Embeddings of Geometries in Finite Projective Spaces

J.A. THAS and H. VAN MALDEGHEM

1 Definitions

A *lax embedding* of a point-line geometry $\mathcal{S}$ with point set P in a projective space $\mathrm{PG}(d, K)$, $d \geq 2$ and K a (not necessarily commutative) field, is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of $\mathrm{PG}(d, K)$ satisfying

(WE1) the set P^θ generates $\mathrm{PG}(d, K)$.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *laxly embedded* in $\mathrm{PG}(d, K)$.

A *polarized embedding* in $\mathrm{PG}(d, K)$ is a lax embedding which also satisfies

(WE2) for any point x of $\mathcal{S}$, the set $X = \{y^\theta : d(x, y) \text{ is not maximal }\}$, with $d(.,.)$ the distance between points in the point graph of $\mathcal{S}$, does not generate $\mathrm{PG}(d, K)$.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *polarly embedded* in $\mathrm{PG}(d, K)$.

A *flat embedding* in $\mathrm{PG}(d, K)$ is a lax embedding which also satisfies

(WE3) for any point x of $\mathcal{S}$, the set $X = \{y^\theta : y \text{ is collinear with } x\}$ is contained in a plane of $\mathrm{PG}(d, K)$.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *flatly embedded* in $\mathrm{PG}(d, K)$.

A *full embedding* in $\mathrm{PG}(d, K)$ is a lax embedding with the additional property that for every line L of $\mathcal{S}$, all points of $\mathrm{PG}(d, K)$ on the line L^θ have an inverse image under θ. In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *fully embedded* in $\mathrm{PG}(d, K)$. In the case of full embeddings we also speak shortly of *embeddings*.

Usually, we simply say that $\mathcal{S}$ is laxly, or polarly, or flatly or fully embedded in $\mathrm{PG}(d, K)$ without referring to θ, that is, we identify the points and lines of $\mathcal{S}$ with their images in $\mathrm{PG}(d, K)$.

In these lectures we will restrict ourselves to finite fields, that is, $K = \mathrm{GF}(q)$. However, several of the problems are solved also for any commutative field K (in particular for polar spaces), or even any non-commutative field (for instance for generalized quadrangles).

The (finite) geometries we will consider are generalized polygons, polar spaces and partial geometries.

2 Some important finite point-line geometries

2.1 Point-line geometries

2.1.1 Generalized polygons

A *generalized n-gon*, $n \geq 2$, or a *generalized polygon*, is a nonempty point-line geometry the incidence graph of which has diameter n (i.e. any two elements are at most at distance n) and girth $2n$ (i.e., the length of any shortest circuit is $2n$; in particular we assume that there is at least one circuit). A *thick* generalized polygon is a generalized polygon for which each element is incident with at least three elements. In this case, the number of points on a line is a constant, say $s+1$, and the number of lines through a point is also a constant, say $t+1$. The pair (s,t) is called the order of the polygon; if $s=t$ we say that the polygon has *order s*. If for a non-thick generalized polygon the number of points on a line is a constant, and the number of lines through a point is a constant, then we say that the generalized polygon has an *order*.

If $\mathcal{S}$ is a finite thick generalized n-gon, then, by the Theorem of Feit and Higman [1964], we have $n \in \{2,3,4,6,8\}$. The digons ($n=2$) are trivial incidence structures (any point is incident with any line), the thick generalized 3-gons are the projective planes (then necessarily $s=t$), and the generalized 4-gons, 6-gons, 8-gons are also called *generalized quadrangles, generalized hexagons, generalized octagons*, respectively.

There is a point-line duality for generalized polygons for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged.

Generalized polygons were introduced by Tits [1959] in his celebrated paper on triality.

There are some equivalent definitions for generalized polygons. Let us mention a rather geometric one (see Van Maldeghem [1998]).

Let $n \geq 2$ be again a natural number. Then a generalized n-gon may be defined as a geometry $\mathcal{S} = (P, B, \mathtt{I})$ with $P \neq \emptyset, B \neq \emptyset$, such that the following two axioms are satisfied:

(GP1) $\mathcal{S}$ contains no ordinary k-gon (as a subgeometry), for $2 \leq k < n$.
(GP2) Any two elements $x, y \in P \cup B$ are contained in some ordinary n-gon in $\mathcal{S}$, a so-called *apartment*.

A generalized n-gon is thick if and only if it satisfies also the following axiom:

(GP3) there exists an ordinary $(n+1)$-gon in $\mathcal{S}$.

2.1.2 Polar spaces

A (finite) point-line geometry is called a *polar space* if it is a (finite) generalized quadrangle, or if it is isomorphic to the geometry formed by the points and lines of a quadric of rank at least two in $\mathrm{PG}(d,q)$ (that is, the quadric contains lines), or if it is isomorphic to the geometry formed by the points and lines of a hermitian variety of rank at least two in $\mathrm{PG}(d,q^2)$, or if it is isomorphic to the geometry formed by the points of $\mathrm{PG}(d,q)$, $d \geq 2$, together with the totally isotropic lines with respect to some symplectic polarity of $\mathrm{PG}(d,q)$ (that are the lines contained in their image for the polarity). Note that the quadric, hermitian variety or symplectic polarity may be singular.

2.1.3 Partial geometries

A (finite) *partial geometry* is a nonempty point-line geometry $\mathcal{S} = (P, B, \mathtt{I})$, with point set P, line set B, and symmetrized incidence $\mathtt{I} \subseteq (P \times B) \cup (B \times P)$, satisfying the following axioms:

(i) any two distinct points are incident with at most one line and any point is incident with a constant number $t+1$ ($t \geq 1$) of lines;

(ii) any two distinct lines are incident with at most one point and any line is incident with a constant number $s+1$ ($s \geq 1$) of points;

(iii) if x is a point and L is a line not incident with x, then there are exactly α point-line pairs (y_i, M_i), $\alpha \geq 1$, for which

$$x\mathtt{I}M_i\mathtt{I}y_i\mathtt{I}L, \quad i = 1, 2, \ldots, \alpha.$$

The integers s, t, α are the *parameters* of the partial geometry; α is also called the *incidence number* of $\mathcal{S}$. The partial geometry $\mathcal{S}$ is often denoted by $pg(s,t,\alpha)$.

The partial geometries can be divided into four (nondisjoint) classes.

(a) The partial geometries with $\alpha = 1$. That are the generalized quadrangles having a constant number of points on any line and a constant number of lines through every point.

(b) The partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, i.e., the $2-(v, s+1, 1)$ designs and their duals.

(c) The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *Bruck nets of order* $s+1$ *and degree* $t+1$. If $t = s+1$, then $\mathcal{S}$ is an affine plane of order $s+1$.

(d) Finally, the so-called *proper* partial geometries having $1 < \alpha < \min(s,t)$.

Just like for generalized polygons there is a point-line duality for partial geometries. Partial geometries were introduced by Bose [1963].

2.1.4 Scheme

If "$\longrightarrow$" means "generalizes to", then, restricting ourselves to thick geometries, we have the following scheme

$$\begin{array}{ccccc} PS & \longleftarrow & GQ & \longrightarrow & PG \\ & & \downarrow & & \\ & & GP & & \end{array}$$

with

GQ: generalized quadrangle

PG: partial geometry

PS: polar space

GP: generalized polygon

2.2 Examples of generalized polygons and partial geometries

2.2.1 Generalized quadrangles

First we describe the GQ with either $s = 1$ or $t = 1$, then we give a brief description of three families of examples known as the classical GQ, and finally we describe a thick nonclassical example.

(a) <u>The grids and dual grids.</u> Any GQ for which each point is incident with 2 lines is called a *grid*. Up to isomorphism any grid $\mathcal{S} = (P, B, \mathtt{I})$ can be described as follows:
$P = \{x_{ij} : i = 0, 1, \cdots, s_1 \text{ and } j = 0, 1, \cdots, s_2\}$, $s_1 > 0$ and $s_2 > 0$,
$B = \{L_0, L_1, \cdots, L_{s_1}, M_0, M_1, \cdots, M_{s_2}\}$,
$x_{ij} \mathtt{I} L_k$ if and only if $i = k$, and $x_{ij} \mathtt{I} M_k$ if and only if $j = k$.

The dual of a grid is called a *dual grid*. If $\mathcal{S}$ is at the same time grid and dual grid, then it is an ordinary quadrangle; this is the motivation for the term "generalized quadrangle".

(b) The classical generalized quadrangles.

(i) Consider a nonsingular quadric Q of rank 2 of the projective space $\mathrm{PG}(d,q)$, with $d=3,4$ or 5. Then the points of Q together with the lines on Q (which are the subspaces of maximal dimension on Q) form a GQ $Q(d,q)$ with parameters, where $|P|=v$ and $|B|=b$,

$s=q$, $t=1$, $v=(q+1)^2$, $b=2(q+1)$, when $d=3$,
$s=t=q$, $v=b=(q+1)(q^2+1)$, when $d=4$,
$s=q$, $t=q^2$, $v=(q+1)(q^3+1)$, $b=(q^2+1)(q^3+1)$, when $d=5$.

Since $Q(3,q)$ is a grid, its structure is trivial. Further, recall that the quadric Q has the following canonical equation:

$X_0X_1+X_2X_3=0$, when $d=3$,
$X_0^2+X_1X_2+X_3X_4=0$, when $d=4$,
$F(X_0,X_1)+X_2X_3+X_4X_5=0$, where $F(X_0,X_1)$ is an irreducible homogeneous quadratic polynomial over $\mathrm{GF}(q)$, when $d=5$.

Remark. For the quadric and the corresponding GQ we often use the same notation. For $d=3$, so Q is hyperbolic, we also denote the GQ by $Q^+(3,q)$; for $d=5$, so Q is elliptic, we also denote the GQ by $Q^-(5,q)$.

(ii) Let H be a nonsingular hermitian variety of the projective space $\mathrm{PG}(d,q^2)$, $d=3$ or 4. Then the points of H together with the lines on H form a GQ $H(d,q^2)$ with parameters

$s=q^2$, $t=q$, $v=(q^2+1)(q^3+1)$, $b=(q+1)(q^3+1)$, when $d=3$,
$s=q^2$, $t=q^3$, $v=(q^2+1)(q^5+1)$, $b=(q^3+1)(q^5+1)$, when $d=4$.

Recall that H has the canonical equation

$$X_0^{q+1}+X_1^{q+1}+\cdots+X_d^{q+1}=0.$$

Remark. For the hermitian variety and the corresponding GQ we often use the same notation.

(iii) The points of $\mathrm{PG}(3,q)$, together with the totally isotropic lines with respect to some nonsingular symplectic polarity, form a GQ $W(q)$ with parameters

$$s=t=q,\ v=b=(q+1)(q^2+1).$$

Recall that the lines of $W(q)$ are the elements of a nonsingular linear complex of lines of $\mathrm{PG}(3,q)$, and that a nonsingular symplectic polarity of $\mathrm{PG}(3,q)$ has the following canonical bilinear form:

$$X_0Y_1-X_1Y_0+X_2Y_3-X_3Y_2.$$

Remark. For $W(q)$ we also use the notations $W_3(q)$ and $W(3,q)$.

(c) A thick nonclassical example. Let O be a hyperoval, that is, a $(q+2)$-arc, of the projective plane $\mathrm{PG}(2,q)$, $q = 2^h$, $h > 1$, and let $\mathrm{PG}(2,q)$ be embedded as a plane in $\mathrm{PG}(3,q)$. Define an incidence structure $T_2^*(O)$ by taking for points just those points of $\mathrm{PG}(3,q)$ not in $\mathrm{PG}(2,q)$, and for lines just those lines of $\mathrm{PG}(3,q)$ which are not contained in $\mathrm{PG}(2,q)$ and meet O (necessarily in a unique point). The incidence is that inherited from $\mathrm{PG}(3,q)$. The incidence structure so defined is a *GQ* with parameters

$$s = q-1,\ t = q+1,\ v = q^3,\ b = q^2(q+2).$$

(d) The other examples. For the other examples, we refer to Thas [1995]. The order of each known *GQ* (thick or with an order) is one of the following:

$(s,1)$ with $s \geq 1$;
$(1,t)$ with $t \geq 1$;
(q,q) with q a prime power;
(q,q^2), (q^2,q) with q a prime power;
(q^2,q^3), (q^3,q^2) with q a prime power;
$(q-1,q+1)$, $(q+1,q-1)$ with q a prime power.

2.2.2 Generalized hexagons and octagons

All known thick finite generalized hexagons and octagons are classical, i.e., they arise in a natural way from Chevalley groups. The classical generalized hexagons can also be defined in a geometric way; the classical generalized octagons do not yet have a simple geometric description (although there exists an elementary algebraic construction), except the non-thick ones.

Let us start with a description of, in principal, all non-thick finite generalized hexagons and octagons.

The non-thick examples.

Consider any projective plane $\mathcal{S} = (P, B, \mathtt{I})$. We define $P' = \{(x,L) : x \in P, L \in B \text{ and } x\mathtt{I}L\}$, $B' = P \cup B$ and $\mathtt{I}'$ the natural inclusion. Then $\mathcal{S}' = (P', B', \mathtt{I}')$ is a (non-thick) generalized hexagon such that every point is incident with exactly two lines. If $\mathcal{S}$ has order q, then $\mathcal{S}'$ has order $(q,1)$. The dual of $\mathcal{S}'$ is a generalized hexagon of order $(1,q)$ and is sometimes called *the double of* $\mathcal{S}$. The hexagon $\mathcal{S}'$ itself is sometimes called the *flag geometry of* $\mathcal{S}$.

The same construction starting with a generalized quadrangle $\mathcal{S}$ yields a generalized octagon $\mathcal{S}'$. If the quadrangle has order (s,t), then the octagon has an order if and only if $s = t$, in which case the order is $(s,1)$.

In particular, one may start with non-classical projective planes and non-classical generalized quadrangles.

There are also other types of non-thick generalized hexagons and octagons, which generalize the grids and dual grids. Up to duality and isomorphism, we may describe these as follows (e.g. the hexagons):

$P = \{x_{i,j} : i = 0, 1, \cdots, s_1 \text{ and } j = 0, 1, \cdots, s_2\} \cup \{y_i : i = 0, 1, \cdots, s_1\}$, $s_1 > 0$ and $s_2 > 0$,
$B = \{L_{i,j} : i = 0, 1, \cdots, s_1 \text{ and } j = 0, 1, \cdots, s_2\} \cup \{M_j : i = 0, 1, \cdots, s_2\}$,
$x_{ij} \mathtt{I} M_k$ if and only if $j = k$; $x_{ij} \mathtt{I} L_{k,m}$ if and only if $i = k$ and $j = m$; $y_i \mathtt{I} L_{k,m}$ if and only if $i = k$, and y_i is never incident with M_k.

As an exercise, you can try to describe a similar example for octagons.

Let us now continue with a description of one class of thick finite classical generalized hexagons (the construction can easily be generalized to the infinite case taking any infinite commutative field instead of $\mathrm{GF}(q)$).

<u>The split Cayley hexagon</u> $H(q)$.

We consider the quadric $Q(6, q)$ in $\mathrm{PG}(6, q)$ given by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The points of $H(q)$ are all points of $Q(6, q)$. The lines of $H(q)$ are certain lines of $Q(6, q)$, namely, those lines of $Q(6, q)$ whose *Grassmann coordinates* satisfy the equations $p_{01} = p_{36}$, $p_{12} = p_{34}$, $p_{20} = p_{35}$, $p_{03} = p_{56}$, $p_{13} = p_{64}$ and $p_{23} = p_{45}$. The order of $H(q)$ is $s = t = q$.

It is convenient to have the following elementary description of $H(2)$ (see Van Maldeghem [20**]). The points are the points, lines and (unordered) point-line pairs of the Fano plane $\mathrm{PG}(2, 2)$. The lines are of two types: (1) the triples $\{p, L, \{p, L\}\}$, where the point p of $\mathrm{PG}(2, 2)$ is incident with the line L of $\mathrm{PG}(2, 2)$, and (2) the triples $\{\{p, L\}, \{a_1, M_1\}, \{a_2, M_2\}\}$, where the points p, a_1, a_2 are the three different points of $\mathrm{PG}(2, 2)$ incident with L, and, dually, L, M_1, M_2 are the three different lines incident with p in $\mathrm{PG}(2, 2)$.

<u>The twisted triality hexagon</u> $T(q^3, q)$.

We consider the *triality quadric* $Q^+(7, q^3)$ with equation

$$X_0X_4 + X_1X_5 + X_2X_6 + X_3X_7 = 0.$$

We call the point $x(x_0, x_1, \ldots, x_7)$ 3-*conjugate* to $y(y_0, y_1, \ldots, y_7)$ if

$$\begin{cases} x_1y_2^q - x_2y_1^q + x_3y_4^q + x_4y_7^q = 0, \\ x_2y_0^q - x_0y_2^q + x_3y_5^q + x_5y_7^q = 0, \\ x_0y_1^q - x_1y_0^q + x_3y_6^q + x_6y_7^q = 0, \\ \\ x_5y_6^q - x_6y_5^q + x_7y_0^q + x_0y_3^q = 0, \\ x_6y_4^q - x_4y_6^q + x_7y_1^q + x_1y_3^q = 0, \\ x_4y_5^q - x_5y_4^q + x_7y_2^q + x_2y_3^q = 0, \\ \\ x_0y_4^q + x_1y_5^q + x_2y_6^q - x_7y_7^q = 0, \\ x_4y_0^q + x_5y_1^q + x_6y_2^q - x_3y_3^q = 0. \end{cases}$$

Note that this is not a symmetric relation. But it has the following property: if x is 3-conjugate to y and to itself, and if y is 3-conjugate to x and to itself, then the line xy of $\mathrm{PG}(7, q^3)$ belongs to $Q^+(7, q^3)$ and every point of that line is 3-conjugate to every other point of that line and to itself. We call such a line *self*-3-*conjugate*. Also, we call a point x of $Q^+(7, q^3)$ *self*-3-*conjugate* if x is 3-conjugate to itself. The self-3-conjugate points and self-3-conjugate lines of $Q^+(7, q^3)$ now form, with the natural incidence, a generalized hexagon $T(q^3, q)$ of order (q^3, q).

If a point x of $Q^+(7, q^3)$ has coordinates in $\mathrm{GF}(q)$, then it is easily seen that it is self-3-conjugate if and only if it lies in the hyperplane with equation $X_3 + X_7 = 0$. The intersection of that hyperplane with $Q^+(7, q^3)$ and with $\mathrm{PG}(7, q)$ is precisely the parabolic quadric $Q(6, q)$ of the previous subsection. We conclude (although some additional calculations are needed for the proof) that $H(q)$ is a subhexagon of $T(q^3, q)$.

The classical generalized octagons do not have such a description. There is a construction with coordinates (see Joswig and Van Maldeghem [1995]), but we will not give it here, because we will not need it. Let us simply remark that the classical generalized octagons are generally called the Ree-Tits octagons and that they arise from the Ree groups ${}^2F_4(q)$, with $q = 2^{2e+1}$. They have order (q, q^2) and are denoted by $O(q)$.

2.2.3 Partial geometries

(a) Designs and nets. It is easy to construct $2-(v, s+1, 1)$ designs. For example, the points and lines of the projective space $\mathrm{PG}(d, q)$, $d \geq 2$, form a $2-((q^{d+1}-1)/(q-1), q+1, 1)$ design. The points and lines of the affine space $\mathrm{AG}(d, q)$, $d \geq 2$, form a $2-(q^d, q, 1)$ design.

If we delete n classes, $0 \leq n \leq q-1$, of parallel lines of $\mathrm{AG}(2, q)$, then the remaining structure is a net of order q and degree $q+1-n$.

Let P be the set of all points of $\mathrm{PG}(d, q)$ which are not contained in a fixed subspace $\mathrm{PG}(d-2, q)$, with $d \geq 2$. Let B be the set of all lines of $\mathrm{PG}(d, q)$ having no point in common with $\mathrm{PG}(d-2,q)$. Finally, let I be the natural incidence. Then $(P,B,\mathtt{I})$ is a partial geometry with parameters

$$s=q, t=q^{d-1}-1, \alpha=q.$$

This dual net will be denoted by H_q^n.

(b) Maximal arcs and partial geometries. In $\mathrm{PG}(2,q)$ any nonempty set of k points may be described as a $\{k;m\}$-arc, where m ($m \neq 0$) is the greatest number of collinear points in the set. For given q and m ($m \neq 0$), k can never exceed $mq-q+m$, and a $\{mq-q+m;m\}$-arc will be called a *maximal arc*. Equivalently, a maximal arc may be defined as a nonempty set of points of $PG(2,q)$ meeting every line in just m points or in none at all. Trivial maximal arcs are the plane $\mathrm{PG}(2,q)$ ($m=q+1$), the affine plane $\mathrm{AG}(2,q)$ obtained by deleting a line L from $\mathrm{PG}(2,q)$ ($m=q$), and a single point ($m=1$)

If K is a $\{mq-q+m;m\}$-arc (i.e. a maximal arc) of $\mathrm{PG}(2,q)$, where $m \geq q$, then it is easy to show that the set

$$K' = \{\text{lines } L \text{ of } \mathrm{PG}(2,q) : L \cap K = \phi\}$$

is a $\{q(q-m+1)/m;q/m\}$-arc (i.e. maximal arc) of the dual plane. Hence, if the plane $\mathrm{PG}(2,q)$ contains a $\{mq-q+m;m\}$-arc, $m \leq q$, then it also contains a $\{q(q-m+1)/m;q/m\}$-arc. It follows that a necessary condition for the existence of a maximal arc, with $m \leq q$, is that m should be a factor of q.

For any m dividing q, with $q=2^h$, there exists a $\{qm-q+m;m\}$-arc in $\mathrm{PG}(2,q)$, see Denniston [1969] and Thas [1974, 1980]. For q odd, no maximal $\{qm-q+m;m\}$-arc, with $1<m<q$, exists; see Ball, Blokhuis and Mazzocca [1997], and Ball and Blokhuis [1998].

Let K be a maximal $\{qm-q+m;m\}$-arc of $\mathrm{PG}(2,q)$, with $m \geq 2$. Then the points of K together with the nonempty intersections $L \cap K$, where L is any line of $\mathrm{PG}(2,q)$, form a $2-(qm-q+m,m,1)$ design. If $m<q$, then the points of $\mathrm{PG}(2,q) \setminus K$ together with the lines having an empty intersection with K form a dual design with parameters $s=q$, $t=(q/m)-1$, $\alpha=q/m$.

Now we describe two classes of proper partial geometries. Let K be a maximal $\{qm-q+m;m\}$-arc of $\mathrm{PG}(2,q)$, $2 \leq m < q$. Let P be the set of all points of $\mathrm{PG}(2,q) \setminus K$, let B be the set of all lines of $\mathrm{PG}(2,q)$ having a nonempty intersection with K, and let I be the natural incidence. Then $\mathcal{S}(K)=(P,B,\mathtt{I})$ is a partial geometry with parameters

$$s=q-m, t=q(m-1)/m, \alpha=(q-m)(m-1)/m.$$

These examples are due to Thas [1973, 1974] and independently to Wallis [1973].

Now we embed the plane $\mathrm{PG}(2,q)$ in the projective space $\mathrm{PG}(3,q)$. Let $P' = \mathrm{PG}(3,q) \setminus \mathrm{PG}(2,q)$, let B' consist of all lines of $\mathrm{PG}(3,q)$ having a unique point in common with K, and let $\mathtt{I}'$ be the natural incidence. Then $T_2^*(K) = (P', B', \mathtt{I}')$ is a partial geometry with parameters

$$s = q-1, t = (q+1)(m-1), \alpha = m-1.$$

These partial geometries are due to Thas [1973, 1974].

Hence there exist partial geometries with parameters as follows:

(i) $s = 2^h - 2^r$, $t = 2^h - 2^{h-r}$, $\alpha = (2^r-1)(2^{h-r}-1)$, with $1 \leq r < h$;

(ii) $s = 2^h - 1$, $t = (2^h+1)(2^r-1)$, $\alpha = 2^r - 1$, with $1 \leq r < h$.

Such a *PG* has either $\alpha = 1$ or is proper. A *PG* of type (i) is a *GQ* if and only if $h = 2$ and $r = 1$. This gives the following model of the unique *GQ* $\mathcal{S}$ with 15 points and 15 lines: points of $\mathcal{S}$ are the 15 points of $\mathrm{PG}(2,4) \setminus K$ with K a given hyperoval of $\mathrm{PG}(2,4)$, lines of $\mathcal{S}$ are the 15 lines of $\mathrm{PG}(2,4)$ intersecting K in exactly 2 points, and incidence is the natural one. A *PG* of type (ii) is a *GQ* if and only if $r = 1$. In this case K is a hyperoval of $PG(2,q), q = 2^h$, and $T_2^*(K)$ is the *GQ* described in 2.2.1 (c).

(c) The other examples For other examples we refer to De Clerck and Van Maldeghem [1995], Mathon [1998] and Mathon (personal communication, 1999).

The parameters of each known proper *PG* are one of the following:

$s = 2^h - 2^r$, $t = 2^h - 2^{h-r}$, $\alpha = (2^r-1)(2^{h-r}-1)$, with $h \neq 2$ and $1 \leq r < h$;

$s = 2^h - 1$, $t = (2^h+1)(2^r-1)$, $\alpha = 2^r - 1$, with $1 < r < h$;

$s = 2^{2h-1} - 1$, $t = 2^{2h-1}$, $\alpha = 2^{2h-2}$, with $h > 1$;

$s = 3^{2n} - 1$, $t = \frac{1}{2}(3^{4n} - 1)$, $\alpha = \frac{1}{2}(3^{2n} - 1)$;

$s = 26$, $t = 27$, $\alpha = 18$;

$s = t = 5$, $\alpha = 2$;

$s = 4$, $t = 17$, $\alpha = 2$;

$s = 8$, $t = 20$, $\alpha = 2$.

2.3 Some properties

2.3.1 Generalized quadrangles

Theorem 2.1 *If $\mathcal{S} = (P, B, \mathtt{I})$ is a GQ of order (s,t), with $|P| = v$ and $|B| = b$, then*

$$v = (s+1)(st+1) \text{ and } b = (t+1)(st+1).$$

Proof. Let L be a fixed line of $\mathcal{S}$ and count in two different ways the number of ordered pairs $(x, M) \in P \times B$ with $x \not\mathrm{I} L$, $x \mathrm{I} M$ and L, M concurrent. There arises $v - s - 1 = (s+1)ts$ or $v = (s+1)(st+1)$. Dually $b = (t+1)(st+1)$. □

Theorem 2.2 *If $\mathcal{S} = (P, B, \mathrm{I})$ is a GQ of order (s,t), then*

$$s + t \text{ divides } st(s+1)(t+1).$$

Proof. The point graph of $\mathcal{S}$ is strongly regular with parameters $v = (s+1)(st+1)$, $k = (t+1)s$, $\lambda = s - 1$ and $\mu = t + 1$. From the theory of strongly regular graphs now follows that $s + t$ divides $st(s+1)(t+1)$. □

Theorem 2.3 (The inequality of Higman [1974]) *Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (s,t). If $s > 1$ and $t > 1$, then $t \leq s^2$, and dually $s \leq t^2$.*

Proof (Cameron [1975]). If (not necessarily distinct) points x, y are collinear, we will write $x \sim y$; otherwise, we write $x \not\sim y$.

Let x, y be two noncollinear points of $\mathcal{S}$. Put $V = \{z \in P : z \not\sim x \text{ and } z \not\sim y\}$, so $|V| = d = (s+1)(st+1) - 2 - 2(t+1)s + (t+1)$. Denote the elements of V by $z_1, z_2, \ldots, z_d$ and let $t_i = |\{u \in P : u \sim x, u \sim y, u \sim z_i\}|$. Count in two different ways the number of ordered pairs (z_i, u), with $u \sim z_i$, $u \sim x$, $u \sim y$, to obtain

$$\sum_i t_i = (t+1)(t-1)s. \tag{1}$$

Next, count in two different ways the number of ordered triples (z_i, u, u'), with $u \neq u'$, $u \sim z_i$, $u \sim x$, $u \sim y$, and $u' \sim z_i$, $u' \sim x$, $u' \sim y$, to obtain

$$\sum_i t_i(t_i - 1) = (t+1)t(t-1). \tag{2}$$

From (1) and (2) it follows that

$$\sum_i t_i^2 = (t+1)(t-1)(s+t).$$

With $d\bar{t} = \sum_i t_i$, $0 \leq \sum_i (\bar{t} - t_i)^2$ simplifies to $d \sum_i t_i^2 - (\sum_i t_i)^2 \geq 0$, which implies $d(t+1)(t-1)(s+t) \geq (t+1)^2(t-1)^2 s^2$, or $t(s-1)(s^2-t) \geq 0$, completing the proof. □

There is an immediate corollary of the proof.

Corollary 2.4 (Bose and Shrikhande [1972]) *If $s > 1$ and $t > 1$, then $s^2 = t$ if and only if $d \sum t_i^2 - (\sum t_i)^2 = 0$ for any pair $\{x, y\}$ of noncollinear points if and only if $t_i = \bar{t}$ for all $i = 1, 2, \cdots, d$ and for any pair $\{x, y\}$ of noncollinear points if and only if for each triple $\{x, y, x\}$ of pairwise noncollinear points there is a constant number of points collinear with x, y, z, in which case this constant number is $s + 1$.*

Remark 2.5 Higman first obtained the inequality $t \leq s^2$ by a complicated matrix-theoretic method. Bose and Shrikhande used the above argument to show that in case $t = s^2$ for each triple $\{x, y, z\}$ of pairwise noncollinear points there is a constant number of points collinear with x, y, z. Cameron apparently first observed that the above technique also provides the inequality.

Example. For the *GQ* $Q(5, q)$ we have $t = q^2 = s^2$. Let $\{x, y, z\}$ be a triple of pairwise noncollinear points of $Q(5, q)$. By Corollary 2.4 there are $q + 1$ points of $Q(5, q)$ collinear with x, y, z. If π is the plane defined by x, y, z in PG$(5, q)$, with $Q(5, q) \subseteq$ PG$(5, q)$, and if θ is the polarity defined by $Q(5, q)$, then the nonsingular conic $\pi^\theta \cap Q(5, q)$ is the set of all points of the *GQ* which are collinear with x, y, z.

For more properties and information on finite generalized quadrangles, we refer to Payne and Thas [1984].

2.3.2 Generalized hexagons and octagons

We start with some general properties of finite generalized polygons. Then, we review some specific properties of the hexagons $H(q)$ and $T(q^3, q)$, and of the octagon $O(q)$.

Theorem 2.6 (Feit & Higman [1964]) *Let $\mathcal{S}$ be a generalized n-gon of order (s, t) with $n \geq 3$. If $\mathcal{S}$ is finite, then one of the following holds:*

(i) $s = t = 1$, *and $\mathcal{S}$ is an ordinary n-gon;*

(ii) $n = 3$, $s = t > 1$, *and $\mathcal{S}$ is a projective plane;*

(iii) $n = 4$ *and the number*

$$\frac{st(1 + st)}{s + t}$$

is an integer;

(iv) $n = 6$, *and if $s, t > 1$, then st is a perfect square. In that case, we put $u = \sqrt{st}$ and $w = s + t$. The number*

$$\frac{u^2(1 + w + u^2)(1 \pm u + u^2)}{2(w \pm u)}$$

is an integer for both choices of signs;

(*v*) $n = 8$, *and if* $s, t > 1$, *then* $2st$ *is a perfect square; in particular* $s \neq t$. *If we put* $u = \sqrt{\frac{st}{2}}$ *and* $w = s + t$, *then the number*

$$\frac{u^2(1 + w + 2u^2)(1 + 2u^2)(1 \pm 2u + 2u^2)}{2(w \pm 2u)}$$

is an integer for both choices of signs;

(*vi*) $n = 12$ *and* $s = 1$ *or* $t = 1$. □

Theorem 2.7 *Let* $\mathcal{S}$ *be a finite generalized* n*-gon of order* (s, t), $s, t > 1$ *and* $n \geq 4$. *Then one of the following holds.*

(*i*) **(Higman [1974]).** $n = 4$ *and* $s \leq t^2$*; dually* $t \leq s^2$*;*

(*ii*) **(Haemers & Roos [1981]).** $n = 6$ *and* $s \leq t^3$*; dually* $t \leq s^3$*;*

(*iii*) **(Higman [1974]).** $n = 8$ *and* $s \leq t^2$*; dually* $t \leq s^2$. □

A very important corollary to the previous results is the following fact, which we already mentioned before.

Corollary 2.8 *Thick finite generalized* n*-gons,* $n \geq 3$, *exist only for* $n \in \{3, 4, 6, 8\}$.

Theorem 2.9 *Let* $\mathcal{S} = (P, B, \mathtt{I})$ *be a finite generalized* n*-gon of order* (s, t), *with* $n \in \{3, 4, 6, 8\}$, *then we have*

$$v = |P| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1 + s)(1 + st) & \text{if } n = 4, \\ (1 + s)(1 + st + s^2t^2) & \text{if } n = 6, \\ (1 + s)(1 + st)(1 + s^2t^2) & \text{if } n = 8. \end{cases}$$

Dually,

$$b = |B| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1 + t)(1 + st) & \text{if } n = 4, \\ (1 + t)(1 + st + s^2t^2) & \text{if } n = 6, \\ (1 + t)(1 + st)(1 + s^2t^2) & \text{if } n = 8. \end{cases}$$

□

Clearly, a generalized polygon of order (s,t) is finite if and only if s and t are finite.

We now mention some properties of the hexagons $H(q)$ and $T(q^3,q)$, and the octagon $O(q)$.

Opposite elements of a generalized polygon are elements lying at maximal distance in the incidence graph. For a generalized n-gon, this is precisely distance n.

Also, a point p of a generalized n-gon $\mathcal{S}$ is called *distance-i-regular*, $2 \le i \le n/2$, if for all points x opposite p, the set of points at distance i from p and $n-i$ from x is determined by any two of its elements. Dually, one defines a *distance-i-regular* line.

Theorem 2.10 *(i) Two points of $H(q)$ are opposite in $H(q)$ if and only if they are not collinear on the quadric $Q(6,q)$.*

(ii) The lines of $H(q)$ through any point x of $H(q)$ are all lines of $Q(6,q)$ through x lying in a certain plane $x^\perp$ of $Q(6,q)$. If x has coordinates $(a_0,a_1,a_2,a_3,a_4,a_5,a_6)$, then the plane $x^\perp$ has equations (one might choose four independent equations out of the following list of eight):

$$\begin{cases} a_1X_0 - a_0X_1 - a_6X_3 + a_3X_6 = 0, \\ a_2X_0 - a_0X_2 + a_5X_3 - a_3X_5 = 0, \\ a_2X_1 - a_1X_2 - a_4X_3 + a_3X_4 = 0, \\ a_3X_3 - a_0X_4 - a_1X_5 - a_2X_6 = 0, \\ a_0X_3 - a_3X_0 + a_6X_5 - a_5X_6 = 0, \\ a_1X_3 - a_3X_1 - a_6X_4 + a_4X_6 = 0, \\ a_2X_3 - a_3X_2 + a_5X_4 - a_4X_5 = 0, \\ a_4X_0 + a_5X_1 + a_6X_2 - a_3X_3 = 0. \end{cases}$$

(iii) All points of $H(q)$ are distance-2-regular, all points and lines of $H(q)$ are distance-3-regular. All lines are distance-2-regular if and only if $q = 3^h$. In that case $H(q)$ is self-dual.

A similar theorem holds for $T(q^3,q)$.

Theorem 2.11 *(i) Two points of $T(q^3,q)$ are opposite in $T(q^3,q)$ if and only if they are not collinear on the quadric $Q^+(7,q^3)$.*

(ii) The lines of $T(q^3,q)$ through any point x of $T(q^3,q)$ lie in a certain plane $x^\perp$ of $Q^+(7,q^3)$. If x has coordinates $(a_0,a_1,a_2,a_3,a_4,a_5,a_6)$, then the plane $x^\perp$ has equa-

tions (one might choose five independent equations out of the following list of sixteen):

$$\begin{cases} a_1^q X_0 - a_0^q X_1 + a_6^q X_3 + a_7^q X_6 = 0, \\ a_0^q X_2 - a_2^q X_0 + a_5^q X_3 + a_7^q X_5 = 0, \\ a_2^q X_1 - a_1^q X_2 + a_4^q X_3 + a_7^q X_4 = 0, \\ a_6^q X_5 - a_5^q X_6 + a_0^q X_7 + a_3^q X_0 = 0, \\ a_4^q X_6 - a_6^q X_4 + a_1^q X_7 + a_3^q X_1 = 0, \\ a_5^q X_4 - a_4^q X_5 + a_2^q X_7 + a_3^q X_2 = 0, \\ a_0^q X_4 + a_1^q X_5 + a_2^q X_6 - a_3^q X_3 = 0, \\ a_4^q X_0 + a_5^q X_1 + a_6^q X_2 - a_7^q X_7 = 0, \\ a_1^{q^2} X_0 - a_0^{q^2} X_1 - a_6^{q^2} X_7 - a_3^{q^2} X_6 = 0, \\ a_0^{q^2} X_2 - a_2^{q^2} X_0 - a_5^{q^2} X_7 - a_3^{q^2} X_5 = 0, \\ a_2^{q^2} X_1 - a_1^{q^2} X_2 - a_4^{q^2} X_7 - a_3^{q^2} X_4 = 0, \\ a_6^{q^2} X_5 - a_5^{q^2} X_6 - a_0^{q^2} X_3 - a_7^{q^2} X_0 = 0, \\ a_4^{q^2} X_6 - a_6^{q^2} X_4 - a_1^{q^2} X_3 - a_7^{q^2} X_1 = 0, \\ a_5^{q^2} X_4 - a_4^{q^2} X_5 - a_2^{q^2} X_3 - a_7^{q^2} X_2 = 0, \\ a_0^{q^2} X_4 + a_1^{q^2} X_5 + a_2^{q^2} X_6 - a_7^{q^2} X_7 = 0, \\ a_4^{q^2} X_0 + a_5^{q^2} X_1 + a_6^{q^2} X_2 - a_3^{q^2} X_3 = 0. \end{cases}$$

(*iii*) *All points of $T(q^3, q)$ are distance-2-regular, all points and lines of $T(q^3, q)$ are distance-3-regular. No line is distance-2-regular.*

Finally we have the following result.

Theorem 2.12 *All points and lines of $O(q)$, $q = 2^{2e+1}$, $e \in \mathbb{N}$, are distance-4-regular. No point or line of $O(q)$ is distance-i-regular for $i = 2, 3$. In fact, there does not exist a thick generalized octagon all points of which are distance-i-regular, with $i = 2, 3$, respectively.*

For more properties and information on generalized polygons, we refer to Thas [1995] and Van Maldeghem [1998].

2.3.3 Polar spaces

The finite *classical* nonsingular polar spaces are:

* $W_d(q)$ (or $W(d, q)$): the polar space formed by the points of $\mathrm{PG}(d, q)$, d odd and $d \geq 3$, together with the totally isotropic lines of a nonsingular symplectic polarity;

* $Q(2d, q)$: the polar space formed by the points and lines of a nonsingular quadric of $\mathrm{PG}(2d, q)$, $d \geq 2$;
* $Q^+(2d+1, q)$: the polar space formed by the points and lines of a nonsingular hyperbolic quadric of $\mathrm{PG}(2d+1, q)$, $d \geq 1$;
* $Q^-(2d+1, q)$: the polar space formed by the points and lines of a nonsingular elliptic quadric of $\mathrm{PG}(2d+1, q)$, $d \geq 2$;
* $H(d, q^2)$: the polar space formed by the points and lines of a nonsingular hermitian variety of $\mathrm{PG}(d, q^2)$, $d \geq 3$.

Theorem 2.13 *The numbers of points of the finite classical nonsingular polar spaces are as follows:*

(i) $|W_d(q)| = (q^{d+1} - 1)/(q-1)$;

(ii) $|Q(2d, q)| = (q^{2d} - 1)/(q-1)$;

(iii) $|Q^+(2d+1, q)| = (q^d + 1)(q^{d+1} - 1)/(q-1)$;

(iv) $Q^-(2d+1, q)| = (q^d - 1)(q^{d+1} + 1)/(q-1)$;

(v) $|H(d, q^2)| = (q^{d+1} + (-1)^d)(q^d - (-1)^d)/(q^2 - 1)$.

Proof. See e.g. Hirschfeld and Thas [1991]. □

2.3.4 Partial geometries

Theorem 2.14 *If $\mathcal{S} = (P, B, \mathtt{I})$ is a PG with parameters s, t, α, with $|P| = v$ and $|B| = b$, then*

$$v = (s+1)(st+\alpha)/\alpha \text{ and } b = (t+1)(st+\alpha)/\alpha.$$

Proof. Let L be a fixed line of $\mathcal{S}$ and count in two different ways the number of ordered pairs $(x, M) \in P \times B$ with $x \not\mathtt{I} L$, $x \mathtt{I} M$ and L, M concurrent. There arises $(v-s-1)\alpha = (s+1)ts$ or $v = (s+1)(st+\alpha)/\alpha$. Dually $b = (t+1)(st+\alpha)/\alpha$. □

Theorem 2.15 *If $\mathcal{S} = (P, B, \mathtt{I})$ is a PG with parameters s, t, α, with $|P| = v$ and $|B| = b$, then*

$$\alpha(s+t+1-\alpha) \text{ divides } st(s+1)(t+1).$$

Proof. The point graph of $\mathcal{S}$ is strongly regular with parameters $v = (s+1)(st+\alpha)/\alpha$, $k = (t+1)s$, $\lambda = s-1+t(\alpha-1)$ and $\mu = (t+1)\alpha$. From the theory of strongly regular graphs it now follows that $\alpha(s+t+1-\alpha)$ divides $st(s+1)(t+1)$. □

Theorem 2.16 (The Krein inequalities) *Let $\mathcal{S} = (P, B, \mathtt{I})$ be a PG with parameters s, t, α. Then the integers s, t and α satisfy the inequalities*

$$(s+1-2\alpha)t \leq (s-1)(s+1-\alpha)^2 \tag{3}$$

and

$$(t+1-2\alpha)s \leq (t-1)(t+1-\alpha)^2. \tag{4}$$

When equality holds in (3), the number of points collinear with three points p_1, p_2, p_3 depends only on the number of collinearities in $\{p_1, p_2, p_3\}$; when equality holds in (4), the number of lines concurrent with three lines L_1, L_2, L_3 depends only on the number of concurrencies in $\{L_1, L_2, L_3\}$.

Proof. The inequalities (3) and (4) are particular cases of the Krein inequalities for strongly regular graphs; see Cameron, Goethals and Seidel [1978]. □

Remark 2.17 For $\alpha = 1$ and $s \neq 1 \neq t$, the inequalities (3) and (4) are the inequalities of Higman for *GQ*.

3 Embeddings of generalized quadrangles

3.1 Introduction

All (fully) embedded finite *GQ* were first determined by Buekenhout and Lefèvre [1974] with a proof most of which is valid in the infinite case. Independently, Olanda [1973, 1977] has given a typically finite proof, and Thas and De Winne [1977] have given a different combinatorial proof under the assumption that the 3-dimensional case is already settled. The infinite case was settled by Dienst [1980a, 1980b]. The main goal of this chapter is to sketch the proof of Buekenhout and Lefèvre. However, because the *GQ* in this course are finite, we have modified their presentation somewhat.

For the subspace of PG(d, s) generated by the pointsets or points $P_1, P_2, \ldots, P_k$ we shall frequently use the notation $\langle P_1, P_2, \ldots, P_k \rangle$; if $P_1, P_2, \ldots, P_k$ are subspaces of PG(d, s) we also use the notation $P_1 P_2 \ldots P_k$.

3.2 The tangent hyperplane

In this section $\mathcal{S} = (P, B, \mathtt{I})$ is a (finite or infinite) *GQ* of order (s, t) (fully) embedded in $\mathrm{PG}(d, s)$, with $d \geq 3$ (remark that any grid embedded in $\mathrm{PG}(d, s)$ has order $(s, 1)$).

Lemma 3.1 *If $t = 1$, then $\mathcal{S}$ is the classical GQ $Q(3, s)$.*

Proof. Here $\mathcal{S}$ is a grid of order $(s, 1)$ which generates $\mathrm{PG}(d, s)$. Hence $d = 3$ and $\mathcal{S} = Q(3, s)$. □

From now on we assume that $t \geq 2$.

Lemma 3.2 *If W is a subspace of $\mathrm{PG}(d, s)$ and if $W \cap B$ denotes the set of all lines of $\mathcal{S}$ in W, then for the substructure $W \cap \mathcal{S} = (W \cap P, W \cap B, \in)$ we have one of the following.*

- (a) *The elements of $W \cap B$ are lines which are incident with a distinguished point of P, and $W \cap P$ consists of the points of P that are incident with these lines.*
- (b) *$W \cap B = \emptyset$ and $W \cap P$ is a set of pairwise noncollinear points of $\mathcal{S}$.*
- (c) *$W \cap \mathcal{S}$ is a subquadrangle of $\mathcal{S}$ which is (fully) embedded in a subspace of W; if W is a hyperplane of $\mathrm{PG}(d, s)$, then $W \cap P$ generates W.*

Proof. Easy. □

If $p \in P$, a *tangent to $\mathcal{S}$ at p* is any line through p such that either $L \in B$ on $L \cap P = \{p\}$. The union of all tangents to $\mathcal{S}$ at p will be called the *tangent set of $\mathcal{S}$ at p*, and we denote it by $\mathcal{S}(p)$. The relation between $\mathcal{S}(p)$ and $p^\perp$, with $p^\perp$ the set of all points of P collinear in $\mathcal{S}$ with p, is: $p^\perp = P \cap \mathcal{S}(p)$. A line L of $\mathrm{PG}(d, s)$ is a *secant to $\mathcal{S}$* if L intersects P in at least two points but is not a member of B.

Lemma 3.3 *If x and y are collinear points of $\mathcal{S}$, then $x^\perp \cap y^\perp$ is the line $\langle x, y \rangle$.*

Proof. Clear. □

Lemma 3.4 *For each $p \in P$, $\langle p^\perp \rangle \subseteq \mathcal{S}(p)$.*

Proof. Without proof. □

Lemma 3.5 *The subspace $\langle p^\perp \rangle$ is a hyperplane of $\mathrm{PG}(d, s)$.*

Proof. Consider a point $x \in P \setminus \langle p^\perp \rangle$. By Lemma 3.2, $\langle p^\perp, x\rangle \cap \mathcal{S}$ is a subquadrangle of $\mathcal{S}$. Clearly this subquadrangle has order (s,t), so it must coincide with $\mathcal{S}$. Hence $\langle p^\perp, x\rangle = \mathrm{PG}(d,s)$, i.e., $\dim\langle p^\perp\rangle = d-1$. □

Lemma 3.6 *The hyperplane $\langle p^\perp \rangle$ is the tangent set $\mathcal{S}(p)$ to $\mathcal{S}$ at p, and is called the tangent hyperplane to $\mathcal{S}$ at p.*

Proof. Without proof. □

Lemma 3.7 *Let x, y, z be three distinct points of $\mathcal{S}$ on a line of $\mathrm{PG}(d,s)$. Then the intersections $\mathcal{S}(x) \cap \mathcal{S}(y)$, $\mathcal{S}(y) \cap \mathcal{S}(z)$, and $\mathcal{S}(x) \cap \mathcal{S}(z)$ coincide.*

Proof. Without proof. □

Lemma 3.8 *Let L be a secant containing three distinct points x, y, y' of P. Then the perspectivity σ of $\mathrm{PG}(d,s)$ with center x and axis $\mathcal{S}(x)$ mapping y onto y' leaves P invariant.*

Proof. Without proof. □

Lemma 3.9 *All secant lines contain the same number of points of $\mathcal{S}$.*

Proof. Let L and L' be secant lines. First suppose L and L' have a point x in common, and let M be any secant line through x. If some M is incident with more than two points of P, then, by Lemma 3.8, we may consider the nontrivial group G of all perspectivities with center x and axis $\mathcal{S}(x)$, leaving P invariant. The group G is regular on the set of points of M in P but different from x, for each M. Hence each secant through x has $1 + |G|$ points of P, so that L and L' have the same number of points of $\mathcal{S}$. If no M is incident with more than two points of P, then clearly L and L' contain two points of $\mathcal{S}$.

Secondly, suppose L and L' do not have any point of P in common, and choose points x, x' of P on L and L', respectively. If x and x' are not collinear, then $\langle x, x'\rangle$ is a secant, so meets P in the same number of points as do L and L', by the previous paragraph. If x and x' are collinear, choose a point $y \in P$ with $x \not\sim y \not\sim x'$, and apply the previous paragraph to the secant lines L, $\langle x, y\rangle$, $\langle x', y'\rangle$, L'. □

3.3 Embedding $\mathcal{S}$ in a polarity: preliminary results

The goal of this section and the next is to extend the mapping $p \mapsto \mathcal{S}(p)$ to a nonsingular polarity of $\mathrm{PG}(d,s)$, i.e., to construct a mapping π such that

(a) for each point x of $\mathrm{PG}(d,s)$, x^π is a hyperplane of $\mathrm{PG}(d,s)$,

(b) for each $x \in P$, $x^\pi = \mathcal{S}(x)$,

(c) $x \in y^\pi$ implies $y \in x^\pi$.

For a point x of $\mathrm{PG}(d,s)$, the *collar* $\mathcal{S}_x$ of $\mathcal{S}$ for x is the set of all points p of $\mathcal{S}$ such that either $p = x$ or $p \neq x$ and the line $\langle p, x\rangle$ is a tangent to $\mathcal{S}$ at p. For example, if $x \in P$, then $\mathcal{S}_x$ is just $x^\perp$. If $x \notin P$, the collar $\mathcal{S}_x$ is the set of points p of P such that $\langle p, x\rangle \cap P = \{p\}$.

For all $x \in \mathrm{PG}(d,s)$ the *polar* x^π of x with respect to $\mathcal{S}$ is the subspace of $\mathrm{PG}(d,s)$ generated by the collar $\mathcal{S}_x$, i.e., $x^\pi = \langle \mathcal{S}_x\rangle$. In particular, if $x \in P$, then $x^\pi = \mathcal{S}(x)$.

Lemma 3.10 *For any point x, let p_1 and p_2 be distinct points of $\mathcal{S}_x$. Then $P \cap \langle p_1, p_2\rangle \subseteq \mathcal{S}_x$.*

Proof. Suppose $p \in P \cap \langle p_1, p_2\rangle$, $p_1 \neq p \neq p_2$. Since $x \in \mathcal{S}(p_1) \cap \mathcal{S}(p_2)$, by Lemma 3.7, also $x \in \mathcal{S}(p)$, hence $p \in \mathcal{S}_x$. □

Lemma 3.11 *Each line L of $\mathcal{S}$ intersects the collar $\mathcal{S}_x$ for each point x of $\mathrm{PG}(d,s)$, in exactly one point, unless each point of L is in $\mathcal{S}_x$.*

Proof. Without proof. □

Lemma 3.12 *Either $x^\pi = \langle \mathcal{S}_x\rangle$ is a hyperplane or $x^\pi = \mathrm{PG}(d,s)$.*

Proof. Without proof. □

Lemma 3.13 *If x^π is a hyperplane, then $\mathcal{S}_x = P \cap x^\pi$.*

Proof. Without proof. □

Lemma 3.14 *Let x be a point of $\mathrm{PG}(d,s)$ and y, y' distinct points of P different from x and not in x^π, which are collinear with x. Then the perspectivity σ of $\mathrm{PG}(d,s)$ with center x and axis x^π mapping y onto y' leaves P invariant.*

Proof. Without proof. □

Lemma 3.15 *Suppose that secant lines to $\mathcal{S}$ have at least three points of P. If x^π is a hyperplane, then either $y \in x^\pi$ implies $x \in y^\pi$, or there is a point z with $z^\pi = \mathrm{PG}(d, s)$ and $\mathcal{S}_z \neq P$.*

Proof. Without proof. □

3.4 The finite case

The arguments given in the previous sections hold also in the case of a projective space of finite dimension $d \geq 3$ over an infinite field. For the remainder of this chapter, however, finiteness is essential. Recall that $\mathcal{S}$ has order (s, t), $s \geq 2$, $t \geq 2$, and denote by $m + 1$ the constant number of points of $\mathcal{S}$ on a secant line.

A pointset K of $\mathrm{PG}(d, s)$, $d \geq 2$ is called a *quadratic set* if

(a) any line of $\mathrm{PG}(d, s)$ intersects K in 0,1,2 or $s + 1$ points;

(b) for each $p \in K$, the union of the lines through p which intersect K either in 1 or $s + 1$ points , together with p, form the *tangent space* $T_p = T_p(K)$ which is either a hyperplane or $\mathrm{PG}(d, s)$.

If $m = 1$, then P is a quadratic set, and then by results of Buekenhout [1976], $\mathcal{S}$ is formed by the points and lines on a nonsingular quadric of rank 2 in $\mathrm{PG}(d, s)$, $d = 4$ or 5.

Hence from now on we assume that $m > 1$ and proceed to establish (a), (b), (c), in 3.3.

Lemma 3.16 *There holds $m = t/s^{d-3}$, and either $d = 3$ or $d = 4$.*

Proof. The secant lines through a point $p \in P$ are the s^{d-1} lines of $\mathrm{PG}(d, s)$ through p which do not lie in the tangent hyperplane $\mathcal{S}(p)$. Hence the total number of points of P is $ms^{d-1} + |p^\perp| = (1 + s)(1 + st)$, implying $m = t/s^{d-3}$. By Higman's inequality we know that $t \leq s^2$, so that $2 \leq m \leq s^2/s^{d-3}$, implying that either $d = 3$ or $d = 4$. □

A subset E of P is called *linearly closed* in P if for all $x, y \in E$, with $x \neq y$, the intersection $\langle x, y\rangle \cap P$ is contained in E. Thus any subset X of P generates a *linear closure* $\bar{X}$ in P.

Lemma 3.17 *Let $\{x_0, x_1, \cdots, x_k\}$ be a set of points of P. Then the linear closure of $\{x_0, x_1, \cdots, x_k\}$ in P is $P \cap \langle x_0, x_1, \cdots, x_k\rangle$.*

Proof. Without proof. □

Lemma 3.18 *We have $\mathcal{S}_x \neq P$.*

Proof. Without proof. □

Lemma 3.19 *The subspace x^π is always a hyperplane.*

Proof. Without proof. □

Relying on Lemma 3.15 this completes the proof that (a), (b), (c) of Section 3.3 hold, so that π is a nonsingular polarity. If we can show that P is the set of absolute points of π, then, since B is the set of all lines of $\mathrm{PG}(d,s)$ which contain x and are contained in $x^\pi \cap P$, where x runs over P, B must be the set of totally isotropic lines of π.

Lemma 3.20 *We have $x \in x^\pi$ if and only if $x \in P$.*

Proof. Without proof. □

This completes the proof of the theorem of Buekenhout and Lefèvre.

Theorem 3.21 (Buekenhout and Lefèvre [1974]) *If $\mathcal{S} = (P, B, \mathtt{I})$ is a GQ which is (fully) embedded in $\mathrm{PG}(d,s)$, $d \geq 3$, then it is obtained in one of the following ways:*

(a) *there is a unitary or symplectic polarity π of $\mathrm{PG}(d,s)$, $d = 3$ or 4, such that P is the set of absolute points of π and B is the set of totally isotropic lines of π;*

(ii) *there is a nonsingular quadric Q of rank 2 in $\mathrm{PG}(d,s)$, $d = 3, 4$ or 5, such that $P = Q$ and B is the set of lines on Q.*

Hence $\mathcal{S}$ must be one of the classical examples described in Section 2.2. We will also call the corresponding embeddings *classical*.

4 Embeddings of polar spaces

4.1 Introduction

Let $\mathcal{S} = (P, B, \mathtt{I})$ be a polar space. A point p of $\mathcal{S}$ is called *singular* if it is collinear (in $\mathcal{S}$) with each point of P. If R is the set of all singular points of $\mathcal{S}$, then R provided with the lines of B in R is a projective space. The set R is called the *radical* of $\mathcal{S}$. If r is the rank of $\mathcal{S}$ and d' is the dimension of the projective space R, then $r - d' - 1$ is called the *nonsingular rank* of $\mathcal{S}$.

Theorem 4.1 (Buekenhout and Lefèvre [1976], Lefèvre-Percsy [1977]) *If the geometry $\mathcal{S} = (P, B, \mathtt{I})$ is a polar space which is (fully) embedded in* $\mathrm{PG}(d, s)$*,* $d \geq 3$*, and if the nonsingular rank of* $\mathcal{S}$ *is at least two, then it is obtained in one of the following ways*

(i) $\mathcal{S}$ *is formed by the points of* $\mathrm{PG}(d, s)$ *together with the totally isotropic lines with respect to some symplectic polarity of* $\mathrm{PG}(d, s)$ *(the symplectic polarity is allowed to be singular);*

(ii) $\mathcal{S}$ *is formed by the points and lines of a quadric of rank at least two in* $\mathrm{PG}(d, s)$ *(the quadric is allowed to be singular)*

(iii) $\mathcal{S}$ *is formed by the points and lines of a hermitian variety of rank at least two in* $\mathrm{PG}(d, s)$*, with* s *a square (the hermitian variety is allowed to be singular).*

Proof. Without proof. □

Remark 4.2 (a) Theorem 3.21 is included in Theorem 4.1. Large parts of the proof of Theorem 3.21 were used by Buekenhout and Lefèvre to prove Theorem 4.1.

(b) As is easily checked, Theorem 4.1 is not valid for nonsingular rank less than two.

(c) By the definition of polar space we knew already that $\mathcal{S}$ is either a *GQ* or isomorphic to one of (*i*), (*ii*), (*iii*).

5 Embeddings of partial geometries

Theorem 5.1 (De Clerck and Thas [1978]) *If* $\mathcal{S} = (P, B, \mathtt{I})$ *is a PG with parameters* s, t, α *which is (fully) embedded in* $\mathrm{PG}(d, s)$*, then one of the following holds:*

(a) $\alpha = s + 1$ *and* $\mathcal{S}$ *is the* $2 - ((s^{d+1} - 1)/(s - 1), s + 1, 1)$ *design formed by all points and all lines of* $\mathrm{PG}(d, s)$*;*

(b) $\alpha = 1$ *and* $\mathcal{S}$ *is a classical GQ;*

(c) $\alpha = t + 1, d = 2$*,* $\mathcal{S}$ *is even,* $\mathrm{PG}(2, s) \setminus P$ *is a maximal* $\{sm - s + m; m\}$*-arc* K *of* $\mathrm{PG}(2, s)$ *with* $m = s/\alpha$ *and* $1 < m < s$*, and* B *consists of all lines of* $\mathrm{PG}(2, s)$ *having an empty intersection with* K*;*

(d) $\alpha = s$*,* $d \geq 2$ *and* $\mathcal{S} = H_s^d$*.*

Proof. If $\alpha = s+1$, then $\mathcal{S}$ is a $2-(v, s+1, 1)$ design. Hence $\mathcal{S}$ consists of all points and all lines of some subspace $\mathrm{PG}(d', s)$ of $\mathrm{PG}(d, s)$. Since $\mathrm{PG}(d, s)$ is generated by $\mathcal{S}$, we have $d = d'$. Consequently $\mathcal{S}$ is the design formed by all points and all lines of $\mathrm{PG}(d, s)$.

If $\alpha = 1$, then by the Theorem of Buekenhout and Lefèvre (Theorem 3.21) the *PG* $\mathcal{S}$ is a classical *GQ*.

Now let $\alpha = t+1$. Since any two lines of $\mathcal{S}$ have a nonempty intersection, and not all lines of B contain a common point, there holds $d = 2$. Each line of $\mathrm{PG}(2, s)$ not in B has exactly $(s+1) - (b/(t+1)) = s/\alpha$ (with $b = |B|$) points in common with $\mathrm{PG}(2, s) \setminus P$. If $\alpha = s$, then $\mathcal{S}$ is the dual of a $2-(s^2, s, 1)$ design, that is, the dual of an affine plane of order s; hence in such a case $\mathcal{S} = H_s^2$. If $1 < m < s$ with $m = s/\alpha$, then $\mathrm{PG}(2, s) \setminus P$ is a maximal $\{sm - s + m; m\}$-arc K and B is the set of all lines of $\mathrm{PG}(2, s)$ having an empty intersection with K; then, by 2.2.3 (b), s is necessarily even.

Let us now suppose that $1 < \alpha < s$ and $\alpha \neq t+1$. As any two lines of $\mathrm{PG}(2, s)$ have a nonempty intersection, we necessarily have $d \geq 3$.

First, let $d = 3$. Suppose that L is a line of $\mathcal{S}$ and that x is a point of $\mathcal{S}$ with $x \notin L$. Let π be the plane of $\mathrm{PG}(3, s)$ containing x and L. The points and lines of $\mathcal{S}$ in π constitute a partial geometry $\mathcal{S}_\pi$ with parameters $t_\pi = \alpha - 1$, $s_\pi = s$, and $\alpha_\pi = \alpha$. Hence the points of π which do not belong to $\mathcal{S}_\pi$ form a maximal $\{s(s\alpha^{-1} + \alpha^{-1} - 1); s\alpha^{-1}\}$-arc of π. Also, s is even. Let M be any line of $\mathrm{PG}(3, s)$ which contains at least two points x', x'' of $\mathcal{S}$. Further, let $M' \in B$, with $M \neq M'$ and $x' \in M'$. Let π' be the plane containing x'' and M'. Since $\pi' \setminus P$ is an $\{s(s\alpha^{-1} + \alpha^{-1} - 1); s\alpha^{-1}\}$-arc of the plane π', we have $|M \cap P| \in \{s+1, s+1-s\alpha^{-1}\}$. Hence each line of $\mathrm{PG}(3, s)$ intersects P in 0, 1, $s+1-s\alpha^{-1}$, or $s+1$ points. Also, the lines of $\mathrm{PG}(3, s)$ contained in P are exactly the lines of $\mathcal{S}$. Now we show that P has no 1-secant, that is, a line containing just one point of P.

Suppose that L is a 1-secant of P, and let x be the common point of L and P. The lines of $\mathcal{S}$ through x are denoted by $M_1, M_2, \cdots, M_{t+1}$. First, let us assume that each plane LM_i contains exactly $s+1$ points of P, with $i = 1, 2, \cdots, t+1$. Since $d = 3$, each line M of B contains at least one point of $P \cap LM_i = M_i$, $i = 1, 2, \cdots, t+1$. Hence $b = (t+1)(st+\alpha)/\alpha = (s+1)t+1$, and so $\alpha = t+1$, a contradiction. Consequently for at least one index i we have $|P \cap LM_i| > s+1$, say $x' \in (P \cap LM_i) \setminus M_i$. By the previous paragraph the line L contains either $s+1-s\alpha^{-1}$ or $s+1$ points of the *PG* $\mathcal{S}_{\pi_i}$, with $\pi_i = x'M_i = LM_i$. Hence $1 \in \{s+1-s\alpha^{-1}, s+1\}$, a contradiction. Therefore P has no 1-secant; that is, each line of $\mathrm{PG}(3, s)$ intersects P in $0, s+1-s\alpha^{-1}$, or $s+1$ points. Next, we show that such a set with $1 < \alpha < s$ cannot exist.

Counting the points of P on all lines of $\mathrm{PG}(3, s)$ containing a fixed point of P, we obtain

$$|P| = 1 + (t+1)s + (s^2+s-t)(s-s\alpha^{-1}). \tag{5}$$

By Theorem 2.14 we also have

$$|P| = (s+1)(st+\alpha)/\alpha. \tag{6}$$

From (5) and (6) it follows that $t = (s+1)(\alpha-1)$. Since $\alpha \neq s+1$ we have $P \neq \mathrm{PG}(3,s)$. Let $y \in \mathrm{PG}(3,s) \setminus P$. Counting the points of P on all lines of $\mathrm{PG}(3,s)$ containing y, we see that $s+1-s\alpha^{-1}$ divides $|P|$. Hence $s\alpha+\alpha-s$ divides

$$(s+1)(s(s+1)(\alpha-1)+\alpha) = (s\alpha+\alpha-s)(s^2+s+1)-s^2.$$

Consequently $s\alpha+\alpha-s$ divides s^2. Let $s = 2^h$, $\alpha = 2^{h'}$, with $0 < h' < h$. Then $2^h+1-2^{h-h'}$ divides $2^{2h-h'}$. Since $(2^h+1-2^{h-h'}, 2^{2h-h'}) = 1$, we necessarily have $2^h+1-2^{h-h'} = 1$, clearly a contradiction.

It has been shown that, for $1 < \alpha < s$ and $\alpha \neq t+1$, we necessarily have $d > 3$.

So let $1 < \alpha < s, \alpha \neq t+1$, and $d > 3$. Let L be a line of $\mathcal{S}$, let π be the plane defined by L and a point x in $P \setminus L$, and let $\mathrm{PG}(3,s)$ be the 3-dimensional space defined by π and a point x' in $P \setminus \pi$. Let x_1 and x_2 be distinct points of P in $\mathrm{PG}(3,s)$. Counting the number of pairs $\{L_1, L_2\}$, with $L_1, L_2 \in B$, L_1 and L_2 in $\mathrm{PG}(3,s)$, $x_1 \in L_1, x_2 \in L_2$, and $L_1 \cap L_2 \neq \phi$, in different ways, it appears that the number of lines of B in $\mathrm{PG}(3,s)$ which contain x_1 is equal to the number of lines of B in $\mathrm{PG}(3,s)$ which contain x_2. It follows that the points and lines of $\mathcal{S}$ in $\mathrm{PG}(3,s)$ constitute a partial geometry $\mathcal{S}'$ with parameters t', $s' = s$, $\alpha' = \alpha$. Since $1 < \alpha' < s'$ and $\alpha' \neq t'+1$ ($\mathcal{S}'$ is not contained in a plane), such a geometry cannot exist.

So the only possibilities are $\alpha = 1, \alpha = s+1, \alpha = t+1$, and $\alpha = s$.

Finally, assume that $\alpha = s$ with $\alpha \neq 1$ and $\alpha \neq t+1$. In this case we have $d \geq 3$. Let L be a line of $\mathcal{S}$, and suppose that the point x of $\mathcal{S}$ is not on L. The points and lines of $\mathcal{S}$ in the plane $\pi = xL$ form a PG with parameters $s' = s$, $t' = \alpha - 1 = s' - 1$, and $\alpha' = \alpha = s'$, that is, the dual of an affine plane of order s. If the line M of $\mathrm{PG}(d,s)$ contains at least two points of P, then M is contained in at least one plane π in which $\mathcal{S}$ induces the dual of an affine plane of order s. Hence each line of $\mathrm{PG}(d,s)$ intersects P in $0, 1, s$ or $s+1$ points, and a line M of $\mathrm{PG}(d,s)$ contains $s+1$ points of P if and only if M belongs to B. If P has no 1-secant, then all planes of $\mathrm{PG}(d,s)$ through a fixed line L of $\mathcal{S}$ contain a point of $P \setminus L$; hence the points of $\mathcal{S}$ in such a plane are the points of a dual affine plane of order s. It follows that

$$|P| = s+1+(s^2-1)(s^{d-1}-1)/(s-1) = s^d + s^{d-1}. \tag{7}$$

Conversely, if $|P| = s^d + s^{d-1}$, then P admits no 1-secant (if such a 1-secant M would exist and $M \cap P = \{y\}$, then, considering all planes of $\mathrm{PG}(d,s)$ containing a line of B through y, we would have $|P| < s^d + s^{d-1}$). Now we show that P has no 1-secant.

First, let $d = 3$. By a reasoning analogous to that in a preceding paragraph, it follows that P has no 1-secant; so $|P| = s^3 + s^2$ for $d = 3$. Now we use induction, and assume that any PG with $\alpha = s$, $\alpha \neq 1$, $\alpha \neq t + 1$, and embedded in $\mathrm{PG}(d - 1, s)$, $d \geq 4$, has no 1-secant. Next, assume that $\mathcal{S} = (P, B, \mathtt{I})$ is a PG embedded in $\mathrm{PG}(d, s)$, $d \geq 4$, with $\alpha = s$, $\alpha \neq 1$, $\alpha \neq t + 1$, which has at least one 1-secant L. let $L \cap P = \{x\}$. Consider a line M of B containing x and $d-2$ points $x_1, x_2, \cdots, x_{d-2}$ of P such that $M, x_1, \ldots, x_{d-2}$ generate a hyperplane $\mathrm{PG}(d - 1, s)$. The geometry induced by $\mathcal{S}$ in $\mathrm{PG}(d - 1, s)$ is a PG $\bar{\mathcal{S}}$ with parameters $\bar{t}$, $\bar{s} = s = \bar{\alpha}$, which is (fully) embedded in $\mathrm{PG}(d-1, s)$. Clearly $\bar{\alpha} \neq 1$, and $\bar{\alpha} \neq \bar{t} + 1$ since $d - 1 > 2$. By the induction hypothesis we have

$$|\bar{P}| = |P \cap \mathrm{PG}(d - 1, s)| = s^{d-1} + s^{d-2}. \tag{8}$$

Let $M_1, M_2, \cdots, M_{t+1}$ be the lines of B through x. The plane LM_i contains $s + 1$ points of $\mathcal{S}$. If the intersection of P and the 3-dimensional space LM_1M_i, with $i > 1$, generates LM_1M_i, then the PG induced by $\mathcal{S}$ in LM_1M_i has no 1-secant, a contradiction. Hence for all $i > 1$, $P \cap LM_1M_i$ is contained in the plane M_1M_i, and consequently $|P \cap LM_1M_i| = s^2 + s$. Let x' be any point of $P \setminus M_1$. Then the plane $x'M_1$ contains s lines of B through x'. The dual affine plane induced by $\mathcal{S}$ in the plane $x'M_1$ contains s lines of B through x, and so x' belongs to at least one of the spaces LM_1M_i, with $i > 1$. It follows that

$$|P| \leq \frac{(s^{d-2} - 1)(s^2 - 1)}{s - 1} + s + 1 = s^{d-1} + s^{d-2}, \tag{9}$$

with $(s^{d-2} - 1)/(s - 1)$ the number of 3-dimensional subspaces containing the plane LM_1. From (8) and (9) it now follows that $P = P \cap \mathrm{PG}(d - 1, s)$, hence $\mathcal{S}$ does not generate $\mathrm{PG}(d, s)$, a contradiction. So P has no 1-secant.

So $|P| = s^d + s^{d-1}$ and each line of $\mathrm{PG}(d, s)$ intersects P in $0, s$ or $s+1$ points. Consequently each line of $\mathrm{PG}(d, s)$ intersects $\mathrm{PG}(d, s) \setminus P$ in 0,1, or $s + 1$ points, and so $\mathrm{PG}(d, s) \setminus P$ is a subspace of $\mathrm{PG}(d, s)$. Since $|P| = s^d + s^{d-1}$, the subspace $\mathrm{PG}(d, s) \setminus P$ has dimension $d - 2$. Let $\mathrm{PG}(d, s) \setminus P = \mathrm{PG}(d - 2, s)$. The lines of B are exactly the lines of $\mathrm{PG}(d, s)$ having an empty intersection with $\mathrm{PG}(d - 2, s)$. We conclude that $\mathcal{S} = H_s^d$, and now the theorem is completely proved. $\square$.

6 Embeddings of the flag geometries of projective planes

6.1 The examples

In this section, we classify all (full) polarized embeddings of generalized hexagons of order $(q, 1)$ in $\mathrm{PG}(d, q)$. Let us first give a description of all examples.

So consider a coordinate system in $PG(2,q)$. A flag in $PG(2,q)$, which is an incident point-line pair, is a pair $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ with $a_0x_0+a_1x_1+a_2x_2=0$ (the coordinates of points are denoted with parentheses; those of lines with square brackets). Let σ be a field automorphism of $GF(q)$. We define as follows a mapping θ_σ from the set of flags of $PG(2,q)$ into the set of points of $PG(8,q)$. The image under θ_σ of the flag $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$, with $a_0x_0+a_1x_1+a_2x_2=0$, is by definition the point $(a_0x_0^\sigma,a_0x_1^\sigma,a_0x_2^\sigma,a_1x_0^\sigma,a_1x_1^\sigma,a_1x_2^\sigma,a_2x_0^\sigma,a_2x_1^\sigma,a_2x_2^\sigma)$ of $PG(8,q)$. In what follows coordinates of a general point of $PG(8,q)$ will be denoted by $X_{00},X_{01},X_{02},X_{10},\ldots,X_{22}$, respectively.

First suppose that σ is not the identity. Then one can check as an exercise that the set of images under θ_σ generates $PG(8,q)$. We now show that the embedding is polarized.

Consider the flag $F=\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ of $PG(2,q)$. Any flag of $PG(2,q)$ not opposite F (viewed as a point of the non-thick generalized hexagon $\mathcal{S}$ which is the flag geometry of $PG(2,q)$) has the form $\{(y_0,y_1,y_2),[b_0,b_1,b_2]\}$ with $b_0y_0+b_1y_1+b_2y_2=0$ and either

$$b_0x_0+b_1x_1+b_2x_2=0 \tag{10}$$

or

$$a_0y_0+a_1y_1+a_2y_2=0. \tag{11}$$

Hence we see that, by multiplying Equation (10) with $y_0^\sigma,y_1^\sigma,y_2^\sigma$, respectively, and first raising Equation (11) to the power σ and then multiplying the result by b_0,b_1,b_2, respectively, the corresponding point $p=(b_iy_j^\sigma)_{i,j=0,1,2}$ of $PG(8,q)$ satisfies either $x_0X_{0j}+x_1X_{1j}+x_2X_{2j}=0$, $j=0,1,2$, or $a_0^\sigma X_{i0}+a_1^\sigma X_{i1}+a_2^\sigma X_{i2}=0$, $i=0,1,2$. Making the appropriate linear combinations (multiplying with a_j^σ and x_i, $i,j=0,1,2$), we see that the point p satisfies the equation

$$\sum_{i,j=0}^{2} a_j^\sigma x_i X_{ij}=0. \tag{12}$$

Remarking that the set of flags containing one fixed point (respectively line) of $PG(2,q)$ is mapped under θ_σ onto the set of points of a line of $PG(8,q)$ — which is immediately checked with an elementary calculation — and identifying every flag of $PG(2,q)$ with its image under θ_σ, we obtain a full polarized embedding of $\mathcal{S}$ in $PG(8,q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $PG(8,q)$) a *semi-classical embedding of* $\mathcal{S}$ *in* $PG(8,q)$ *(with respect to* σ*)*.

It is easily seen that the group $\mathbf{PGL}_3(q)$ acts in a natural way as an automorphism group and as a subgroup of $\mathbf{PGL}_9(q)$ on the embedding.

Now suppose σ is the identity. Then all points of the image of θ_{id} belong to the hyperplane $\mathrm{PG}(7,q)$ with equation $X_{00}+X_{11}+X_{22}=0$. Also, the points of $\mathcal{S}^{\theta_{\mathrm{id}}}$ not opposite a given point $(a_0x_0,a_0x_1,\ldots,a_2x_2)$ of $\mathcal{S}^{\theta_{\mathrm{id}}}$ are contained in the hyperplane with equation (and this follows immediately from Equation (12))

$$\sum_{i,j=0}^{2} a_jx_iX_{ij}=0. \tag{13}$$

Now we note that the hyperplane with equation (13) is always distinct from $\mathrm{PG}(7,q)$. Indeed, the conditions $a_jx_i=0$, $i,j=0,1,2$, $i\neq j$, readily imply that, without loss of generality, we may assume $a_0=x_0=1$ and $a_1=a_2=x_1=x_2=0$, contradicting the fact that we have a flag. Hence, as before, identifying every flag of $\mathrm{PG}(2,q)$ with its image under θ_{id}, we obtain a full polarized embedding of $\mathcal{S}$ in $\mathrm{PG}(7,q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $\mathrm{PG}(7,q)$) a *natural embedding of* $\mathcal{S}$ *in* $\mathrm{PG}(7,q)$.

By another elementary calculation, one easily sees that the intersection of all hyperplanes with equation (13) is the point k with coordinates $x_{ii}=1$, $x_{ij}=0$, $i,j\in\{0,1,2\}$, $j\neq i$. This point lies in $\mathrm{PG}(7,q)$ if and only if the characteristic of $\mathrm{GF}(q)$ is equal to 3. Hence, in this case, we can project the polarly embedded generalized hexagon $\mathcal{S}$ from k onto some hyperplane $\mathrm{PG}(6,q)$ of $\mathrm{PG}(7,q)$ not containing k to obtain a full polarized embedding of $\mathcal{S}$ in the 6-dimensional projective space $\mathrm{PG}(6,q)$. We call this embedding also a *natural embedding of* Γ.

The exceptional behaviour over fields with characteristic 3 is in conformity with the special behaviour of classical generalized hexagons over such fields (the hexagons $H(q)$, $q=3^e$, are self-dual, as remarked before).

Hence we see that with every Desarguesian projective plane $\Pi\cong\mathrm{PG}(2,q)$, there corresponds a full polarized embedding of the corresponding generalized hexagon $\mathcal{S}$ in $\mathrm{PG}(7,q)$, and if $q=3^e$, then there is an additional full polarized embedding of Γ in $\mathrm{PG}(6,q)$.

Remark. Everything in this section can be generalized to the infinite case without notable change.

6.2 Some lemmas

Suppose that the flag geometry $\mathcal{S}=(P,B,\mathtt{I})$ of any projective plane Π is polarly and fully embedded in $\mathrm{PG}(d,q)$, for some $d>1$. We then have the following results.

In a generalized hexagon, we denote the set of points not opposite a given point x by $x^{\perp\!\!\!\perp}$. Also, the set of points not opposite every point of a given line L is denoted by $L^{\perp\!\!\!\perp}$.

The following lemma is easy to prove.

Lemma 6.1 *For every point x of $\mathcal{S}$, the space $\zeta_x = \langle x^{\perp\!\!\perp} \rangle$ is a hyperplane which does not contain any point opposite x in $\mathcal{S}$. In particular, $\zeta_x \neq \zeta_y$ for $x, y \in P$ with $x \neq y$. Also, for every line $L \in B$, the space $\xi_L = \langle L^{\perp\!\!\perp} \rangle$ is at most $(d-2)$-dimensional and $P \cap \xi_L = L^{\perp\!\!\perp}$.*

An almost direct consequence of Lemma 6.1 is the following lemma.

Lemma 6.2 *Every apartment Σ of $\mathcal{S}$ generates a 5-dimensional subspace of* $\mathrm{PG}(d, q)$.

Proof. Suppose Σ generates a projective subspace of dimension < 5. Then there is a point x of Σ which is contained in the subspace generated by the other five points of Σ. Hence, if y is opposite x in Σ, it follows that $x \in \zeta_y$, contradicting Lemma 6.1. $\square$

The following lemma is the key lemma to show an upper bound for d.

Lemma 6.3 *Let U be any subspace of* $\mathrm{PG}(d, q)$ *containing an apartment Σ of $\mathcal{S}$. Then the points x of $\mathcal{S}$ in U for which $x^\perp \subseteq U$ together with the lines of $\mathcal{S}$ in U form a subhexagon $\mathcal{S}'$ of $\mathcal{S}$. Let L, M be two concurrent lines of Σ and let x, y be two points not contained in Σ but incident with respectively L and M. If U contains $x^\perp$ and $y^\perp$, then $\mathcal{S}'$ has some order $(s, 1)$, $1 < s \leq q$.*

Proof. If two lines L, M of $\mathcal{S}$ belong to U, and if these correspond to either two points or two lines of the projective plane Π, then also the line N of $\mathcal{S}$ which corresponds to the joining line or intersection point of the two points or lines, respectively, in Π, belongs to U. Hence we obtain a subplane of Π and that corresponds to a subhexagon $\mathcal{S}'$ of $\mathcal{S}$. The last statement is obvious and follows from the theory of projective planes. $\square$

By Lemma 6.2 we know that $d \geq 5$. Now we show that $d \geq 6$.

Lemma 6.4 *We have $d \geq 6$.*

Proof. Suppose by way of contradiction that $d = 5$. For any line L, the space ξ_L is at most 3-dimensional. Suppose it is 2-dimensional. Then two lines at distance 4 in $\mathcal{S}$ and concurrent with L meet in ξ_L, a contradiction. Hence ξ_L is 3-dimensional, for all lines L of $\mathcal{S}$.

Now let L and M be two opposite lines of $\mathcal{S}$. If $\xi_L \cup \xi_M$ is contained in a 4-space, then the intersection $\xi_L \cap \xi_M$ contains a plane, which meets L in some point of $\mathcal{S}$; hence ξ_M contains a point of L, which must then be non-opposite every point of M, a contradiction. Hence $\xi_L \cup \xi_M$ generates $\mathrm{PG}(5, q)$ and hence $\xi_L \cap \xi_M$ is a line K. Note that the set of points of $\mathrm{PG}(5, q)$ on K is precisely the set $L^{\perp\!\!\perp} \cap M^{\perp\!\!\perp}$. Let N be a line of $\mathcal{S}$ at distance 4 from M and opposite L. Then $L^{\perp\!\!\!\perp} \cap N^{\perp\!\!\!\perp}$ forms a line K' in $\mathrm{PG}(5,q)$ which meets K in a point (on the unique line of $\mathcal{S}$ concurrent with both M and N), and which meets every line of $\mathcal{S}$ concurrent with L in precisely one point. For exactly one element, this point coincides with the intersection of K and K'. Hence at least q lines of $\mathcal{S}$ concurrent with L are contained in the plane π generated by K and K', and consequently two of these lines meet non-trivially, a contradiction.

The lemma is proved. □

Lemma 6.5 *The projective plane* Π *is isomorphic to* $\mathrm{PG}(2,q)$.

Proof. Without proof. □

We now show an upper bound for d.

Lemma 6.6 *We have* $d \leq 8$.

Proof. Consider an apartment Σ in $\mathcal{S}$. We consider a coordinatization over $\mathrm{GF}(q)$ of Π with respect to the triangle T in Π corresponding to Σ (we use homogeneous coordinates). The point in Π with coordinates $(1,1,1)$ is some line E in $\mathrm{PG}(d,q)$. Now let r be some generator of the multiplicative group of $\mathrm{GF}(q)$, and let R be the line of $\mathrm{PG}(d,q)$ which corresponds to the point of Π with coordinates $(1,r,0)$. It is easily seen that the triangle T and the points $(1,1,1)$ and $(1,r,0)$ generate the whole plane Π. Hence, $\mathcal{S}$ is contained in the subspace generated by Σ, E, R. Consequently $\mathrm{PG}(d,q)$ must be generated by Σ, E, R. Since R meets Σ in some point, and since Σ generates a subspace of dimension 5, we see that $d \leq 5+2+1=8$. □

The examples above actually show that the restrictions $6 \leq d \leq 8$ are the best we can do, since there are examples for $d = 6,7,8$.

6.3 The classification

We have the following theorem, due to Thas and Van Maldeghem [1999a, 20**a, 20**b, 20**c, 20**d].

Theorem 6.7 *If the generalized hexagon* $\mathcal{S}$ *of order* $(q,1)$ *is fully embedded in* $\mathrm{PG}(d,q)$*, and if the embedding is polarized, or if* $d \geq 8$*, then it is a classical or semi-classical embedding of the Desarguesian plane* $\mathrm{PG}(2,q)$ *in either* $\mathrm{PG}(6,q)$*, or* $\mathrm{PG}(7,q)$*, or* $\mathrm{PG}(8,q)$.

Proof. Without proof. □

7 Embeddings of generalized hexagons

As we already noted, the hexagons $H(q)$ and $T(q^3, q)$ admit a full, polarized flat embedding in $\mathrm{PG}(6, q)$ and $\mathrm{PG}(7, q^3)$, respectively. We call these embeddings *classical*. Moreover, if q is even, then we may project $H(q)$ from the nucleus of the corresponding (parabolic) quadric onto a hyperplane $\mathrm{PG}(5, q)$ not containing the nucleus. In this way, we obtain a full polarized flat embedding of $H(q)$, q even, which we also call *classical*.

Theorem 7.1 (Thas and Van Maldeghem [1996]) *Let the generalized hexagon $\mathcal{S}$ be fully embedded in $\mathrm{PG}(d, q)$, and suppose that the embedding is polarized and flat. Then $\mathcal{S}$ is isomorphic to either $H(q)$ or $T(q, \sqrt[3]{q})$ and the embedding is classical.*

Proof. Without proof. $\square$

Now consider the classical embeddings of $H(q)$. Taking Grassmann coordinates, we obtain a set of points in $\mathrm{PG}(20, q)$ which corresponds bijectively with the set of points of $H(q)^D$. Moreover, since the classical embeddings are flat, we obtain a full embedding of $H(q)^D$ in some subspace $\mathrm{PG}(d, q)$ of $\mathrm{PG}(20, q)$. One can show that $d = 13$ and that the embedding is polarized (but, of course, not flat). We call this embedding the classical embedding of $H(q)^D$. For q even, we obtain the same embedding of $H(q)^D$ starting from the classical embedding of $H(q)$ in $\mathrm{PG}(5, q)$.

We have now the following characterization of this embedding.

Theorem 7.2 (Thas and Van Maldeghem [20f])** *If $H(q)^D$ is fully embedded in $\mathrm{PG}(d, q)$, $d \geq 13$, then $d = 13$ and the embedding is classical (and hence polarized).*

Proof. We present a rough sketch. For $q = 2$, one shows a more general result by direct computation (see Theorem 8.3 below). Now let $q \geq 3$. We use the fact that every two opposite lines L, M of $H(q)^D$ are contained in a unique subhexagon $\mathcal{H}(L, M)$ of order $(q, 1)$. We then select four different lines L_0, L_1, L_2, L_3 of $H(q)^D$ such that there exist two points x, y at distance 3 from all these lines. Then one shows that $H(q)^D$ is generated (in the sense that two distinct collinear points generate all points on the joining line) by the three subhexagons $\mathcal{H}(L_0, L_1)$, $\mathcal{H}(L_1, L_2)$ and $\mathcal{H}(L_2, L_3)$. If one of these generates an 8-dimensional or 6-dimensional subspace of $\mathrm{PG}(d, q)$, then one shows $d \leq 12$, a contradiction. So these three subhexagons generate a 7-dimensional space, and moreover, the embeddings in these spaces are polarized. Then one shows that the embedding of $H(q)^D$ is polarized, and using this fact, one can show that the embeddings of the three subhexagons mentioned above can be suitably chosen and determine completely and uniquely the embedding of $H(q)^D$. $\square$

Finally we mention the following result.

Theorem 7.3 (Thas and Van Maldeghem [1996]) *No full flat polarized embedding of a thick generalized octagon in a projective space exists.*

8 Polarized, flat and lax embeddings of generalized polygons

A way to obtain lax polarized embeddings which are not full is to start from a full polarized embedding and extend the field of the ambient projective space. Let us refer to this method by saying that *the embedding is obtained from a full embedding by field extension.*

One can hope that all lax polarized embeddings of a certain class of geometries are obtained from full embeddings by field extension. This is sometimes true, but not always. First, let us mention two cases where it is true.

The following theorem was proved in case $d = 3$ by Lefèvre-Percsy [1981].

Theorem 8.1 (Thas and Van Maldeghem [1998a]) *Let $\mathcal{S}$ be a finite thick generalized quadrangle of order (s, t) laxly embedded in the projective space* $\mathrm{PG}(d, q)$. *Suppose the embedding is polarized. Then either the embedding is obtained from a full embedding of a classical generalized quadrangle by field extension, or $\mathcal{S}$ is isomorphic to $W(2)$, the unique generalized quadrangle of order* 2, *and the embedding is a unique one in a projective* 4*-space over an odd characteristic finite field.*

The embedding of $W(2)$ referred to in the last part of the statement is the following.

Let x_1, x_2, x_3, x_4, x_5 be the consecutive vertices of a proper pentagon in $W(2)$. Let $\mathbb{K}$ be any field and identify x_i, $i \in \{1, 2, 3, 4, 5\}$, with the point $(0, \ldots, 0, 1, 0, \ldots, 0)$ of $\mathrm{PG}(4, \mathbb{K})$, where the 1 is in the ith position. Identify the unique point y_{i+3} of $W(2)$ on the line $x_i x_{i+1}$ and different from both x_i and x_{i+1}, with the point $(0, \ldots, 0, 1, 1, 0, \ldots, 0)$ of $\mathrm{PG}(4, \mathbb{K})$, where the 1's are in the ith and the $(i + 1)$th position (subscripts are taken modulo 5). Finally, identify the unique point z_i of the line $x_i y_i$ (it is easy to see that this is indeed a line of $W(2)$) different from both x_i and y_i, with the point whose coordinates are all 0 except in the ith position, where the coordinate is -1, and in the positions $i - 2$ and $i + 2$, where it takes the value 1 (again subscripts are taken modulo 5). It is an elementary excercise to check that this defines a weak embedding of $W(2)$ in $\mathrm{PG}(4, \mathbb{K})$.

Theorem 8.2 (Thas and Van Maldeghem [1998b]) *If $\mathcal{H}$ is a thick finite generalized hexagon laxly embedded in* $\mathrm{PG}(d, q)$, *and if the embedding is both flat and polarized, then $d \in \{5, 6, 7\}$, $\mathcal{H}$ is a classical generalized hexagon, and the embedding arises from a natural embedding by field extension.*

Despite the previous theorem, there exists an analogue for $H(2)$ of the embedding of $W(2)$ in $\mathrm{PG}(4, \mathbb{K})$, $\mathbb{K}$ any field. Let us give a description.

Let $\{x_0, x_1, \ldots, x_6\}$ be the points of $\mathrm{PG}(2,2)$ and let $\{L_7, L_8, \ldots, L_{13}\}$ be the lines of $\mathrm{PG}(2,2)$. The fourteen points and lines of $\mathrm{PG}(2,2)$ are fourteen points of $H(2)$. We identify the point x_i with the 14-tuple $(0, \ldots, 0, 1, 1, 0, \ldots, 0)$, where the 1s are in the $(i+1)$st and $(i+2)$nd positions, and we identify the line L_i with the 14-tuple $(0, \ldots, 0, 1, -1, 0, \ldots, 0)$, where the 1 is in the $(i+1)$st position, and the -1 either in the $(i+2)$nd position (if $i < 13$), or in the first position (if $i = 13$). We identify a flag $\{x_i, L_j\}$ with the 14-tuple obtained by summing the 14-tuples x_i and L_j. Finally, let $\{x_i, L_j\}$ be a non-incident point-line pair. Then there are exactly three points $x_{i_1}, x_{i_2}, x_{i_3}$ unequal x_i which are not incident with L_j, and there are exactly three lines $L_{j_1}, L_{j_2}, L_{j_3}$ uequal L_j which are not incident with x_i. If the set of points incident with L_j is $\{x_{i'_1}, x_{i'_2}, x_{i'_3}\}$, and if the set of lines incident with x_i is $\{L_{j'_1}, L_{j'_2}, L_{j'_3}\}$, then we identify the pair $\{x_i, L_j\}$ with the 14-tuple

$$\frac{1}{2}(x_{i_1} + x_{i_2} + x_{i_3} - x_{i'_1} - x_{'i_2} - x_{i'_3} - x_i + L_{j_1} + L_{j_2} + L_{j_3} - L_{j'_1} - L_{j'_2} - L_{j'_3} - L_j)$$

(we compute this as integers; the result is inside the set of integers). This identification gives us an embedding of $H(2)$ in $\mathrm{PG}(13, \mathbb{K})$, for any field $\mathbb{K}$. One can check as an exercise that this embedding is polarized.

There also exists such an embedding of $H(2)^D$ in $\mathrm{PG}(13, \mathbb{K})$. We will not give an explicit description. We have the following theorem.

Theorem 8.3 (Thas and Van Maldeghem [20f])** *Every lax embedding of $H(2)$ in $\mathrm{PG}(d, \mathbb{K})$, $\mathbb{K}$ any field, $d \geq 13$, is projectively equivalent with the embedding described above. Similarly for $H(2)^D$.*

Proof. Without proof. □

All polarized lax embeddings of quadrangles in arbitrary projective space (over any skew field) are classified by Steinbach and Van Maldeghem [1999, 20**]. They do not all arise from field extensions. There are infinitely many counterexamples, but they are all classified.

The ultimate question for embeddings of polygons, however, is: can one classify all lax embeddings of generalized polygons? In the previous theorems, there always was an extra condition, going from full, to polarized, flat, or the polygon being classical. In the finite case, it turns out that we always need an extra condition, but one of the weakest additional conditions is a condition on the dimension of the ambient projective space, possibly combined with a condition on the order of the polygon. For quadrangles, here is everything that is known put together in one theorem. The proofs of the distinct cases are very different and sometimes quite involved.

Theorem 8.4 (Thas and Van Maldeghem [20e])** *If the generalized quadrangle $\mathcal{S}$ of order (s,t), $s,t > 1$, is laxly embedded in* $\mathrm{PG}(d,q)$, *then* $d \leq 5$. *Furthermore we have the following.*

(i) *If $d = 5$, then $\mathcal{S} \cong Q(5,s)$. Either the embedding is obtained by field extension of a classical embedding, or the embedding is obtained by field extension of an embedding of $Q(5,2)$ in* $\mathrm{PG}(5,q)$, *with q an odd prime number (the latter embedding is not polarized and it is unique up to a special linear transformation; if $q = 3$, then it is full in an appropriate affine space). In all cases, the full automorphism group of $\mathcal{S}$ is induced by* $\mathbf{PGL}_6(q)$.

(ii) *If $d = 4$, then $s \leq t$.*

(a) *If $s = t$, then $\mathcal{S} \cong Q(4,s)$. Either the embedding is obtained by field extension of a classical embedding, or the embedding is obtained by field extension of an embedding of $Q(4,2)$ in* $\mathrm{PG}(4,q)$, *with q an odd prime number (and the latter is polarized and unique up to a linear transformation; if $q = 3$, then it is full in an appropriate affine space), or the embedding is obtained by field extension of an embedding of $Q(4,3)$ in* $\mathrm{PG}(4,q)$, *with* $q \equiv 1 \bmod 3$ *and with q either an odd prime number or the square of a prime number p with* $p \equiv -1 \bmod 3$ *(the latter embedding is not polarized, and it is unique up to a special linear transformation; if $q = 4$, then it is full in an appropriate affine space). In all cases, the full automorphism group of $\mathcal{S}$ is induced by* $\mathbf{PGL}_5(q)$, *except in the last case, where the group* $\mathbf{PSp}_4(3)$ *(which is a proper subgroup of the full automorphism group of $Q(4,3)$) acting naturally as an automorphism group on* $W(3)$ *(which is dual to $Q(4,3)$) is induced on $\mathcal{S}$ by* $\mathbf{PGL}_5(q)$.

(b) *If $t = s+2$, then $s = 2$ and $\mathcal{S} \cong Q(5,2)$.*

(c) *If $t^2 = s^3$, then $\mathcal{S} \cong H(4,s)$ and the embedding is obtained from a classical embedding by field extension.*

(d) *If $\mathcal{S}$ is classical or dual classical, then either we have case* (a) *or case* (c), *or $\mathcal{S} \cong Q(5,s)$ and arises from a projection of an embedding of $\mathcal{S}$ in* $\mathrm{PG}(5,q)$ *(see (i)).*

(iii) *If $d = 3$ and $s = t^2$, then $\mathcal{S} \cong H(3,s)$ and the embedding is obtained from a classical embedding by field extension.*

(iv) *If $d = 3$ and $\mathcal{S}$ is classical or dual classical, but not isomorphic to $W(s)$, with s odd, then either we have case (iii), or the embedding arises from projecting an embedding described in (i) or (ii) above.*

For generalized hexagons, we have the following result.

Theorem 8.5 (Thas and Van Maldeghem [1998b]) *(i) If $\mathcal{H}$ is a thick generalized hexagon laxly embedded in* $\mathrm{PG}(d,q)$*, and if the embedding is flat and polarized, then* $d \in \{5,6,7\}$*,* $\mathcal{H}$ *is a classical generalized hexagon, and the embedding arises from a classical embedding by field extension.*

(ii) If the thick generalized hexagon $\mathcal{H}$ *of order* (s,t) *is flatly and fully embedded in* $\mathrm{PG}(d,s)$*, then* $d \in \{4,5,6,7\}$ *and* $t \leq s$*. Also, if* $d = 7$*, then* $\mathcal{H} \cong T(s, \sqrt[3]{s})$ *and the embedding is the classical one. If* $d = 6$ *and* $t^5 > s^3$*, then* $\mathcal{H} \cong H(s)$ *and the embedding is the classical one. If* $d = 5$ *and* $s = t$*, then* $\mathcal{H} \cong H(s)$*, with* s *even, and the embedding is the natural one.*

(iii) If the thick generalized hexagon $\mathcal{H}$ *of order* (s,t) *is flatly lax embedded in* $\mathrm{PG}(d,q)$*, then* $d \leq 7$*. Also, if* $d = 7$*, then the embedding is also polarized, and hence we can apply (i). If* $d = 6$*, and if* $\mathcal{H}$ *is classical or dual classical with* $s \neq t^3$*, then* $\mathcal{H} \cong H(s)$ *and the embedding is polarized, and hence we can apply (i) again.*

(iv) If the thick generalized hexagon $\mathcal{H}$ *of order* (s,t) *is laxly embedded in* $\mathrm{PG}(d,q)$*, and if the embedding is polarized, then* $d \geq 5$*. Also, if* $d = 5$*, then the embedding is also flat,* s *is even, and hence we can apply (i). If* $d = 6$*, if the embedding is full and if* q *is odd, then* $\mathcal{H}$ *is a classical embedding of* $H(q)$ *in* $\mathrm{PG}(6,q)$*.*

Finally, we mention a fairly strong result for finite polar spaces. It classifies all laxly embedded polar spaces of rank at least 3 in a finite projective space of dimension at least 3, except if the polar space is isomorphic to a symplectic one over a field of odd characteristic, where the ambient projective space must have dimension at least 4.

Theorem 8.6 (Thas and Van Maldeghem [1998c]) *Assume that* $\mathcal{S}$ *is a polar space of rank at least 3 which is laxly embedded in* $\mathrm{PG}(d,q)$*.*

(i) If $d \geq 3$ *and if* $\mathcal{S}$ *is isomorphic either to one of* $Q^+(n,s)$*,* $Q^-(n,s)$*,* $Q(n,s)$*,* $H(n,s)$*, or to* $W(n,s)$ *with, in the latter case,* s *even, then there exists a* $\mathrm{PG}(n,q)$ *containing* $\mathrm{PG}(d,q)$ *and a* $\mathrm{PG}(n-d-1,q)$ *of* $\mathrm{PG}(n,q)$ *skew to* $\mathrm{PG}(d,q)$ *such that* $\mathcal{S}$ *is the projection from* $\mathrm{PG}(n-d-1,q)$ *onto* $\mathrm{PG}(d,q)$ *of a polar space* $\widetilde{\mathcal{S}} \cong \mathcal{S}$ *which is fully embedded in a subspace* $\mathrm{PG}(n,s)$ *of* $\mathrm{PG}(n,q)$ *(and hence classified by* Theorem 4.1*).*

(ii) If $d \geq 4$ *and if* $\mathcal{S}$ *is isomorphic to* $W(2m+1,s)$*,* $m \geq 2$ *and* s *odd, then there exists a* $\mathrm{PG}(2m+1,q)$ *containing* $\mathrm{PG}(d,q)$ *and a* $\mathrm{PG}(2m-d,q)$ *of* $\mathrm{PG}(2m+1,q)$ *skew to* $\mathrm{PG}(d,q)$ *such that* $\mathcal{S}$ *is the projection from* $\mathrm{PG}(2m-d,q)$ *onto* $\mathrm{PG}(d,q)$ *of a polar space* $\widetilde{\mathcal{S}} \cong W(2m+1,s)$ *which is fully and naturally embedded in a subspace* $\mathrm{PG}(2m+1,s)$ *of* $\mathrm{PG}(2m+1,q)$*.*

9 Open cases and conjectures

The most interesting cases left over are the following (and we sometimes phrase conjectural statements):

(*i*) Classify the lax embeddings of $W(s)$, with s odd, in $\mathrm{PG}(3,q)$.

(*ii*) Classify the full embeddings of $H(q)$ in $\mathrm{PG}(d,q)$. Determine the upper bound for d ($d \leq 13$?).

(*iii*) Classify the full embeddings of $H(q)^D$ in $\mathrm{PG}(d,q)$, $d < 13$. Show they all arise from projecting the classical embedding of $H(q)^D$ in $\mathrm{PG}(13,q)$. Classify the lax embeddings of $H(s)^D$ in $\mathrm{PG}(13,q)$. Show they are polarized and all arise from a classical embedding by field extension.

(*iv*) Classify the full embeddings of hexagons of order $(q,1)$ in $\mathrm{PG}(6,q)$. Show that they either are polarized, and hence classified, or arise by projecting a semi-classical embedding in $\mathrm{PG}(7,q)$. Classify the lax polarized embeddings of such hexagons. Show they arise from classical or semiclassical embeddings by field extensions.

(*v*) Classify the polarized embeddings of octagons of order $(q,1)$ in $\mathrm{PG}(d,q)$. Use this to classify (polarized) embeddings of Ree-Tits octagons in $\mathrm{PG}(d,q)$, for appropriate d.

(*vi*) Classify (polarized) embeddings of $T(q,q^3)$ in $\mathrm{PG}(25,q)$, for q even, and in $\mathrm{PG}(27,q)$, for q odd.

Some partial answers are already obtained by the authors.

References

[1998] **S. Ball** and **A. Blokhuis**, An easier proof of the maximal arcs conjecture, *Trans. Amer. Math. Soc.* **126** (1998), 3377 – 3380.

[1997] **S. Ball, A. Blokhuis** and **F. Mazzocca**, Maximal arcs in Desarguesian planes of odd order do not exist, *Combinatorica* **17** (1997), 31 – 47.

[1963] **R. C. Bose**, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math* **13** (1963), 389 – 419.

[1972] **R. C. Bose** and **S. S. Shrikhande**, Geometric and pseudo-geometric graphs $(q^2+1, q+1, 1)$, *J. Geometry* **2** (1972), 75 – 94.

[1976] **F. Buekenhout**, Characterizations of semi quadrics: A survey, **in** *Teorie combinatorie* (ed. B. Segre), Roma Accad. Naz. Lincei (1976), 393 – 421.

[1974] **F. Buekenhout** and **C. Lefèvre**, Generalized quadrangles in projective spaces, *Arch. Math.* **25** (1974), 540 – 552.

[1976] **F. Buekenhout** and **C. Lefèvre**, Semi-quadratic sets in projective spaces, *J. Geom.* **7** (1976), 17 – 42.

[1975] **P. J. Cameron**, Partial quadrangles, *Quart. J. Math. Oxford Ser.* (2) **26** (1975), 61 – 73.

[1978] **P. J. Cameron, J. M. Goethals** and **J. J. Seidel**, Strongly regular graphs having strongly regular subconstituents, *J. Algebra* **55** (1978), 257 – 280.

[1978] **F. De Clerck** and **J. A. Thas**, Partial geometries in finite projective spaces, *Arch. Math.* **30** (1978), 537 – 540.

[1995] **F. De Clerck** and **H. Van Maldeghem**, Some classes of rank 2 geometries, **in** Handbook of Incidence Geometry, Buildings and Foundations (ed. F. Buekenhout), Chapter 10, North-Holland (1995), 433 – 475.

[1969] **R. H. F. Denniston**, Some maximal arcs in finite projective planes, *J. Combin. Theory Ser.* A **6** (1969), 317 – 319.

[1980a] **K. J. Dienst**, Verallgemeinerte Vierecke in Pappusschen projektiven Räumen, *Geom. Dedicata* **9** (1980), 199 – 206.

[1980b] **K. J. Dienst**, Verallgemeinerte Vierecke in projektiven Räumen, *Arch. Math.* (Basel) **35** (1980), 177 – 186.

[1964] **W. Feit** and **G. Higman**, The non-existence of certain generalized polygons, *J. Algebra* **1** (1964), 114 – 131.

[1981] **W. H. Haemers** and **C. Roos** An inequality for generalized hexagons, *Geom. Dedicata* **10** (1981), 219 – 222.

[1974] **D. G. Higman**, Invariant relations, coherent configurations and generalized polygons, **in** *"Combinatorics"* (1974, ed. M. Hall and J. H. Van Lint), D. Reidel, Dordrecht (1974), 347 – 363.

[1991] **J. W. P. Hirschfeld** and **J. A. Thas**, *General Galois Geometries*, Oxford University Press, Oxford, 1991.

[1995] **M. Joswig** and **H. Van Maldeghem**, An essay on the Ree octagons, *J. Alg. Combin.* **4** (1995), 145 – 164.

[1977] **C. Lefèvre-Percsy**, Semi-quadriques en tant que sous-ensembles des espaces projectifs, *Bull. Soc. Math. Belg.* **29** (1977), 175 – 183.

[1981] **C. Lefèvre-Percsy**, Quadrilatères généralisés faiblement plongés dans PG$(3, q)$, *European J. Combin.* **2** (1981), 249 – 255.

[1998] **R. Mathon**, A new family of partial geometries, *Geom. Dedicata* **73** (1998), 11 – 19.

[1973] **D. Olanda**, Sistemi rigati immersi in uno spazio proiettivo, *Rend. Accad. Naz. Lincei* **62** (1977), 489 – 499.

[1977] **D. Olanda**, Sistemi rigati immersi in uno spazio proiettivo, *Ist. Mat. Univ. Napoli Rel.* **26** (1973), 1 – 21.

[1984] **S. E. Payne** and **J. A. Thas**, *Finite Generalized Quadrangles*, Pitman *Res. Notes Math. Ser.* **110**, London, Boston, Melbourne, 1984.

[1999] **A. Steinbach** and **H. Van Maldeghem**, Generalized quadrangles weakly embedded of degree > 2 in projective space, *Forum Math.* **11** (1999), 139 – 176.

[20**] **A. Steinbach** and **H. Van Maldeghem**, Generalized quadrangles weakly embedded of degree 2 in projective space, to appear in *Pacific J. Math.*

[1973] **J. A.Thas**, Construction of partial geometries, *Simon Stevin* **46** (1974), 95 – 98.

[1974] **J. A.Thas**, Construction of maximal arcs and partial geometries, *Geom. Dedicata* **3** (1974), 61 – 64.

[1980] **J. A. Thas**, Construction of maximal arcs and dual ovals in translation planes, *European J. Combin.* **1** (1980), 189 – 192.

[1995] **J. A. Thas**, Generalized polygons, **in** *Handbook of Incidence Geometry, Buildings and Foundations*, (ed. F.Buekenhout), Chapter 9, North-Holland (1995), 383 – 431.

[1977] **J. A. Thas** and **P. De Winne**, Generalized quadrangles in finite projective spaces, *J. Geom.* **10** (1977), 126 – 137.

[1996] **J. A. Thas** and **H. Van Maldeghem**, Embedded thick finite generalized hexagons in projective space, *J. London Math. Soc. (2)* **54** (1996), 566 – 580.

[1998a] **J. A. Thas** and **H. Van Maldeghem**, Generalized quadrangles weakly embedded in finite projective space, *J. Statist. Plann. Inference* **73** (1998), 353 – 361.

[1998b] **J. A. Thas** and **H. Van Maldeghem**, Flat lax and weak lax embeddings of finite generalized hexagons, *European J. Combin.* **19** (1998), 733 – 751.

[1998c] **J. A. Thas** and **H. Van Maldeghem**, Lax embeddings of polar spaces in finite projective spaces, *Forum Math.* **11** (1999), 349 – 367.

[1999a] **J. A. Thas** and **H. Van Maldeghem**, On embeddings of the flag geometries of projective planes in finite projective spaces, *Des. Codes Cryptogr.* **17** (1999), 97 – 104.

[20**a] **J. A. Thas** and **H. Van Maldeghem**, Classification of embeddings of the flag geometries of projective planes in finite projective spaces, Part I, to appear in *J. Combin. Theory Ser.* A.

[20**b] **J. A. Thas** and **H. Van Maldeghem**, Classification of embeddings of the flag geometries of projective planes in finite projective spaces, Part 2, to appear in *J. Combin. Theory Ser.* A.

[20**c] **J. A. Thas** and **H. Van Maldeghem**, Classification of embeddings of the flag geometries of projective planes in finite projective spaces, Part 3, to appear in *J. Combin. Theory Ser.* A.

[20**d] **J. A. Thas** and **H. Van Maldeghem**, Some remarks on embeddings of the flag geometries of projective planes in finite projective spaces, to appear in *J. Geom.*

[20**e] **J. A. Thas** and **H. Van Maldeghem**, Lax embeddings of generalized quadrangles, to appear in *Proc. London Math. Soc.* (3).

[20**f] **J. A. Thas** and **H. Van Maldeghem**, Full embeddings of the finite dual split Cayley hexagons, manuscript in preparation.

[1959] **J. Tits**, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13 – 60.

[1998] **H. Van Maldeghem**, *Generalized Polygons*, Birkhäuser-Verlag, *Monographs in Mathematics* **93**, Basel, Boston, Berlin, 1998.

[20**] **H. Van Maldeghem**, An elementary construction of the split Cayley hexagon $H(2)$, to appear in *Atti Sem. Mat. Fis. Univ. Modena* .

[1973] **W. D. Wallis**, Configurations arising from maximal arcs, *J. Combin. Theory Ser.* A **15** (1973), 115 – 119.