Matrix techniques for strongly regular graphs and related geometries

Supplement to the notes of W. Haemers

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1 Eigenvalues of a regular graph

We give the proof of a result mentioned on page 1, line -5 in [3].

Theorem 1.1 *Let A be the adjacency matrix of a regular graph of degree k, and let ρ be an eigenvalue of A. Then $|\rho| \leq k$. If $\rho = k$, then the corresponding eigenvector is the all-one vector $\vec{1}$, or the graph is disconnected.*

Proof. Suppose that $\{1, \ldots, v\}$ is the vertex set of the graph, and let

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_v \end{bmatrix}$$

be an eigenvector corresponding to the eigenvector ρ; w.l.o.g. we may assume that

$$\max_{i \in \{1, \ldots, v\}} |u_i| = 1$$

(otherwise we divide the vector by an appropriate scalar), so w.l.o.g. we have $u_j = 1$ for a certain $j \in \{1, \ldots, v\}$. The absolute value $|(A\vec{u})_j|$ of the j-th component of $A\vec{u}$ is at most $\sum_{i \sim j} |u_i|$; since the absolute values of all components of $\vec{u}$ are less than or equal to 1, we have $\sum_{i \sim j} |u_i| \leq k$. On the other hand $|(A\vec{u})_j|$ must be equal to $|\rho u_j| = |\rho|$, from which we obtain $|\rho| \leq k$.
If $\rho = k$, then we have $\sum_{i \sim j} u_i = k u_j = k$, so $u_i = 1$ for all vertices i which are adjacent

to j. We can repeat the procedure for all these i; in the case of a connected graph we find, after a finite number of steps, that $\vec{u} = \vec{1}$. $\square$

Exercise
Let G be a regular graph of degree k and with adjacency matrix A. Prove that

1. the multiplicity of the eigenvalue k is exactly the number of connected components of G.

2. $-k$ is an eigenvalue of A if and only if G is bipartite.

The following lemma will be used throughout these notes.

Lemma 1.2 *The $(v \times v)$-matrix J with all entries equal to 1 has eigenvalues 0 and v, with multiplicities respectively $v-1$ and 1. The $(v \times v)$-identity matrix I had eigenvalue 1 with multiplicity v. All $(1 \times v)$-vectors are eigenvectors of I.*

2 The friendship property; polarities in projective planes

A special case of the property $\lambda = \mu$ in a strongly regular graph, mentioned on page 2, line -5 in [3], is $\lambda = \mu = 1$. A graph with this property is said to have the **friendship property**: each pair of vertices has exactly one common neighbour. We will now determine all graphs with this property.

Theorem 2.1 *The only regular graph having the friendship property is the triangle.*

Proof. Suppose that G is a graph with the friendship property, that A is its adjacency matrix and that it is regular of degree k. The diagonal entries of $A^2 = AA^T$ are all equal to k because of the regularity of G; the off-diagonal entries of A^2 are all equal to 1 because of the friendship property. This yields

$$A^2 - (k-1)I = J.$$

Hence, an eigenvalue ρ of A different from k must satisfy $\rho^2 = k - 1$ (see lemma 1.2), so $\rho = \pm\sqrt{k-1}$. Now we recall that k has multiplicity 1, and that A has a zero diagonal and hence trace equal to zero; if f denotes the multiplicity of $\sqrt{k-1}$, we find

$f = \frac{1}{2}\left(v - 1 - \frac{k}{\sqrt{k-1}}\right)$. It is clear that f can only be an integer if $k = 2$, and that this value of k corresponds to the triangle. $\square$

We can consider the adjacency matrix A of a graph G as the incidence matrix of a (necessarily symmetric) design. The symmetry of the matrix learns us that this design has a polarity, which has no absolute points as the diagonal of A is zero. If G is regular of degree k and satisfies the friendship property, then two points of the design are contained in exactly one block and two blocks intersect in exactly one point; each block contains exactly k points and each point is contained in exactly k blocks. As a consequence, the design must be a projective plane of order $k-1$. We just proved, however, that the only possible value for k is 2; the corresponding projective plane is the triangle and hence is degenerate (in a non-degenerate projective plane there exist four points of which no three are collinear). Thus we have proved the following

Corollary 2.2 *A polarity of a non-degenerate projective plane has at least one absolute point.*

We can also determine the maximal number of absolute points of a polarity of a projective plane using matrix techniques.

Theorem 2.3 *A polarity in a projective plane of order $t \geq 2$ has at most $t\sqrt{t}+1$ absolute points.*

Proof. Let A be the incidence matrix of a projective plane of order $t \geq 2$ with a polarity; it immediately follows that A is symmetric. Since every pair of points determines a unique line, we have $A^2 = J + tI$, so the eigenvalues of A are $t+1$, $\sqrt{t}$ and $-\sqrt{t}$. The number of absolute points is precisely the number of ones on the diagonal of A.
First we suppose that all points are absolute. If an off-diagonal element a_{ij}, $i \neq j$, of A were equal to 1, then we would have $a_{ii} = a_{ij} = a_{ji} = a_{jj} = 1$, which would mean that there are two different lines (labelled by i and j) containing a pair of points (labelled by i and j), clearly a contradiction. Hence A must be the identity matrix, but this contradicts $t \geq 2$. We conclude that not all points are absolute.
An appropriate permutation yields the following form for the matrix A:

$$A = \begin{bmatrix} I & C \\ C^T & D \end{bmatrix},$$

where I is an identity matrix of size c, say, and D is a matrix with zero diagonal; note that c is the number of absolute points of the polarity. The quotient matrix B which corresponds to this partition (see page 13 in [3]) is

$$B = \begin{bmatrix} 1 & t \\ \frac{tc}{t^2+t+1-c} & t+1-\frac{tc}{t^2+t+1-c} \end{bmatrix}$$

(tc is the number of ones in the submatrix C (or C^T), so $\frac{tc}{t^2+t+1-c}$ is the average row sum in C^T). One easily calculates that the eigenvalues of B are $\mu_1 = t+1$ and $\mu_2 = 1 - \frac{tc}{t^2+t+1-c}$; the theory of interlacing learns us that the smallest eigenvalue μ_2 of B must be greater than or equal to the smallest eigenvalue of A. This yields

$$c \leq t\sqrt{t} + 1.$$

□

Now we deal with a situation in which this bound is met. Let φ be a Hermitian polarity in a projective plane of square order t, and let $\mathcal{U}$ be the set of absolute points of φ; then $|\mathcal{U}| = t\sqrt{t}+1$. We consider the incidence structure $\mathcal{H}$ consisting of all absolute points and all non-absolute lines of φ, which corresponds to the submatrix C of A. A closer look at the matrices A, B and C tells us that each two points of $\mathcal{H}$ are contained in exactly one line of $\mathcal{H}$, that each line of $\mathcal{H}$ contains $\sqrt{t}+1$ points of $\mathcal{H}$ and that each point of $\mathcal{H}$ is contained in t lines of $\mathcal{H}$. Consequently $\mathcal{H}$ is a $2-(t\sqrt{t}+1, \sqrt{t}+1, 1)$-design, i.e. a unital.

3 The line graph of a graph

Let G be a regular connected graph. The **line graph** of G is by definition the graph G' whose vertices are the edges of G; two edges of G are adjacent if they have a vertex in common. We will investigate whether G' can be a strongly regular graph.

Theorem 3.1 *Let G be a regular connected graph of degree k. Then the line graph of G is strongly regular if and only if one of the following holds:*

1. *G is the pentagon*
2. *G is the complete bipartite graph $K_{k,k}$*
3. *G is the complete graph K_{k+1}*

Proof. Let A be the adjacency matrix of G, let C be the adjacency matrix of the line graph G' of G and let N be the **vertex-edge incidence matrix** of G. It is clear that

$$\begin{aligned} NN^T &= A + kI, \\ N^T N &= C + 2I; \end{aligned}$$

as a consequence, an eigenvalue λ of A corresponds to an eigenvalue $\lambda + k - 2$ of C. Furthermore an easy counting argument shows that, if $k > 2$, the number of edges of G is greater than the number of vertices of G.
If $k = 1$, we see that G is the union of disjoint edges, so the line graph G' is void (and hence not strongly regular). If $k = 2$, G is a polygon and G' is isomorphic to G; theorem 1.3 in [3] learns us that the pentagon is the only polygon which is a non-trivial strongly regular graph.
¿From now on we may assume $k > 2$, so the matrix N has more columns than rows. As a consequence the matrix $N^T N$ has an eigenvalue 0, and C has an eigenvalue -2. Since k is an eigenvalue of A, $2k - 2$ is an eigenvalue of C; an eigenvalue $\lambda \neq k$ of A yields an eigenvalue $\lambda + k - 2 \neq 2k - 2$ of C. Now we have to consider two cases:

1. If $-k$ is an eigenvalue of A, then G is bipartite; it is easily proved that the multiplicity of $-k$ is 1. That means that A has an eigenvalue $\lambda \neq \pm k$ with multiplicity $v - 2$, where v is the number of vertices of G. By the fact that every vector orthogonal to the eigenvectors corresponding to the eigenvalues k and $-k$ must be an eigenvector corresponding to the eigenvalue λ, it is easily seen that λ must be 0. This implies that G is the complete bipartite graph $K_{k,k}$. So we see that C has eigenvalues $2k - 2$, $k - 2$ and -2, and that G' is isomorphic to the $k \times k$ lattice graph L_k, an srg $(k^2, 2k - 2, k - 2, 2)$ which is uniquely determined by its parameters if $k \neq 4$ (see [2]).

2. If $-k$ is not an eigenvalue of A, then G is not bipartite and A has precisely two distinct eigenvalues k and λ, where $|\lambda| \neq k$. It can easily be proved (by considering the trace of A) that λ must be equal to -1; this implies that G is the complete graph K_{k+1}. Therefore C has eigenvalues $2k - 2$, $k - 3$ and -2 and G' is the triangular graph $T(k + 1)$, an srg $\left(\frac{k(k+1)}{2}, 2k - 2, k - 1, 4\right)$ which is uniquely determined by its parameters if $k \neq 7$ (see [2]).

□

4 (Partial) linear spaces and their point and line graphs.

Definition
An incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ a set of points, $\mathcal{L}$ a set of lines and I an incidence relation, is a **partial linear space** of order (s, t) if the following axioms are satisfied:

1. Each line is incident with $s + 1$ points;

2. Each point is incident with $t + 1$ lines;

3. Every two different points are incident with at most one line.

A partial linear space is called a **linear space** if every two different points of $\mathcal{P}$ are incident with exactly one line of $\mathcal{L}$.

Notation
The number of points of a partial linear space is denoted by v, the number of lines is denoted by b.

Definition
A **partial geometry** $\mathrm{pg}(s, t, \alpha)$ is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ that satisfies the following axiom:

4. If $p \in \mathcal{P}$, $L \in \mathcal{L}$ then there are α elements $L_1, \dots, L_\alpha$ of $\mathcal{L}$ and α elements $p_1, \dots, p_\alpha$ of $\mathcal{P}$ such that $p \mathrel{\mathrm{I}} L_i$, $p_i \mathrel{\mathrm{I}} L_i$ and $p_i \mathrel{\mathrm{I}} L$ (for $1 \leq i \leq \alpha$).

A **generalized quadrangle** is a $\mathrm{pg}(s, t, 1)$.

Lemma 4.1 (De Bruijn-Erdős) *Let $\mathcal{S}$ be a linear space. Assume that there exists no line of $\mathcal{S}$ that contains all points of $\mathcal{S}$. Then $b \geq v$.*

Proof. Let N be the incidence matrix of the linear space $\mathcal{S}$. Consider the matrix NN^T. The non-diagonal elements of this matrix are all equal to 1, because of the definition of a linear space. Hence

$$NN^T = J + D \tag{1}$$

with D a diagonal matrix. If there is a point of $\mathcal{S}$ that lies on exactly one line of $\mathcal{S}$, then we easily see that $\mathcal{S}$ contains only that line and that all the points of $\mathcal{S}$ are incident with this line. This is in contradiction with the assumptions. So we know that through every point of $\mathcal{S}$ there are at least two lines. ¿From (1) it follows that all the diagonal entries of the matrix D are greater than 0. So the matrix D is positive definite. The matrix J is a positive semidefinite matrix. Furthermore the sum of a positive definite and a positive semi-definite matrix is positive definite. This proves that $J+D$ is positive definite. But then $J+D$ is non-singular and so $J+D$ has rank v. A theorem of linear algebra states that if C is a $(f \times g)$-matrix over the real numbers and D is a $(g \times h)$-matrix over the real numbers, then rank $(C) \geq$ rank (CD). There follows that

$$b \geq \text{rank } (N) \geq \text{rank } (NN^T) = v.$$

This proves the lemma. □

Definition
Let $\mathcal{S}$ be a partial linear space. The **point graph** of $\mathcal{S}$ is the graph with vertices the points of $\mathcal{S}$, two different vertices are adjacent whenever they are collinear. The **line graph** of $\mathcal{S}$ is the graph with vertices the lines of $\mathcal{S}$, two different vertices are adjacent whenever they are concurrent.

Let N be the incidence matrix of a partial linear space $\mathcal{S}$ of order (s,t), let A be the adjacency matrix of the point graph of $\mathcal{S}$ and let C be the adjacency matrix of the line graph of $\mathcal{S}$. Then the following relations hold:

$$NN^T = A + (t+1)I \qquad (2)$$

$$N^TN = C + (s+1)I \qquad (3)$$

Lemma 4.2 *The matrices NN^T and N^TN have the same non-zero eigenvalues (with the same multiplicities).*

Proof. Let $\vec{u}$ be an eigenvector of N^TN corresponding to the eigenvalue $\lambda \neq 0$. Then $N^TN\vec{u} = \lambda\vec{u}$. Multiplying both sides of this equation on the left by N yields $NN^TN\vec{u} = N\lambda\vec{u} = \lambda(N\vec{u})$. Suppose that $N\vec{u} = \vec{0}$; if we multiply this equation on the left by N^T we obtain $N^TN\vec{u} = \lambda\vec{u} = \vec{0}$, a contradiction since $\lambda \neq 0$. So $N\vec{u}$ is a (non-zero) eigenvector of the matrix NN^T corresponding to the eigenvalue λ. Analogously we find for each eigenvector of NN^T a eigenvector of N^TN corresponding to the same non-zero eigenvalue. Thus we have proved that NN^T and N^TN have the same eigenvalues. Now let λ be a non-zero eigenvalue of N^TN with multiplicity m, and let $\{u_1, \ldots, u_m\}$ be a basis for the space of eigenvectors corresponding to λ. Then

$\{Nu_1, \ldots, Nu_m\}$ is a set of eigenvectors of NN^T corresponding to λ; it is easily seen that this set must be linearly independent (otherwise $\lambda = 0$), and that every eigenvector of NN^T corresponding to λ can be represented as a linear combination of elements of $\{Nu_1, \ldots, Nu_m\}$. As a consequence the non-zero eigenvalues of NN^T and N^TN have the same multiplicities. $\square$

We consider a partial linear space $\mathcal{S}$ with point graph G and line graph G'; we will investigate whether G and G' can both be strongly regular.

Theorem 4.3 *Let $\mathcal{S}$ be a partial linear space with incidence matrix N, point graph G and line graph G'; suppose that G and G' are strongly regular graphs. Then one of the following holds:*

1. *N is a square matrix of full rank, $s = t$, and G and G' have the same parameters.*
2. *$\mathcal{S}$ is a partial geometry.*

Proof. Let A (respectively C) be the adjacency matrix of G (respectively G'). By relations (2) and (3) and lemma 4.2 an eigenvalue λ of A corresponds to an eigenvalue $\lambda + t - s$ of C. A necessary and sufficient condition for G and G' to be strongly regular is that A and C have exactly two restricted eigenvalues. Two distinct situations occur:

1. Neither NN^T nor N^TN has an eigenvalue 0
   Then N must be a square matrix of full rank, and consequently $s = t$. This also means that A and C have the same eigenvalues (with the same multiplicities), so G and G' are strongly regular graphs with the same parameters.

2. NN^T or N^TN has an eigenvalue 0
   Then both NN^T and N^TN must have an eigenvalue 0, otherwise either A or C would have an extra eigenvalue. As a consequence N cannot have full rank, or equivalently, must have an eigenvalue 0. As NN^T has an eigenvalue 0, A has an eigenvalue $-t-1$; furthermore we know that G is regular of degree $k := (t+1)s$. If l denotes the smallest eigenvalue of A, we have $|l| \geq t+1$; from subsection 3.3 in [3] it follows that the maximal number of vertices in a clique of G is

$$\begin{aligned} 1 - \frac{k}{l} &= 1 + \frac{k}{|l|} \\ &\leq 1 + \frac{k}{t+1} \\ &= s + 1. \end{aligned}$$

As there obviously exist cliques of size $s+1$ in G, namely the lines of the partial linear space, we find that $l = -t-1$ and that the lines of the partial linear space are maximal cliques. Subsection 3.3 in [3] implies that there exists a constant α such that each vertex outside a maximal clique in G is adjacent to exactly α vertices of the clique, i.e. our partial linear space is a partial geometry.

□

If $\mathcal{S}$ is a partial geometry $\mathrm{pg}(s,t,\alpha)$, it is clear that (in the notation of theorem 4.3) G and G' are strongly regular, that A has eigenvalue $-t-1$ and that C has eigenvalue $-s-1$. Consequently NN^T and N^TN have an eigenvalue 0, so N has no full rank. Thus we have proved the following

Corollary 4.4 *Let $\mathcal{S}$ be a partial linear space with incidence matrix N, point graph G and line graph G'; suppose that G and G' are strongly regular graphs. Then $\mathcal{S}$ is a partial geometry if and only if N has no full rank.*

Example
An example of case 1 in theorem 4.3 is obtained by considering the adjacency matrix of an $\mathrm{srg}\,(v,k,0,1)$ (see theorem 1.3 in [3]) as the incidence matrix of an incidence structure. One easily sees that this incidence structure must be a partial linear space, that the matrix is square and that it has full rank (because it has no eigenvalue 0).

Remark
We again use the notation of theorem 4.3. The α-property of a partial geometry $\mathrm{pg}(s,t,\alpha)$ can be expressed in matrix form by considering the product AN: if the point labelled by i is incident with the line labelled by j, then $(AN)_{ij}$ is equal to s, if not, then $(AN)_{ij}$ is equal to α. Thus we find $AN = sN + \alpha(J-N)$; using relation 2 we obtain

$$NN^TN = (t+s+1-\alpha)N + \alpha J. \tag{4}$$

Equation (4) can be multiplied on the right (respectively left) with N^T to prove that G (respectively G') is a strongly regular graph.

In a partial geometry $\mathrm{pg}(s,t,\alpha)$ with $\alpha < s+1$ a pair of points does not necessarily determine a line. We will try to add lines to the partial geometry such that every two points become collinear, or equivalently, such that we obtain a 2-design. If it is possible to find such lines, then we say that the partial geometry is **embeddable** in a 2-design.

The following theorem (see [1]) gives a necessary condition for the parameters of a partial geometry to be embeddable in a 2-design. To prove this theorem, we need the following lemma:

Lemma 4.5 *Let $\mathcal{S}$ be a partial linear space with incidence matrix N and adjacency matrix A. Suppose that the point graph G of $\mathcal{S}$ is a strongly regular graph. Then one of the following holds:*

1. *$\mathcal{S}$ is a partial geometry;*
2. *$b \geq v$.*

If $b = v$ then $\det(A + (t+1)I)$ *is a square.*

Proof. Suppose first that $b = v$. In this case the matrix N is symmetric. ¿From (2) we know that $NN^T = A + (t+1)I$. So $\det(A + (t+1)I) = (\det N)^2$. This proves that $\det(A + (t+1)I)$ is a square.
Next, let $b < v$. In the same way as in the proof of lemma 4.1, we find that $v > b \geq \text{rank}\,(N) \geq \text{rank}\,(NN^T)$. So NN^T has rank less than v. From (2) there follows in this case that A must have an eigenvalue $-t-1$. But then $-t-1$ should be a solution of the eigenvalue equation of A, which is equal to $X^2 + (\mu - \lambda)X + \mu - k = 0$ since by assumption G is a strongly regular graph. So we proved that

$$(t+1)^2 - (\mu - \lambda)(t+1) + \mu - k = 0. \qquad (5)$$

¿From (5) we see that $t+1 \mid \mu$ (since $k = (t+1)s$). So $\mu = (t+1)\alpha$ for some nonnegative integer α. Substituting this value in (5), we get $\lambda = s - 1 + t(\alpha - 1)$. So we see that G has the parameters of the point graph of a $\text{pg}(s, t, \alpha)$. Since the lines are maximal cliques, our partial linear space was in fact a partial geometry. □

Theorem 4.6 *Suppose the partial geometry $\mathcal{S}$ is embeddable in a 2-design and suppose that $\mathcal{S}$ is not a 2-design $\mathcal{D}$ (i.e. $\alpha \neq s+1$). Let G be the point graph of $\mathcal{S}$. Then either $\alpha = t$ and the coclique size of G is equal to the clique size of G, or*

$$\alpha \leq \frac{t(s+1)}{s+t+1}.$$

Proof. We have to add new lines to the partial geometry $\mathcal{S}$ until every two points of $\mathcal{S}$ are exactly on one line. Suppose that the number of lines of $\mathcal{S}$ is equal to b' and the number of lines of $\mathcal{D}$ is b. The number of points of $\mathcal{S}$ (and so also of $\mathcal{D}$) is v.

We now apply the previous lemma to the noncollinearity graph $G^{'}$ of $\mathcal{S}$ (this is the graph with vertices the points of $\mathcal{S}$, two different vertices are adjacent whenever they are not collinear in $\mathcal{S}$). It is clear that this graph is strongly regular since it is the complement of the point graph of $\mathcal{S}$ which is a strongly regular graph. So the lemma tells us that either $b - b^{'} \geq v$ or $G^{'}$ is the point graph of a partial geometry. In the first case, after some calculation the inequality reduces to $\alpha(s+t+1) \leq t(s+1)$ (it is easy to check that the number of lines b of $\mathcal{D}$ is equal to $\frac{(st+\alpha)(st+t+\alpha)}{\alpha^2}$). In the second case, remark first that the lines we want to add should be cocliques of the point graph of $\mathcal{S}$ since otherwise there exist points of $\mathcal{S}$ that are on a line of $\mathcal{S}$ and on a new line, so they are on two different lines of the 2-design $\mathcal{D}$, a contradiction. Since $G^{'}$ is the point graph of a partial geometry, the new lines should be maximal cliques of $G^{'}$. This means that the new lines are maximal cocliques of G. So the maximal size of a coclique of G must be equal to the maximal size of a clique of G. It is easy to prove that in this case $t = \alpha$ (see [3]). □

5 Pseudo-geometric graphs.

Definition
A strongly regular graph Γ is called **pseudo-geometric** if its parameters have the following form:

$$v = (s+1)\left(\frac{st+\alpha}{\alpha}\right) \qquad k = s(t+1)$$
$$\lambda = t(\alpha-1)+s-1 \qquad \mu = \alpha(t+1)$$

These are exactly the parameters of the point graph of a partial geometry $pg(s,t,\alpha)$. We call this partial geometry (if it exists or not) the partial geometry corresponding to the pseudo-geometric graph.
If Γ is the point graph of at least one partial geometry, then we call Γ *geometric*.

Problem
¿From the definition of a pseudo-geometric graph we know that it is strongly regular. So any pseudo-geometric graph has three eigenvalues, namely $k = s(t+1)$, $-t-1$ and $s-\alpha$. The interesting problem is to find out whether a pseudo-geometric graph indeed comes from a partial geometry or not. To solve this problem, one has to find back the lines of the geometry out of the pseudo-geometric graph. In the graph the lines are cliques of size $s+1$, such that any two adjacent points of the geometry are in exactly one of the chosen cliques and such that through every point there are $t+1$ cliques. It

is not necessary to check the last condition of a partial geometry, because the cliques we choose are maximal so we know that every point outside the clique is adjacent to a constant number of points of the clique (see theorem 3.3 in [3]). However, it is not always obvious to find the right set of cliques to form the lines of the partial geometry.

Lemma 5.1 *Let A be the adjacency matrix of a strongly regular graph G. Then the eigenvectors of A different from $r\vec{1} = (1, 1, \ldots, 1)$, r an integer, are the same as the eigenvectors of J corresponding to the eigenvalue 0, with J the matrix with all entries equal to 1.*

Proof. It is easy to see that $AJ = s(t+1)J$ and $JA = s(t+1)J$. Let $\vec{u}$ be an eigenvector of A corresponding to the eigenvalue λ_u. So $A\vec{u} = \lambda_u \vec{u}$. Multiplying both sides of this equality by J we get $JA\vec{u} = J\lambda_u\vec{u}$, and thus $s(t+1)J\vec{u} = \lambda_u J\vec{u}$. This means that $\lambda_u = s(t+1)$ or $J\vec{u} = 0$. However, $\lambda_u \neq s(t+1)$ because $\vec{u}$ is not a multiple of $\vec{1}$. So $J\vec{u} = 0$ and we conclude that $\vec{u}$ is an eigenvector of J corresponding to the eigenvalue 0. $\square$

Theorem 5.2 *Let G be a pseudo-geometric graph. Assume that the partial geometry corresponding to G is a generalized quadrangle $GQ(s,t)$. Then we have*

$$t \leq s^2$$

and if equality holds, then G is geometric.

Proof. Since G is pseudo-geometric, it has exactly three eigenvalues $s(t+1)$, $-t-1$ and $s-1$ with multiplicities respectively $1, \frac{s^2(st+1)}{s+t}$ and $\frac{st(s+1)(t+1)}{s+t}$ (see theorem 1.2 in [3]).
Let J be the $(v \times v)$ matrix with all entries equal to 1 and let I be the $(v \times v)$ identity matrix. Define the matrix E by

$$E = -(s+1)A + (s^2-1)I + J.$$

¿From the eigenvalues of A we can calculate the eigenvalues of E, and their multiplicities follow from the multiplicities of the eigenvalues of A. Namely,

- We know that A has eigenvalue $s(t+1)$ with multiplicity 1. The corresponding eigenvector to this eigenvalue is the all one vector $\vec{1} = (1, 1, \ldots, 1)^T$. From $J\vec{1} = v\vec{1}$ we see that $\vec{1}$ is an eigenvector of the matrix J with corresponding eigenvalue v. From lemma (1.2) it follows that $\vec{1}$ is an eigenvector of the matrix

I with corresponding eigenvalue 1. Now from the definition of the matrix E follows that

$$\begin{aligned} E\vec{1} &= -(s+1)A\vec{1} + (s^2-1)I\vec{1} + J\vec{1} \\ &= -(s+1)s(t+1) + (s^2-1)1 + v \\ &= 0. \end{aligned}$$

This means that $\vec{1}$ is also an eigenvector of the matrix E with multiplicity 1 and corresponding eigenvalue 0;

- Let $\vec{a}$ be an eigenvector of A corresponding to the eigenvalue $s-1$. Then $\vec{a}$ is an eigenvector of I with eigenvalue 1 and an eigenvector of J with eigenvalue 0 (see lemma (1.2) and lemma(5.1)). There follows that

$$\begin{aligned} E\vec{a} &= -(s+1)A\vec{a} + (s^2-1)I\vec{a} + J\vec{a} \\ &= -(s+1)(s-1) + (s^2-1)1 \\ &= 0 \end{aligned}$$

So also each eigenvector of A corresponding to the eigenvalue $s-1$ is an eigenvector of E corresponding to the eigevalue 0;

- For the eigenvalue $-t-1$ of A one uses the same arguement as in the previous cases. We find that each eigenvector of A corresponding to the eigenvalue $-t-1$ of A is an eigenvector of E corresponding to the eigenvalue $(s+1)(s+t)$.

So the eigenvalues of the matrix E are 0 and $(s+1)(s+t)$ with multiplicities respectively $1+\frac{st(s+1)(t+1)}{s+t}$ and $\frac{s^2(st+1)}{s+t}$.
Now assume the matrix A has the following form:

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & a_{11} & a_{12} & \cdots & a_{1,s(t+1)} & a_{1,s(t+1)+1} & & \cdots & a_{1,v} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{s(t+1),1} & a_{s(t+1),2} & \cdots & a_{s(t+1),s(t+1)} & & & \cdots & a_{s(t+1),v} \\ 0 & a_{s(t+1)+1,1)} & & \cdots & & & & \cdots & a_{s(t+1)+1,v} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{v,1} & & \cdots & & & & \cdots & a_{v,v} \end{bmatrix}$$

(if necessary we change the order of the rows and coloms of A to bring it in this form). Define the following $(s(t+1) \times s(t+1))$ submatrix A_{11} of A:

$$\begin{bmatrix} a_{11} & \cdots & a_{1,s(t+1)} \\ \vdots & & \vdots \\ a_{s(t+1),1} & \cdots & a_{s(t+1),s(t+1)} \end{bmatrix}.$$

To the matrix A_{11} there corresponds a subgraph of G. We call this subgraph G_{11}. From the choice of A_{11} we see that G_{11} is the subgraph of G consisting of all the vertices of G adjacent to one given vertex, where two vertices are adjacent in G_{11} if they are adjacent in G. Since the graph G is strongly regular, the graph G_{11} is regular of degree $\lambda = s-1$. So every vertex of G_{11} has $s-1$ neighbours, which implies that every (connected) component of G_{11} has at least s vertices. Furthermore as G_{11} contains $s(t+1)$ vertices, we see that G_{11} has at most $t+1$ components.
Let $E_{11} := -(s+1)A_{11} + (s^2-1)I_{s(t+1),s(t+1)} + J_{s(t+1),s(t+1)}$, with $I_{s(t+1),s(t+1)}$ the $(s(t+1) \times s(t+1))$-identity matrix and $J_{s(t+1),s(t+1)}$ the $(s(t+1) \times s(t+1))$-matrix with all entries equal to 1.
Since the multiplicity of the eigenvalue $(s+1)(s+t)$ of the matrix E is $\frac{s^2(st+1)}{s+t}$, the matrix E_{11} has eigenvalue 0 at least $s(t+1) - \frac{s^2(st+1)}{s+t}$ times. So the matrix A_{11} has eigenvalue $s-1$ at least $1 + s(t+1) - \frac{s^2(st+1)}{s+t}$ times (note that the eigenvalue of A_{11} that corresponds to $\vec{1}$ also equals $s-1$ and it was not included yet).
We already mentioned that the graph G_{11} is regular of degree $\lambda = s-1$. So from part 1 of the exercise in section 1 we know that A_{11} has at least $1 + s(t+1) - \frac{s^2(st+1)}{s+t}$ components. However, we proved above that A_{11} has at most $t+1$ components. So we get the inequality

$$1 + s(t+1) - \frac{s^2(st+1)}{s+t} \leq t+1.$$

After some calculation one finds that $t \leq s^2$.

If $t = s^2$, we see from the proof of the inequality that G_{11} can be partitioned in cliques of size s. So we found the maximal cliques in the graph G that represent the lines of the generalized quadrangle. This implies that the graph G is geometric. □

6 Spreads

If G is a strongly regular graph on v vertices, regular of degree k and with smallest eigenvalue l, we know from subsection 3.3 in [3] that the clique bound is $K := 1 - \frac{k}{l}$ and

that the coclique bound is $\overline{K} := \frac{-vl}{k-l}$. We note that $K\overline{K} = v$. A **spread** in G is defined as a partition of G into maximal cliques, also called **Delsarte cliques**. This, of course, corresponds to a partition of the complement $\overline{G}$ of G into **Delsarte cocliques**, i.e. to a colouring of $\overline{G}$ with $\frac{v}{K} = \overline{K}$ colours. Subsection 3.4 in [3] learns us that this number of colours meets the Hoffman bound. Thus we conclude that a Hoffman colouring of $\overline{G}$ corresponds to a spread in G.
Now we assume that G is the point graph of a partial geometry. If the partial geometry has a spread, then it is clear that G has a spread as well, but the converse is not necessarily true: for instance, the partial geometry $\mathrm{pg}(2, 1, 2)$ (which is indeed unique for these parameters) has no spread, while its point graph has several spreads.
A spread of a partial geometry $\mathrm{pg}(s, t, \alpha)$ corresponds to a coclique of size $\frac{st+\alpha}{\alpha}$ in the line graph; if such a coclique in the line graph exists, then we obtain a spread of the partial geometry.

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