

(iii) if $(x,y) \in R_i$, then also $(y,x) \in R_i$, for all $x, y \in X$ and $i \in \{0, ..., d\}$;

(iv) for any $(x,y) \in R_k$ the number $p_{ij}^k$ of $z \in X$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ depends only on $i$, $j$ and $k$.

The numbers $p_{ij}^k$ are called the *intersection numbers* of the association scheme. The above definition is the original definition of Bose & Shimamoto [6]; it is what Delsarte [16] calls a symmetric association scheme. In Delsarte's more general definition, (iii) is replaced by:

(iii') for each $i \in \{0, ..., d\}$ there exists a $j \in \{0, ..., d\}$ such that $(x, y) \in R_i$ implies $(y, x) \in R_j$,

(iii") $p_{ij}^k = p_{ji}^k$, for all $i, j, k \in \{0, ..., d\}$.

Define $n = |X|$, and $n_i = p_{ii}^0$. Clearly, for each $i \in \{1, ..., d\}$, $(X, R_i)$ is a simple graph which is regular of degree $n_i$.

**Theorem 2.1** *The intersection numbers of an association scheme satisfy*

*(i)* $p_{0j}^k = \delta_{jk},\ p_{ij}^0 = \delta_{ij} n_j,\ p_{ij}^k = p_{ji}^k,$

*(ii)* $\sum_i p_{ij}^k = n_j,\ \ \sum_j n_j = n,$

*(iii)* $p_{ij}^k n_k = p_{ik}^j n_j,$

*(iv)* $\sum_l p_{ij}^l p_{kl}^m = \sum_l p_{kj}^l p_{il}^m.$

**Proof.** (i)-(iii) are straightforward. The expressions at both sides of (iv) count quadruples $(w, x, y, z)$ with $(w, x) \in R_i$, $(x, y) \in R_j$, $(y, z) \in R_k$, for a fixed pair $(w, z) \in R_m$. □

It is convenient to write the intersection numbers as entries of the so-called *intersection matrices* $L_0, \dots, L_d$:

$$(L_i)_{kj} = p_{ij}^k.$$

Note that $L_0 = I$. ¿From the definition it is clear that an association scheme with two classes is the same as a pair of complementary strongly regular graphs. If $(X, R_1)$ is strongly regular with parameters $(v, k, \lambda, \mu)$, then the intersection matrices of the scheme are

$$L_1 = \begin{bmatrix} 0 & k & 0 \\ 1 & \lambda & k-\lambda-1 \\ 0 & \mu & k-\mu \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & v-k-1 \\ 0 & k-\lambda-1 & v-2k+\lambda \\ 1 & k-\mu & v-2k+\mu-2 \end{bmatrix}.$$

We see that (iii) generalises 1.2(i).

## 2.2 The Bose-Mesner algebra

The relations $R_i$ of an association scheme are described by their adjacency matrices $A_i$ of order $n$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{whenever } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $A_i$ is the adjacency matrix of the graph $(X, R_i)$. In terms of the adjacency matrices, the axioms (i)-(iv) become

(i) $\sum_{i=0}^{d} A_i = J$,

(ii) $A_0 = I$,

(iii) $A_i = A_i^\top$, for all $i \in \{0, \ldots, d\}$,

(iv) $A_i A_j = \sum_k p_{ij}^k A_k$, for all $i, j, k \in \{0, \ldots, d\}$.

¿From (i) we see that the 0-1 matrices $A_i$ are linearly independent, and by use of (ii)-(iv) we see that they generate a commutative $(d+1)$-dimensional algebra $\mathcal{A}$ of symmetric matrices with constant diagonal. This algebra was first studied by Bose & Mesner [5] and is called the Bose-Mesner algebra of the association scheme.

Since the matrices $A_i$ commute, they can be diagonalized simultaneously (see Marcus & Minc [31]), that is, there exist a matrix $S$ such that for each $A \in \mathcal{A}$, $S^{-1}AS$ is a diagonal matrix. Therefore $\mathcal{A}$ is semisimple and has a unique basis of minimal idempotents $E_0, \ldots, E_n$ (see Burrow [11]). These are matrices satisfying

$$E_i E_j = \delta_{ij} E_i, \ \sum_{i=0}^{d} E_i = I.$$

The matrix $\frac{1}{n}J$ is a minimal idempotent (idempotent is clear, and minimal follows since rk $J = 1$). We shall take $E_0 = \frac{1}{n}J$. Let $P$ and $\frac{1}{n}Q$ be the matrices relating our two bases for $\mathcal{A}$:

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \ E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

Then clearly

$$PQ = QP = nI.$$

It also follows that

$$A_j E_i = P_{ij} E_i,$$

which shows that the $P_{ij}$ are the eigenvalues of $A_j$ and that the columns of $E_i$ are the corresponding eigenvectors. Thus $\mu_i = \text{rk } E_i$ is the multiplicity of the eigenvalue $P_{ij}$ of $A_j$ (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$). We see that $\mu_0 = 1$, $\sum_i \mu_i = n$, and $\mu_i = \text{trace } E_i = n(E_i)_{jj}$ (indeed, $E_i$ has only eigenvalues 0 and 1, so rk $E_k$ equals the sum of the eigenvalues).

**Theorem 2.2** *The numbers $P_{ij}$ and $Q_{ij}$ satisfy*

*(i)* $P_{i0} = Q_{i0} = 1,\ P_{0i} = n_i,\ Q_{0i} = \mu_i,$

*(ii)* $P_{ij}P_{ik} = \sum_{l=0}^{d} p_{jk}^{l} P_{il},$

*(iii)* $\mu_i P_{ij} = n_j Q_{ji},\ \sum_i \mu_i P_{ij} P_{ik} = n n_j \delta_{jk},\ \sum_i n_i Q_{ij} Q_{ik} = n \mu_j \delta_{jk},$

*(iv)* $|P_{ij}| \leq n_j,\ |Q_{ij}| \leq \mu_j.$

**Proof.** Part (i) follows easily from $\sum_i E_i = I = A_0$, $\sum_i A_i = J = nE_0$, $A_i J = n_i J$, and trace $E_i = \mu_i$. Part (ii) follows from $A_j A_k = \sum_l p_{jk}^{l} A_l$. The first equality in (iii) follows from $\sum_i n_j Q_{ji} P_{ik} = n n_j \delta_{jk} = \text{trace } A_j A_k = \sum_i \mu_i P_{ij} P_{ik}$, since $P$ is nonsingular; this also proves the second equality, and the last one follows since $PQ = nI$. The first inequality of (iv) holds because the $P_{ij}$ are eigenvalues of the $n_j$-regular graphs $(X, R_j)$. The second inequality then follows by use of (iii). □

Relations (iii) are often referred to as the *orthogonality relations*, since they state that the rows (and columns) of $P$ (and $Q$) are orthogonal with respect to a suitable weight function.

If $d = 2$, and $(X, R_1)$ is strongly regular with parameters $(v, k, \lambda, \mu)$, the matrices $P$ and $Q$ are

$$P = \begin{bmatrix} 1 & k & v-k-1 \\ 1 & r & -r-1 \\ 1 & s & -s-1 \end{bmatrix},\quad Q = \begin{bmatrix} 1 & f & g \\ 1 & fr/k & gs/k \\ 1 & -f\frac{r+1}{v-k-1} & -g\frac{s+1}{v-k-1} \end{bmatrix},$$

where $r$, $s$, $f$ and $g$ can be expressed in terms of $v$, $k$, $\lambda$ by use of 1.2.

In general the matrices $P$ and $Q$ can be computed from the intersection numbers of the scheme, as follows from the following

**Theorem 2.3** *For $i = 0, \dots, d$, the intersection matrix $L_j$ has eigenvalues $P_{ij}$ $(0 \leq i \leq d)$.*

**Proof.** Theorem 2.2(ii) yields

$$P_{ij} \sum_k P_{ik}(P^{-1})_{km} = \sum_{k,l} P_{il}(L_j)_{lk}(P^{-1})_{km},$$

hence $PL_jP^{-1} = \text{diag }(P_{0j}, \dots, P_{dj})$. □

Thanks to this theorem, it is relatively easy to compute $P$, $Q$ $(= \frac{1}{n}P^{-1})$ and $\mu_i$ $(= Q_{0i})$. It is also possible to express $P$ and $Q$ in terms of the (common) eigenvectors of the $L_j$. Indeed, $PL_jP^{-1} = \text{diag }(P_{0j}, \dots, P_{dj})$ implies that the rows of $P$ are left eigenvectors and the columns of $Q$ are right eigenvectors. In particular, $\mu_i$ can be computed from the right eigenvector $u_i$ and the left eigenvector $v_i^\top$, normalized such that $(u_i)_0 = (v_i)_0 = 1$, by use of $\mu_i u_i^\top v_i = n$. Clearly, each $\mu_i$ must be an integer. These are the *rationality conditions* for an association scheme. As we saw in the case of a strongly regular graph, these conditions can be very powerful.

## 2.3 The Krein parameters

The Bose-Mesner algebra $\mathcal{A}$ is not only closed under ordinary matrix multiplication, but also under componentwise (Hadamard, Schur) multiplication (denoted $\circ$). Clearly $\{A_0, \dots, A_d\}$ is the basis of minimal idempotents with respect to this multiplication. Write

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{d} q_{ij}^k E_k.$$

The numbers $q_{ij}^k$ thus defined are called the *Krein parameters*. (Our $q_{ij}^k$ are those of Delsarte, but differ from Seidel's [35] by a factor $n$.) As expected, we now have the analogue of 2.1 and 2.2.

**Theorem 2.4** *The Krein parameters of an association scheme satisfy*

*(i)* $q_{0j}^k = \delta_{jk},\ q_{ij}^0 = \delta_{ij}\mu_j,\ q_{ij}^k = q_{ji}^k,$

*(ii)* $\sum_i q_{ij}^k = \mu_j,\ \ \sum_j \mu_j = n,$

*(iii)* $q_{ij}^k \mu_k = q_{ik}^j \mu_j,$

*(iv)* $\sum_l q^l_{ij} q^m_{kl} = \sum_l q^l_{kj} q^m_{il}$,

*(v)* $Q_{ij} Q_{ik} = \sum_{l=0}^{d} q^l_{jk} Q_{il}$,

*(vi)* $n\mu_k q^k_{ij} = \sum_l n_l Q_{li} Q_{lj} Q_{lk}$.

**Proof.** Let $\sum(A)$ denote the sum of all entries of the matrix $A$. Then $JAJ = \sum(A)J$, $\sum(A \circ B) = \text{trace } AB^\top$ and $\sum(E_i) = 0$ if $i \neq 0$, since then $E_i J = nE_i E_0 = 0$. Now (i) follows by use of $E_i \circ E_0 = \frac{1}{n} E_i$, $q^0_{ij} = \sum(E_i \circ E_j) = \text{trace } E_i E_j = \delta_{ij}\mu_j$, and $E_i \circ E_j = E_j \circ E_i$, respectively. Equation (iv) follows by evaluating $E_i \circ E_j \circ E_k$ in two ways, and (iii) follows from (iv) by taking $m = 0$. Equation (v) follows from evaluating $A_i \circ E_j \circ E_k$ in two ways, and (vi) follows from (v), using the orthogonality relation $\sum_l n_l Q_{lm} Q_{lk} = \delta_{mk}\mu_k n$. Finally, by use of (iii) we have

$$\mu_k \sum_j q^k_{ij} = \sum_j q^j_{ik}\mu_j = n \cdot \text{trace } (E_i \circ E_k) = n \sum_l (E_i)_{ll} (E_k)_{ll} = \mu_i \mu_k,$$

proving (ii). □

The above results illustrate a dual behaviour between ordinary multiplication, the numbers $p^k_{ij}$ and the matrices $A_i$ and $P$ on the one hand, and Schur multiplication, the numbers $q^k_{ij}$ and the matrices $E_i$ and $Q$ on the other hand. If two association schemes have the property that the intersection numbers of one are the Krein parameters of the other, then the converse is also true. Two such schemes are said to be (formally) dual to each other. One scheme may have several (formal) duals, or none at all (but when the scheme is invariant under a regular abelian group, there is a natural way to define a dual scheme, cf. Delsarte [16]). In fact usually the Krein parameters are not even integers. But they cannot be negative. These important restrictions, due to Scott [34] are the so-called Krein conditions.

**Theorem 2.5** *The Krein parameters of an association scheme satisfy* $q^k_{ij} \geq 0$ *for all* $i, j, k \in \{0, \ldots, d\}$.

**Proof.** The numbers $\frac{1}{n} q^k_{ij}$ $(0 \leq k \leq d)$ are the eigenvalues of $E_i \circ E_j$ (since $(E_i \circ E_j)E_k = \frac{1}{n} q^k_{ij} E_k$). On the other hand, the Kronecker product $E_i \otimes E_j$ is positive semidefinite, since each $E_i$ is. But $E_i \circ E_j$ is a principal submatrix of $E_i \otimes E_j$, and therefore is positive semidefinite as well, i.e., has no negative eigenvalue. □

The Krein parameters can be computed by use of equation 2.4(vi). This equation also shows that the Krein condition is equivalent to

$$\sum_l n_l Q_{li} Q_{lj} Q_{lk} \geq 0 \quad \text{for all } i, j, k \in \{0, \ldots, d\}.$$

In case of a strongly regular graph we obtain

$$q_{11}^1 = \frac{f^2}{v}\left(1 + \frac{r^3}{k^2} - \frac{(r+1)^3}{(v-k-1)^2}\right) \geq 0,$$

$$q_{22}^2 = \frac{g^2}{v}\left(1 + \frac{s^3}{k^2} - \frac{(s+1)^3}{(v-k-1)^2}\right) \geq 0$$

(the other Krein conditions are trivially satisfied in this case), which is equivalent to the result mentioned in the previous section.

Neumaier [32] generalized Seidel's absolute bound to association schemes, and obtained the following.

**Theorem 2.6** *The multiplicities $\mu_i$ $(0 \leq i \leq d)$ of an association scheme with $d$ classes satisfy*

$$\sum_{q_{ij}^k \neq 0} \mu_k \leq \begin{cases} \mu_i \mu_j & \text{if } i \neq j, \\ \frac{1}{2}\mu_i(\mu_i + 1) & \text{if } i = j. \end{cases}$$

**Proof.** The left hand side equals rk $(E_i \circ E_j)$. But rk $(E_i \circ E_j) \leq$ rk $(E_i \otimes E_j) =$ rk $E_i \cdot$rk $E_j = \mu_i \mu_j$. And if $i = j$, then rk $(E_i \circ E_i) \leq \frac{1}{2}\mu_i(\mu_i + 1)$. Indeed, if the rows of $E_i$ are linear combinations of $\mu_i$ rows, then the rows of $E_i \circ E_i$ are linear combinations of the $\mu_i + \frac{1}{2}\mu_i(\mu_i - 1)$ rows that are the elementwise products of any two of these $\mu_i$ rows. □

For strongly regular graphs with $q_{11}^1 = 0$ we obtain Seidel's bound: $v \leq \frac{1}{2}f(f+3)$. But in case $q_{11}^1 > 0$, Neumaier's result states that the bound can be improved to $v \leq \frac{1}{2}f(f+1)$.

## 2.4 Distance regular graphs

Consider a connected simple graph with vertex set $X$ of diameter $d$. Define $R_i \subset X^2$ by $(x, y) \in R_i$ whenever $x$ and $y$ have graph distance $i$. If this defines an association scheme, then the graph $(X, R_1)$ is called *distance-regular.* The corresponding association scheme is called *metric.* By the triangle inequality, $p_{ij}^k = 0$ if $i+j < k$ or $|i-j| > k$. Moreover, $p_{ij}^{i+j} > 0$. Conversely, if the intersection numbers of an association scheme satisfy these conditions, then $(X, R_1)$ is easily seen to be distance-regular. The *intersection array* of a distance-regular graph is the following array of relevant intersection numbers.

$$\{p_{1,1}^0, p_{1,2}^1, \ldots, p_{1,d}^{d-1}; p_{1,0}^1, p_{1,1}^2, \ldots, p_{1,d-1}^d\} \; .$$

Many of the association schemes that play a rôle in combinatorics are metric. Strongly regular graphs are obviously metric. The line graph of the Petersen graph and the Hoffman-Singleton graph are easy examples of distance-regular graphs that are not strongly regular.

Any $k$-regular graph of diameter $d$ has at most

$$1 + k + k(k-1) + \ldots + k(k-1)^{d-1}$$

vertices, as is easily seen. Graphs for which equality holds are called *Moore graphs.* Moore graphs are distance-regular, and those of diameter 2 were dealt with in Theorem 1.3. Using the rationality conditions Damerell [15] and Bannai & Ito [2] showed:

**Theorem 2.7** *A Moore graph with diameter* $d >= 3$ *is a* $(2d+1)$*-gon.*

A strong non-existence result of the same nature is the theorem of Feit & G. Higman [18] about finite generalized polygons. A *generalized* $m$*-gon* is a point-line geometry such that the incidence graph is a connected, bipartite graph of diameter $m$ and girth $2m$. It is called *regular* of order $(s, t)$ for certain (finite or infinite) cardinal numbers $s$, $t$ if each line is incident with $s+1$ points and each point is incident with $t+1$ lines. (It is not difficult to prove that if each point is on at least three lines, and each line has at least three points (and $m < \infty$), then the geometry is necessarily regular, and in fact $s = t$ in case $m$ is odd.) ¿From such a regular generalized $m$-gon of order $(s, t)$, where $s$ and $t$ are finite and $m >= 3$, we can construct a distance-regular graph with valency $s(t+1)$ and diameter $d = \lfloor \frac{m}{2} \rfloor$ by taking the collinearity graph on the points.

**Theorem 2.8** *A finite generalized* $m$*-gon of order* $(s, t)$ *with* $s > 1$ *and* $t > 1$ *satisfies* $m \in \{2, 3, 4, 6, 8\}$.

Proofs of this theorem can be found in Feit & Higman [18], Brouwer, Cohen & Neumaier [7] and Van Maldeghem [37]; again the rationality conditions do the job. The Krein conditions yield some additional information:

**Theorem 2.9** *A finite regular generalized $m$-gon with $s > 1$ and $t > 1$ satisfies $s \le t^2$ and $t \le s^2$ if $m = 4$ or 8; it satisfies $s \le t^3$ and $t \le s^3$ if $m = 6$.*

This result is due to Higman [24] and Haemers & Roos [23]. For each $m \in \{2, 3, 4, 6, 8\}$ infinitely many generalized $m$-gons exist. (For $m = 2$ we have trivial structures - the incidence graph is complete bipartite; for $m = 3$ we have (generalized) projective planes; an example of a generalized 4-gon of order (2,2) with collinearity graph $T(6)$ can be described as follows: the points are the pairs from a 6-set, and the lines are the partitions of the 6-set into three pairs, with obvious incidence.)

Many association schemes have the important property that the eigenvalues $P_{ij}$ can be expressed in terms of orthogonal polynomials. An association scheme is called *$P$-polynomial* if there exist polynomials $f_k$ of degree $k$ with real coefficients, and real numbers $z_i$ such that $P_{ik} = f_k(z_i)$. Clearly we may always take $z_i = P_{i1}$. By the orthogonality relation 2.2(iii) we have

$$\sum_i \mu_i f_j(z_i) f_k(z_i) = \sum_i \mu_i P_{ij} P_{ik} = n n_j \delta_{jk},$$

which shows that the $f_k$ are orthogonal polynomials.

**Theorem 2.10** *An association scheme is metric if and only if it is $P$-polynomial.*

**Proof.** Let the scheme be metric. Theorem 2.1 gives

$$A_1 A_i = p_{1i}^{i-1} A_{i-1} + p_{1i}^{i} A_i + p_{1i}^{i+1} A_{i+1}.$$

Since $p_{1i}^{i+1} \neq 0$, $A_{i+1}$ can be expressed in terms of $A_1$, $A_{i-1}$ and $A_i$. Hence for each $j$ there exists a polynomial $f_j$ of degree $j$ such that

$$A_j = f_j(A_1).$$

Using this we have

$$P_{ij} E_i = A_j E_i = f_j(A_1) E_i = f_j(A_1 E_i) E_i = f_j(P_{i1}) E_i,$$

hence $P_{ij} = f_j(P_{i1})$.

Now suppose that the scheme is $P$-polynomial. Then the $f_j$ are orthogonal polynomials, and therefore they satisfy a 3-term recurrence relation (see Szegö [36] p.42)

$$\alpha_{j+1} f_{j+1}(z) = (\beta_j - z) f_j(z) + \gamma_{j-1} f_{j-1}(z).$$

Hence

$$P_{i1}P_{ij} = -\alpha_{j+1}P_{ij+1} + \beta_j P_{ij} + \gamma_{j-1}P_{ij-1} \quad \text{for } i = 0, \ldots, d.$$

Since $P_{i1}P_{ij} = \sum_l p^l_{1j} P_{il}$ and $P$ is nonsingular, it follows that $p^l_{1j} = 0$ for $|l - j| > 1$. Now the full metric property easily follows by induction. □

This result is due to Delsarte [16] (Theorem 5.6, p.61). There is also a result dual to this theorem, involving so-called $Q$-polynomial and cometric schemes. However, just as the intersection numbers $p^k_{ij}$ have a combinatorial interpretation while the Krein parameters $q^k_{ij}$ do not, the metric schemes have the combinatorial description of distance-regular graphs, while there is no combinatorial interpretation for the cometric property. For more information on $P$- and $Q$-polynomial association schemes, see Delsarte [16], Bannai & Ito [3] and Brouwer, Cohen & Neumaier [7].

# 3 Matrix tools

## 3.1 Partitions

Suppose $A$ is a symmetric real matrix whose rows and columns are indexed by $X = \{1, \ldots, n\}$. Let $\{X_0, \ldots, X_d\}$ be a partition of $X$. The *characteristic matrix* $S$ is the $n \times (d+1)$ matrix whose $j^{\text{th}}$ column is the characteristic vector of $X_j$ ($j = 0, \ldots, d$). Define $k_i = |X_i|$ and $K = \text{diag}(k_0, \ldots, k_d)$. Let $A$ be partitioned according to $\{X_0, \ldots, X_d\}$, that is

$$A = \begin{bmatrix} A_{0,0} & \ldots & A_{0,d} \\ \vdots & & \vdots \\ A_{d,0} & \ldots & A_{d,d} \end{bmatrix},$$

wherein $A_{i,j}$ denotes the submatrix (block) of $A$ formed by rows in $X_i$ and the columns in $X_j$. Let $b_{i,j}$ denote the average row sum of $A_{i,j}$. Then the matrix $B = (b_{i,j})$ is called the *quotient matrix*. We easily have

$$KB = S^\top AS, \; S^\top S = K.$$

If the row sum of each block $A_{i,j}$ is constant then the partition is called *regular* and we have $A_{i,j}\mathbf{1} = b_{i,j}\mathbf{1}$ for $i, j = 0, \ldots, d$, so

$$AS = SB.$$

The following result is well-known and often applied, see [5],[10].

**Lemma 3.1** *If, for a regular partition, $v$ is an eigenvector of $B$ for an eigenvalue $\lambda$, then $Sv$ is an eigenvector of $A$ for the same eigenvalue $\lambda$.*

**Proof.** $Bv = \lambda v$ implies $ASv = SBv = \lambda Sv$. □

Suppose $A$ is the adjacency matrix of a connected graph $\Gamma$. Let $\gamma$ be a vertex of $\Gamma$ with local diameter $d$ and let $X_i$ denote the number of points at distance $i$ from $\gamma$ ($i = 0, \ldots, d$). Then $\{X_0, \ldots, X_d\}$ is called the *distance partition* of $\Gamma$ around $\gamma$. Note that in this case we can compute $K$ from $B$, since $k_0 = 1$, $k_i b_{i,i+1} = k_{i+1} b_{i+1,i}$ and $b_{i+1,i} \neq 0$ for $i = 0, \ldots, d-1$. If the distance partition is regular, $\Gamma$ is called *distance-regular around* $\gamma$ and the quotient matrix $B$ is a tridiagonal matrix, called the *intersection matrix* of $\Gamma$ with respect to $\gamma$. If $\Gamma$ is distance-regular around each vertex with the same intersection matrix, then $\Gamma$ is (by definition) a distance-regular graph with intersection matrix $B$ and intersection array

$$\{b_{0,1}, \ldots, b_{d-1,d}; b_{1,0}, \ldots, b_{d,d-1}\}.$$

Clearly the intersection array determines the intersection matrix, because $B$ has constant row sum $k (= k_1 = b_{0,1})$. Lemma 3.1 gives that for a distance-regular graph $\Gamma$, the eigenvalues of its intersection matrix $B$ are also eigenvalues of its adjacency matrix $A$. In fact, the distinct eigenvalues of $\Gamma$ are precisely the eigenvalues of $B$ as we saw in Theorem 2.3.

## 3.2 Interlacing

Consider two sequences of real numbers: $\lambda_1 \geq \ldots \geq \lambda_n$, and $\mu_1 \geq \ldots \geq \mu_m$ with $m < n$. The second sequence is said to *interlace* the first one whenever

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \text{ for } i = 1, \ldots, m.$$

The interlacing is *tight* if there exist an integer $k \in [0, m]$ such that

$$\lambda_i = \mu_i \text{ for } 1 \leq i \leq k \text{ and } \lambda_{n-m+i} = \mu_i \text{ for } k+1 \leq i \leq m.$$

If $m = n-1$, the interlacing inequalities become $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_m \geq \lambda_n$, which clarifies the name. Godsil [19] reserves the name 'interlacing' for this particular case and calls it generalized interlacing otherwise. Throughout, the $\lambda_i$'s ans $\mu_i$'s will be eigenvalues of matrices $A$ and $B$, respectively. Basic to eigenvalue interlacing is

Rayleigh's principle, a standard (and easy to prove) result from linear algebra, which can be stated as follows. Let $u_1, \ldots, u_n$ be an orthonormal set of eigenvectors of the real symmetric matrix $A$, such that $u_i$ is a $\lambda_i$-eigenvector (we use this abbreviation for an eigenvector corresponding to the eigenvalue $\lambda_i$). Then

$$\frac{u^\top A u}{u^\top u} \geq \lambda_i \text{ if } u \in \langle u_1, \ldots, u_i \rangle \text{ and}$$

$$\frac{u^\top A u}{u^\top u} \leq \lambda_i \text{ if } u \in \langle u_1, \ldots, u_{i-1} \rangle^\perp .$$

In both cases, equality implies that $u$ is a $\lambda_i$-eigenvector of $A$.

**Theorem 3.1** *Let $S$ be a real symmetric matrix such that $S^\top S = I$ and let $A$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Define $B = S^\top A S$ and let $B$ have eigenvalues $\mu_1 \geq \ldots \geq \mu_m$ and respective eigenvectors $v_1 \ldots v_m$.*

*(i) The eigenvalues of $B$ interlace those of $A$.*

*(ii) If $\mu_i = \lambda_i$ or $\mu_i = \lambda_{n-m+1}$ for some $i \in [1, m]$, then $B$ has a $\mu_i$-eigenvector $v$ such that $Sv$ is a $\mu_i$-eigenvector of $A$.*

*(iii) If for some integer $l$, $\mu_i = \lambda_i$, for $i = 0, \ldots, l$ (or $\mu_i = \lambda_{n-m+i}$ for $i = l, \ldots, m$), then $Sv_i$ is a $\mu_i$-eigenvector of $A$ for $i = 1, \ldots, l$ (respectively $i = l, \ldots, m$).*

*(iv) If the interlacing is tight, then $SB = AS$.*

**Proof.** With $u_1, \ldots, u_n$ as above, for each $i \in [1, m]$, take a nonzero vector $s_i$ in

$$\langle v_1, \ldots, v_i \rangle \cap \left\langle S^\top u_1, \ldots, S^\top u_{i-1} \right\rangle^\perp . \tag{1}$$

Then $Ss_i \in \langle u_1, \ldots, u_{n-1} \rangle^\perp$, hence by Rayleigh's principle,

$$\lambda_i \geq \frac{(Ss_i)^\top A (Ss_i)}{(Ss_i)^\top (Ss_i)} = \frac{s_i{}^\top B s_i}{s_i{}^\top s_i} \geq \mu_i,$$

and similarly (or by applying the above inequality to $-A$ and $-B$) we get $\lambda_{n-m+1} \leq \mu_i$, proving (i).

If $\lambda_i = \mu_i$, then $s_i$ and $Ss_i$ are $\lambda_i$-eigenvectors of $B$ and $A$, respectively, proving (ii).

We prove (iii) by induction on $l$. Assume $Sv_i = u_i$ for $i = 1, \ldots, l-1$. Then we may take $s_l = v_l$ in 1, but in proving (ii) we saw that $Ss_l$ is a $\lambda_l$-eigenvector of $A$. (The statement between parentheses follows by considering $-A$ and $-B$.) Thus we have (iii).

Let the interlacing be tight. Then by (iii), $Sv_1, \ldots, Sv_m$ is an orthonormal set of eigenvectors of $A$ for the eigenvalues $\mu_1, \ldots, \mu_m$. So we have $SBv_i = \mu_i Sv_i = ASv_i$, for $i = 1, \ldots, m$. Since the vectors $v_i$ form a basis, it follows that $SB = AS$. □

If we take $S = [I \ O]^\top$, then $B$ is just a principal submatrix of $A$ and we have the following corollary.

**Corollary 3.1** *If $B$ is a principal submatrix of a symmetric matrix $A$, then the eigenvalues of $B$ interlace the eigenvalues of $A$.*

Suppose rows and columns of $A$ are partitioned with characteristic matrix $\tilde{S}$ and quotient matrix $\tilde{B}$

**Corollary 3.2** *Let $\tilde{B}$ be the quotient matrix of a symmetric matrix $A$ whose rows and columns are partitioned according to a partitioning $\{X_1, \ldots, X_m\}$.*

*(i) The eigenvalues of $\tilde{B}$ interlace the eigenvalues of $A$.*

*(ii) If the interlacing is tight, then the partition is regular.*

**Proof.** Put $S = \tilde{S}K^{-\frac{1}{2}}$, where $K = \text{diag}(|X_1|, \ldots, |X_m|)$. Then the eigenvalues of $B = S^\top AS$ interlace those of $A$. This proves (i), because $B$ and $\tilde{B} = K^{-\frac{1}{2}}BK^{\frac{1}{2}}$ have the same spectrum. If the interlacing is tight, then $SB = AS$, hence $A\tilde{S} = \tilde{S}\tilde{B}$, and the partition is regular. □

Theorem 3.1.(i) is a classical result; see Courant & Hilbert [13]. For the special case of a pricipal submatrix (Corollary 3.1), the result even goes back to Cauchy and is therefore often referred to as Cauchy interlacing. Interlacing for the quotient matrix (Corollary 3.2) is especially applicable to combinatorial structures (as we shall see). Payne (see, for instance, [33]) has applied the extremal inequalities $\lambda_1 \geq \mu_i \geq \lambda_n$ to finite geometries several times. He contributes the method to Higman and Sims and therefore calls it the Higman-Sims technique.

**Lemma 3.2** *Let $M$ be a symmetric $v \times v$ matrix with a symmetric partition*

$$M = \begin{bmatrix} M_1 & N \\ N^\top & M_2 \end{bmatrix},$$

*where $M_1$ has order $v_1$ (say). Suppose $M$ has just two distinct eigenvalues $r$ and $s$ $(r > s)$ with multiplicities $f$ and $v - f$. Let $\lambda_1 \geq \ldots \geq \lambda_{v_1}$ be the eigenvalues of $M_1$ and let $\mu_1 \geq \ldots \geq \mu_{v-v_1}$ be the eigenvalues of $M_2$. Then $r \geq \lambda_i \geq s$ for $i = 1, \ldots, v_1$, and*

$$\mu_i = \begin{cases} r \text{ if } 1 \leq i \leq f - v_1, \\ s \text{ if } f + 1 \leq i \leq v - v_1, \\ r + s - \lambda_{f-i+1} \text{ otherwise.} \end{cases}$$

**Proof.** The inequalities $r \geq \lambda_i \geq s$ and also the first two lines of the formulas for $\mu_i$ follow from Corollary 3.1. We have $(M - rI)(M - sI) = O$. With the given block structure of $M$ this gives $N^\top M_1 + M_2 N^\top - (r + s)N^\top = O$. Suppose that $\lambda_i \neq r, s$, let $V$ be the corresponding eigenspace and let $\{v_1, \ldots, v_m\}$ be a basis for $V$. We claim that $B = \{N^\top v_1, \ldots, N^\top v_m\}$ is independent. Suppose not. Then $N^\top v = \mathbf{0}$ for some $v \in V$, $v \neq \mathbf{0}$ which implies that $M \left[\begin{smallmatrix} v \\ \mathbf{0} \end{smallmatrix}\right] = \lambda_i \left[\begin{smallmatrix} v \\ \mathbf{0} \end{smallmatrix}\right]$, that is, $\lambda_i$ is an eigenvalue of $M$, a contradiction. Now $N^\top M_1 + M_2 N^\top = (r+s)N^\top$ gives $M_2(N^\top v_i) = (r+s-\lambda_i)N^\top v_i$, thus $B$ is an independent set of eigenvectors of $M_2$ for the eigenvalue $r+s-\lambda_i$. This almost proves the lemma. Only the numbers of $\mu_i$'s that are equal to $r$ or $s$ are not determined, but these follow from $\sum \mu_i + \sum \lambda_i = \text{trace } M_1 + \text{trace } M_2 = \text{trace } M = fr + (v-f)s$. □

## 3.3 Applications

Let $G$ be a graph on $n$ vertices (undirected, simple, and loopless) having an adjacency matrix $A$ with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. The size of the largest coclique (= independent set of vertices) of $G$ is denoted by $\alpha(G)$. Both Corollaries 3.1 and 3.2 lead to a bound for $\alpha(G)$.

**Theorem 3.2** $\alpha(G) \leq |\{i|\lambda_i \geq 0\}|$ *and* $\alpha(G) \leq |\{i|\lambda_i \leq 0\}|$.

**Proof.** $A$ has a principal submatrix $B = O$ of size $\alpha = \alpha(G)$. Corollary 3.1 gives $\lambda_\alpha \geq \mu_\alpha = 0$ and $\lambda_{n-\alpha-1} \leq \mu_1 = 0$. □

**Theorem 3.3** *If $G$ is regular of degree $k$, then*

$$\alpha(G) \leq n\frac{-\lambda_n}{k-\lambda_n},$$

*and if a coclique $C$ meets this bound, then every vertex not in $C$ is adjacent to precisely $-\lambda_n$ vertices of $C$.*

**Proof.** We apply Corollary 3.2. The coclique gives rise to a partition of $A$ with quotient matrix

$$B = \begin{bmatrix} 0 & k \\ \frac{k\alpha}{n-\alpha} & k - \frac{k\alpha}{n-\alpha} \end{bmatrix},$$

where $\alpha = \alpha(G)$. $B$ has eigenvalues $\mu_1 = k = \lambda_1$ (the row sum) and $\mu_2 = -k\alpha/(n-\alpha)$ (since trace $B = k+\mu_2$) and so $\lambda_n \leq \mu_2$ gives the required inequality. If equality holds, then $\mu_2 = \lambda_n$, and since $\mu_1 = \lambda_1$, the interlacing is tight and hence the partition is regular. □

The first bound is due to Cvetković [14]. The second bound is an unpublished result of Hoffman. If in a strongly regular graph both bounds are tight, we have a special structure.

**Theorem 3.4** *Let $G$ be a strongly regular graph with eigenvalues $k$ (degree), $r$ and $s$ ($r > s$) and multiplicities $1$, $f$ and $g$, respectively. Suppose that $G$ is not complete multi-partite (i.e. $r \neq 0$) and let $C$ be a coclique in $G$.*

*(i)* $|C| \leq g$,

*(ii)* $|C| \leq ns/(s-k)$,

*(iii) if $|C| = g = ns/(s-k)$, then the subgraph $G'$ of $G$ induced by the vertices which are not in $C$, is strongly regular with eigenvalues $k' = k+s$ (degree), $r' = r$ and $s' = r+s$ and respective multiplicities $1$, $g-1$ and $f-g+1$.*

**Proof.** (i) and (ii) follow from Theorem 3.2 and 3.3. Assume $|C| = g = ns/(s-k)$, then Theorem 3.3 gives that $G'$ is regular of degree $k+s$. Next we apply Lemma 3.2 to $M = A - \frac{k-r}{n}J$, where $A$ is the adjacency matrix of $G$. Since $G$ is regular, $A$ and $J$ commute and therefore $M$ has eigenvalues $r$ and $s$ with multiplicities $f+1$ and $g$, respectively. We take $M_1 = -\frac{k-r}{n}J$ of size $|C| = g$ and $M_2 = A' - \frac{k-r}{n}J$, where $A'$ is the adjacency matrix of $G'$. Lemma 3.2 gives the eigenvalues of $M_2$: $r$ ($f+1-g$ times,

$s$ (0 times), $r+s$ ($g-1$ times) and $r+s+g(k-r)/n$ (1 time). Since $G'$ is regular of degree $k+s$ and $A'$ commutes with $J$ we obtain the required eigenvalues for $A'$. By Theorem 1.1 $G'$ is strongly regular. □

For instance, an $(m-1)$-coclique in the complement of the triangular graph $T(m)$ is tight for both bounds and the graph on the remaining vertices is the complement of $T(m-1)$.

## 3.4 Chromatic number

A colouring of a graph $G$ is a partition of its vertices into cocliques (colour classes). Therefore the number of colour classes, and hence the chromatic number $\chi(G)$ of $G$ is bounded below by $\frac{n}{\alpha(G)}$. Thus upper bounds for $\alpha(G)$ give lower bounds for $\chi(G)$. For instance if $G$ is regular of degree $k=\lambda_1$ Theorem 3.3 implies that $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$. This bound remains however valid for non-regular graphs.

**Theorem 3.5**

*(i) If $G$ is not the empty graph then* $\chi(G) \geq 1 - \dfrac{\lambda_1}{\lambda_n}$ .

*(ii) If $\lambda_2 > 0$ then* $\chi(G) \geq 1 - \dfrac{\lambda_{n-\chi(G)+1}}{\lambda_2}$ .

**Proof.** Let $X_1, \ldots, X_\chi$ ($\chi = \chi(G)$) denote the colour classes of $G$ and let $u_1, \ldots, u_n$ be an orthonormal set of eigenvectors of $A$ (where $u_i$ corresponds to $\lambda_i$). For $i = 1, \ldots, \chi$ let $s_i$ denote the restriction of $u_1$ to $X_i$, that is

$$(s_i)_j = \begin{cases} (u_1)_j & \text{if } j \in X_i \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

and put $\tilde{S} = [s_1 \ \cdots \ s_\chi]$ (if some $s_i = 0$ we delete it from $\tilde{S}$ and proceed similarly) and $D = \tilde{S}^\top \tilde{S}$, $S = \tilde{S} D^{-\frac{1}{2}}$ and $B = S^\top A S$. Then $B$ has zero diagonal (since each colour class corresponds to a zero submatrix of $A$) and an eigenvalue $\lambda_1$ ($d = D^{\frac{1}{2}}\mathrm{j}$ is a $\lambda_1$-eigenvector of $B$). Moreover Corollary 3.2 gives that the remaining eigenvalues of $B$ are at least $\lambda_n$. Hence

$$0 = \mathrm{trace}(B) = \mu_1 + \ldots + \mu_\chi \geq \lambda_1 + (\chi - 1)\lambda_n \ ,$$

which proves (i), since $\lambda_n < 0$. The proof of (ii) is similar, but a bit more complicated. With $s_1, \ldots, s_\chi$ as above, choose a non-zero vector $s$ in

$$\langle u_{n-\chi+1}, \ldots, u_n \rangle \cap \langle s_1, \ldots, s_\chi \rangle^\perp \ .$$

The two spaces have non-trivial intersection since the dimensions add up to $n$ and $u_1$ is orthogonal to both. Redefine $s_i$ to be the restriction of $s$ to $X_i$, and let $\tilde{S}, D, S$, and $d$ be analogous to above. Put $A' = A - (\lambda_1 - \lambda_2)u_1 u_1^\top$, then the largest eigenvalue of $A'$ equals $\lambda_2$, but all other eigenvalues of $A$ are also eigenvalues of $A'$ with the same eigenvectors. Define $B = S^\top A' S$. Now $B$ has again zero diagonal (since $u_1^\top S = 0$). Moreover, $B$ has smallest eigenvalue $\mu_\chi \leq \lambda_{n-\chi+1}$, because

$$\mu_\chi \leq \frac{d^\top B d}{d^\top d} = \frac{s^\top A' s}{s^\top s} \leq \lambda_{n-\chi+1} \ .$$

So interlacing gives

$$0 = \text{trace}(B) = \mu_1 + \ldots + \mu_\chi \leq \lambda_{n-\chi+1} + (\chi - 1)\lambda_2 \ .$$

Since $\lambda_2 > 0$, (ii) follows. □

The first inequality is due to Hoffman [25]. The proof given here seems to be due to the author [20] and is a customary illustration of interlacing, see for example Lovász [30] (problem 11.21) or Godsil [19] (p.84). In [21] more inequalities of the above kind are given. But only the two treated here turned out to be useful. The condition $\lambda_2 > 0$ is not strong; only the complete multipartite graphs, possibly extended with some isolated vertices have $\lambda_2 \leq 0$. The second inequality looks a bit awkward, but can be made more explicit if the smallest eigenvalue $\lambda_n$ has large multiplicity $m_n$, say. Then (ii) yields $\chi \geq \min\{1 + m_n, 1 - \frac{\lambda_n}{\lambda_2}\}$ (indeed, if $\chi \leq m_n$, then $\lambda_n = \lambda_{n-\chi+1}$, hence $\chi \geq 1 - \frac{\lambda_n}{\lambda_2}$). For strongly regular graphs with $\lambda_2 > 0$ it is shown in [21], by use of the absolute bound (Theorem 2.6), that the minimum is always taken by $1 - \frac{\lambda_n}{\lambda_2}$, except for the pentagon. So we have

**Corollary 3.3** *If $G$ is a strongly regular graph, not the pentagon or a complete multipartite graph, then*

$$\chi(G) \geq 1 - \frac{\lambda_n}{\lambda_2} \ .$$

For example if $G$ is the complement of the triangular graph $T(m)$ then $G$ is strongly regular with eigenvalues $\lambda_1 = \frac{1}{2}(m-2)(m-3)$, $\lambda_2 = 1$ and $\lambda_n = 3 - m$ (for $m \geq 4$). The above bound gives $\chi(G) \geq m-2$, which is tight, whilst Hoffman's lower bound (Theorem 3.5(i)) equals $\frac{1}{2}m$. On the other hand, if $m$ is even, Hoffman's bound is tight for the complement of $G$ whilst the above bound is much less.

# 4 The (81,20,1,6) strongly regular graph

Let $\Gamma = (X, E)$ be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$. Then $\Gamma$ (that is, its adjacency matrix) has spectrum $\{20^1, 2^{60}, -7^{20}\}$, where the exponents denote multiplicities. We will show that up to isomorphism there is a unique such graph $\Gamma$. More generally we give a short proof for the fact (due to Ivanov & Shpectorov [28] that a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2+1)(q-1), q-2, q(q-1))$ that is the collinearity graph of a partial quadrangle (that is, in which all maximal cliques have size $q$) is the second subconstituent of the collinearity graph of a generalized quadrangle $GQ(q, q^2)$. In the special case $q = 3$ this will imply our previous claim, since $\lambda = 1$ implies that all maximal cliques have size 3, and it is known (see Cameron, Goethals & Seidel [12]) that there is a unique generalized quadrangle $GQ(3, 9)$ (and this generalized quadrangle has an automorphism group transitive on the points).

Let us first give a few descriptions of our graph on 81 vertices.

A. Let $X$ be the point set of $AG(4, 3)$, the 4-dimensional affine space over $\mathbf{F}_3$, and join two points when the line connecting them hits the hyperplane at infinity (a $PG(3, 3)$) in a fixed elliptic quadric $Q$. This description shows immediately that $v = 81$ and $k = 20$ (since $|Q| = 10$). Also $\lambda = 1$ since no line meets $Q$ in more than two points, so that the affine lines are the only triangles. Finally $\mu = 6$, since a point outside $Q$ in $PG(3, 3)$ lies on 4 tangents, 3 secants and 6 exterior lines with respect to $Q$, and each secant contributes 2 to $\mu$. We find that the group of automorphisms contains $G = 3^4 \cdot PGO_4^- \cdot 2$, where the last factor 2 accounts for the linear transformations that do not preserve the quadratic form $Q$, but multiply it by a constant. In fact this is the full group, as will be clear from the uniqueness proof.

B. A more symmetric form of this construction is found by starting with $X = \mathbf{1}^\perp / \langle \mathbf{1} \rangle$ in $\mathbf{F}_3^6$ provided with the standard bilinear form. The corresponding quadratic form ($Q(x) = \mathrm{wt}(x)$, the number of nonzero coordinates of $x$) is elliptic, and if we join two vertices $x+\langle \mathbf{1} \rangle$, $y+\langle \mathbf{1} \rangle$ of $X$ when $Q(x-y)=0$, i.e., when their difference has weight 3, we find the same graph as under A. This construction shows that the automorphism group contains $G = 3^4 \cdot (2 \times \mathrm{Sym}\ (6)) \cdot 2$, and again this is the full group.

C. There is a unique strongly regular graph $G$ with parameters $(112, 30, 2, 10)$, the collinearity graph of the unique generalized quadrangle with parameters $GQ(3, 9)$. Its second subconstituent is an $(81, 20, 1, 6)$ strongly regular graph, and hence isomorphic to our graph $\Gamma$. (See Cameron, Goethals & Seidel [12].) We find that Aut $\Gamma$ contains (and in fact it equals) the point stabilizer in $U_4(3) \cdot D_8$ acting on $GQ(3, 9)$.

D. The graph $\Gamma$ is the coset graph of the truncated ternary Golay code $C$: take the $3^4$ cosets of $C$ and join two cosets when they contain vectors differing in only one place.

E. The graph $\Gamma$ is the Hermitean forms graph on $\mathbf{F}_9^2$; more generally, take the $q^4$ matrices $M$ over $\mathbf{F}_{q^2}$ satisfying $M^\top = \overline{M}$, where $^-$ denotes the field automorphism $x \to x^q$ (applied entrywise), and join two matrices when their difference has rank 1. This will give us a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2+1)(q-1), q-2, q(q-1))$.

F. The graph $\Gamma$ is the graph with vertex set $\mathbf{F}_{81}$, where two vertices are joined when their difference is a fourth power. (This construction was given by Van Lint & Schrijver [29].)

Now let us embark upon the uniqueness proof. Let $\Gamma = (X, E)$ be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2+1)(q-1), q-2, q(q-1))$ and assume that all maximal cliques (we shall just call them lines) of $\Gamma$ have size $q$. Let $\Gamma$ have adjacency matrix $A$. Using the spectrum of $A$ - it is $\{k^1, (q-1)^f, (q-1-q^2)^g\}$, where $f = q(q-1)(q^2+1)$ and $g = (q-1)(q^2+1)$ - we can obtain some structure information. Let $\mathbf{T}$ be the collection of subsets of $X$ of cardinality $q^3$ inducing a subgraph that is regular of degree $q-1$.

1. **Claim.** *If $T \in \mathbf{T}$, then each point of $X \setminus T$ is adjacent to $q^2$ points of $T$.*
   Look at the matrix $B$ of average row sums of $A$, with sets of rows and columns partitioned according to $\{T, X \setminus T\}$. We have

$$B = \begin{bmatrix} q-1 & q^2(q-1) \\ q^2 & k-q^2 \end{bmatrix}$$

with eigenvalues $k$ and $q-1-q^2$, so interlacing is tight, and by Corollary 3.2(ii) it follows that the row sums are constant in each block of $A$.

2. **Claim.** *Given a line $L$, there is a unique $T_L \in \mathbf{T}$ containing $L$.*
   Let $Z$ be the set of vertices in $X \setminus L$ without a neighbour in $L$. Then $|Z| = q^4 - q - q(k-q+1) = q^3 - q$. Let $T = L \cup Z$. Each vertex of $Z$ is adjacent to $q\mu = q^2(q-1)$ vertices with a neighbour in $L$, so $T$ induces a subgraph that is regular of degree $q-1$.

3. **Claim.** *If $T \in \mathbf{T}$ and $x \in X \setminus T$, then $x$ is on at least one line $L$ disjoint from $T$, and $T_L$ is disjoint from $T$ for any such line $L$.*
   The point $x$ is on $q^2+1$ lines, but has only $q^2$ neighbours in $T$. Each point of $L$ has $q^2$ neighbours in $T$, so each point of $T$ has a neighbour on $L$ and hence is not in $T_L$.

4. **Claim.** *Any $T \in \mathbf{T}$ induces a subgraph $\Delta$ isomorphic to $q^2K_q$.*
   It suffices to show that the multiplicity $m$ of the eigenvalue $q-1$ of $\Delta$ is (at least) $q^2$ (it cannot be more). By interlacing we find $m >= q^2 - q$, so we need some additional work. Let $M := A - (q-1/q^2)J$. Then $M$ has spectrum $\{(q-1)^{f+1}, (q-1-q^2)^g\}$, and we want that $M_T$, the submatrix of $M$ with rows and columns indexed by $T$, has eigenvalue $q-1$ with multiplicity (at least) $q^2-1$, or, equivalently (by Lemma 3.2), that $M_{X\setminus T}$ has eigenvalue $q-1-q^2$ with multiplicity (at least) $q-2$. But for each $U \in \mathbf{T}$ with $U \cap T = \emptyset$ we find an eigenvector $x_U = (2-q)\chi_U + \chi_{X\setminus(T\cup U)}$ of $M_{X\setminus T}$ with eigenvalue $q-1-q^2$. A collection $\{x_U | U \in \mathbf{U}\}$ of such eigenvectors cannot be linearly dependent when $\mathbf{U} = \{U_1, U_2, \ldots\}$ can be ordered such that $U_i \not\subset \bigcup_{j<i} U_j$ and $\bigcup \mathbf{U} \neq X \setminus T$, so we can find (using Claim 3) at least $q-2$ linearly independent such eigenvectors, and we are done.

5. **Claim.** *Any $T \in \mathbf{T}$ determines a unique partition of $X$ into members of $\mathbf{T}$.*
   Indeed, we saw this in the proof of the previous step.

Let $\Pi$ be the collection of partitions of $X$ into members of $\mathbf{T}$. We have $|\mathbf{T}| = q(q^2+1)$ and $|\Pi| = q^2+1$. Construct a generalized quadrangle $GQ(q,q^2)$ with point set $\{\infty\} \cup \mathbf{T} \cup X$ as follows: The $q^2+1$ lines on $\infty$ are $\{\infty\} \cup \pi$ for $\pi \in \Pi$. The $q^2$ remaining lines on each $T \in \mathbf{T}$ are $\{T\} \cup L$ for $L \subset T$. It is completely straightforward to check that we really have a generalized quadrangle $GQ(q,q^2)$.

This reader is mainly a composition of the papers [8], [9], [21] and [22].

# References


[1] Aschbacher, M., The non-existence of rank three permutation groups of degree 3250 and subdegree 57, *J. Algebra* **19** (1971), 538-540.

[2] Bannai E. & T. Ito, On finite Moore graphs, *J. Fac. Sci. Univ. Tokyo, Sect IA* **20** (1973), 191-208.

[3] Bannai, E & T. Ito, *Algebraic Combinatorics I: association Schemes,* Benjamin/Cummings, London, 1984.

[4] Biggs, N.L., *Algebraic Graph Theory,* Cambridge Tracts in Mathematics, Vol 67, Cambridge Univ. Press, Cambridge, 1974.

[5] Bose, R.C. & D.M. Mesner, On linear associative algebras corresponding to association schemes of partially balanced designs, *Ann. Math. Statist.* **30** (1959), 21-38.

[6] Bose, R.C. & T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *J. Am. Statist. Assoc.* **47** (1952), 151-184.

[7] Brouwer, A.E., A.M. Cohen & A. Neumaier, *Distance-Regular Graphs,* Springer-Verlag, 1989.

[8] Brouwer, A.E. & W.H. Haemers, Association Schemes, in: *Handbook of Combinatorics* Eds. R. Graham, M. Grötschel & L. Lovász, Elsevier, Amsterdam, 1995, pp. 747-771.

[9] Brouwer, A.E. & W.H. Haemers, Structure and uniqueness of the $(81, 20, 1, 6)$ strongly regular graph, *Discrete Math.* **106/107** (1992), 77-82.

[10] Brouwer, A.E. & J.H. van Lint, Strongly regular graphs and partial geometries, in: *Enumeration and designs*, Proc. Silver Jubilee Conf. on Combinatorics, Waterloo 1982, eds. D.M. Jackson & S.A. Vanstone, Academic Press, Toronto. pp. 85-122.

[11] Burrow, M. *Representation theory of finite groeps,* Academic press, New York, 1965.

[12] Cameron, P.J., J.-M. Goethals & J.J Seidel, Strongly regular graphs having strongly regular subconstituents, *J. Algebra* **55** (1978), 257-280.

[13] Courant, R. & D. Hilbert, *Methoden der Mathematischen Physik,* Springer, Berlin, 1924.

[14] Cvetković, D.M., Graphs and their spectra, *Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **354-356** (1971), 1-50.

[15] Damerell, R.M., On Moore graphs, *Proc. Cambridge Philos. Soc.* **74** (1973), 227-236.

[16] Delsarte, Ph., An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.* **10** (1973).

[17] Delsarte, Ph., J.-M. Goethals & J.J. Seidel, Bound for systems of lines and Jacobi polynomials, *Philips Res. Rep.* **30** (1975), 91*-105*

[18] Feit, W & G. Higman, The non-existence of certain generalized polygons, *j. Algebra* **1** (1964), 114-131.

[19] Godsil, C.D., *Algebrais Combinatorics,* Chapman and Hall, New York, 1993.

[20] W.H. Haemers, *Eigenvalue methods,* in: *Packing and covering in combinatorics* (A. Schrijver ed.), Math. Centre Tract 106, Mathematical Centre, Amsterdam, 1979, pp. 15-38.

[21] Haemers, W.H. *Eigenvalue techniques in design and graph theory* (thesis Technical University Eindhoven 1979), Math. Centre Tract 121, Mathematical Centre, Amsterdam, 1980.

[22] Haemers, W.H., Interlacing eigenvalues and graphs, *Linear Alg. Appl.* **226-228** (1995), 593-616.

[23] Haemers, W.H. & C. Roos, An inequality for generalized hexagons, *Geom. Dedicata* **10**, 219-222.

[24] Higman, D.G., Partial geometries, generalized quadrangles and strongly regular graphs, in: *Atti Convegno di Geometria, Combinatoria e sue Applicazioni, Perugia 1970,* ed. A. Berlotti (1971) pp. 263-293.

[25] A.J. Hoffman, *On eigenvalues and colourings of graphs,* in: *Graph theory and its applications,* (B. Harris, ed.), Acad. Press, New York, 1970, pp. 79-91.

[26] Hoffman, A.J. & R.R. Singleton, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497-504.

[27] Hubaut, X., Strongly regular graphs, *Discret Math.* **13** (1975), 357-381.

[28] Ivanov, A.A. & S.V. Shpectorov, The association schemes of dual polar spaces of type ${}^2A_{2d-1}(p^f)$ are characterized by their parameters if $d > 3$, *Linear Alg. Appl.* **114** (1989), 113-139.

[29] Lint, J.H. van & A. Schrijver, Constructions of strongly regular graphs, two-weight codes and partial geometries by finite fields, *Combinatorica* **1** (1981), 63-73.

[30] L. Lovász, *Combinatorial problems and exercises,* North Holland, Amsterdam, 1979.

[31] Marcus, M & H. Minc, *Introduction to linear algebra.* Macmillan, New York, 1965.

[32] Neumaier, A., New inequalities for the parameters of an association scheme, in : *Combinatorics and Graph Theory, Proc. Calcutta 1980, Lecture Notes in Math.* Vol 885, ed. S.B. Rao, Springer, Berlin, pp. 365-367.

[33] Payne, S.E., An inequality for generalized quadrangles, *Proc. Amer. Math. Soc.* **71** (1978), 147-152.

[34] Scot Jr, L.L., A condition on Higman's parameters, *Notices Amer. Math. Soc.* **20**, A-97 (1973) 701-20-45.

[35] Seidel, J.J., Strongly regular graphs, in: *Surveys in Combinatorics, Proc. 7th Bitish Combinatorial Conf.* LMSLN ser. Vol 38, ed. B. Bollobás, Cambridge Univ. Press, Cambridge, 1979, pp.157-180.

[36] Szegö, G., *Orthogonal Polynomials,* AMS Colloq. Publ. **23**, 1939 (1st ed.), 1975 (4th ed.).

[37] Van Maldeghem, H. *Generalized Polygons*, Birkhäuser, Basel, 1998.