(Partial) t-spreads and minimal t-covers in finite projective spaces

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Abstract

These notes discuss (partial) t-spreads and minimal t-covers of finite projective spaces. When a t-spread in $\mathrm{PG}(N,q)$ exists, we focus on the problem of the extendability of partial t-spreads, with small deficiency, to t-spreads. When no t-spreads exist in $\mathrm{PG}(N,q)$, then we discuss the problem of finding the largest cardinality for which partial t-spreads exist, and we also discuss the smallest cardinality of a t-cover of $\mathrm{PG}(N,q)$. If a t-spread in $\mathrm{PG}(N,q)$ exists, then we also study minimal t-covers of $\mathrm{PG}(N,q)$ with small excess. We concentrate on the common nature of all of these problems, the techniques used for solving these problems, and the links of these problems with blocking sets in $\mathrm{PG}(2,q)$.

1 t-Spreads of projective spaces

1.1 Definition and Existence

Definition 1.1 Let $\mathcal{P}$ be a projective space. A *t-spread* of $\mathcal{P}$ is a set $\mathcal{S}$ of t-dimensional subspaces of $\mathcal{P}$ which partitions $\mathcal{P}$. That is, every point of $\mathcal{P}$ is contained in exactly one t-space of $\mathcal{S}$.

If $\mathcal{P} = \mathrm{PG}(N,q)$ is a finite projective space, then a t-spread can only exist if the number of points of a t-space divides the number of points of the whole space, i.e. if $(\frac{q^{t+1}-1}{q-1}) \mid (\frac{q^{N+1}-1}{q-1})$ and hence $(q^{t+1}-1) \mid (q^{N+1}-1)$, which holds if and only if $(t+1) \mid (N+1)$. André [1] showed that this necessary condition is also sufficient.

Theorem 1.2 *In* $\mathrm{PG}(N,q)$, *a t-spread exists if and only if* $t+1$ *divides* $N+1$.

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In these notes, particular attention will be given to 1-spreads in $\mathrm{PG}(3,q)$. In $\mathrm{PG}(3,q)$, we will also simply talk about *spreads* instead of 1-spreads.

We give two basic constructions of t-spreads:

Construction 1. Suppose that $(t+1) \mid (N+1)$. Consider the fields $F_0 = \mathrm{GF}(q)$, $F_1 = \mathrm{GF}(q^{t+1})$ and $F_2 = \mathrm{GF}(q^{N+1})$. Then $F_0 \subseteq F_1 \subseteq F_2$. The field F_2 is an $(N+1)$-dimensional vector space V over F_0. The subspaces of V form the projective space $\mathcal{P} = \mathrm{PG}(N,q)$. The field F_1 is a $(t+1)$-dimensional subspace of V and hence a t-dimensional (projective) subspace of $\mathcal{P}$. The same holds for all cosets $a \cdot F_1$ ($a \in F_2$). These cosets partition the multiplicative group of F_2. Hence they form a t-spread of $\mathcal{P}$.

Construction 2. Suppose again that $(t+1) \mid (N+1)$. Embed the space $\mathcal{P} = \mathrm{PG}(N,q)$ into the space $\bar{\mathcal{P}} = \mathrm{PG}(N,q^{t+1})$. Then the following can be shown: the t-dimensional subspaces of $\mathcal{P}$ cover (in their extensions to $\bar{\mathcal{P}}$) all points of $\bar{\mathcal{P}}$. Furthermore, there exists a $\left(\frac{N+1}{t+1} - 1\right)$-dimensional subspace U of $\bar{\mathcal{P}}$ such that no point of U is contained in a $(t-1)$-dimensional subspace of $\mathcal{P}$. Hence each point $P \in U$ is contained in a unique t-dimensional subspace of $\mathcal{P}$. This latter subspace is the space $\langle P, P^{\sigma}, P^{\sigma^2}, \ldots, P^{\sigma^t}\rangle$, where σ is the automorphism of $\bar{\mathcal{P}}$ induced by the Frobenius-automorphism $x \mapsto x^q$. These t-dimensional subspaces form a t-spread of $\mathcal{P}$ (see [7]).

Both constructions can be done also in an infinite projective space $\mathrm{PG}(N,K)$, provided that there exists a field extension of degree $t+1$ of K. For instance, this gives a 1-spread in $\mathrm{PG}(3,\mathbb{R})$.

Both constructions yield regular t-spreads as defined in the following subsection.

Topics which will be investigated in detail in these notes are *maximal partial t-spreads* and *minimal t-covers* in $\mathrm{PG}(N,q)$. To situate these substructures, we define them in the following definitions.

Definition 1.3 (1) A *partial t-spread* $\mathcal{S}$ of $\mathrm{PG}(N,q)$ is a set of t-dimensional subspaces of $\mathrm{PG}(N,q)$ which are pairwise disjoint.
(2) A *maximal partial t-spread* $\mathcal{S}$ of $\mathrm{PG}(N,q)$ is a set of t-dimensional subspaces of $\mathrm{PG}(N,q)$ which cannot be extended to a larger partial t-spread.
(3) Let $\mathcal{S}$ be a partial t-spread of $\mathrm{PG}(N,q)$, $(t+1)|(N+1)$, of cardinality $(q^{N+1}-1)/(q^{t+1}-1)-\delta$. Then the parameter δ is called the *deficiency* of $\mathcal{S}$.
(4) A point of $\mathrm{PG}(N,q)$ is called a *hole* with respect to a partial t-spread $\mathcal{S}$ when it does not belong to an element of $\mathcal{S}$.

Definition 1.4 (1) A *t-cover* $\mathcal{C}$ of $\mathrm{PG}(N,q)$ is a set of t-dimensional subspaces of $\mathrm{PG}(N,q)$ which cover all of the points of $\mathrm{PG}(N,q)$.
(2) A t-cover $\mathcal{C}$ is *minimal* when no proper subset of it is still a t-cover.

We will also denote a 1-cover of $\mathrm{PG}(3,q)$ by a *cover*.

1.2 Regular and geometric t-spreads

Let $\mathcal{S}$ be a t-spread in $\mathcal{P} = \mathrm{PG}(N,q)$. Let $\mathcal{L}$ be the set of $(2t+1)$-dimensional subspaces of $\mathcal{P}$ spanned by pairs of elements of $\mathcal{S}$. If for each $S \in \mathcal{S}$ and each $L \in \mathcal{L}$, either $S \subset L$ or $S \cap L = \emptyset$, then the t-spread $\mathcal{S}$ is called *geometric* or *normal*. This means that $\mathcal{S}$ induces a t-spread in every element of $\mathcal{L}$.

The notion "geometric" is due to Baer [2] and is justified by the following theorem [2, 2.3].

Theorem 1.5 *Let $\mathcal{S}$ be a geometric t-spread in $\mathcal{P} = \mathrm{PG}(N,q)$, where $N+1 = k(t+1)$. Then the incidence structure $(\mathcal{S}, \mathcal{L}, I)$, where $\mathcal{L}$ is the set of $(2t+1)$-dimensional subspaces of $\mathcal{P}$ spanned by pairs of elements of $\mathcal{S}$ and I is the inclusion, is a projective space isomorphic to* $\mathrm{PG}(k-1, q^{t+1})$.

This projective space can be seen very well in Construction 2. Here it corresponds (via intersection) to the projective space U.

In the case $k = 2$, each hyperplane of $\mathcal{P}$ contains exactly one element of $\mathcal{S}$ (proof: counting). Hence the concept of a t-spread in $\mathrm{PG}(2t+1,q)$ is self-dual. For $k > 2$, the geometry of subspaces spanned by elements of $\mathcal{S}$ is self-dual.

For the definition of a regular t-spread, we need first the notion of a t-regulus.

Let U_1, U_2, U_3 be three mutually disjoint t-dimensional subspaces of $\mathcal{P} = \mathrm{PG}(2t+1,q)$. Then for every point $P \in U_1$, the space $\langle P, U_2 \rangle$ intersects U_3 in a point, hence there is a unique line through P intersecting U_2 and U_3. This means that there are $\frac{q^{t+1}-1}{q-1}$ transversal lines through U_1, U_2, U_3. Furthermore, there are $q+1$ mutually disjoint t-spaces $U_1, \ldots, U_{q+1}$ being transversal to these lines. The set of these $q+1$ spaces is called a *t-regulus*.

If $t = 1$, then a 1-regulus is one of the two sets of generators of a hyperbolic quadric and the two sets of generators of a hyperbolic quadric are called *opposite reguli*.

Definition 1.6 A geometric t-spread $\mathcal{S}$ of a projective space $\mathcal{P}$ is called *regular*, if for any $U_1, U_2, U_3 \in \mathcal{S}$ with $U_3 \in \langle U_1, U_2 \rangle$, the t-regulus defined by U_1, U_2, U_3 is completely contained in $\mathcal{S}$.

Theorem 1.7 (Bruck [13]) *For any N, t, with $(t+1) \mid (N+1)$, there is (up to projective collineation) a unique regular t-spread of* $\mathrm{PG}(N,q)$.

Hence Constructions 1 and 2 yield the same t-spreads.

In the important case $t = 1$, one can easily get non-regular 1-spreads from a regular one by replacing some reguli by their opposite reguli. A spread which can be constructed in this way is called *semi-regular.*

A spread is called *aregular* if it contains no regulus. We give one example [34, 17.3.3].

Let $q = p^h$, $h \geq 2$, and let $x^{p+1} + bx - c$ be a polynomial without roots in $F = \mathrm{GF}(q)$. Then the set

$$\mathcal{S} = \{\langle(1,0,0,0),(0,1,0,0)\rangle\} \cup \{\langle(z,y,1,0),(cy^p, z^p + by^p, 0, 1)\rangle \mid\mid (y,z) \in F^2\}$$

is an aregular spread of $\mathrm{PG}(3,q)$.

1.3 t-Spreads and Translation Planes

The following construction is due to André [1] and Bruck and Bose [14].

Let $\mathcal{S}$ be a t-spread in $\mathcal{P} = \mathrm{PG}(N,q)$ with $N+1 = k(t+1)$. We embed $\mathcal{P}$ as a hyperplane into $\mathcal{P}' = \mathrm{PG}(N+1,q)$. Consider the following incidence structure $\mathcal{A}$.

- The points are the points of $\mathcal{P}' \setminus \mathcal{P}$.
- The lines are the $(t+1)$-spaces of $\mathcal{P}'$ intersecting $\mathcal{P}$ in an element of $\mathcal{S}$.
- Incidence is inclusion.

This incidence structure $\mathcal{A}$ is a *net of order* q^{t+1} *and of degree* $(q^{N+1}-1)/(q^{t+1}-1)$. This means that the lines contain q^{t+1} points, that every point lies on $(q^{N+1}-1)/(q^{t+1}-1)$ lines, and that there exist $(q^{N+1}-1)/(q^{t+1}-1)$ *parallel classes* of pairwise disjoint lines, such that lines of distinct parallel classes intersect in one point. Every such parallel class consists of all the lines of $\mathcal{A}$ passing through a common element of $\mathcal{S}$.

If instead of a t-spread $\mathcal{S}$ of $\mathcal{P} = \mathrm{PG}(N,q)$ with $N+1 = k(t+1)$, a partial t-spread of deficiency δ is used, then a net $\mathcal{A}$ of order q^{t+1} and degree $(q^{N+1}-1)/(q^{t+1}-1) - \delta$ is obtained.

A net $\mathcal{A}$ of order k and degree $k+1$ is an affine plane of order k. The *deficiency* of a net of order k and degree r is equal to $\delta = k + 1 - r$.

When $\mathcal{S}$ is a geometric t-spread, then the incidence structure $\mathcal{A}$ is an affine space of dimension k and order q^{t+1}. This space can be extended to a projective space by adding the elements of $\mathcal{S}$ as ideal points and the spaces $\langle S_1, S_2\rangle$ $(S_1, S_2 \in \mathcal{S})$ as ideal lines.

For $q > 2$, the space constructed in this way is desarguesian if and only if the t-spread $\mathcal{S}$ is regular (Bruck and Bose [15, 12.1]). In particular, the following theorem holds.

Theorem 1.8 *Any geometric t-spread in* $\mathrm{PG}(N,q)$*, where* $N+1=k(t+1)$ *and* $k \geq 3$*, is regular.*

The most important case of the above construction is the case $N=2t+1$. In this case, we get a *translation plane* Π, that is an affine plane whose translation group is transitive on the points. Here, a translation is a (P,l_∞)-elation, i.e. a collineation fixing all parallel classes and fixing one parallel class elementwise. The translations of Π are simply the translations of $\mathcal{P}' \setminus \mathcal{P}$.

Conversely, every translation plane arises from the construction above, and hence corresponds to some t-spread of a projective space (Bruck and Bose [14]).

The correspondence to translation planes was one of the main reasons for the consideration of t-spreads. Another motivation was a paper of Bruck [13] who showed that a big set of parallelisms of a projective space could lead to a projective plane whose order is not a prime power. Here, a *parallelism* of $\mathrm{PG}(3,q)$ is a set of spreads such that every line of the space is contained in exactly one of the spreads.

1.4 Connection with unitals

In the case $(k,t)=(2,1)$, the correspondence given above can be used to construct unitals in the translation plane Π. Here, a *unital* in a plane of order q^2 is a set $\mathcal{U}$ of q^3+1 points such that each line contains either 1 or $q+1$ points of $\mathcal{U}$.

For this, let $s \in \mathcal{S}$ and $P \in s$. Let $\mathcal{O}$ be an ovoid in $\mathcal{P}'/P$, i.e. a set of q^2+1 lines through P no three of which are coplanar, such that $\mathcal{O} \cap \mathcal{P} = s$. (That is, $\mathcal{P}/P$ is the tangent plane to the ovoid in s/P.)

Let $\mathcal{U}'$ be the set of points of $\mathcal{O} \setminus \mathcal{P}$. Then $\mathcal{U} := \mathcal{U}' \cup \{s\}$ is a unital of Π. A unital constructed in this way is called *Buekenhout-Metz unital* [19, 40].

There exists a slight modification of this construction. Suppose that the spread $\mathcal{S}$ contains a regulus R. Let $\mathcal{Q}$ be a parabolic quadric in $\mathcal{P}'$ intersecting $\mathcal{P}$ in the points of R. Then the set of points of Π corresponding to points of $\mathcal{Q}$ is a unital. A unital constructed in this way is called *Buekenhout unital.*

If the spread $\mathcal{S}$ is regular, these constructions yield unitals in $\mathrm{PG}(2,q^2)$. These are the only known unitals in desarguesian planes.

1.5 Connection with Ovoids of the Klein quadric

Let $\mathcal{Q} = O^+(5,q)$ be a hyperbolic quadric in $\mathrm{PG}(5,q)$ (a so-called *Klein quadric*). On $\mathcal{Q}$, there are two classes M_1, M_2 of planes such that two planes are in the same class if and only if they intersect in a point. Consider the following geometry.

- The points are the planes of M_1.
- The lines are the points of $\mathcal{Q}$.
- The "planes" are the planes of M_2.
- Incidence is inclusion.

Then this geometry is isomorphic to $\mathcal{P} = \mathrm{PG}(3,q)$. Conversely, the lines of $\mathcal{P}$ correspond to the points of a Klein quadric via *Plücker coordinates*: if we assign to the line $\langle (x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\rangle \in \mathcal{P}$ the coordinate point $\langle (x_0y_1 - x_1y_0, x_0y_2 - x_2y_0, x_0y_3 - x_3y_0, x_1y_2 - x_2y_1, x_3y_1 - x_1y_3, x_2y_3 - x_3y_2)\rangle \in \mathrm{PG}(5,q)$, this gives a bijective mapping of the lines of $\mathcal{P}$ onto the points of a Klein quadric [34, 15.2, 15.4].

What happens to spreads of $\mathcal{P}$ under this correspondence? A spread is a set $\mathcal{S}$ of lines such that every point of $\mathcal{P}$ is contained in a unique element of $\mathcal{S}$ and every plane of $\mathcal{P}$ contains a unique element of $\mathcal{S}$. Hence a spread $\mathcal{S}$ corresponds to a set $\mathcal{S}'$ of points on the Klein quadric $\mathcal{Q}$ such that every plane of $\mathcal{Q}$ contains a unique point of $\mathcal{S}'$. Such a point set is called *ovoid*. Hence there is a one-to-one correspondence between spreads of $\mathrm{PG}(3,q)$ and ovoids of the Klein quadric. For instance, a regular spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$ defines an ovoid equal to a 3-dimensional elliptic quadric of $\mathcal{Q}$.

A partial spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$ containing k lines defines a set $\mathcal{S}'$ of k points of $\mathcal{Q}$ such that every plane of $\mathcal{Q}$ intersects $\mathcal{S}'$ in at most one point. Such a set of k points of $\mathcal{Q}$ is called a *k-cap* of $\mathcal{Q}$.

1.6 Blocking sets in $\mathrm{PG}(2,q)$

Definition 1.9 (1) A *blocking set* in $\mathrm{PG}(2,q)$ is a set of points intersecting every line of $\mathrm{PG}(2,q)$.
(2) A blocking set is called *trivial* when it contains a line of $\mathrm{PG}(2,q)$.
(3) A blocking set is called *minimal* when no proper subset of it is still a blocking set.

In the study of blocking sets, a main problem is to find and characterize the smallest non-trivial blocking sets of $\mathrm{PG}(2,q)$.

In the next theorems, we collect the most important characterization results on small non-trivial blocking sets.

Theorem 1.10 (1) (Bruen [16]) *The smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$, *q square, have cardinality $q+\sqrt{q}+1$ and are equal to Baer subplanes* $\mathrm{PG}(2,\sqrt{q})$.
(2) (Blokhuis, Storme, Szőnyi [11]) *In* $\mathrm{PG}(2,q)$, *q square, $q=p^h, h>2, p\geq 5$, p prime, every non-trivial blocking set K of cardinality $|K| < q+q^{2/3}+1$ contains a Baer subplane.*

(3) (Szőnyi [44]) *In* $\mathrm{PG}(2,q)$, $q = p^2$, p *prime, every non-trivial blocking set* K *of cardinality* $|K| < 3(q+1)/2$ *contains a Baer subplane.*
(4) (Blokhuis [8]) *In* $\mathrm{PG}(2,q)$, q *prime,* $q > 2$, $|K| \geq 3(q+1)/2$ *for every non-trivial blocking set* K.
(5) (Blokhuis, Storme, Szőnyi [11]) *In* $\mathrm{PG}(2,q)$, q *non-square,* $q = p^h, h > 2, p \geq 5$, p *prime,* $|K| \geq q + q^{2/3} + 1$ *for every non-trivial blocking set* K.

Theorem 1.11 (Polverino [41], Polverino and Storme [43]) (a) *In* $\mathrm{PG}(2,p^3)$, $p = p_0^h$, $h \geq 1$, h *odd,* p_0 *prime,* $p_0 \geq 5$, $p \neq 5$, *the smallest possible size for a non-trivial minimal blocking set is* $p^3 + p^2 + 1$ *and such a blocking set is projectively equivalent to the set*

$$K = \{(x, Tr(x), 1) || x \in \mathrm{GF}(p^3)\} \cup \{(x, Tr(x), 0) || \ x \in \mathrm{GF}(p^3) \setminus \{0\}\},$$

with $Tr : \mathrm{GF}(p^3) \to \mathrm{GF}(p) : x \mapsto x + x^p + x^{p^2}$ *the trace function from* $\mathrm{GF}(p^3)$ *to* $\mathrm{GF}(p)$.

(b) *In* $\mathrm{PG}(2,p^3)$, $p = p_0^h$, $h \geq 1$, h *odd,* p_0 *prime,* $p_0 \geq 5$, $p \neq 5$, *the second smallest possible size for a non-trivial minimal blocking set is* $p^3 + p^2 + p + 1$ *and such a blocking set is projectively equivalent to the set*

$$K = \{(x, x^p, 1) || x \in \mathrm{GF}(p^3)\} \cup \{(x, x^p, 0) || x \in \mathrm{GF}(p^3) \setminus \{0\}\}.$$

Theorem 1.12 (Polverino and Storme [43]) *In* $\mathrm{PG}(2,p^3)$, $p = p_0^h$, $h \geq 2$, h *even,* p_0 *prime,* $p_0 \geq 5$, *the smallest non-trivial minimal blocking sets are:*

(a) *A Baer subplane* $\mathrm{PG}(2,p^{3/2})$ *of cardinality* $p^3 + p^{3/2} + 1$.

(b) *A minimal blocking set of size* $p^3 + p^2 + 1$ *and such a blocking set is projectively equivalent to the set*

$$K = \{(x, Tr(x), 1) || x \in \mathrm{GF}(p^3)\} \cup \{(x, Tr(x), 0) || \ x \in \mathrm{GF}(p^3) \setminus \{0\}\},$$

with Tr *the trace function from* $\mathrm{GF}(p^3)$ *to* $\mathrm{GF}(p)$.

(c) *A minimal blocking set of size* $p^3 + p^2 + p + 1$ *and such a blocking set is projectively equivalent to the set*

$$K = \{(x, x^p, 1) || x \in \mathrm{GF}(p^3)\} \cup \{(x, x^p, 0) || x \in \mathrm{GF}(p^3) \setminus \{0\}\}.$$

Remark 1.13 A particular property of the minimal blocking sets of sizes $p^3 + p^2 + 1$ and $p^3 + p^2 + p + 1$ is that they are characterised to be projected subgeometries $\mathrm{PG}(3,p)$ in the plane $\mathrm{PG}(2,p^3)$. Consider the subgeometry $\mathrm{PG}(3,p)$ naturally embedded in $\mathrm{PG}(3,p^3)$. Consider a point r of $\mathrm{PG}(3,p^3)$ not belonging to the subgeometry $\mathrm{PG}(3,p)$ and project $\mathrm{PG}(3,p)$ from this point r onto a plane not passing through r. If r belongs to a line l of the subgeometry, then the projection has $p^3 + p^2 + 1$ points and is the minimal blocking set of this cardinality in $\mathrm{PG}(2,p^3)$. If r does not belong to a line of the subgeometry, then the projection has $p^3 + p^2 + p + 1$ points, and is the minimal blocking set of this cardinality in $\mathrm{PG}(2,p^3)$.

1.7 First link between blocking sets in PG$(2, q)$ and geometric t-spreads

Consider a regular t-spread $\mathcal{S}$ in PG(N, q), $N+1 = 3(t+1)$. Then Theorem 1.5 shows that this geometric t-spread can be used to construct a projective plane isomorphic to PG$(2, q^{t+1})$. The points of this projective plane are the elements of $\mathcal{S}$ and the lines correspond to the $(2t+1)$-dimensional subspaces of PG(N, q) defined by two elements of $\mathcal{S}$. These latter subspaces form the set $\mathcal{L}$ of Theorem 1.5.

Let Π be a $(t+1)$-dimensional subspace of PG(N, q). Consider the set of elements of $\mathcal{S}$ which are intersected by Π and consider the set of points B that they define in PG$(2, q^{t+1})$. Then this latter set B is a blocking set of PG$(2, q^{t+1})$ as Π intersects every element of $\mathcal{L}$ in at least one point.

This idea to construct blocking sets in PG$(2, q)$ was first mentioned by Lunardon [36] who called these blocking sets *linear*. The blocking sets discussed in Remark 1.13 are examples of linear blocking sets.

This method of constructing blocking sets has led to many new infinite classes of non-trivial blocking sets. For concrete examples of such new blocking sets, we refer to Polverino and Polito [42].

The same construction can also be used to construct blocking sets in projective spaces of higher dimension.

2 t-Covers and partial t-spreads in PG(N, q), where $t+1$ does not divide $N+1$

2.1 Elementary bounds and examples

If $t+1$ does not divide $N+1$, then a t-spread of $\mathcal{P} = $ PG(N, q) does not exist. In this case the question arises, what are the biggest partial t-spreads of $\mathcal{P}$ (i.e. sets of mutually disjoint t-spaces), and what are the smallest t-covers of $\mathcal{P}$ (i.e. sets of t-spaces covering all the points of $\mathcal{P}$). Let $N = k(t+1)+r-1$, where $1 \le r \le t$. The value of r plays a crucial role in our question.

Proposition 2.1 *Let* $N = k(t+1)+r-1$, *where* $1 \le r \le t$, *and let* $\mathcal{P} = $ PG(N, q).

(a) *A partial t-spread of* $\mathcal{P}$ *contains at most* $q^r \cdot \frac{q^{k(t+1)}-1}{q^{t+1}-1}$ *elements.*

(b) *A t-cover of* $\mathcal{P}$ *contains at least* $q^r \cdot \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$ *elements.*

Proof: The number of points of $\mathcal{P}$ is $\frac{q^{k(t+1)+r}-1}{q-1} = q^r\frac{q^{k(t+1)}-1}{q-1} + \frac{q^r-1}{q-1}$, where the first term is a multiple of $\frac{q^{t+1}-1}{q-1}$, the number of points of a t-dimensional subspace. From this the assertion follows. □

We shall give an example due to Beutelspacher [5] achieving this bound in the case of t-covers and being close to it in the case of partial t-spreads.

Lemma 2.2 *Let U be an s-dimensional subspace of $\mathcal{P} = \mathrm{PG}(s+t+1,q)$, where $s \geq t$. Then there is a set $\mathcal{S}$ of t-dimensional subspaces of $\mathcal{P}$ which do not intersect U such that every point of $\mathcal{P}\setminus U$ is contained in exactly one element of $\mathcal{S}$.*

Proof: Let $\mathcal{P}' = \mathrm{PG}(2s+1,q) \supseteq \mathcal{P}$, and let $\mathcal{S}'$ be an s-spread of $\mathcal{P}'$ which contains U. Each element of $\mathcal{S}'\setminus\{U\}$ intersects $\mathcal{P}$ in a t-dimensional subspace. (As it is disjoint to U, the intersection cannot be bigger.) The set of these intersections is the set $\mathcal{S}$ we are looking for. □

Corollary 2.3 *Let U be an $(r+t)$-dimensional subspace of $\mathcal{P} = \mathrm{PG}(k(t+1)+r-1,q)$, where $1 \leq r \leq t$ and $k > 1$. Then there is a set $\mathcal{S}$ of t-dimensional subspaces of $\mathcal{P}$ which do not intersect U such that every point of $\mathcal{P}\setminus U$ is contained in exactly one element of $\mathcal{S}$.*

Proof: Apply the previous lemma for $s = t+r, 2(t+1)+r-1, \ldots, (k-1)(t+1)+r-1$. □

Corollary 2.4 *Let $t,k \geq 1$ and $1 \leq r \leq t$. In $\mathcal{P} = \mathrm{PG}(k(t+1)+r-1,q)$ there is a partial t-spread $\mathcal{S}$ with $|\mathcal{S}| = q^r\cdot\frac{q^{k(t+1)}-1}{q^{t+1}-1} - q^r + 1$.*

Proof: Add one t-dimensional subspace of U to the set of Corollary 2.3. □

Corollary 2.5 *Let $t,k \geq 1$ and $1 \leq r \leq t$. In $\mathcal{P} = \mathrm{PG}(k(t+1)+r-1,q)$ there is a t-cover $\mathcal{S}$ with $|\mathcal{S}| = q^r\cdot\frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$.*

Proof: Consider the set $\mathcal{S}_0$ of Corollary 2.3. Let W be a $(t-r)$-dimensional subspace of U, and let $\mathcal{S}_1$ be a set of t-dimensional subspaces containing W which form an $(r-1)$-spread in the quotient geometry U/W (of dimension $2r-1$). The set $\mathcal{S} := \mathcal{S}_0 \cup \mathcal{S}_1$ is a t-cover of the given cardinality. □

The preceding corollary shows that the bound of Proposition 2.1(b) is sharp.

For the t-cover constructed in Corollary 2.5, there is a particular subspace W of dimension $t-r$ such that every point of $\mathcal{P} \setminus W$ is contained in exactly one element of $\mathcal{S}$, whereas every point of W is contained in exactly q^r+1 elements of $\mathcal{S}$.

Eisfeld showed that each t-cover $\mathcal{S}$ reaching equality in Proposition 2.1 looks like the t-cover constructed in Corollary 2.5.

Theorem 2.6 (Eisfeld [21]) *Let $\mathcal{S}$ be a t-cover of $\mathrm{PG}(k(t+1)+r-1, q)$, $1 \le r \le t$, of cardinality $q^r \cdot \frac{q^{k(t+1)}-1}{q^{t+1}-1}+1$.*

Then there is exactly one $(t-r)$-dimensional space W such that every point of $\mathcal{P} \setminus W$ is contained in exactly one element of $\mathcal{S}$, whereas every point of W is contained in exactly q^r+1 elements of $\mathcal{S}$.

2.2 Upper bounds for partial t-spreads

The following bound is by now the best known upper bound on the size of a partial t-spread of $\mathrm{PG}(N, q)$. Part (a) is essentially due to Beutelspacher [4, 4.1], part (b) is due to Drake and Freeman [20, Cor. 8]. The theorem shows that Corollary 2.4 is sharp in the case $r=1$. For $r>1$, it closes half of the gap between Corollary 2.4 and Proposition 2.1(a).

Theorem 2.7 *Let $\mathcal{S}$ be a partial t-spread in $\mathcal{P}=\mathrm{PG}(d, q)$, where $d=k(t+1)-1+r$, $1 \le r \le t$. Let $|\mathcal{S}|=q^r \cdot \frac{q^{k(t+1)}-1}{q^{t+1}-1}-s$. Then*

(a) $s \ge q-1$.

(b) $s > \frac{q^r-1}{2}-\frac{q^{2r-t-1}}{5}$.

Furthermore, there exists an example with $s=q^r-1$.

Proof: The last assertion follows from Corollary 2.4.

Recall that a point $P \in \mathcal{P}$ is called a *hole*, if it is not contained in an element of $\mathcal{S}$. We prove the theorem in several steps.

Step 1: *The number of holes is $\frac{q^r-1}{q-1}+s \cdot \frac{q^{t+1}-1}{q-1}$.*

For, the number of holes is

$$|\mathcal{P}|-|\mathcal{S}| \cdot \frac{q^{t+1}-1}{q-1}=\frac{q^{k(t+1)+r}-1}{q-1}-q^r \cdot \frac{q^{k(t+1)}-1}{q-1}+s \cdot \frac{q^{t+1}-1}{q-1}.$$

Step 2: *The number of holes in a hyperplane is congruent* $\frac{q^r-1}{q-1} + s \cdot \frac{q^t-1}{q-1} \bmod q^t$.

Let H be a hyperplane of $\mathcal{P}$. Then an element of $\mathcal{S}$ intersects H either in $\frac{q^{t+1}-1}{q-1}$ or in $\frac{q^t-1}{q-1}$ points, which is in any case $\frac{q^t-1}{q-1} \bmod q^t$ points. As $|\mathcal{S}| \equiv q^r - s \bmod q^t$, the number of points of H contained in an element of $\mathcal{S}$ is $\frac{q^t-1}{q-1} \cdot |\mathcal{S}| \equiv (q^r - s) \cdot \frac{q^t-1}{q-1} \bmod q^t$. Hence the number of holes in H is congruent to $\frac{q^d-1}{q-1} + (s - q^r) \cdot \frac{q^t-1}{q-1} \equiv \frac{q^r-1}{q-1} + s \cdot \frac{q^t-1}{q-1} \bmod q^t$.

Step 3: *(a) holds.*

Suppose that $s \leq q - 2$. By step 2, any hyperplane H contains at least $\frac{q^r-1}{q-1} + s \cdot \frac{q^t-1}{q-1}$ holes. Counting the incident pairs (H, P), where H is a hyperplane of $\mathcal{P}$ and P is a hole, we conclude that $\left(\frac{q^r-1}{q-1} + s \cdot \frac{q^{t+1}-1}{q-1}\right) \cdot \frac{q^d-1}{q-1} \geq \frac{q^{d+1}-1}{q-1} \cdot \left(\frac{q^r-1}{q-1} + s \cdot \frac{q^t-1}{q-1}\right)$, which is equivalent to $s(q^d - q^t) \geq q^d(q^r - 1)$. This gives a contradiction.

Step 4: *(b) holds.*

We can restrict ourselves to the case $k \geq 2$. (Otherwise $|\mathcal{S}| \leq 1$.) For $i \in \mathbb{N}$, let λ_i be the number of hyperplanes with i holes. Then double-counting yields the following identities:

$$\begin{aligned}
\sum_i \lambda_i &= \frac{q^{k(t+1)+r} - 1}{q - 1} \\
\sum_i i\lambda_i &= \left(\frac{q^r - 1}{q - 1} + s\frac{q^{t+1} - 1}{q - 1}\right) \cdot \frac{q^{k(t+1)+r-1} - 1}{q - 1} \\
\sum_i i(i-1)\lambda_i &= \left(\frac{q^r - 1}{q - 1} + s\frac{q^{t+1} - 1}{q - 1}\right)\left(\frac{q^r - 1}{q - 1} + s\frac{q^{t+1} - 1}{q - 1} - 1\right)\frac{q^{k(t+1)+r-2} - 1}{q - 1}
\end{aligned}$$

By step 2,

$$\sum_i \left(i - \frac{q^r - 1}{q - 1} - s \cdot \frac{q^t - 1}{q - 1}\right)\left(i - \frac{q^r - 1}{q - 1} - s \cdot \frac{q^t - 1}{q - 1} + q^t\right)\lambda_i \geq 0.$$

Inserting the values of $\sum \lambda_i$, $\sum i\lambda_i$ and $\sum i(i-1)\lambda_i$, the assertion follows after some calculations.

□

Conjecture: The sharp bound is $s \geq q^r - 1$.

2.3 Maximal partial 1-spreads of sizes close to the upper bound

In Section 3, we will show that a partial t-spread in $\mathrm{PG}(2t+1, q)$ having a small deficiency can always be extended to a t-spread. In other words: there is a gap between the number of elements of the largest partial t-spreads (namely the t-spreads) and the number of elements of the second largest maximal partial t-spreads. The following theorem shows that such a gap does not exist for partial 1-spreads in $\mathrm{PG}(2k, q)$.

Theorem 2.8 *In* PG$(2k,q)$ $(k \geq 2, q \geq 4)$*, there exist maximal partial 1-spreads of size* $q \cdot \frac{q^{2k}-1}{q^2-1} - s$ *for all* $s \in \mathbb{N}$ *with* $q-1 \leq s \leq 2q-3$*.*

These maximal partial 1-spreads are constructed by taking the partial 1-spread constructed in Corollary 2.4 and replacing some reguli by the opposite reguli.

One can apply the same argument to the construction in Corollary 2.5. This yields on the one hand several non-isomorphic covers of the minimal cardinality, and on the other hand we get minimal 1-covers whose cardinality is slightly bigger than the lower bound.

3 Extendability of spreads in PG$(3,q)$

3.1 Maximal partial spreads

From Theorem 1.2, we know that it is possible to construct 1-spreads in PG$(3,q)$. These spreads contain exactly q^2+1 pairwise disjoint lines.

The problem of constructing spreads in PG$(3,q)$ is a problem which has been studied intensively. A second problem to which a lot of attention has been paid is the problem of the cardinalities k for which a *maximal partial spread* of PG$(3,q)$, of size k, exists.

The main problem studied regarding maximal partial spreads is the determination of the size of the largest maximal partial spreads of PG$(3,q)$, or equivalently, the determination of the smallest positive integer δ for which there exists a maximal partial spread of size $q^2+1-\delta$ in PG$(3,q)$.

Presently, the largest known maximal partial spreads have size q^2-q+2. Their construction is based on the following idea [17], explained in the proof of Theorem 3.1.

Theorem 3.1 *In* PG$(3,q)$*,* $q>3$*, there exists a maximal partial spread of size* q^2-q+2*, so of deficiency* $\delta = q-1$*.*

Proof: In Section 1.2, it was shown that there exists in PG$(3,q)$ a spread $\mathcal{S}$ which is not regular. Hence, for such a spread there exists a line l, not belonging to $\mathcal{S}$, in PG$(3,q)$ such that the set of lines $\mathcal{L}$ of $\mathcal{S}$ intersecting l does not form a regulus.

Then $\mathcal{S}' = (\mathcal{S} \setminus \mathcal{L}) \cup \{l\}$ is a partial spread of size q^2-q+1 of PG$(3,q)$. This partial spread $\mathcal{S}'$ can only be extended to a larger partial spread by the lines of PG$(3,q)$ intersecting all $q+1$ lines of $\mathcal{L}$.

If there were two such lines l', l'', then including the line l, the lines of $\mathcal{L}$ would belong to the complementary regulus defined by l, l', l'', but this contradicts our assumptions.

Hence, there is at most one line l' extending $\mathcal{S}'$ to a larger partial spread.

This shows that PG$(3,q)$ contains a maximal partial spread of size q^2-q+1 or of size q^2-q+2.

Results of Bruen and Thas [18], Freeman [25], Jungnickel [35] show that it is always possible to construct maximal partial spreads of size q^2-q+2. $\square$

Now the question arises whether maximal partial spreads of deficiency $\delta < q-1$ exist. No such examples have been found. This had led to the conjecture that q^2-q+2 is the largest size of a maximal partial spread of positive deficiency.

The obvious way to prove this conjecture is to show that all maximal partial spreads of size bigger than q^2-q+2 are extendable to a spread.

This problem has received a lot of attention. We now present the best known results. These have been obtained by using results on blocking sets in PG$(2,q)$.

In the next section, we give the important link between maximal partial spreads in PG$(3,q)$ and blocking sets in PG$(2,q)$.

3.2 Link between blocking sets and maximal partial spreads

Let $\mathcal{S}$ denote a partial spread of PG$(3,q)$ with $q^2+1-\delta$ lines.

Definition 3.2 A *poor* plane of $\mathcal{S}$ is a plane of PG$(3,q)$ not containing a line of $\mathcal{S}$, and a *rich* plane of $\mathcal{S}$ is a plane containing a line of $\mathcal{S}$.

Theorem 3.3 *A maximal partial spread $\mathcal{S}$ of deficiency δ has $\delta(q+1)$ holes and $\delta(q+1)$ poor planes.*

Proof: The point set of PG$(3,q)$ can be partitioned into q^2+1 lines. So, if a spread $\mathcal{S}$ has deficiency δ, then there are $\delta(q+1)$ points not covered by the lines of $\mathcal{S}$. So, there are $\delta(q+1)$ holes.

Since the definition of maximal partial spread is self-dual, there are also $\delta(q+1)$ planes not containing an element of $\mathcal{S}$. So, there are $\delta(q+1)$ poor planes. $\square$

If E is a plane, then $H(E)$ denotes the set of holes of E.

We first state a lemma with particular properties of lines not in $\mathcal{S}$, and of poor and rich planes. The results in Parts (a), (b) and (c) are valid for arbitrary partial spreads of PG$(3,q)$ with $q^2+1-\delta$ lines.

Lemma 3.4 (a) *If l is a line that is not in $\mathcal{S}$, then the number of holes on l and the number of poor planes on l are the same.*

(b) *If E is a poor plane, then $|H(E)| = q + \delta$, and if E is a rich plane, then $|H(E)| = \delta$.*

(c) *If E is a poor plane, then $H(E)$ is a blocking set of E.*

Proof: (a) If t is the number of lines of $\mathcal{S}$ that meet l, then l has $q + 1 - t$ holes. Furthermore, t of the planes on l contain a line of $\mathcal{S}$; so $q + 1 - t$ planes on l are poor.

(b) Suppose that E is a poor plane, then all the lines of $\mathcal{S}$ intersect E in a point. Since $\mathcal{S}$ contains $q^2 + 1 - \delta$ lines, $|H(E)| = q + \delta$. A similar argument proves the result for a rich plane.

(c) Since E is poor, Part (a) shows that every line of E contains a hole. This shows that $H(E)$ is a blocking set of E. □

Part (c) of the preceding lemma implies immediately that the deficiency δ of a maximal partial spread is greater than or equal to the value ϵ, where $q + \epsilon$ is the cardinality of the smallest non-trivial blocking set of PG$(2, q)$.

Theorem 3.5 *Let $\mathcal{S}$ be a maximal partial spread of* PG$(3, q)$*, of deficiency δ. Then $\delta \geq \epsilon$, with $q + \epsilon$ the cardinality of the smallest non-trivial blocking set of* PG$(2, q)$.

Proof: Suppose that $\delta < \epsilon$. Consider a poor plane E of $\mathcal{S}$. Then the set of holes $H(E)$ constitutes a blocking set in E which necessarily must contain a line l since the number of holes $q + \delta$ is smaller than the cardinality of the smallest non-trivial blocking set in a plane.

Then l can be used to extend $\mathcal{S}$ to a larger maximal partial spread. This contradicts the definition of maximal partial spread (Definition 1.3). □

Corollary 3.6 *Let $\mathcal{S}$ be a maximal partial spread of* PG$(3, q)$*, of positive deficiency δ. Then*
(1) $\delta \geq \sqrt{q} + 1$ *when q is square,*
(2) $\delta \geq q^{2/3} + 1$ *when q is non-square, $q = p^h, h > 2, p \geq 5$, p prime,*
(3) $\delta \geq (q + 3)/2$ *when q is an odd prime.*

Proof: This follows from the preceding theorem, using the bounds of Theorem 1.10. □

Since the sets of holes in the poor planes of a maximal partial spread form non-trivial blocking sets in these poor planes, results giving concrete information regarding the description of these blocking sets are of great use for studying maximal partial spreads. We will now make use of such information for q square.

For the remainder of this section, $\mathcal{S}$ denotes a maximal partial spread of PG$(3, q)$, q square, with $q^2 + 1 - \delta$ lines, where $\delta \leq (q + 1)/2$ satisfies the property that every non-trivial blocking set with at most $q + \delta$ points contains a Baer subplane. For bounds on δ, see Theorem 1.10 (2), (3).

Then, when q is square, the following results are known.

Lemma 3.7 *Let $\mathcal{S}$ be a maximal partial spread of deficiency δ, where δ is a value for which it is known that every non-trivial blocking set of* PG$(2, q)$ *of size $q + \delta$ contains a Baer subplane.*

If E is a poor plane, then $H(E)$ contains a unique Baer subplane $B(E)$ of E.

Proof: Immediate from the preceding lemma and from the assumptions made. □

The preceding lemma implies that the set of holes of $\mathcal{S}$ contains $\delta(q + 1)$ distinct Baer subplanes. Where can these Baer subplanes of holes come from? A plausible explanation might be that the set of holes of $\mathcal{S}$ contains subgeometries PG$(3, \sqrt{q})$. This is indeed the case, as the following theorem shows.

Theorem 3.8 (Metsch and Storme [39]) *Suppose that δ is an integer and q is a square prime power, $q > 4$, such that*

(1) $2\delta \leq q + 1$,
(2) *every non-trivial blocking set of* PG$(2, q)$, *q square, with at most $q + \delta$ points contains a Baer subplane.*

Let $\mathcal{S}$ be a maximal partial spread of PG$(3, q)$ *with $q^2 + 1 - \delta$ lines, then*

(a) $\delta = s(\sqrt{q} + 1)$ *for an integer s,*
(b) *the set of holes of $\mathcal{S}$ is the disjoint union of s Baer subgeometries* PG$(3, \sqrt{q})$.

The preceding result shows that only a discrete spectrum of values δ can exist for which there can be a maximal partial spread of deficiency δ.

The question is how this limited number of possibilities for δ can be eliminated. The following weight argument of Blokhuis and Metsch [9] eliminates the possibility $\delta = \sqrt{q}+1$ for $q > 4$.

Theorem 3.9 *Consider the affine space* $\mathrm{AG}(d, q^s)$, $s \geq 2$, *and define on this affine space the weight function* f *by*

$$f : \mathrm{AG}(d, q^s) \to \mathrm{GF}(q^s) : (x_1, \ldots, x_d) \mapsto (\prod_{i=1}^{d} x_i)^{q-1}.$$

Then this weight function has the following properties:
(1) *The weight of* $\mathrm{AG}(d, q^s)$ *is zero.*
(2) *If* $d(q-1) < q^s - 1$, *then the weight of every affine subspace of* $\mathrm{AG}(d, q^s)$ *is zero.*
(3) *If* e *is an integer,* $1 \leq e \leq d$, *then the weight of* $\mathrm{AG}(e, q)$, *naturally embedded in* $\mathrm{AG}(d, q)$, *is non-zero.*

Theorem 3.10 (Blokhuis and Metsch [9]) *In* $\mathrm{PG}(3, q)$, $q > 4$, q *square, there does not exist a maximal partial spread of deficiency* $\delta = \sqrt{q} + 1$.

Proof: If such a maximal partial spread would exist, then it would be possible to partition $\mathrm{PG}(3, q)$ into $q^2 - \sqrt{q}$ pairwise disjoint lines, and in a subgeometry $\Pi = \mathrm{PG}(3, \sqrt{q})$.

Consider a hyperplane H intersecting Π in a Baer subplane. Then the affine space $\mathrm{AG}(3, q) = \mathrm{PG}(3, q) \setminus H$ can be partitioned into $q^2 - \sqrt{q}$ affine lines and one affine subspace $\mathrm{AG}(3, \sqrt{q})$.

Apply now the weight argument of the preceding theorem. The weight of the affine lines is zero, but the weight of the affine subgeometry is non-zero. Hence, the sum of all the weights of the affine points is not zero. But this contradicts Theorem 3.9 (1). □

The preceding proof cannot be used for $q = 4$. Surprisingly, the preceding theorem even is not valid for $q = 4$. It is possible to construct a spread of $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$. The first example was by Mesner [37], the classification of all such maximal partial spreads is by van Dam [45].

Theorem 3.11 (van Dam [45]) *There exist exactly three distinct maximal partial spreads of size 14 in* $\mathrm{PG}(3, 4)$.

Hence, summarizing the preceding results by using the bounds of Theorem 1.10 (2), (3), the following corollary is obtained:

Corollary 3.12 (a) *Suppose that* δ *is an integer and* q *square,* $q = p^h, h > 2, p \geq 5$, p *prime, such that* $\delta < q^{2/3} + 1$.

If $\mathcal{S}$ *is a maximal partial spread of* $\mathrm{PG}(3, q)$ *with* $q^2 + 1 - \delta$ *lines, then* $\delta = s(\sqrt{q} + 1)$ *for an integer* $s \geq 2$ *and the set of points of* $\mathrm{PG}(3, q)$ *not covered by* $\mathcal{S}$ *is the disjoint union of* s *Baer subgeometries* $\mathrm{PG}(3, \sqrt{q})$.

(b) *Suppose that δ is an integer and $q = p^2$, p prime, $q > 4$, such that $2\delta \leq q + 1$.*

If $\mathcal{S}$ is a maximal partial spread of PG$(3, q)$ *with $q^2 + 1 - \delta$ lines, then $\delta = s(\sqrt{q} + 1)$ for an integer $s \geq 2$ and the set of points of* PG$(3, q)$ *not covered by $\mathcal{S}$ is the disjoint union of s Baer subgeometries.*

Remark 3.13 Since it is conjectured that $q^2 - q + 2$ is the largest size of maximal partial spreads in PG$(3, q)$, $q > 3$, which are not equal to spreads, the general idea is that no maximal partial spreads exist satisfying the conditions of the preceding corollary. This problem is still open.

Note that the upper bound $\delta \leq (q+1)/2$ of (b) is practically half of the deficiency $\delta = q-1$ of the largest known maximal partial spreads (Theorem 3.1).

Note that all the results on the extendability of partial spreads of PG$(3, q)$ to spreads can be retranslated via the Klein correspondence into results on the extendability of caps on the Klein quadric to ovoids (Section 1.5).

As an example, we state the extendability result of the preceding corollary into this form.

Corollary 3.14 (a) *Suppose that δ is an integer and q square, $q = p^h, h > 2, p \geq 5$, p prime, such that $\delta < q^{2/3} + 1$. Let $\mathcal{Q}$ be the Klein quadric of* PG$(5, q)$.

If $\mathcal{K}$ is a k-cap of $\mathcal{Q}$, with $k = q^2 + 1 - \delta$, which cannot be extended to a larger cap, then $\delta = s(\sqrt{q} + 1)$ for an integer $s \geq 2$.

(b) *Suppose that δ is an integer and $q = p^2$, p prime, $q > 4$, such that $2\delta \leq q + 1$.*

If $\mathcal{K}$ is a k-cap of $\mathcal{Q}$, with $k = q^2 + 1 - \delta$, which cannot be extended to a larger cap, then $\delta = s(\sqrt{q} + 1)$ for an integer $s \geq 2$.

3.3 Maximal partial spreads in PG$(3, p^3)$

Theorems 1.11 and 1.12 give the complete classification of all blocking sets of PG$(2, p^3)$ of size at most $p^3 + p^2 + p + 1$. This information has led to the following results on the deficiency of maximal partial spreads.

Theorem 3.15 (Metsch and Storme [39]) *Let $\mathcal{S}$ be a maximal partial spread of* PG$(3, p^3)$, *p non-square, $p = p_0^h$, $h \geq 1$, p_0 prime, $p_0 \geq 5$, $p \neq 5$, of deficiency $\delta \leq p^2 + p + 1$. Then $\delta = p^2 + p + 1$ and the set of holes forms a projected subgeometry* PG$(5, p)$ *in* PG$(3, p^3)$.

Theorem 3.16 (Metsch and Storme [39]) *Let $\mathcal{S}$ be a maximal partial spread of* PG$(3, p^3)$, *$p = p_0^h$, $h \geq 2$, h even, p_0 prime, $p_0 \geq 5$, of deficiency $\delta \leq p^2 + p + 1$.*

Then, (1) $\delta \equiv 0 \pmod{p^{3/2} + 1}$, $\delta \geq 2(p^{3/2} + 1)$, and the set of holes is the union of disjoint subgeometries PG$(3, p^{3/2})$, *or (2) $\delta = p^2 + p + 1$ and the set of holes forms a projected subgeometry* PG$(5, p)$ *in* PG$(3, p^3)$.

One could state that the conclusions of Theorem 3.15 and Theorem 3.16 (2) on the set of holes, of a maximal partial spread of deficiency $\delta = p^2+p+1$, are not surprising taking into consideration the characterizations of the blocking sets in Theorems 1.11 and Theorem 1.12 (b), (c) as projected subgeometries $\mathrm{PG}(3,p)$ (Remark 1.13).

3.4 Spectrum of the sizes of maximal partial spreads

In Theorem 3.1, it was stated that there exist maximal partial spreads of size q^2-q+2 in $\mathrm{PG}(3,q)$, $q>3$. Regarding other possible values for the deficiency of maximal partial spreads, the following important results are known.

Theorem 3.17 (Glynn [26]) *Let $\mathcal{S}$ be a maximal partial spread of* $\mathrm{PG}(3,q)$ *of size k, then $k \geq 2q$.*

Theorem 3.18 (a) (Heden [31, 32, 33]) *In* $\mathrm{PG}(3,q)$, *q odd, $q \geq 7$, for every integer k in the interval $[(q^2+1)/2+6, q^2-q+2]$, there exists a maximal partial spread of size k.*
(b) (Barát, Heden, Storme [3]) *In* $\mathrm{PG}(3,q)$, *q even, $q > q_0$, for every integer k in the interval $[(5q^2+q+16)/8, q^2-q+2] \setminus \{q^2-q\}$, there exists a maximal partial spread of size k.*

4 Partial t-spreads in $\mathrm{PG}(N,q)$, $(t+1)|(N+1)$

4.1 General results

When $(t+1)|(N+1)$, it is possible to partition the point set of $\mathrm{PG}(N,q)$ into $(q^{N+1}-1)/(q^{t+1}-1)$ pairwise disjoint subspaces of dimension t; so t-spreads in $\mathrm{PG}(N,q)$ exist.

As for 1-spreads in $\mathrm{PG}(3,q)$, here in $\mathrm{PG}(N,q)$, the problem arises of the deficiency of the largest maximal partial t-spread, not equal to a t-spread. In the proofs of Section 3, the holes played a crucial role in determining the lower bounds on the deficiency δ of a maximal partial spread. Also for the problem of the deficiency δ of the largest maximal partial t-spread in $\mathrm{PG}(N,q)$, the holes provide a way to determine a lower bound on δ.

Lemma 4.1 *Let $\mathcal{S}$ be a partial t-spread of* $\mathrm{PG}(N,q)$ *of deficiency $\delta < q$. Let $\mathcal{F}$ be the set of holes of $\mathcal{S}$, then $\mathcal{F}$ is a set of cardinality $\delta(q^{t+1}-1)/(q-1)$ intersecting every hyperplane in at least $\delta(q^t-1)/(q-1)$ points.*

Proof: Consider a t-spread in $\mathrm{PG}(N,q)$. Then every hyperplane Π of $\mathrm{PG}(N,q)$ is the disjoint union of a number α of t-dimensional subspaces contained in $\mathcal{S}$, and a number β of $(t-1)$-dimensional intersections with elements in $\mathcal{S}$ but not lying in Π.

These numbers are constants since they are the solutions to the system of equations

$$\begin{aligned} \alpha+\beta &= \frac{q^{N+1}-1}{q^{t+1}-1} \\ \alpha\left(\frac{q^{t+1}-1}{q-1}\right)+\beta\left(\frac{q^{t}-1}{q-1}\right) &= \frac{q^{N}-1}{q-1}. \end{aligned}$$

For a partial t-spread $\mathcal{S}$ of $\mathrm{PG}(N,q)$ of deficiency $\delta<q$, it is possible to prove that every hyperplane of $\mathrm{PG}(N,q)$ contains at most α elements of $\mathcal{S}$; implying that every hyperplane has at least $\delta(q^t-1)/(q-1)$ holes. □

In coding theoretical papers of Hamada and Helleseth discussing problems on codes meeting the Griesmer bound, the concept of *minihypers* is used which is of particular interest for the study of maximal partial t-spreads, when one links the statement of the preceding lemma to the definition of minihypers.

Definition 4.2 (Hamada and Tamari [30]) *Let $\mathcal{F}$ be a set of f points in $\mathrm{PG}(N,q)$, $N\geq 2$. If $\mathcal{F}$ intersects every hyperplane of $\mathrm{PG}(N,q)$ in at least m points, and there is a hyperplane sharing exactly m points with $\mathcal{F}$, then $\mathcal{F}$ is called an $\{f,m;N,q\}$-minihyper.*

The principal results of Hamada and Helleseth [28], and of Hamada and Maekawa [29] state the following results on minihypers which immediately imply results on the deficiency δ of a maximal partial t-spread in $\mathrm{PG}(N,q)$.

Theorem 4.3 *Let $v_i=(q^i-1)/(q-1)$. Let t,q,h and λ_i, $i=1,\ldots,h$, be any integers such that $t\geq 2, h\geq 1$, $q>(h-1)^2$ and $0\leq\lambda_1\leq\cdots\leq\lambda_h<t$.*
(1) If $t<\lambda_{h-1}+\lambda_h+1$, there is no $\{\sum_{i=1}^h v_{\lambda_i+1},\sum_{i=1}^h v_{\lambda_i};t,q\}$-minihyper.
(2) If $t\geq\lambda_{h-1}+\lambda_h+1$, then $\mathcal{F}$ is a $\{\sum_{i=1}^h v_{\lambda_i+1},\sum_{i=1}^h v_{\lambda_i};t,q\}$-minihyper if and only if $\mathcal{F}$ is the union of h pairwise disjoint λ_i-dimensional subspaces of $\mathrm{PG}(t,q)$.

Corollary 4.4 *Let $\mathcal{F}$ be a $\{\delta v_{t+1},\delta v_t;N,q\}$-minihyper, with $2t+1\leq N$, and with $\delta<\sqrt{q}+1$, then $\mathcal{F}$ is the union of δ pairwise disjoint t-dimensional subspaces.*

The preceding corollary and Lemma 4.1 now imply the following result regarding the deficiency of a maximal partial t-spread in $\mathrm{PG}(N,q)$.

Theorem 4.5 *Let $\mathcal{S}$ be a maximal partial t-spread in $\mathrm{PG}(N,q)$ of deficiency $\delta>0$. Then $\delta\geq\sqrt{q}+1$.*

Theorem 3.5 stated that the deficiency of a maximal partial spread in PG$(3, q)$ always is greater than or equal to ϵ, where $q + \epsilon$ is the cardinality of the smallest non-trivial blocking sets in PG$(2, q)$.

For q square, $\epsilon = \sqrt{q} + 1$ (Theorem 1.10) which is also the lower bound on δ in Theorem 4.5, but for q non-square, $\epsilon \geq q^{2/3} + 1$ and this is a bigger value than the lower bound of the preceding theorem.

But also for arbitrary t-spreads in PG(N, q), $(t+1)|(N+1)$, it is possible to use results on blocking sets in PG$(2, q)$ to obtain results on the deficiency of partial t-spreads. This link, used by Govaerts and Storme, led to the following result.

Theorem 4.6 (Govaerts and Storme [27]) *Let $\mathcal{S}$ be a maximal partial t-spread in* PG(N, q), $(t+1)|(N+1)$, *of deficiency $\delta > 0$. Then $\delta \geq \epsilon$, with $q + \epsilon$ the cardinality of the smallest non-trivial blocking sets in* PG$(2, q)$.

Using the results of Theorem 1.10, we obtain the following lower bounds on the deficiency δ.

Corollary 4.7 *Let $\mathcal{S}$ be a maximal partial t-spread in* PG(N, q), $(t+1)|(N+1)$, *of deficiency $\delta > 0$. Then*

(1) $\delta \geq \sqrt{q} + 1$ *when q is square,*
(2) $\delta \geq q^{2/3} + 1$ *when $q = p^h$, p prime, $p > 3$, $h > 1$ odd,*
(3) $\delta \geq (q+3)/2$ *when q is an odd prime.*

In Corollary 3.12, for spreads in PG$(3, q)$, q square, great improvements to the preceding results were obtained by using the fact that non-trivial blocking sets in PG$(2, q)$ of size equal to, or slightly bigger than, $q + \sqrt{q} + 1$, contain a Baer subplane.

This fact was used by Ferret and Storme to obtain a first improvement to the preceding theorem, for q square.

Theorem 4.8 (Ferret and Storme [24]) *Let $\mathcal{S}$ be a maximal partial t-spread in* PG(N, q), $(t+1)|(N+1)$, *$q = p^h$, p prime, h even, $q > q_0$. Then $\delta \geq 2\sqrt{q}$.*

4.2 Partial t-spreads in PG$(2t+1, q)$

The preceding section presented lower bounds for partial t-spreads of PG(N, q), when $(t+1)|(N+1)$. For partial t-spreads in PG$(2t+1, q)$, an extra technique can be used to obtain strong results on the deficiency of partial t-spreads in PG$(2t+1, q)$. This technique is based on the link between partial t-spreads of PG$(2t+1, q)$ and nets, as defined in Section 1.3.

Theorem 4.9 (a) (Metsch [38, Corollary 2.6]) *Suppose that $\mathcal{N}$ is a net of order k and of deficiency δ. If $3k > 8\delta^3 - 18\delta^2 + 8\delta + 4$, then $\mathcal{N}$ can be completed to an affine plane of order k.*
(b) (Jungnickel [35]) *Suppose that $\mathcal{N}$ is a net of order k and of deficiency δ satisfying $(\delta-1)^2 < k$, and assume that $\mathcal{N}$ can be extended to an affine plane $\mathcal{A}$. Then* $\mathrm{Aut}(\mathcal{N}) \leq \mathrm{Aut}(\mathcal{A})$. *Also, if $\mathcal{N}$ is a translation net with translation group G, then $\mathcal{A}$ is a translation plane which admits G as a translation group.*

These results form the basis for the following extendability results on partial t-spreads in $\mathrm{PG}(2t+1,q)$.

Theorem 4.10 (Metsch and Storme [39]) *Let $\mathcal{S}$ be a maximal partial t-spread of* $\mathrm{PG}(2t+1,q)$, *q square, $t \geq 2$, with deficiency δ satisfying $3q^{t+1} > 8\delta^3 - 18\delta^2 + 8\delta + 4$.*

If also $\delta \leq q-2$ when $q=p^2$, p prime, and if $\delta \leq (q-1)/(p^{h'}-1)-1$, with $q=p^{2h}$, $h>1$, and with h' the largest divisor of $2h$ smaller than h, then $\delta \equiv 0 \pmod{\sqrt{q}+1}$ and the set of holes is the disjoint union of subgeometries $\mathrm{PG}(2t+1,\sqrt{q})$.

Also, $\delta \geq 2(\sqrt{q}+1)$ when $(2t+1)(\sqrt{q}-1) < q-1$.

Theorem 4.11 (Metsch and Storme [39]) *Let $\mathcal{S}$ be a maximal partial t-spread of* $\mathrm{PG}(2t+1,q)$, *$q=p^{3h}$, $h \geq 1$, p prime, q non-square, with deficiency δ satisfying $3q^{t+1} > 8\delta^3 - 18\delta^2 + 8\delta + 4$.*

If $\delta \leq q-2$ when $q=p^3$, p prime, and if $\delta \leq (q-1)/(p^{h'}-1)-1$, with h' the largest divisor smaller than h of $3h$ when $h>1$, then $\delta \equiv 0 \pmod{q^{2/3}+q^{1/3}+1}$ and the set of holes is the disjoint union of projected subgeometries $\mathrm{PG}(3t+2,\sqrt[3]{q})$ *in* $\mathrm{PG}(2t+1,q)$.

The preceding results can be generalized to the following theorem.

Theorem 4.12 (Metsch and Storme [39]) (a) *Let $q=p^r$, p prime, $r>3$ prime. Let $\mathcal{S}$ be a maximal partial t-spread of* $\mathrm{PG}(2t+1,q)$, *with deficiency δ satisfying $3q^{t+1} > 8\delta^3 - 18\delta^2 + 8\delta + 4$ and $\delta \leq q-2$.*

Then $\delta \equiv 0 \pmod{p^{r-1}+p^{r-2}+\cdots+p+1}$ and the set of holes is the disjoint union of projected subgeometries $\mathrm{PG}(r(t+1)-1,p)$ *in* $\mathrm{PG}(2t+1,q)$.
(b) *Let $q=p^{rh}$, p prime, r the largest non-trivial divisor of rh, $h>1$. Let h' be the largest divisor of rh smaller than r. Let $\mathcal{S}$ be a maximal partial t-spread of* $\mathrm{PG}(2t+1,q)$, *with deficiency δ satisfying $3q^{t+1} > 8\delta^3 - 18\delta^2 + 8\delta + 4$ and $\delta \leq (q-1)/(p^{h'}-1)-1$.*

Then $\delta \equiv 0 \pmod{p^{r(h-1)}+p^{r(h-2)}+\cdots+p^r+1}$ and the set of holes is the disjoint union of projected subgeometries $\mathrm{PG}(h(t+1)-1,p^r)$ *in* $\mathrm{PG}(2t+1,q)$.

5 Minimal t-covers in $\mathrm{PG}(N,q)$, $(t+1)|(N+1)$

For $t+1$ a divisor of $N+1$, it is possible to construct t-spreads, that is, to partition the point set of $\mathrm{PG}(N,q)$ into $(q^{N+1}-1)/(q^{t+1}-1)$ pairwise disjoint subspaces of dimension t (Theorem 1.2).

In Sections 3 and 4, we studied partial t-spreads in $\mathrm{PG}(N,q)$, $(t+1)|(N+1)$, of small deficiency δ. We proved that they are contained in a t-spread of $\mathrm{PG}(N,q)$.

This latter problem suggests an alternative problem.

Take a t-spread of $\mathrm{PG}(N,q)$ and add to this t-spread an other t-dimensional subspace. Then a new set of t-dimensional subspaces is obtained which covers the point set of $\mathrm{PG}(N,q)$.

But the question one can pose is:

Is every set $\mathcal{C}$ containing $(q^{N+1}-1)/(q^{t+1}-1)+1$ t-dimensional subspaces covering all the points of $\mathrm{PG}(N,q)$, the union of a t-spread and one extra t-dimensional subspace?

Or a bit more general, if we consider a set $\mathcal{C}$ of *slightly more* than $(q^{N+1}-1)/(q^{t+1}-1)$ t-dimensional subspaces which cover all the points of $\mathrm{PG}(N,q)$, does $\mathcal{C}$ contain a t-spread?

This latter question was first studied by Blokhuis, O'Keefe, Payne, Storme and Wilbrink [10] who studied minimal 1-covers in $\mathrm{PG}(3,q)$.

To explain the known results, we state the necessary definitions for this section.

Definition 5.1 Let $\mathcal{C}$ be a t-cover of $\mathrm{PG}(k(t+1)-1,q)$.
(1) The *excess* of a minimal t-cover $\mathcal{C}$ in $\mathrm{PG}(k(t+1)-1,q)$ is equal to the number of elements in this cover minus $(q^{k(t+1)}-1)/(q^{t+1}-1)$.
(2) A point of $\mathrm{PG}(N,q)$ is called a *multiple point* or an *excess point* of $\mathcal{C}$ when it lies on at least two elements of the t-cover $\mathcal{C}$.
(3) The *excess* of a point with respect to a t-cover $\mathcal{C}$ is the number of elements of the t-cover passing through this point, minus one.
(4) The *weight of a subspace* Π *with respect to a given t-cover* is the minimum of the excesses of the points belonging to Π.

Definition 5.2 *Let $\mathcal{L}$ be a collection of t-dimensional subspaces of* $\mathrm{PG}(N,q)$*, where each t-dimensional space is accorded a positive integer, called its* weight*. The set of points which lie on at least one element of $\mathcal{L}$ is called the* sum *of the t-spaces $\mathcal{L}$. Furthermore, the* weight *of a point P in the sum of t-spaces $\mathcal{L}$ is the sum of the weights of the elements of $\mathcal{L}$ passing through P.*

The following theorem of [10], and its application for regular spreads, shows that there exist minimal 1-covers in PG$(3,q)$, having all possible small excesses.

Theorem 5.3 *Let $\mathcal{S}$ be a spread of* PG$(3,q)$ *and let $\mathcal{R} = \cup_{i=1}^m R_i$ be the union of m reguli contained in $\mathcal{S}$, with $R_i \nsubseteq \cup_{j\neq i} R_j$, for all $i = 1,\ldots,m$. For $i = 1,\ldots,m$, let R_i^{opp} be the opposite regulus to R_i. If $\mathcal{R}$ contains $m(q+1)-r$ lines of $\mathcal{S}$, then $\mathcal{M} = (\mathcal{S}\setminus\mathcal{R})\cup_{i=1}^m R_i^{\mathrm{opp}}$ is a minimal 1-cover of* PG$(3,q)$ *of excess r.*

Proof: It is immediate that $\mathcal{M}$ is a 1-cover, and that no line of $\mathcal{M}\cap\mathcal{S}$ can be deleted to obtain a smaller 1-cover. Suppose that $\mathcal{M}\setminus\{l\}$ is a 1-cover, where l is a line of R_i^{opp} for some i. Then each point of l lies on a line of a regulus R_j^{opp}, for some $j \neq i$. Thus $R_i \subseteq \cup_{j\neq i} R_j$; a contradiction. Since all opposite reguli R_i^{opp} are pairwise disjoint, $|\mathcal{M}| = q^2+1-(m(q+1)-r)+m(q+1)$; so $\mathcal{M}$ has excess r. □

If $\mathcal{S}$ is the regular spread, then since every three lines of $\mathcal{S}$ lie in a regulus contained in $\mathcal{S}$, it is possible to construct a great variety of minimal 1-covers of this kind.

Note also that the condition of Theorem 5.3 is trivially fulfilled for $m < (q+3)/2$ since two distinct reguli share at most two lines.

Example 5.4 (1) In a regular spread $\mathcal{S}$, it is possible to construct through a given line l of $\mathcal{S}$ exactly q reguli sharing pairwise only that line l. Choosing $\mathcal{R}$ to be the union of $r+1$ of these reguli gives a minimal 1-cover of excess r for each $0 \leq r \leq q-1$.

(2) Let $\mathcal{S}$ be the regular spread in PG$(3,q)$ for q odd, and let $\mathcal{R}$ be the union of three reguli in $\mathcal{S}$, pairwise meeting in a line but with no common line (such a set $\mathcal{R}$ exists; under the Klein correspondence it is the union of three conics, pairwise tangent in three distinct points, on a 3-dimensional elliptic quadric contained in the Klein quadric). This provides a different example of a minimal 1-cover of PG$(3,q), q$ odd, of excess 3.

(3) Let $\mathcal{S}$ be a spread comprising $q+1$ reguli which share exactly two lines l_1 and l_2. Choosing $\mathcal{R}$ to be the union of m of these reguli gives a minimal 1-cover of excess $2(m-1)$ for each $1 \leq m \leq q+1$, that is, excesses $0, 2, 4, \ldots, 2q$ occur.

All the examples of minimal 1-covers constructed in this section have the property that the collection of multiple points forms a union of lines of PG$(3,q)$ which is also a sum of lines of PG$(3,q)$. In the following theorem, we show that this is a feature of minimal 1-covers of small excess. Remark that blocking sets again appear in the formulation of this theorem on 1-covers.

Theorem 5.5 (Blokhuis, O'Keefe, Payne, Storme and Wilbrink [10]) *Let $\mathcal{C}$ be a minimal 1-cover of* PG$(3,q)$, *with $|\mathcal{C}| = q^2+1+r$ and with $q+r$ smaller than the size of the smallest non-trivial blocking set in* PG$(2,q)$. *Then the multiple points form a sum of lines, where the weight of a line in this sum is equal to the weight of this line with respect to the 1-cover, and with the sum of the weights of the lines equal to r.*

Recently, Govaerts and Storme [27] extended this latter theorem to arbitrary minimal t-covers in $\mathrm{PG}(k(t+1)-1,q)$.

Theorem 5.6 *Let $\mathcal{C}$ be a minimal t-cover of $\mathrm{PG}(k(t+1)-1,q)$, with $|\mathcal{C}| = (q^{k(t+1)}-1)/(q^{t+1}-1)+r$ and with $q+r$ smaller than the size of the smallest non-trivial blocking set in $\mathrm{PG}(2,q)$. Then the multiple points form a sum of t-dimensional spaces, where the weight of a t-space in this sum is equal to the weight of this t-space with respect to the t-cover, and with the sum of the weights of the t-dimensional spaces equal to r.*

For particular values of k, t and q, examples of minimal t-covers with small excess have been found [27]. This shows that there are indeed minimal t-covers satisfying the hypotheses of the preceding theorem.

6 General covering and blocking problems in projective spaces

We considered already the question of determining the smallest t-covers of a finite projective space $\mathcal{P}$, i.e. the smallest sets of t-dimensional subspaces covering all points (i.e. 0-dimensional subspaces) of the space. There are several generalizations of this problem:

- For $t > s$, what are the smallest sets $\mathcal{S}$ of t-dimensional subspaces covering all s-dimensional subspaces of $\mathcal{P}$? (That is, every s-dimensional subspace of $\mathcal{P}$ is to be contained in one element of $\mathcal{S}$.)
- In particular, does there exist a set $\mathcal{S}$ of t-dimensional subspaces such that every s-dimensional subspace of $\mathcal{P}$ is contained in exactly one element of $\mathcal{S}$?
- Dually, for $t < s$, what are the smallest sets $\mathcal{S}$ of t-dimensional subspaces blocking all s-dimensional subspaces? (That is, every s-dimensional subspace of $\mathcal{P}$ contains an element of $\mathcal{S}$.)

The following results are known on these questions. (We do not state explicitly the dual versions of these results.)

Theorem 6.1 (Bose and Burton [12]) *Let $\mathcal{B}$ be a set of points of $\mathcal{P} = \mathrm{PG}(N,q)$ such that each k-dimensional subspace of $\mathcal{P}$ contains a point of $\mathcal{B}$. Then $|\mathcal{B}| \geq \frac{q^{N-k+1}-1}{q-1}$ with equality if and only if $\mathcal{B}$ is the set of points of an $(N-k)$-dimensional subspace of $\mathcal{P}$.*

A generalization of this theorem is the following one:

Theorem 6.2 (Beutelspacher and Ueberberg [6]) *Let $\mathcal{S}$ be a family of t-dimensional subspaces of $\mathcal{P} = \mathrm{PG}((t+1)r+k-1, q)$, where $t \geq 1, r \geq 2$ and $k \geq 0$. Suppose that every $(tr+k)$-dimensional subspace of $\mathcal{P}$ contains at least one element of $\mathcal{S}$. Then $|\mathcal{S}| \geq \frac{q^{(t+1)r}-1}{q^{t+1}-1}$, where equality holds if and only if $\mathcal{S}$ is a geometric t-spread in a $((t+1)r-1)$-dimensional subspace of $\mathcal{P}$.*

The theorem of Bose and Burton solves the case $t = 0$ of the third question, while the preceding theorem solves the case $s(t+1) \geq t(N+1)$. The case $t = 1, 2s = N$, being closest to the case handled by Beutelspacher and Ueberberg, is much more complicated.

Example 6.3 Consider the space $\mathcal{P} = \mathrm{PG}(2s, q)$. Let $Q \in \mathcal{P}$ be a point and F a set of $(q^{2s}-1)/(q^2-1)$ planes through Q forming a geometric spread in the quotient geometry $\mathcal{P}/Q$. Fix an $(s+1)$-space U through Q. Let $\mathcal{S}$ consist of the $(q^{s+1}-1)/(q-1)$ lines of U through Q together with the $q^2(q^{2s}-1)/(q^2-1)$ lines that lie in a plane of F but do not contain Q. Then every s-space contains a line of $\mathcal{S}$.

Theorem 6.4 (Eisfeld and Metsch [22]) *Suppose that $\mathcal{S}$ is a set of lines in $\mathrm{PG}(2s, q)$, such that every s-dimensional subspace contains a line of $\mathcal{S}$. Then*

$$|\mathcal{S}| \geq \frac{q^{2s+2}-q^2}{q^2-1} + \frac{q^{s+1}-1}{q-1} = (q^{2s}+q^{2s-2}+q^{2s-4}+\cdots+q^2)+(q^s+q^{s-1}+q^{s-2}+\cdots+1).$$

The only examples where equality holds are given in Example 6.3.

Metsch recently found some bounds for $t = 1$, $N > 2s$ and sufficiently big values of q. Here, equality holds for similar examples.

Astonishingly, there is nearly nothing known about the second question. It is not even known if in some projective space $\mathcal{P}$ there exists a set $\mathcal{S}$ of planes such that each line of $\mathcal{P}$ is contained in exactly one plane of $\mathcal{S}$. One can only show:

Proposition 6.5 *Let $\mathcal{S}$ be a set of planes of $\mathcal{P} = \mathrm{PG}(N, q)$ such that each line of $\mathcal{P}$ is contained in exactly one plane of $\mathcal{S}$. Then $N \equiv 2$ or $0 \bmod 6$.*

Proof: Each plane covers q^2+q+1 lines. Hence the number of lines of $\mathcal{P}$ must be a multiple of q^2+q+1. Hence $(q^2+q+1) \mid (q^{N+1}-1)(q^N-1)$, whence $3 \mid (N+1)$ or $3 \mid N$.

On the other hand, given a point $P \in \mathcal{P}$, the set of planes of $\mathcal{S}$ through P is a 1-spread in the quotient space $\mathcal{P}/P$. Hence $N-1$ is odd. From this the assertion follows. $\square$

One can also join the question of covering and blocking. What is the smallest number of points and planes blocking all lines of $\mathrm{PG}(3, q)$?

Example 6.6 Let E be a plane of $\mathcal{P} = \mathrm{PG}(3, q)$ and let P be a point of E. Let $l_1, \ldots, l_{q+1}$ be the lines of E through P. Choose $k \in \{1, \ldots, q\}$. Let $\mathcal{S}_1$ be the set of points, different from P, on the lines $l_1, \ldots, l_k$. Let $\mathcal{S}_2$ be the set of planes, different from E, through one of the lines $l_{k+1}, \ldots, l_{q+1}$. Then each line of $\mathcal{P}$ is either contained in a plane of $\mathcal{S}_2$ or does contain a point of $\mathcal{S}_1$, that is, the set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ covers all lines and has size $q(q+1)$.

Theorem 6.7 (Eisfeld, Storme and Sziklai [23]) *Let $\mathcal{S}$ be a 1-cover of $\mathcal{P} = \mathrm{PG}(3, q)$ by points and planes, that is, $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ is a set of points and $\mathcal{S}_2$ is a set of planes of $\mathcal{P}$ such that each line of $\mathcal{P}$ either is contained in a plane of $\mathcal{S}_2$ or contains a point of $\mathcal{S}_1$. Then $|\mathcal{S}| \geq q^2 + q$.*

Furthermore, the only examples where equality holds are given in Example 6.6.

Similar examples can be given if we want to cover t-spaces in $\mathrm{PG}(2t+1, q)$ by points and hyperplanes. However, the optimality of these examples has not been proved yet.

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