Constructions and Characterizations of (Semi)partial Geometries

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Chapter 1

Introduction

The purpose of this course is to give an overview of results on (α, β)-geometries with an emphasize on *(semi)partial geometries.*
Before we treat these geometries, we want to give a quick overview of some general concepts we will need in the rest of this course. This will also give us the opportunity to fix the notations we are using, although sometimes the notation is not consistent over the whole course.

1.1 Generalities on geometries

An *incidence structure* (a terminology going back to Dembowski [49]) is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}(\neq \emptyset)$ the set of points, $\mathcal{L}(\neq \emptyset)$ the set of lines and a symmetric incidence relation $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. The elements of I will also be called the *flags* of $\mathcal{S}$. In this course both the sets $\mathcal{P}$ and $\mathcal{L}$ will be finite. In a lot of cases, lines will be subsets of the point set $\mathcal{P}$ and the incidence I will be the symmetrized containment.
The *dual* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the incidence structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathrm{I}^D)$ with $\mathcal{P}^D = \mathcal{L}$, $\mathcal{L}^D = \mathcal{P}$, and I^D=I.
Isomorphisms (or collineations), anti-isomorphisms (or correlations), automorphisms, anti-automorphisms, involutions and polarities of incidence structures are defined in the usual way, and we will not explicitely define them here.
In more recent standard works (for instance in [19]) an incidence structure is called a *rank* 2 *geometry* $\mathcal{S}$ or a $\{0, 1\}$-geometry (note the curly brackets!), being a special case of what is commonly known as a *rank n geometry*, where n sets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{n-1}$ are involved with an incidence relation between any two of them. Elements of $\mathcal{P}_i$ are then called elements of *type i*. As we are only dealing with 2 sets $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_1 = \mathcal{L}$ we will rather use the Dembowski terminology. We will sometimes call $\mathcal{S}$ for short a *geometry.*
An incidence structure $\mathcal{S}$ is called a *partial linear space*, if each point is on at least 2 lines, if all lines have at least two points and if any two distinct

points in $\mathcal{P}$ are incident with at most one line, or equivalently, if any two distinct lines are incident with at most one point. Some authors call this a *semi-linear space*. Lines incident with only 2 points, are called *thin lines*. If all lines are thin lines, then $\mathcal{S}$ is called a *thin partial linear space*. If all lines are incident with at least 3 points and if every point is incident with at least 3 lines, the partial linear space is called *thick*. Two points are said to be *collinear* if they are incident with a common line. Note that a point is collinear to itself. Dually, two lines are said to be *concurrent* if they are incident with a common point. We will denote collinear points x and y (resp. concurrent lines L and M) by $x \sim y$, (resp., $L \sim M$) while $x \not\sim y$ (resp., $L \not\sim M$) means that x and y are not collinear (resp., L and M are not concurrent). If $x \sim y$ (resp., $L \sim M$) we may also say that x (resp., L) is *orthogonal* or *perpendicular* to y (resp., M). The line (resp., point) which is incident with distinct collinear points x, y (resp., distinct concurrent lines L, M) is denoted by xy (resp., LM or $L \cap M$). We will often use the notation $x^\perp$ for the set of points collinear to a point x (hence $x \in x^\perp$). Dually, $L^\perp$ will denote the set of lines concurrent to a line L.
If any two different points are collinear, then $\mathcal{S}$ is called a *linear space*.
In this course we will mainly deal with a quite special class of partial linear spaces (grids and dual grids, see below, being counterexamples). They will have the next two properties:

($\mathcal{S}_1$): Each point is incident with $t+1$ $(t \geq 1)$ lines.

($\mathcal{S}_2$): Each line is incident with $s+1$ $(s \geq 1)$ points.

A partial linear space $\mathcal{S}$ satisfying these two properties will be called a partial linear space of *order* (s,t), if $s=t$, then the partial linear space will be said to have order s. Let (x, L) be an anti-flag of $\mathcal{S}$, i.e. x is a point and L is a line of $\mathcal{S}$, such that x is not incident with L. We denote by $\alpha(x, L)$ the number of points on L collinear with x, or equivalently the number of lines through x concurrent with L. We will sometimes call $\alpha(x, L)$ the *incidence number* of the anti-flag (x, L). Of special interest here, will be those partial linear spaces of order (s,t) in which $\alpha(x, L)$ can take only a few values.
A *grid* (resp., *dual grid*) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P} = \{x_{ij} \mid i = 0, \ldots, s_1 \text{ and } j = 0, \ldots, s_2\}, s_1 > 0 \text{ and } s_2 > 0$ (resp., $\mathcal{L} = \{L_{ij} \mid i = 0, \ldots, t_1 \text{ and } j = 0, \ldots, t_2\}, t_1 > 0 \text{ and } t_2 > 0$, $\mathcal{L} = \{L_0, \ldots, L_{s_1}, M_0, \ldots, M_{s_2}\}$ (resp., $\mathcal{P} = \{x_0, \ldots, x_{t_1}, y_0, \ldots, y_{t_2}\}$), $x_{ij} \mathrm{~I~} L_k$ if and only if $i = k$ (resp., $L_{ij} \mathrm{~I~} x_k$ if and only if $i = k$), and $x_{ij} \mathrm{~I~} M_k$ if and only if $j = k$ (resp., $L_{ij} \mathrm{~I~} y_k$ if and only if $j = k$). Note that s_1, s_2 (resp., t_1, t_2) are not necessarily equal; if $s_1 = s_2 = s$ (resp., $t_1 = t_2 = t$) we will call it a grid of order s (resp., a dual grid of order t).
For a lot of the examples we will encounter in this course the points and lines are points and lines of a projective or affine space. To be more precise, an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ will be called *projective* if $\mathcal{P}$ is a subset

of the point set of some projective space $\mathrm{PG}(d,q)$, $\mathcal{L}$ is a set of lines of $\mathrm{PG}(d,q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathrm{PG}(d,q)$. We also say that $\mathcal{S}$ is *embedded* in $\mathrm{PG}(d,q)$. If $\mathrm{PG}(d',q)$ is the subspace of $\mathrm{PG}(d,q)$ generated by all points of $\mathcal{P}$, then we say that $\mathrm{PG}(d',q)$ is the *ambient space* of $\mathcal{S}$.
In the same way we define $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ to be *affine* or to be *embedded* in the finite affine space $\mathrm{AG}(d,q)$ if $\mathcal{P}$ is a subset of the point set of $\mathrm{AG}(d,q)$, $\mathcal{L}$ is a set of lines of $\mathrm{AG}(d,q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathrm{AG}(d,q)$. If $\mathrm{AG}(d',q)$ is the subspace of $\mathrm{AG}(d,q)$ generated by all points of $\mathcal{P}$, then we say that $\mathrm{AG}(d',q)$ is the *ambient space* of $\mathcal{S}$.
A special type of affine embedding is the so-called *linear representation* of a geometry of order (s,t) in $\mathrm{AG}(n{+}1, s{+}1)$. It is an embedding of $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ in $\mathrm{AG}(n+1, s+1)$ such that the line set $\mathcal{L}$ of $\mathcal{S}$ is a union of parallel classes of lines of $\mathrm{AG}(n+1, s+1)$ and so the point set $\mathcal{P}$ of $\mathcal{S}$ is the point set of $\mathrm{AG}(n+1, s+1)$. Then the lines of $\mathcal{S}$ define a set of points $\mathcal{K}$ of size $t+1$ in the hyperplane at infinity $\Pi_\infty = \mathrm{PG}(n, s+1)$ of $\mathrm{AG}(n+1, s+1)$. If (x, L) is an anti-flag of $\mathcal{S}$, then the line $xL \cap \Pi_\infty$ intersects $\mathcal{K}$ in $\alpha(x,L)+1$ points. A linear representation of an incidence structure $\mathcal{S}$ in $\mathrm{AG}(n+1, s+1)$ will be denoted by $T_n^*(\mathcal{K})$. We shall give several examples in this course.

1.2 Graphs

1.2.1 Some general definitions from graph theory

A finite *graph* $\Gamma = (X, E)$ is a structure consisting of a set $X (\neq \emptyset)$ with v elements and a set E of unordered pairs of X. The elements of X are called the *vertices* of the graph Γ, while the elements of E are called the *edges*. If x and y are two different vertices such that $\{x, y\} \in E$, then x and y are called *adjacent* and we write $x \sim y$; if $\{x, y\} \notin E$ then we denote this by $x \not\sim y$. Remark that $x \not\sim x$. If E is the set of all unordered pairs of X then Γ is called the *complete graph on v vertices* and is denoted by K_v. The *complement* Γ^C of a graph $\Gamma = (X, E)$ is the graph $\Gamma^C = (X^C, E^C)$ with $X^C = X$ and $E^C = X^{|2|} \setminus E$. The *line graph* $L(\Gamma)$ of a graph Γ is the graph with vertices the edges of Γ, two distinct edges being adjacent if and only if they have a common vertex.
A *path* of length m from x to y, is a sequence of vertices $x = x_0, x_1, x_2, \ldots, x_m = y$ in the graph, such that $x_i \neq x_{i+2}$, $0 \leq i \leq m-2$ and $x_i \sim x_{i+1}$, $0 \leq i \leq m-1$. If $x = y$ then any such path of length at least 3 will be called a *circuit*. Two distict vertices x and y of a graph Γ are at *distance* $\delta(x,y)$, provided there exists a path of length $\delta(x,y)$ between these vertices and there exists no shorter one; by definition a vertex has distance 0 from itself. A vertex has distance 1 from all its adjacent vertices. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from x. For convenience we will use $\Gamma(x)$

for the set $\Gamma_1(x)$. If $|\Gamma(x)| = k$ for all vertices x, then Γ is called *regular* of *valency* k. A graph is *connected* if and only if for any two distinct vertices x and y, there is at least one path connecting these 2 vertices. The *diameter* d of a connected graph Γ is the maximum value of the distance function $\delta(x, y)$. The *girth* of a graph Γ having at least one circuit, is the length of its shortest circuit.
For any graph Γ of diameter d, and vertex set $\{x_1, \ldots, x_v\}$, the *distance matrices* A_h, $h = 0, \ldots, d$, are the $v \times v$ matrices defined as follows:

$$(A_h)_{ij} = \begin{cases} 1 & \text{if } \delta(x_i, x_j) = h \\ 0 & \text{otherwise.} \end{cases}$$

Given a partial linear space $\mathcal{S}$, one may define the *point graph* or *collinearity graph* $\Gamma(\mathcal{S})$, by taking as vertices the points of $\mathcal{S}$. Two different vertices are adjacent whenever they are collinear. Remark that we are using the same symbol ($\sim$) for the collinearity relation as for the adjacency relation, although a point x is collinear to itself but not adjacent to itself. A geometry $\mathcal{S}$ is connected whenever its point graph $\Gamma(\mathcal{S})$ is. From now on, we will only deal with connected geometries.
The *incidence graph* $\mathcal{I}(\mathcal{S})$ is the graph with vertices the elements of $\mathcal{P} \cup \mathcal{L}$, and 2 vertices are adjacent if and only if the corresponding elements are incident; hence edges of $\mathcal{I}(\mathcal{S})$ are the flags of $\mathcal{S}$. Unlike the case of the collinearity graph, the geometry is completely determined (up to duality) by its incidence graph. Obviously, two vertices of the same type (i.e. either points or lines) in the incidence graph are connected by paths of even length. In particular a circuit in an incidence graph has even length and hence the girth is an even positive integer, say $2n$. By definition, n is called the *gonality* or *geometric girth* of $\mathcal{S}$. A *geodesic* (*based at* x) is a path γ in the incidence graph starting in x and such that the length of γ is equal to the distance $\delta(x, y)$, where y is the last element of γ. A *maximal geodesic* is a geodesic that is not properly contained in another one. The *local diameter* $d(x)$ is the length of the longest geodesic based at x, whether x be a point or a line. The *point-diameter* d_p (resp. *line-diameter* d_l) of $\mathcal{S}$ is the greatest value taken by $d(x)$ for x a point (resp. a line). The *diameter* d of a geometry $\mathcal{S}$ is the diameter of the incidence graph $\mathcal{I}(\mathcal{S})$, hence it is the largest of the two numbers d_p, d_l.
Finally, the *flag graph* of a geometry $\mathcal{S}$ is the graph with as vertices the flags of $\mathcal{S}$, where two flags are adjacent whenever they share exactly one element. The *flag-diameter* d^* of $\mathcal{S}$ is the diameter of the flag graph.

1.2.2 Strongly regular graphs

A regular graph Γ is called a strongly regular graph (notation srg(v, k, λ, μ)) provided:

1. any two vertices x and y, $x \sim y$, are both adjacent to a constant number λ of vertices (independent of the choice of the adjacent pair $\{x, y\}$);

2. any two distinct vertices x and y, $x \not\sim y$, are both adjacent to a constant number μ of vertices (independent of the choice of the non-adjacent pair $\{x, y\}$).

As we will exclude disconnected graphs and their complements, we may assume $0 < \mu < k < v - 1$. It is easy to check that the complement of a $\mathrm{srg}(v, k, \lambda, \mu)$ is a $\mathrm{srg}(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$.
The distance matrix $A_1 = A$ (or the $(0, 1)$ adjacency matrix) of a $\mathrm{srg}(v, k, \lambda, \mu)$ satisfies

$$AJ = kJ, \qquad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where J is the all-one matrix. Hence A has the valency k as an eigenvalue with multiplicity 1, and two other eigenvalues r and l ($r > 0$ and $l < 0$) with $r + l = \lambda - \mu$ and $rl = \mu - k$.
There are some necessary conditions known for the existence of a $\mathrm{srg}(v, k, \lambda, \mu)$. We will summarize the most important ones in the next theorem. For the proofs and more information on strongly regular graphs we refer to [8], [129], [25], [103], [27] and [14].

Theorem 1.2.1
If Γ is a $\mathrm{srg}(v, k, \lambda, \mu)$ then:

1. $k(k - \lambda - 1) = \mu(v - k - 1)$.

2. *The multiplicities of the eigenvalues r and l of A are respectively*

$$f = \frac{-k(l + 1)(k - l)}{(k + rl)(r - l)} \quad \text{and} \quad g = \frac{k(r + 1)(k - r)}{(k + rl)(r - l)};$$

   they clearly have to be integers.

3. *The eigenvalues $r > 0$ and $l < 0$ are both integers, except for one family of graphs, the so-called conference graphs, which are $\mathrm{srg}(2k + 1, k, \frac{k}{2} - 1, \frac{k}{2})$. For a conference graph the number of vertices can be written as a sum of two squares, and the eigenvalues are $\frac{-1 \pm \sqrt{v}}{2}$.*

4. *The Krein conditions:*

$$(r + 1)(k + r + 2rl) \leq (k + r)(l + 1)^2,$$

$$(l + 1)(k + l + 2rl) \leq (k + l)(r + 1)^2.$$

5. *The absolute bound:*

$$v \leq \frac{1}{2}f(f+3), \qquad v \leq \frac{1}{2}g(g+3).$$

A lot of examples of strongly regular graphs are known, see for instance [73] and [14]. We shall give here a short description of some examples which are important for the rest of this course.

1. The pentagon $Pn(5)$ is the unique srg$(5, 2, 0, 1)$.

2. The line graph of the complete graph K_n is called a triangular graph and is denoted by $T(n)$. This graph is a srg$(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$. If $n \neq 8$ then every strongly regular graph Γ with these parameters is indeed a triangular graph. If $n = 8$ there are exactly three non-isomorphic graphs with the same parameters but not triangular; these graphs are known as the graphs of Chang [29], see also [100].

3. The strongly regular graph $T(5)^C$ is better known as the Petersen graph $Pe(10)$; it is the unique srg$(10, 3, 0, 1)$. This graph can also be constructed by taking as vertices the 10 points of a Desargues configuration, two vertices being adjacent if they are not on a line of the Desargues configuration.

4. The Clebsch graph $Cl(16)$ is a srg$(16, 5, 0, 2)$. There is only one graph with these parameters (easy exercise). The graph can be constructed as follows. Let C be the set $\{1, 2, 3, 4, 5\}$. The vertices of the graph are the set C and the subsets of cardinality 1 and 2. The vertex C is adjacent to the 5 singletons $\{i\}$, a singleton $\{i\}$ is adjacent to C and to all the pairs containing i, a pair $\{i, j\}$ is adjacent to the 2 singletons $\{i\}$ and $\{j\}$ and to the 3 pairs of C that are disjoint from $\{i, j\}$. Other constructions of this graph are known. Another simple construction goes as follows. The vertices of the graph are the elements of GF(16), two vertices being adjacent whenever their difference is a 3rd power in GF(16). The name comes from the fact that this graph can also be constructed from the set of the 16 lines on the Clebsch quartic surface (see [30] or [33])

5. The graph *HoS*(50) of Hoffman and Singleton [72]. A lot of constructions of this graph are known. This graph is a srg$(50, 7, 0, 1)$, and is uniquely defined by its parameters. We shall give only one construction which will be useful later on. It is commonly known that there is a bijection between the 35 unordered triples of a 7-set and the 35 lines of PG(3,2), such that lines intersect if and only if the corresponding triples have exactly one element in common. The graph *HoS*(50) is constructed as follows. The vertices are the 15 points together with

the 35 lines of PG(3,2). Points are mutually non-adjacent. A point is adjacent to a line whenever the point lies on that line. Two lines are adjacent whenever the corresponding two triples are disjoint.

6. The Higman-Sims family (see for instance [73]).

   These graphs are constructed using the Steiner system $S(3,6,22)$. This Steiner system is the uniquely defined extension of the projective plane $PG(2,4)$. This Steiner system has as other parameters $b = 77, r = 21, \lambda_2 = 5$, and two different blocks intersect in 0 or 2 points.

   (a) The Higman-Sims graph $HS(100)$.
   Take as vertices of the graph a symbol ∞, together with the 22 points and the 77 blocks of $S(3,6,22)$. The symbol ∞ is adjacent to all the 22 points but to no block. Points are never adjacent and a point is adjacent to a block whenever it is contained in that block. Two blocks are adjacent whenever they are disjoint. This graph is a $\mathrm{srg}(100,22,0,6)$, and is uniquely defined by its parameters, see [58].

   (b) The Higman-Sims graph $HS(77)$.
   This graph is the subgraph of $HS(100)$ defined on the set $\Gamma_2(\infty)$ of the vertices that are not adjacent to ∞. It is a $\mathrm{srg}(77,16,0,4)$, and is uniquely defined by its parameters, see [10].

   (c) The graph $Gew(56)$ of Gewirtz [57].
   Delete from $S(3,6,22)$ all the 21 blocks through a fixed point. Take as vertices the 56 other blocks, two blocks being adjacent whenever they are disjoint. This graph is a $\mathrm{srg}(56,10,0,2)$, and is also uniquely defined by its parameters (see also [12]).

1.3 (α,β)-geometries

An (α,β)-geomety $\mathcal{S}$ is a connected partial linear space of order (s,t), such that for any anti-flag (x,L) the incidence number $\alpha(x,L)$ can have at most 2 values α or β.
If $\alpha = \beta$, then $\mathcal{S}$ is known under the name *partial geometry* and was introduced by Bose [8]. The numbers s, t and α are called the *parameters* of $\mathcal{S}$, and we will use the notation $\mathrm{pg}(s,t,\alpha)$.
It is an easy excercise to prove that the point graph $\Gamma(\mathcal{S})$ of a $\mathrm{pg}(s,t,\alpha)$ $\mathcal{S}$ is a

$$\mathrm{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

Remarks

1. If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial geometry with parameters s, t, α, then the dual structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathrm{I}^D) = (\mathcal{L}, \mathcal{P}, \mathrm{I})$, is a partial geometry with parameters $s^D = t$, $t^D = s$ and $\alpha^D = \alpha$.

2. $|\mathcal{P}| = v = (s+1)\dfrac{(st+\alpha)}{\alpha}$ and $|\mathcal{L}| = b = (t+1)\dfrac{(st+\alpha)}{\alpha}$.

3. The partial geometries can be divided into four (non-disjoint) classes.

   (a) The partial geometries with $\alpha = 1$, that are the generalized quadrangles. See for instance [97] for more information on generalized quadrangles. Chapter 2 will treat these geometries in detail.

   (b) The partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, i.e. the 2–$(v, s+1, 1)$ designs and their duals. We refer to any standard book on design theory for more details.

   (c) The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *nets of order* $s+1$ *and degree* $t+1$, introduced by Bruck [16] (note that we are here in conflict with our former definition of order).

   (d) Finally, the so–called *proper* partial geometries with $1 < \alpha < \min(s, t)$. We shall mainly deal with this class of partial geometries in Chapter 3.

A second class of (α, β)-geometries $\mathcal{S}$ that we will treat in Chapter 4.2 are the ones with $\beta = 0$. However the point graph $\Gamma(\mathcal{S})$ of a $(0, \alpha)$ geometry is not necessarily a strongly regular graph. If $\Gamma(\mathcal{S})$ is strongly regular then the geometry is called a *semipartial geometry* and was introduced by Debroey and Thas [46]. A special case of such geometries corresponds to $\alpha = 1$; these geometries are called *partial quadrangles* and were introduced by Cameron [24] as a generalization of the generalized quadrangles (hence the name, although in our context we better should call them *semigeneralized quadrangles*). The $(0, s)$-geometries are known under the name of *copolar spaces* (see [62]). The $(1, s+1)$-geometries are called *polar spaces*, and were introduced and classified by [53]. Under mild nonsingularity conditions, also the copolar spaces are classified, see Theorem 4.8.1. This is not the case for general semipartial geometries as we will see in Chapter 4.2.
The partial and semipartial geometries can be regarded as generalizations of the generalized quadrangles. But generalized quadrangles, together with the projective planes, can also be seen as special cases of the more general *(weak) generalized polygons*. These incidence structures are incidence structures whose diameter is equal to the gonality, but we will not treat them as such.

Chapter 2

Generalized Quadrangles

2.1 Axioms and definitions

A (finite) generalized quadrangle (GQ) is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I}$ $)$ of order (s,t) such that for any anti-flag (x, L), the incidence number $\alpha(x, L) = 1$. Generalized quadrangles were introduced by Tits [123] in his celebrated work on triality.

Conform the definitions given in Chapter 1, the GQ is said to have *order* (s,t); if $s = t, \mathcal{S}$ is said to have *order* s. We will often call s and t the *parameters* of the GQ. Note that the dual of a GQ of order (s,t) is a GQ of order (t,s). Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

The *trace* of a pair $\{x, y\}$ of distinct points is defined to be the set $x^\perp \cap y^\perp$ and is denoted $\mathrm{tr}(x,y)$ or $\{x,y\}^\perp$. We have $|\{x,y\}^\perp| = s+1$ or $t+1$ according as $x \sim y$ or $x \not\sim y$. More generally, if $A \subseteq \mathcal{P}$, A "perp" is defined by $A^\perp = \cap\{x^\perp \,|\, x \in A\}$. For $x \neq y$, the *span* of the pair $\{x,y\}$ is $\mathrm{sp}(x,y) = \{x,y\}^{\perp\perp} = \{u \in P \,|\, u \in z^\perp \, \forall z \in x^\perp \cap y^\perp\}$. If $x \not\sim y$, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and y.

If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x,y\}^{\perp\perp}| = t+1$, we say that the pair $\{x,y\}$ is *regular*. The point x is *regular* provided $\{x,y\}$ is regular for all $y \in \mathcal{P}$, $y \neq x$. A point x is *coregular* provided each line incident with x is regular.

The pair $\{x,y\}$, $x \not\sim y$, is *antiregular* provided $|z^\perp \cap \{x,y\}^\perp| \leq 2$ for all $z \in \mathcal{P} \setminus \{x,y\}$. A point x is *antiregular* provided $\{x,y\}$ is antiregular for all $y \in \mathcal{P} \setminus x^\perp$.

For $x \neq y$, the *closure* of the pair $\{x,y\}$ is $\mathrm{cl}(x,y) = \{z \in \mathcal{P} \,|\, z^\perp \cap \{x,y\}^{\perp\perp} \neq \emptyset\}$.

A *triad* (of points) is a triple of pairwise noncollinear points. Given a triad $T = \{x,y,z\}$, *a center* of T is just a point of $T^\perp$. We say T is *acentric*, *centric*, or *unicentric* according as $|T^\perp|$ is zero, positive, or equal to 1.

2.2 Restrictions on the parameters

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order (s, t), then the point graph $\Gamma(\mathcal{S})$ is a

$$\mathrm{srg}((s+1)(st+1), s(t+1), s-1, t+1),$$

while $\Gamma(\mathcal{S}^D)$ is a

$$\mathrm{srg}((t+1)(st+1), t(s+1), t-1, s+1).$$

Hence Theorem 1.2.1 yield some necessary conditions that we summarize in the next theorem.

Theorem 2.2.1
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ *of order (s, t) then*

(i) $|\mathcal{P}| = v = (s+1)(st+1)$*;*

(ii) $|\mathcal{L}| = b = (t+1)(st+1)$*;*

(iii) The eigenvalues of the adjacency matrix A of $\Gamma(\mathcal{S})$ are $k = s(t+1)$, $r = s-1$, $l = -t-1$; moreover r has multiplicity $f = st(s+1)(t+1)/(s+t)$, hence $s+t$ divides $st(s+1)(t+1)$.

Also the Krein condition for the graph $\Gamma(\mathcal{S})$ yields a necessary condition. For a GQ this condition is known as the inequality of Higman [65]. We will give a proof of this inequality using the so-called *variance trick*.

Theorem 2.2.2 (Inequality of Higman [65])
Assume $\mathcal{S}$ is a GQ *of order (s, t). If $s > 1$ and $t > 1$, then $t \leq s^2$, and dually $s \leq t^2$.*

Proof. Let x, y be two noncollinear points of $\mathcal{S}$. Put $V = \{z \in \mathcal{P} \mid z \not\sim x$ and $z \not\sim y\}$, so $|V| = d = (s+1)(st+1) - 2 - 2(t+1)s + (t+1)$. Denote the elements of V by $z_1, \ldots, z_d$ and let $t_i = |\{u \in \{x, y\}^\perp \mid u \sim z_i\}|$. Count in different ways the number of ordered pairs $(z_i, u) \in V \times \{x, y\}^\perp$ with $u \sim z_i$ to obtain

$$\sum_i t_i = (t+1)(t-1)s. \tag{2.1}$$

Next count in two different ways the number of ordered triples $(z_i, u, u') \in V \times \{x, y\}^\perp \times \{x, y\}^\perp$ with $u \neq u'$, $u \sim z_i$, $u' \sim z_i$, to obtain

$$\sum_i t_i(t_i - 1) = (t+1)t(t-1). \tag{2.2}$$

From (2.1) and (2.2) it follows that $\sum\limits_i t_i^2 = (t+1)(t-1)(s+t)$.

With $d\bar{t} = \sum\limits_i t_i$, the expression $0 \leq \sum\limits_i (\bar{t}-t_i)^2$ simplifies to $d\sum\limits_i t_i^2 - (\sum\limits_i t_i)^2 \geq 0$, which implies $d(t+1)(t-1)(s+t) \geq (t+1)^2(t-1)^2s^2$, or $t(s-1)(s^2-t) \geq 0$, completing the proof. ■

There is an immediate corollary of the proof.

Theorem 2.2.3 ([9])
Let $\mathcal{S}$ be a generalized quadrangle of order (s,t), $s \neq 1 \neq t$. Then $t = s^2$ if and only if for each triad $\{x, y, z\}$ there is a constant number of points collinear with x, y and z. This constant number of points is $s+1$.

Remark

Higman [65] first proved Theorem 2.2.2 by a complicated matrix-theoretic method. The argument given here was used by Bose and Shrikhande [9] to show that in case $t = s^2 > 1$ each triad has $1+s$ centers, and Cameron [24] apparently first observed that the above technique also provides the inequality.

Theorem 2.2.4
If $s \neq 1, t \neq 1, s \neq t^2$ and $t \neq s^2$, then $t \leq s^2 - s$ and dually $s \leq t^2 - t$.

Proof. Suppose $s \neq 1$ and $t \neq s^2$. By Theorem 2.2.2 we have $t = s^2 - x$ with $x > 0$. By Theorem 2.2.1 $(s+s^2-x)|s(s^2-x)(s+1)(s^2-x+1)$. Hence modulo $s+s^2-x$ we have $0 \equiv x(-s)(-s+1) \equiv x(x-2s)$. If $x < 2s$, then $s+s^2-x \leq x(2s-x)$ forces $x \in \{s, s+1\}$. Consequently $x = s, x = s+1$, or $x \geq 2s$, from which it follows that $t \leq s^2 - s$. ■

Remark

1. If $\mathcal{S}$ $(s \neq 1 \neq t)$ admits either a regular or antiregular pair then $s \geq t$.

2. If $\mathcal{S}$ has order s and if $\mathcal{S}$ admits an antiregular pair then s is odd.

2.3 The classical finite generalized quadrangles

We give a brief description of three families of generalized quadrangles, known as the *classical generalized quadrangles*, all of which are associated with classical groups and were first recognized as generalized quadrangles by Tits; see, e.g., [49].

(i) Consider a nonsingular quadric $Q^+(3,q)$ (respectively, $Q(4,q)$ and $Q^-(5,q)$) of Witt index 2, that is, of projective index 1, in the projective space $\mathrm{PG}(3,q)$ (respectively, $\mathrm{PG}(4,q)$ and $\mathrm{PG}(5,q)$). Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters

$$s = q, t = 1, v = (q+1)^2, b = 2(q+1), \text{ for } Q^+(3,q),$$

$$s = t = q, v = b = (q+1)(q^2+1), \text{ for } Q(4,q),$$

$$s = q, t = q^2, v = (q+1)(q^3+1), b = (q^2+1)(q^3+1), \text{ for } Q^-(5,q).$$

In each of the three cases for the quadric and the corresponding GQ the same notation will be used.

The GQ $Q^+(3,q)$ is a grid of order q. Further, recall that these quadrics have the following canonical equations:

$$\begin{aligned} Q^+(3,q) &: X_0X_1 + X_2X_3 = 0, \\ Q(4,q) &: X_0^2 + X_1X_2 + X_3X_4 = 0, \\ Q^-(5,q) &: F(X_0,X_1) + X_2X_3 + X_4X_5 = 0, \end{aligned}$$

where F is an irreducible homogeneous polynomial in X_0, X_1 over $\mathrm{GF}(q)$.

(ii) Let $H(d,q^2)$ be a nonsingular Hermitian variety of the projective space $\mathrm{PG}(d,q^2)$, $d = 3$ or 4. Then the points of $H(d,q^2)$, together with the lines on $H(d,q^2)$, form a generalized quadrangle with parameters:

$$s = q^2, t = q, v = (q^2+1)(q^3+1), b = (q+1)(q^3+1), \text{ when } d = 3,$$

$$s = q^2, t = q^3, v = (q^2+1)(q^5+1), b = (q^3+1)(q^5+1), \text{ when } d = 4.$$

Remember that $H(d,q^2)$ has the canonical equation

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_d^{q+1} = 0.$$

In each of these two cases for the Hermitian variety and the corresponding GQ the same notation will be used.

(iii) The points of $\mathrm{PG}(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle $W_3(q)$, shortly $W(q)$, with parameters

$$s = t = q, v = b = (q+1)(q^2+1).$$

Remember that the lines of $W_3(q)$ are the elements of a general linear complex of lines of PG$(3,q)$, see e.g. [69], and that a symplectic polarity of PG$(3,q)$ has the following canonical bilinear form :

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

Theorem 2.3.1 ([97])

(a) $Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) is self-dual if and only if q is even.

(b) $Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$.

Proof. (a) Let $Q^+(5,q)$ be the Klein quadric of the lines of PG$(3,q)$; see [69]. Then $Q^+(5,q)$ is a hyperbolic quadric of PG$(5,q)$. The image of $W(q)$ on $Q^+(5,q)$ is the intersection of $Q^+(5,q)$ with a nontangent hyperplane PG$(4,q)$ of PG$(5,q)$. The nonsingular quadric $Q^+(5,q) \cap$ PG$(4,q)$ of PG$(4,q)$ is denoted by $Q(4,q)$. The lines of $W(q)$ which are incident with a given point form a flat pencil of lines, hence their images on $Q^+(5,q)$ form a line of $Q(4,q)$. Now it easily follows that $W(q)$ is isomorphic to the dual of $Q(4,q)$.
Now consider the nonsingular quadric $Q(4,q)$ of PG$(4,q)$. Let L_0 and L_1 be nonconcurrent lines of $Q(4,q)$. Then the 3-space L_0L_1 intersects $Q(4,q)$ in a hyperbolic quadric having reguli $\{L_0, L_1, \ldots, L_q\}$ and $\{M_0, M_1, \ldots, M_q\}$. In $Q(4,q)$ we have $L_i \sim M_j, i,j = 0, \ldots, q$, so $\{L_0, L_1\}$ is a regular pair of lines of $Q(4,q)$. Hence all lines of $Q(4,q)$ are regular. As for q odd no point of $Q(4,q)$ is regular, $Q(4,q)$ (and also $W(q)$) is not self-dual. So let q be even. The tangent 3-spaces of $Q(4,q)$ all meet in one point n, the nucleus of $Q(4,q)$; see [71]. From n we project $Q(4,q)$ onto a PG$(3,q)$ not containing n. This yields a bijection of the point set of $Q(4,q)$ onto PG$(3,q)$, mapping the $(q+1)(q^2+1)$ lines of $Q(4,q)$ onto $(q+1)(q^2+1)$ lines of PG$(3,q)$. Since the $q+1$ lines of $Q(4,q)$ which are incident with a given point are contained in a tangent 3-space of $Q(4,q)$, they are mapped onto the elements of a flat pencil of lines of PG$(3,q)$. Hence the images of the lines of $Q(4,q)$ constitute a general linear complex of lines of PG$(3,q)$, i.e. they are the totally isotropic lines with respect to a symplectic polarity of PG$(3,q)$. It follows that $Q(4,q) \cong W(q)$, and consequently $Q(4,q)$ (and $W(q)$) are self-dual.
(b) Let $Q^-(5,q)$ be an elliptic quadric, i.e. a nonsingular quadric of projective index 1, in PG$(5,q)$. Extend PG$(5,q)$ to PG$(5,q^2)$. Then the extension of $Q^-(5,q)$ is a hyperbolic quadric $Q^+(5,q^2)$, i.e. a nonsingular quadric of projective index 2, in PG$(5,q^2)$. Hence $Q^+(5,q^2)$ is the Klein quadric of the lines of PG$(3,q^2)$. So to $Q^-(5,q)$ in $Q^+(5,q^2)$ there corresponds a set $\mathcal{V}$ of lines in PG$(3,q^2)$. To a given line L of the GQ $Q^-(5,q)$ there correspond $q+1$ lines of PG$(3,q^2)$ that all lie in a plane and pass through a point x. Let $\mathcal{H}$ be the set of points on the lines of $\mathcal{V}$. Then with each point of $Q^-(5,q)$

there corresponds a line of $\mathcal{V}$, and with each line L of $Q^-(5,q)$ there corresponds a point x of $\mathcal{H}$. With distinct lines L, L' of $Q^-(5,q)$ correspond distinct points x, x' of $\mathcal{H}$ (a plane of $Q^+(5,q^2)$ contains at most one line of $Q^-(5,q)$). Since a point y of $Q^-(5,q)$ is on q^2+1 lines of $Q^-(5,q)$, these q^2+1 lines are mapped onto the q^2+1 points of the image of y. Hence we obtain an anti-isomorphism from $Q^-(5,q)$ onto the structure $(\mathcal{H}, \mathcal{V}, \mathrm{I})$, where I is the natural incidence relation. So $(\mathcal{H}, \mathcal{V}, \mathrm{I})$ is a GQ of order (q^2, q) embedded in $\mathrm{PG}(3,q^2)$. But now by a celebrated result of Buekenhout and Lefèvre [20] (see also Section 2.6) the GQ $(\mathcal{H}, \mathcal{V}, \mathrm{I})$ must be $H(3,q^2)$. ■

The proof just given is in [97]. An algebraic proof of the same theorem was given by Bruen and Hirschfeld [17].

Remark

Tits [124] proves that $W(q)$ is self-polar if and only if $q = 2^{2h+1}, h \geq 0$.

2.4 Nonclassical finite generalized quadrangles

2.4.1 The trivial nonclassical generalized quadrangles, grids and dual grids

In Section 2.1 grids and dual grids were defined. The GQ $\mathcal{S}$ of order (s,t) has $t=1$ if and only if it is a grid for which all lines are incident with the same number $s+1$ of points; the GQ $\mathcal{S}$ of order (s,t) has $s=1$ if and only if it is a dual grid for which all the points are incident with the same number $t+1$ of lines.

2.4.2 The nonclassical examples of Tits

The earliest known nontrivial nonclassical examples of GQ were discovered by Tits and first appeared in [49].
Let $d=2$ (respectively, $d=3$) and let O be an oval (respectively, an ovoid) of $\mathrm{PG}(d,q)$; for the definitions of oval and ovoid, see [68], [69] and [119]. Further, let $\mathrm{PG}(d,q)$ be embedded as a hyperplane in $\mathrm{PG}(d+1,q)$. Define points as

(i) the points of $\mathrm{PG}(d+1,q) \setminus \mathrm{PG}(d,q)$,

(ii) the hyperplanes X of $\mathrm{PG}(d+1,q)$ for which $|X \cap O| = 1$, and

(iii) one new symbol (∞).

Lines are defined as

(a) the lines of $\mathrm{PG}(d+1,q)$ which are not contained in $\mathrm{PG}(d,q)$ and meet O (necessarily in a unique point), and

(b) the points of O.

Incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(d+1,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O in it. The point (∞) is incident with no line of type (a) and all lines of type (b). It is an easy exercise to show that the incidence structure so defined is a GQ with parameters

$$s=t=q, v=b=(q+1)(q^2+1), \text{ when } d=2,$$

$$s=q, t=q^2, v=(q+1)(q^3+1), b=(q^2+1)(q^3+1), \text{ when } d=3.$$

If $d=2$, the GQ is denoted by $T_2(O)$; if $d=3$, the GQ is denoted by $T_3(O)$. If no confusion is possible, these quadrangles are also denoted by $T(O)$.

2.4.3 The examples of Ahrens and Szekeres; Hall, Jr. and Payne

For each prime power q, Ahrens and Szekeres [1] constructed GQ of order $(q-1,q+1)$. For q even, these examples were found independently by Hall [64]. Then a construction was found by Payne [88] which included all these examples and for q even produced some additional ones (see also [89], [93] and [97]). These examples yield the only known cases, with $s>1$ and $t>1$, in which s and t are not powers of the same prime.

The construction of Ahrens and Szekeres [1] and Hall [64], for q even

Let O be a hyperoval (see [68] and [119]), i.e. a $(q+2)$-arc, of the projective plane $\mathrm{PG}(2,q)$, $q=2^h$, and let $\mathrm{PG}(2,q)$ be embedded as a plane in $\mathrm{PG}(3,q)$. As each line of $\mathrm{PG}(2,q)$ is either an exterior line or a 2-secant of O, the linear representation $T_2^*(O)$ yields a GQ with parameters

$$s=q-1, t=q+1, v=q^3, b=q^2(q+2).$$

The construction of Ahrens and Szekeres [1], for q odd

Let the elements of $\mathcal{P}$ be the points of the affine 3-space $\mathrm{AG}(3,q)$ over $\mathrm{GF}(q)$, q odd. Elements of $\mathcal{L}$ are the following curves of $\mathrm{AG}(3,q)$:

(i) $x=\sigma, y=a, z=b$,

(ii) $x=a, y=\sigma, z=b$,

(iii) $x=c\sigma^2-b\sigma+a, y=-2c\sigma+b, z=\sigma$.

Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence I is the natural one. Then $(\mathcal{P},\mathcal{L},\mathrm{I})$ is a GQ of order $(q-1,q+1)$. It will be denoted by $AS(q)$.

The construction of Payne [88]

Let x be a regular point of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order s, $s > 1$. Then $\mathcal{P}'$ is defined to be the set $\mathcal{P} \setminus x^{\perp}$. The elements of $\mathcal{L}'$ are of two types: the elements of type (a) are the lines of $\mathcal{L}$ which are not incident with x; the elements of type (b) are the hyperbolic lines $\{x, y\}^{\perp\perp}$, $y \not\sim x$. Now let us define the incidence I'. If $y \in \mathcal{P}'$ and $L \in \mathcal{L}'$ is of type (a), then $y \,\mathrm{I}'\, L$ if and only if $y \,\mathrm{I}\, L$; if $y \in \mathcal{P}'$ and $L \in \mathcal{L}'$ is of type (b), then $y \,\mathrm{I}'\, L$ if and only if $y \in L$. The incidence structure $P(\mathcal{S}, x) = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ so defined is a GQ of order $(s-1, s+1)$.
A quick look at the examples of order s in 2.3 and 2.4.2 reveals that regular points and regular lines arise in the following cases (see [97]):

- all lines of $Q(4,q)$ are regular;
- the points of $Q(4,q)$ are regular if and only if q is even;
- all points of $W(q)$ are regular;
- the lines of $W(q)$ are regular if and only if q is even;
- the unique point (∞) of type (iii) of $T_2(O)$ is regular if and only if q is even;
- all lines of type (b) of $T_2(O)$ are regular.

It is easily seen that for q even the GQ $P(T_2(O), (\infty))$ is the GQ $T_2^*(\overline{O})$, with $\overline{O}$ the hyperoval containing the oval O. Further, $P(W(q), x)$, with x any point of $\mathrm{PG}(3,q)$, has the following nice description: points are the points of $\mathrm{PG}(3,q)$ not in the plane $\pi_x = x^{\zeta}$, with ζ the symplectic polarity defining $W(q)$; lines are the lines of $W(q)$ not in π_x together with the lines of $\mathrm{PG}(3,q)$ through x but not in π_x; incidence is the natural one.

2.4.4 Generalized quadrangles as group coset geometries

The following construction method for GQ was introduced by Kantor [75]. Let G be a finite group of order $s^2t, 1 < s, 1 < t$, together with a family $J = \{A_i \mid 0 \le i \le t\}$ of $1+t$ subgroups of G, each of order s. Assume furthermore that for each $A_i \in J$, there exists a subgroup A_i^* of G, of order st, containing A_i. Put $J^* = \{A_i^* \mid 0 \le i \le t\}$ and define as follows an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I}) = \mathcal{S}(G, J)$.
Points are of three kinds:

(i) the elements of G;

(ii) the right cosets A_i^*g, $A_i^* \in J^*$, $g \in G$;

(iii) a symbol (∞).

Lines are of two kinds:

(a) the right cosets $A_i g$, $A_i \in J$, $g \in G$;

(b) the symbols $[A_i]$, $A_i \in J$.

A point g of type (i) is incident with each line $A_i g$, $A_i \in J$; a point $A_i^* g$ of type (ii) is incident with $[A_i]$ and with each line $A_i h$ contained in $A_i^* g$; the point (∞) is incident with each line $[A_i]$ of type (b).
Then Kantor [75] proved that the following holds: $\mathcal{S}(G, J)$ is a GQ of order (s, t) provided

K1. $A_i A_j \cap A_k = \{1\}$, for i, j, k distinct, and

K2. $A_i^* \cap A_j = \{1\}$, for $i \neq j$.

If the conditions K1 and K2 are satisfied, then one easily sees that

$$A_i^* = \cup\{A_i g \mid A_i g = A_i \text{ or } A_i g \cap A_j = \emptyset \text{ for all } A_j \in J\},$$

so that A_i^* is uniquely defined by A_i.
Suppose K1 and K2 are satisfied. For any $h \in G$ let us define θ_h by

$$g^{\theta_h} = gh, (A_i g)^{\theta_h} = A_i gh,$$

$$(A_i^* g)^{\theta_h} = A_i^* gh, [A_i]^{\theta_h} = [A_i], (\infty)^{\theta_h} = (\infty),$$

with $g \in G, A_i \in J$, $A_i^* \in J^*$. Then θ_h is an automorphism of $\mathcal{S}(G, J)$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h \mid h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (i).
If K1 and K2 are satisfied, then J is called a 4-*gonal family for* G.
Let $\mathbb{F} = \mathrm{GF}(q)$, q any prime power. Let $f : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$ be a fixed symmetric, nonsingular biadditive map. Put $G = \{(\alpha, c, \beta) \mid \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by:

$$(\alpha, c, \beta).(\alpha', c', \beta') = (\alpha + \alpha', c + c' + f(\beta, \alpha'), \beta + \beta').$$

This makes G into a group whose center is $C = \{(0, c, 0) \in G \mid c \in \mathbb{F}\}$. Suppose that for each $u \in \mathbb{F}$ there is an additive map $\delta_u : \mathbb{F}^2 \to \mathbb{F}^2$ and a map $g_u : \mathbb{F}^2 \to \mathbb{F}$ for which

$$g_u(\alpha + \beta) - g_u(\alpha) - g_u(\beta) = f(\alpha^{\delta_u}, \beta) = f(\beta^{\delta_u}, \alpha),$$

for all $\alpha, \beta \in \mathbb{F}^2, u \in \mathbb{F}$. With such a setup, we can define a family of subgroups of G by:

$$A(u) = \{(\alpha, g_u(\alpha), \alpha^{\delta_u}) \mid \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(0,0,\beta) \in G \mid \beta \in \mathbb{F}^2\}.$$

Then put $J = \{A(u) \mid u \in \mathbb{F} \cup \{\infty\}\}$ and $J^* = \{A^*(u) \mid u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha^{\delta_u}) \mid \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(0,c,\beta) \in G \mid \beta \in \mathbb{F}^2\}.$$

Necessary and sufficient conditions were worked out in [92] (or see Section 10.4 of [97]) for J to be a 4-gonal family.

Theorem 2.4.1
J is a 4-gonal family for G if and only if

(i) $\delta(u,r) : \alpha \to \alpha^{\delta_u} - \alpha^{\delta_r}$ *is bijective for* $u \neq r$,

(ii) $g_u(\alpha) = g_r(\alpha)$, $u \neq r$, *implies* $\alpha = 0$,

(iii) if u, r, and v are distinct, then $\gamma = 0$ is the only solution to

$$g_u(\gamma^{\delta^{-1}(u,v)}) - g_v(\gamma^{\delta^{-1}(u,v)}) + g_v(-\gamma^{\delta^{-1}(v,r)}) - g_r(-\gamma^{\delta^{-1}(v,r)}) = 0.$$

Let $\mathcal{C} = \{A_u \mid u \in \mathbb{F}\}$ be a set of q distinct 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0,0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$, and then define $g_u(\alpha) = \alpha A_u \alpha^T$ and $\alpha^{\delta_u} = \alpha K_u$ for $\alpha \in \mathbb{F}^2$. Then necessarily $f(\alpha,\beta) = \alpha\beta^T$ for all $\alpha, \beta \in \mathbb{F}^2$. With G, $A(u)$, $A^*(u)$, J as above, the following theorem is a combination of results of [92], [94] and [78].

Theorem 2.4.2
The set J is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan.

In particular, let $\mathcal{C} = \{A_u \mid u \in \mathbb{F}\}$ be a set of q upper triangular 2×2-matrices over $\mathbb{F}$, with

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix},$$

$x_u, y_u, z_u, u \in \mathbb{F}$. For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{2.3}$$

is a nonsquare of $\mathbb{F}$ whenever $r, u \in \mathbb{F}, r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$\operatorname{tr}((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{2.4}$$

whenever $r, u \in \mathbb{F}, r \neq u$. (Here $\operatorname{tr}(w)$ is the trace of w over the prime subfield, so that $\operatorname{tr}(w) = 1$ if and only if $X^2 + X + w = 0$ has no solution in $\mathbb{F}$.)

In [117] it was shown that (2.3) and (2.4) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathrm{PG}(3, q)$ to define a *flock* of the quadratic cone $X_0 X_1 = X_2^2$, that is, to intersect the cone in q mutually disjoint nonsingular conics. An example of a flock is given by taking all planes through an exterior line and intersecting the cone in a conic. Such a flock is called a *linear flock.*

Theorem 2.4.3

To any flock of the quadratic cone of $\mathrm{PG}(3, q)$ *there corresponds a* GQ *of order* (q^2, q). *The flock is linear if and only if the corresponding* GQ *is isomorphic to the classical* GQ $H(3, q^2)$

Remarks

1. Exploiting the relationship between flocks of quadratic cones and GQ many new classes of GQ were discovered.

2. A flock of a quadric Q in $\mathrm{PG}(3, q)$ (Q might be elliptic, hyperbolic or conical) yield a translation plane. The construction has been found independently by Walker [130] and Thas (unpublished). Exploiting this relationship between flocks and translation planes, many new classes of each of these objects were discovered (consequently also several new classes of GQ were discovered).

The Roman generalized quadrangles of Payne

The last examples of 4-gonal families were deduced by Payne [95], [96] from the semifield planes of Ganley [55], but more than just the ideas of Thas [117] is required. These are the only examples of GQ known to arise from 4-gonal families but not from q-clans via the procedure just described.

Let $q = 3^h$ and let n be a nonsquare of $\mathrm{GF}(q)$. With the notations of this section, put

$$g_u(\gamma) = \gamma \begin{pmatrix} u & 0 \\ 0 & -nu \end{pmatrix} \gamma^T + (\gamma \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \gamma^T)^{\frac{1}{3}} + (\gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}u \end{pmatrix} \gamma^T)^{\frac{1}{9}},$$

$$\alpha^{\delta_u} = u\alpha,$$

and

$$f(\alpha,\beta)=\alpha\begin{pmatrix}-1 & 0\\ 0 & -n\end{pmatrix}\beta^T+(\alpha\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\beta^T)^{\frac{1}{3}}+(\alpha\begin{pmatrix}0 & 0\\ 0 & n^{-1}\end{pmatrix}\beta^T)^{\frac{1}{9}}.$$

Then Payne [95], [96] shows that the corresponding set J is a 4-gonal family, thus giving a GQ of order (q^2,q). As these results were obtained while he was visiting the University of Rome, Payne calls these GQ the *Roman generalized quadrangles*.

The construction of Knarr [80]

Let $F=\{C_1,C_2,\cdots,C_q\}$ be a flock of the quadratic cone K with vertex x_0 of PG$(3,q)$, with q odd. The plane of C_i is denoted by π_i, $i=1,2,\cdots,q$. Let K be embedded in the nonsingular quadric $Q(4,q)$ of PG$(4,q)$. The polar line of π_i with respect to $Q(4,q)$ is denoted by L_i; let

$$L_i\cap Q(4,q)=\{x_0,x_i\},\, i=1,2,\cdots,q.$$

Then no point of $Q(4,q)$ is collinear with all three of $x_0,x_i,x_j, 1\leq i<j\leq q$. In [2] it is proved that it is also true that no point of $Q(4,q)$ is collinear with all three of $x_i,x_j,x_k, 0\leq i<j<k\leq q$. Such a set U of $q+1$ points of $Q(4,q)$ will be called a BLT-*set* in $Q(4,q)$, following a suggestion of Kantor [79]. Since the GQ $Q(4,q)$ is isomorphic to the dual of the GQ $W(q)$, to a BLT-set in $Q(4,q)$ corresponds a set V of $q+1$ lines of $W(q)$ with the property that no line of $W(q)$ is concurrent with three distinct lines of V; such a set V will also be called a BLT-*set*.
To F corresponds a GQ $\mathcal{S}$ of order (q^2,q). Knarr [80] proves that $\mathcal{S}$ is isomorphic to the following incidence structure.
Start with a symplectic polarity ζ of PG$(5,q)$. Let $p\in$ PG$(5,q)$ and let PG$(3,q)$ be a 3-dimensional subspace of PG$(5,q)$ for which $p\notin$ PG$(3,q)\subset p^\zeta$. In PG$(3,q)$ ζ induces a symplectic polarity ζ', and hence a GQ $W(q)$. Let V be a BLT-set of the GQ $W(q)$ and construct a geometry $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ as follows.

Points:

(i) p;

(ii) lines of PG$(5,q)$ not containing p but contained in one of the planes $\pi_t=pL_t$, with L_t a line of the BLT-set V;

(iii) points of PG$(5,q)$ not in p^ζ.

Lines:

(a) planes $\pi_t=pL_t$, with $L_t\in V$;

(b) totally isotropic planes of ζ not contained in p^ζ and meeting some π_t in a line (not through p).

The incidence relation I is just the natural incidence inherited from $\mathrm{PG}(5,q)$. Then Knarr [80] proves that $\mathcal{S}$ is a GQ of order (q^2,q) isomorphic to the GQ arising from the flock F.

2.4.5 Isomorphisms

For a proof of the following theorem we refer to [96] and [97].

Theorem 2.4.4

(a) The GQ $T_2(O)$ *is isomorphic to the classical* GQ $Q(4,q)$ *if and only if* O *is an irreducible conic; it is isomorphic to* $W(q)$ *if and only if* q *is even and* O *is a conic.*

(b) The GQ $T_2^*(O)$ *and* $AS(q)$ *are isomorphic to the respective* GQ $P(T_2(O'),(\infty))$, *with* $O' = O \setminus \{x\}$ *and* $x \in O$, *and* $P(W(q),y)$, *with* y *any point of* $W(q)$.

(c) The GQ $T_3(O)$ *is isomorphic to* $Q^-(5,q)$ *if and only if* O *is an elliptic quadric of* $\mathrm{PG}(3,q)$.

(d) Apart from a few small values of q, *no nonclassical* GQ *arising from a flock is isomorphic to a previously known* GQ.

Remark

If q is odd, then the oval O is a conic (see [68]), implying $T_2(O) \cong Q(4,q)$. In such a case $T_2(O)$ is not self-dual. If q is even and O is a conic, then $T_2(O)$, which is isomorphic to $Q(4,q)$, is self-dual. The problem of determining all ovals for which $T_2(O)$ is self-dual has been solved (cf. [52], and [122], [97]). For a survey of the known ovals of $\mathrm{PG}(2,q)$, q even, see [119].
If q is odd, then the ovoid O is an elliptic quadric (see [69]), implying $T_3(O) \cong Q^-(5,q)$. For q even, one other class of ovoids is known; these ovoids only exist for $q = 2^{2e+1}$, $e \geq 1$, and are called *Tits-Suzuki ovoids* (see [69]).

2.5 Generalized quadrangles with small parameters

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite GQ of order (s,t), $1 < s \leq t$.

2.5.1 $s = 2$

By Theorems 2.2.1 and 2.2.2 $s+t$ divides $st(s+1)(t+1)$, and $t \leq s^2$. Hence $t \in \{2,4\}$.

Theorem 2.5.1
Up to isomorphism there is only one GQ *of order* 2 *and only one* GQ *of order* $(2,4)$.

Corollary

The GQ $W(2)$ *and* $Q(4,2)$ *are self-dual and mutually isomorphic. The* GQ $AS(3)$ *is isomorphic to the* GQ $Q^{-}(5,2)$.

It is easy to show that the GQ of order 2 is unique. The uniqueness of the GQ of order $(2,4)$ was proved independently at least five times, by Dixmier and Zara [51], Seidel [101], Shult [104], Thas [110] and Freudenthal [54].

Two interesting models of the GQ *of order* 2

(1) Let O be a hyperoval, i.e. a 6-arc of PG$(2,4)$. Points of the GQ are the points of PG$(2,4)$ not on O, lines of the GQ are the lines of PG$(2,4)$ intersecting O, incidence is the natural one.

(2) The following construction was apparently first discovered by Sylvester [107]. A *duad* is an unordered pair $ij = ji$ of distinct integers from among $1, 2, \ldots, 6$. A *syntheme* is a set $\{ij, kl, mn\}$ of three duads for which i, j, k, l, m, n are distinct. It is routine to verify that Sylvester's syntheme-duad geometry with duads playing the role of points, synthemes playing the role of lines, and containment as the incidence relation, is the unique GQ of order 2.

Two interesting models of the GQ *of order* $(2,4)$

(1) In [97] the following construction of the GQ of order $(2,4)$ is given. In addition to the duads and synthemes given above, let $1, 2, \ldots, 6$ and $1', 2', \ldots, 6'$ denote twelve additional points, and let $\{i, ij, j'\}$, $1 \leq i, j \leq 6$, $i \neq j$, denote thirty additional lines. It is easy to verify that the 27 points and 45 lines just constructed yield a representation of the unique GQ of order $(2,4)$.

(2) Points of the GQ are the 27 lines on a general cubic surface V in PG$(3,\mathbb{C})$, lines of the GQ are the 45 tritangent planes of V, and incidence is inclusion (for the properties of the general cubic surface in PG$(3,\mathbb{C})$ we refer, e.g., to Baker [1921-1934]); see also [69] where a full chapter is devoted to cubic surfaces).

2.5.2 $s = 3$

Again by Theorems 2.2.1 and 2.2.2 we have $t \in \{3, 5, 6, 9\}$.

Theorem 2.5.2
Any GQ *of order* $(3,5)$ *must be isomorphic to the* GQ $T_2^*(O)$ *arising from a hyperoval in* PG$(2,4)$*, any* GQ *of order* $(3,9)$ *must be isomorphic to* $Q^-(5,3)$*, and a* GQ *of order* 3 *is isomorphic to either* $W(3)$ *or to its dual* $Q(4,3)$*. Finally, there is no* GQ *of order* $(3,6)$*.*

The uniqueness of the GQ of order $(3,5)$ was proved by Dixmier and Zara [51], the uniqueness of the GQ of order $(3,9)$ was proved independently by Dixmier and Zara [51] and Cameron (see [122]), the determination of all GQ of order 3 is due independently to Dixmier and Zara [51] and Payne [90]. Dixmier and Zara [51] proved that there is no GQ of order $(3,6)$. All these proofs, some of them simplified or streamlined, are also contained in [97].

2.5.3 $s = 4$

Using Section 2.2 it is easy to check that $t \in \{4,6,8,11,12,16\}$. Nothing is known about $t = 11$ or $t = 12$. In the other cases unique examples are known, but the uniqueness question is settled only in the case $t = 4$.

Theorem 2.5.3
A GQ *of order* 4 *must be isomorphic to* $W(4)$*.*

The proof of this fact that appears in [97] is that of [91], with a gap filled in by Tits.

2.6 Generalized quadrangles in finite projective and affine spaces

2.6.1 Generalized quadrangles in finite projective spaces

The following fundamental and beautiful theorem is due to Buekenhout and Lefèvre.

Theorem 2.6.1 ([20])
A projective GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *with ambient space* PG(d,q) *must be obtained in one of the following ways:*

(i) there is a nonsingular quadric $Q(d,q)$ *of Witt index* 2 *in* PG(d,q)*,* $d = 3, 4$ *or* 5*, such that* $\mathcal{P}$ *is the set of points of* $Q(d,q)$ *and* $\mathcal{L}$ *is the set of lines on* $Q(d,q)$*;*

(ii) there is a nonsingular Hermitian variety $H(d,q^2)$ *in* PG(d,q^2)*,* $d = 3$ *or* 4*, such that* $\mathcal{P}$ *is the set of points of* $H(d,q^2)$ *and* $\mathcal{L}$ *is the set of lines on* $H(d,q^2)$*;*

(iii) $d = 3$*,* $\mathcal{P}$ *is the set of all points of* PG$(3,q)$ *and* $\mathcal{L}$ *is the set of all totally isotropic lines with respect to some symplectic polarity of* PG$(3,q)$*.*

Hence $\mathcal{S}$ must be one of the classical examples described in Section 2.3.

2.6.2 Generalized quadrangles in finite affine spaces

All GQ embedded in $\mathrm{AG}(d, q)$ were determined by Thas [112]; the theorem on the embedding in $\mathrm{AG}(3, q)$ was proved independently by Bichara [7].
We note that in contrast with the projective case, there arise five nontrivial "sporadic" cases in the finite affine case.

Remark

Much more can be said on generalized quadrangles. For instance there are a lot of characterization theorems known, but this should bring us too far and we refer to [97] for more on this subject.

Chapter 3

Partial geometries

3.1 Pseudo-geometric (s,t,α)-graphs

We quickly recall the definition. A (finite) *partial geometry* $\mathrm{pg}(s,t,\alpha)$ is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t) such that for each anti-flag (x, L) the incidence number $\alpha(x, L)$ is a constant $\alpha(\neq 0)$. The numbers s, t and α are called the *parameters* of $\mathcal{S}$. This incidence structure was introduced by Bose [8]. As already remarked in Chapter 1 the point graph $\Gamma(\mathcal{S})$ of a partial geometry $\mathcal{S}$ is a

$$\mathrm{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

(If $\alpha = s+1$, the graph is a complete graph).
Each strongly regular graph Γ having parameters of this form with $t \geq 1$, $s \geq 1$, $1 \leq \alpha \leq s+1$ and $1 \leq \alpha \leq t+1$ is called a *pseudo–geometric* (s,t,α)–graph. If the graph Γ is indeed the point graph of at least one partial geometry, then Γ is called *geometric*.
If we translate the necessary conditions for strongly regular graphs in Theorem 1.2.1 in terms of the parameters of a pseudo–geometric (s,t,α)–graph, then this yields the following theorem (generalizing the results of Theorems 2.2.1 and 2.2.2).

Theorem 3.1.1
If Γ is a pseudo–geometric (s,t,α)–graph, then

$$r = s-\alpha,\ l = -t-1,\ f = \frac{st(s+1)(t+1)}{\alpha(s+t+1-\alpha)}.$$

Moreover:

1. *v is an integer, hence $\alpha | (s+1)st$.*

2. *The multiplicities of the eigenvalues of the adjacency matrix are integers, hence*

$$\alpha(s+t+1-\alpha)|st(s+1)(t+1).$$

3. *The Krein inequalities for strongly regular graphs are satisfied, hence*

$$(s+1-2\alpha)t \leq (s-1)(s+1-\alpha)^2.$$

Given a pseudo–geometric (s,t,α)–graph Γ, the problem is to find a subset $\mathcal{L}$ of cliques of size $s+1$, such that any two adjacent vertices of Γ are in exactly one element of $\mathcal{L}$.

Lemma 3.1.1
Let Γ be a pseudo-geometric (s,t,α)-graph with $s>t$. If it is possible to find in Γ a set of cliques $\mathcal{L}=\{K_1,\dots,K_b\}$ such that any two adjacent vertices are contained in exactly one element of of $\mathcal{L}$ and such that every vertex is contained in $t+1$ elements of $\mathcal{L}$, then Γ is geometric.

Proof. We shall prove that each of the cliques $K_1,K_2,\dots,K_b$ contains exactly $s+1$ vertices and that if q is any vertex not contained in any clique K_i $(1\leq i\leq b)$, then there are exactly α vertices in K_i which are joined to q. This will show that if the vertices of Γ are taken as points, and the cliques $K_1,\dots,K_b$ as lines, then we have a partial geometry $\mathrm{pg}(s,t,\alpha)$.
Note that we work with the usual parameters of a strongly regular graph.
Let p be any vertex. Without loss of generality we can take $K_1,\dots,K_{t+1}$ to contain p. Any two cliques of $\mathcal{L}$ intersect in at most one point, hence the sets $K_1\setminus\{p\},K_2\setminus\{p\},\dots,K_{t+1}\setminus\{p\}$ are disjoint and they must contain between them all the $k=(t+1)s$ vertices which are adjacent to p. So the average number of vertices in those $(t+1)$ cliques of the set $K_1,\dots,K_{t+1}$ is $s+1$. Hence there exists a clique containing at least $s+1$ vertices.
Let K_j be such a clique and let us take a subset K of K_j such that K contains exactly $s+1$ vertices. Let $\bar{K}$ be the subset of vertices not contained in K (hence this set has size $v-(s+1)$). Let $g(x)$ be the number of vertices in $\bar{K}$ which have exactly x vertices in K adjacent to it, then

$$\sum_{x=0}^{s+1} g(x)=|\bar{K}|=(s+1)\frac{st+\alpha}{\alpha}-(s+1)=(s+1)\frac{st}{\alpha}.$$

Let us count the number of pairs (p,p') where $p\in K$, $p'\in\bar{K}$ and p is adjacent to p', in two ways. Each vertex in K has s vertices adjacent to it in K. Hence each vertex in K has got $k-s$ vertices adjacent to it and contained in $\bar{K}$. Thus the number of such pairs is

$$|K|(k-s)=(s+1)(k-s)=(s+1)(s(t+1)-s)=(s+1)st.$$

On the other hand, there are $g(x)$ vertices in $\bar{K}$ which have exactly x vertices in K adjacent to it. These vertices contribute $xg(x)$ to our counting, hence

$$\sum_{x=0}^{s+1} xg(x) = (s+1)st.$$

Hence the average value of x is

$$\bar{x} = \frac{\sum xg(x)}{\sum g(x)} = \frac{(s+1)st}{(s+1)\frac{st}{\alpha}} = \alpha.$$

Now let us count in two ways the number of triplets (p, p', p'') where (p, p') is an ordered pair of vertices in K and p'' is a vertex in $\bar{K}$ which is adjacent to both p and p', in two ways. Since p and p' have $s-1$ mutually adjacent vertices in K, they have $\lambda - (s-1)$ common adjacent vertices in $\bar{K}$. Hence the number of such triples is

$$(s+1)s(\lambda - s + 1) = (s+1)s(s-1+t(\alpha-1) - s + 1) = (s+1)st(\alpha - 1).$$

On the other hand, each of the $g(x)$ vertices in $\bar{K}$ which have x adjacent vertices in K, contributes $x(x-1)g(x)$ to our counting. Hence

$$\sum_{x=0}^{s+1} x(x-1)g(x) = (s+1)st(\alpha-1).$$

Thus

$$\sum_{x=0}^{s+1} g(x)x^2 = (s+1)st(\alpha-1) + (s+1)st = (s+1)st\alpha.$$

Now we have

$$\begin{aligned}
\sum_{x=0}^{s+1} g(x)(x-\alpha)^2 &= \sum x^2 g(x) - 2\alpha \sum xg(x) + \alpha^2 \sum g(x) \\
&= (s+1)st\alpha - 2\alpha(s+1)st + \alpha^2(s+1)\frac{st}{\alpha} \\
&= 0,
\end{aligned}$$

which is only possible if x is constant and equal to α. Hence every vertex of $\bar{K}$ has exactly α adjacent vertices in K.
If K_j contains any vertex p' other than those already contained in K, then p' belongs to $\bar{K}$, and therefore has exactly α adjacent vertices in K. But each point of K is adjacent to p' (because $p' \in K_j$) so $\alpha = s+1$ which contradicts $1 \leq \alpha \leq t+1 < s+1$ (note that $t < s$). This shows that none of the cliques containing p contains more than $s+1$ vertices, but we saw

earlier in this proof that the average number of vertices in those cliques is $s+1$. Thus each of them contains exactly $s+1$ vertices. And since each clique contains at least one vertex, each of the cliques $K_1, \ldots, K_b$ contains exactly $s+1$ vertices. Also, if p' is a point not contained in K_i $(1 \leq i \leq b)$ then there are exactly α vertices in K_i adjacent to p'. This completes the proof of the lemma. ∎

Definition

A *claw* $[p, S]$ of a pseudo-geometric graph Γ is an ordered pair consisting of a vertex p, the *vertex of the claw* and a non-empty set S of vertices distinct from p such that every vertex in S is joined to p in Γ but no two vertices in S are joined in Γ. The *order* of the claw $[p, S]$ is defined as $\eta = |S|$.

Theorem 3.1.2
Let Γ be a pseudo-geometric (s,t,α) graph.
(i) If

1. $s > t(2t+1)(\alpha-1)$*;*
2. *two distinct vertices of Γ are contained in at most one grand clique;*
3. *there exists no claw of order $t+2$ in Γ;*

then Γ is geometric.
(ii) If

1. $s+1 > (t+1)\alpha + t(\alpha-1)(2t+1)$*;*
2. *there exists no claw of order $t+2$ in Γ;*

then Γ is geometric.
(iii) If

$$s+1 > \frac{1}{2}\big((t+1)t + \alpha(t+2)(t^2+1)\big),$$

then Γ is geometric.

Remark

The third part in theorem 3.1.2 is known as Bose's theorem. This condition is very strong, so in general we need other methods to determine if a pseudo-geometric graph is geometric or not. Attempts to construct a partial geometry from a pseudo–geometric (s, t, α)–graph Γ were in most cases unsuccessful; we refer for instance to [106], to [43] and to [40]; an exception however is the sporadic partial geometry of Haemers [60] (see 3.6).
An important improvement of Bose's theorem was proved by Neumaier. We will not give the proof.

Theorem 3.1.3 (Neumaier [87])
Let Γ be a strongly regular graph with the parameters v, k, λ, μ whose adjacency matrix has eigenvalues k, r, l. Suppose that l is an integer less than -1, and that

$$r \leq \frac{1}{2} l(l+1)(\mu+1) - 1.$$

Then Γ is the point graph of a partial geometry whose parameters satisfy $\alpha = t+1$ or $\alpha = t$.

Theorem 3.1.4 ([26])
If Γ is a pseudo-geometric (s, t, α)–graph satisfying the Bose-inequality, then $t \leq 2\alpha - 1$.

Proof. If a pseudo-geometric graph satisfies the Bose inequality, then it is geometric, hence it has a dual which is a $\mathrm{pg}(s, t, \alpha)$. By theorem 3.1.1 we get for this $\mathrm{pg}(s, t, \alpha)$:

$$(t+1-2\alpha)s \leq (t-1)(t+1-\alpha)^2 \tag{3.1}$$

Suppose $t+1-2\alpha \geq 0$ then by Bose's inequality $2(s+1) > t(t+1) + \alpha(t+2)(t^2+1)$ we get

$$(t+1-2\alpha)(t(t+1) + \alpha(t+2)(t^2+1) < 2(s+1)(t+1-2\alpha),$$

hence

$$(t+1-2\alpha)(t(t+1) + \alpha(t+2)(t^2+1) - 2) < 2s(t+1-2\alpha).$$

Which is by (3.1) less or equal to

$$2(t-1)(t+1-\alpha)^2. \tag{3.2}$$

Suppose $t+1-2\alpha \geq 3$ then

$$\begin{aligned} (t+1-2\alpha)(t(t+1) + \alpha(t+2)(t^2+1) - 2) &> 3(t+2)(t^2+1) \\ &= 3(t^3+t+2t^2+2), \end{aligned}$$

and

$$\begin{aligned} 2(t-1)(t+1-\alpha)^2 &< 2(t-1)(t+1)^2 \\ &= 2(t^3+t+2t^2-t^2-2t-1) \\ &< 2(t^3+t+2t^2+2). \end{aligned}$$

Hence $(t+1-2\alpha)(t(t+1)+\alpha(t+2)(t^2+1)-2) > 2(t-1)(t+1-\alpha)$ which contradicts (3.2). Suppose $t+1-2\alpha = 1$ then by (3.2) we get

$$t(t+1) + \alpha(t+2)(t^2+1) - 2 < 2(t-1)(t+1-2\alpha+\alpha)^2,$$

and $t = 2\alpha$ hence,

$$2\alpha(2\alpha+1) + \alpha(2\alpha+2)(4\alpha^2+1) - 2 < 2(2\alpha-1)(1+2\alpha+\alpha^2),$$

hence

$$2\alpha^2 + \alpha + 4\alpha^4 + 2\alpha^2 + 2\alpha^3 + 3\alpha < 2\alpha + 4\alpha^2$$

which is obviously a contradiction. Suppose $t+1-2\alpha = 2$ then by (3.1) we get

$$2[t(t+1) + \alpha(t+2)(t^2+1) - 2] < 2(t-1)(t+1-2\alpha+\alpha)^2$$

and $t = 2\alpha - 1$ hence

$$(2\alpha+1)(2\alpha+2) + \alpha(2\alpha+3)(4\alpha^2+4\alpha+2) - 2 < 2\alpha(4+4\alpha+\alpha^2).$$

Hence $2\alpha+14\alpha^2+19\alpha^3+8\alpha^4+3\alpha+1 < 0$ which obviously is a contradiction. This completes the proof. ■

3.2 Adjusting the definition of pseudo-geometric graphs

Note that we already proved the following theorem implicitly in lemma 3.1.1.

Theorem 3.2.1
Let Γ be a pseudo-geometric (s,t,α)–graph. If C is an $(s+1)$–clique in Γ, then any point outside C has exactly α neighbours in C.

Proof. Let C be a clique of size $s+1$ in Γ and let $p_1, \ldots, p_{v-s-1}$ be the vertices of Γ outside the clique C. Let p_i have x_i neighbours in C. Note that

$$v - s - 1 = \frac{(s+1)st}{\alpha}.$$

Count in two ways ordered pairs (p,q) where $p \in \Gamma \setminus C$ and $q \in C$, then

$$\sum_{i=1}^{v-s-1} x_i = (s+1)(k-(s+1)+1) = (s+1)st$$

since for a vertex q in C, there are $k-(s+1)+1$ neighbours in $\Gamma \setminus C$. Now count in two ways ordered triples (p, q_1, q_2) where $p \in \Gamma \setminus C$ and $q_1, q_2 \in C$, then

$$\sum_{i=1}^{v-s-1} x_i(x_i-1) = (s+1)s(\lambda-(s+1-2)) = (s+1)st(\alpha-1)$$

hence

$$\sum_{i=1}^{v-s-1} x_i^2 = (s+1)st\alpha.$$

Using these results we obtain

$$\sum_{i=1}^{v-s-1} (x_i - \alpha)^2 = (s+1)st\alpha - 2\alpha(s+1)st + \alpha^2 \frac{(s+1)st}{\alpha} = 0$$

which is only possible if $x_i = \alpha$ for $i = 1, \ldots, v - s - 1$. This proves the theorem. ■

Corollary 3.2.1
If for a pseudo-geometric (s, t, α)–graph, $\alpha < s + 1$, then there is no clique bigger than $s + 1$.

Proof. If C' were such a clique, then C' would contain a subclique C of size $s + 1$. Hence any point of $C' \setminus C$ would be joined to all $s + 1$ points of C which contradicts theorem 3.2.1 and our assumption $\alpha < s + 1$. ■

Theorem 3.2.2
A pseudo-geometric (s, t, α) graph Γ is geometric if every edge is contained in a unique clique of size $s + 1$.

Proof. Take for the lines all of those cliques of size $s + 1$.
If two lines would meet in more than one point, then there is an edge which is contained in two cliques of size $s + 1$ which contradicts our assumption. Because of theorem 3.2.1 also the α-condition is satisfied. So we only need to check if the number of lines through a given point is a constant. Take a vertex x of Γ, then we know that $|\Gamma(x)| = k$. Because no two lines meet in more than one point, the set $|\Gamma(x)|$ can be partitioned into sets of size s which are those cliques of size $s + 1$ through x, minus our vertex x. Hence $t + 1 = \frac{k}{s}$. ■

Remark

Cameron, Goethals and Seidel [26] proved that any pseudo–geometric $(s, s^2, 1)$–graph is geometric (see also [59]). It is not known whether this theorem also holds for pseudo–geometric (s, t, α)–graphs satisfying the Krein equality in the case $\alpha > 1$. There exists a pseudo–geometric $(4, 27, 2)$–graph, the McLaughlin graph. This graph does satisfy the Krein equality; however although several attempts have been made, for instance by van Lint [127], it is not known whether this graph is geometric or not. For the moment there is no $\mathrm{pg}(4, 27, 2)$ known.

3.3 Partial geometries and maximal arcs

3.3.1 Maximal arcs in projective planes

In a finite projective plane π of order q (not necessarily Desarguesian) each set $\mathcal{K}(\neq \emptyset)$ of k points can be seen as a $\{k; d\}$-arc, where $d(\neq 0)$ is the maximum number of points of $\mathcal{K}$ on a line of the plane. Given q and d, then $k \leq (q+1)(d-1)+1$. If equality holds the set $\mathcal{K}$ is called a *maximal arc of degree* d. Equivalently such an arc $\mathcal{K}$ can be defined as a non-empty set of points in the plane, such that any line intersects $\mathcal{K}$ in 0 or d points. If $d = 1$, then of course $\mathcal{K}$ is a point, if $d = q$ then $\mathcal{K}$ is an affine plane of order q obtained by deleting a line L of π. If $\mathcal{K}$ is a maximal arc of degree d, $d \leq q$, then one easily shows that $\overline{\mathcal{K}} = \{$lines L of π such that $L \cap \mathcal{K} = \emptyset\}$ is a $\{q(q-d+1)/d; q/d\}$-arc, i.e. a maximal arc of degree q/d, in the dual projective plane $\overline{\pi}$. Hence, if π is Desarguesian then a maximal arc of degree d, $d \leq q$, yields one of degree q/d.
If $\mathcal{K}$ is a maximal arc of degree d, $d \leq q$, in a plane π of order q then $d|q$. However this is not a sufficient condition. In [5] it has been proved that a Desarguesian plane of odd order q cannot contain a maximal arc of degree $1 < d < q$ (see also [4] for a short proof). This result generalizes an old result of Cossu [32] who proved that PG$(2,9)$ has no $\{21; 3\}$-arc, and of Thas [111] who proved the nonexistence of maximal arcs of degree 3 in Desarguesian projective planes of order $q = 3^h (h > 1)$. By computer Denniston has proved that none of the 3 non-Desarguesian planes of order 9 has a maximal arc of degree 3. Desarguesian planes of order $q = 2^h$ do have maximal arcs of degree d for any divisor d of q.

Theorem 3.3.1 ([50])
Assume $x^2 + bx + 1$ is an irreducible quadratic form over GF(q), $q = 2^h$, *and let F_λ, $\lambda \in$ GF$(q) \cup \{\infty\}$, be a conic in* PG$(2,q)$ *with equation $x_0^2 + bx_0x_1 + x_1^2 + \lambda x_2^2 = 0$. Let H be a subgroup of order $d = 2^m$ of the additive group* GF(q), $+$. *Then the set $\mathcal{K}$ of points $\cup F_\lambda, \lambda \in H$, is a maximal arc of degree d in* PG$(2,q)$.

Proof. F_0 is the point $(0,0,1)$ while F_∞ is the degenerate conic $x_2 = 0$. If $\lambda \in$ GF$(q) \setminus \{0\} =$ GF$^*(q)$, then F_λ is a conic with nucleus $(0,0,1)$. The conics F_λ, $\lambda \in$ GF$(q) \cup \{\infty\}$, partition the points of PG$(2,q)$. $\mathcal{K}$ is the union of $d-1$ pairwise disjoint nondegenerate conics, and their common nucleus $(0,0,1)$. If L is a line through the nucleus, then L has exactly one point in common with each F_λ, hence $|L \cap \mathcal{K}| = d$. If L is a line not passing through the nucleus, then the q points of $L \setminus (x_2 = 0)$ are on $q/2$ conics of $\cup F_\lambda$, $\lambda \in$ GF(q). If $a_0x_0 + a_1x_1 + x_2 = 0$ is the equation of L, then L has no element in common with the conic F_λ, $\lambda \in$ GF(q), if and only if $(1+\lambda a_0^2)x^2 + bx + (1 + \lambda a_1^2) = 0$ has no solution in GF(q). This implies that $G = \{\lambda \in$ GF$(q) \| F_\lambda \cap L = \emptyset\}$ is a subgroup of index 2 in GF$(q), +$. Hence the elements $\lambda \in \mathrm{GF}(q)$ such that $|F_\lambda \cap L| = 2$ are in the coset $G' \neq G$ of G in $\mathrm{GF}(q), +$. If H is a subgroup of G, then $G' \cap H = \emptyset$ and hence $|L \cap \mathcal{K}| = 0$. If H is not a subgroup of G, then $\mathrm{GF}(q), + = HG$ and $H \cap G$ has index 2 in H; hence $|G \cap H| = |G' \cap H| = d/2$, and $|L \cap \mathcal{K}| = d$. This implies that $\mathcal{K}$ is a maximal arc of degree d. ■

Remark

Note that the maximal arcs of degree 2 in a plane of order 2^h are the hyperovals. Thas [109], [113] has constructed maximal arcs in $\mathrm{PG}(2, 2^h)$ which are not isomorphic to the Denniston-arcs; moreover he constructed maximal arcs in some translation planes.

3.3.2 The partial geometry $\mathcal{S}(\mathcal{K})$

Let $\mathcal{K}$ be a maximal arc of degree d in a projective plane π of order q. We define the incidence structure $\mathcal{S}(\mathcal{K}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows. The points of $\mathcal{S}(\mathcal{K})$ are the points of π that are not contained in $\mathcal{K}$. The lines of $\mathcal{S}(\mathcal{K})$ are the lines of π that are incident with d points of $\mathcal{K}$. The incidence is the one of π. Then $\mathcal{S}(\mathcal{K})$ is a partial geometry $\mathrm{pg}(q-d, q-q/d, q-q/d-d+1)$. This infinite family was constructed by Thas [108], [109] and independently by Wallis [131].

Remarks

As there exist $\{2^{h+m} - 2^h + 2^m; 2^m\}$–arcs, $0 < m < h$, in projective planes of order 2^h there exist partial geometries $\mathcal{S}(\mathcal{K})$ with parameters $s = 2^h - 2^m$, $t = 2^h - 2^{h-m}$, $\alpha = (2^m - 1)(2^{h-m} - 1)$. This is a generalized quadrangle if and only if $h = 2$, and then it is the unique quadrangle of order 2.
Assume $m = h - 1, h \geq 2$. Then the point graph of $\mathcal{S}(\mathcal{K})$ is $T(2^h + 2)^C$, the complement of the triangular graph $T(2^h + 2)$. Hence $T(2^h + 2)^C$ is a geometric $(2^{h-1}, 2^h - 2, 2^{h-1} - 1)$–graph. Although these graphs are uniquely defined by their parameters, this does not imply that the geometry is unique. For instance, Mathon [83] proved by computer that there exist exactly two partial geometries $\mathrm{pg}(4, 6, 3)$ (point graph is $T(10)^C$). One is of type $\mathcal{S}(\mathcal{K})$, with $\mathcal{K}$ the unique hyperoval in $\mathrm{PG}(2, 8)$, the other one is not of that type. All the complements of the triangular graphs $T(2n)$ are pseudo–geometric $(n - 1, 2(n - 2), n - 2)$–graphs. However it is possible to prove that $T(8)^C$ and the complements of the Chang graphs (having the same parameters as $T(8)^C$) are not geometric ([36]). On the other hand in [81] it has been proved that $T(12)^C$ is not geometric, while, again by computer, R. Mathon (private communication) proved that the graph $T(18)^C$ yields at least 70 non-isomorphic $\mathrm{pg}(8, 14, 7)$.

3.3.3 The partial geometry $T_2^*(\mathcal{K})$

Let $\mathcal{K}$ be a maximal arc of degree $d = 2^m$ in the projective plane $\mathrm{PG}(2, 2^h)$. As $\mathcal{K}$ has only exterior lines and d-secants, it will yield a linear representation $T_2^*(\mathcal{K})$ of a partial geometry $\mathrm{pg}(2^h - 1, (2^h + 1)(2^m - 1), 2^m - 1)$ in $\mathrm{AG}(3, q)$. This infinite family was constructed for the first time in [108], [109], and generalizes the construction of Ahrens and Szekeres [1] and Hall [64] of the generalized quadrangle $T_2^*(\mathcal{O})$, $\mathcal{O}$ a hyperoval (see page 15).

3.4 The partial geometries $\mathbf{PQ}^+(4n-1, q), q = 2$ or $q = 3$

3.4.1 Some properties of hyperbolic quadrics in $\mathrm{PG}(2m-1, q)$

Let $\mathrm{Q}^+ = \mathrm{Q}^+(2m-1, q)$, $m \geq 2$, be the hyperbolic quadric in $\mathrm{PG}(2m-1, q)$ (the quadric with projective index $m - 1$). The set of maximal totally isotropic or singular subspaces on a hyperbolic quadric Q^+ is divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$. Two maximal totally isotropic or singular subspaces on the quadric are in the same family if and only if the codimension of their intersection has the parity of $m - 1$ (see [71] for more details on quadrics).

Assume q is odd, and let x and y be two points of $\mathrm{PG}(2m - 1, q) \setminus \mathrm{Q}^+$. Then x and y are called equivalent if and only if there exists a point $z \in \mathrm{PG}(2m - 1, q) \setminus \mathrm{Q}^+$ such that the lines xz and yz are tangent lines of Q^+. This relation can also be defined as follows. Embed Q^+ in the non-singular hyperquadric Q of $\mathrm{PG}(2m, q)$. The pole of $\mathrm{PG}(2m - 1, q)$ with respect to Q is denoted by p. Then x and y are equivalent if and only if the lines xp and yp are both secants or are both exterior lines of Q. The proof that this relation is indeed an equivalence relation was given by Thas [115]. There are two equivalence classes E_1 and E_2. For some i, $\mathrm{Q}^+ \cup E_i$ is the projection of the non-singular hyperquadric Q of $\mathrm{PG}(2m, q)$, from the point p onto $\mathrm{PG}(2m - 1, q)$.

3.4.2 The partial geometry $\mathbf{PQ}^+(4n - 1, 2)$

In [39] an infinite class of partial geometries was constructed as follows. Define a spread Σ of the non-singular hyperbolic quadric $\mathrm{Q}^+ = \mathrm{Q}^+(4n-1, 2)$, $n \geq 2$, in $\mathrm{PG}(4n - 1, 2)$ to be a (maximal) set of $2^{2n-1} + 1$ disjoint $(2n - 1)$–dimensional spaces on Q^+. Let Σ be a spread of $\mathrm{Q}^+ = \mathrm{Q}^+(4n - 1, 2)$ and let $\mathcal{L}$ be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n - 1, 2) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ the set of points of $\mathrm{PG}(4n - 1, 2)$ not on the quadric, $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n - 1, 2)$ is a $\mathrm{pg}(2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$.

If $n = 2$, then $\mathrm{PQ}^+(7,2)$ is a $\mathrm{pg}(7,8,4)$. Cohen [31] was the first to construct a partial geometry with these parameters using the root system E_8. In [61] a $\mathrm{pg}(8,7,4)$ was constructed using coding theory. Kantor [77] proved that $\mathrm{PQ}^+(7,2)$ and the dual of the geometry of Haemers–van Lint are isomorphic. Later on Tonchev [125] showed by computer that the model of Cohen and the dual of the geometry of Haemers–van Lint are isomorphic. In [40] this isomorphism is proved without the use of a computer.
Remark that non-isomorphic spreads of the quadric $\mathrm{Q}^+(4n-1,2)$ will produce non-isomorphic partial geometries. If $2n-1$ is composite then $\mathrm{Q}^+(4n-1,2)$ has non-isomorphic spreads, and probably this is true for all $n > 2$ (see [76]).

3.4.3 The partial geometry $\mathbf{PQ}^+(4n-1,3)$

For $q = 3$ an analogous construction is given by Thas [115]. Again let Σ be a spread of $\mathrm{Q}^+ = \mathrm{Q}^+(4n-1,3)$ and let $\mathcal{L}$ be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1,3) = (\mathcal{P},\mathcal{L},\mathrm{I})$ with $\mathcal{P}$ one of the sets E_i, and with $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space L^* of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1,3)$ is a partial geometry with parameters $s = 3^{2n-1}-1$, $t = 3^{2n-1}$, $\alpha = 2 \cdot 3^{2n-2}$.
Up to now it is only known that $\mathrm{Q}^+(7,3)$ has a spread. This yields a $\mathrm{pg}(26,27,18)$ $\mathrm{PQ}^+(7,3)$ $(v = 1080, b = 1120)$.

3.4.4 Partial geometries derived from $\mathbf{PQ}^+(4n-1,q)$, $q = 2$ or $q = 3$

Replaceable spreads of a partial geometry

Let Φ be a spread of a $\mathrm{pg}(s,t,\alpha)$ $\mathcal{S}$, i.e. a set of $st/\alpha+1$ lines partitioning the point set, we will refer to Φ as a pg-spread. Assume $t > 1$ and let L be any line of $\mathcal{S} \setminus \Phi$. Let Φ_L be the set of $s+1$ lines of Φ intersecting L. We call L *regular with respect to* Φ if there exists a set of $s+1$ lines $\mathcal{L}^* = \{L_0 = L, L_1, \ldots, L_s\}$ that partitions the point set $\mathcal{P}(\Phi_L)$ of Φ_L.

Lemma 3.4.1
If L is a regular line with respect to the pg*-spread Φ of a* $\mathrm{pg}(s,t,\alpha)$ *$\mathcal{S}$, then $\alpha = t$ (hence $\mathcal{S}$ is a net and $\mathcal{P} = \mathcal{P}(\Phi_L)$) or $t \geq s+1$. Moreover if $t = s+1$ then every line M not being an element of the spread Φ neither of $\mathcal{L}^*$ intersects $\mathcal{P}(\Phi_L)$ in α points. Conversely, if every line M not being an element of the spread Φ neither of $\mathcal{L}^*$ intersects $\mathcal{P}(\Phi_L)$ in α points then $t = s+1$.*

Proof. Assume that a line M_i, not being an element of the pg-spread Φ neither of $\mathcal{L}^*$, intersects $\mathcal{P}(\Phi_L)$ in a_i points. Note that there are $d =$

$\frac{t(st+\alpha)}{\alpha} - (s+1)$ such lines M_i. By counting the ordered pairs (p, M_i), $p \in \mathcal{P}(\Phi_L)$, $p \mathrm{~I~} M_i$, $i = 1, \ldots, d$, in two different ways, we get

$$\sum_{i=1}^{d} a_i = (s+1)^2(t-1).$$

Counting the ordered triples (p, p', M_i), p, $p' \in \mathcal{P}(\Phi_L)$, $p \mathrm{~I~} M_i \mathrm{~I~} p'$, $i = 1, \ldots, d$, we get

$$\sum_{i=1}^{d} a_i(a_i - 1) = (s+1)^2 s(\alpha - 1).$$

Using the variance trick, we find

$$s(\alpha - t)^2(t - s - 1) \geq 0$$

Hence either $\alpha = t$ and then $\mathcal{P} = \mathcal{P}(\Phi_L)$, hence $a_i = s+1$, $i = 1, \ldots, d$, or $t \geq s+1$. If $t = s+1$ then $a_i = \frac{\sum a_i}{d} = \alpha$.
Conversely, assume that every line M_i, $i = 1, \ldots, d$ not being an element of the pg-spread Φ neither of $\mathcal{L}^*$ intersects $\mathcal{P}(\Phi_L)$ in α points. We count in two different ways the number of ordered pairs (p, M_i) with p a point of $\mathcal{P}(\Phi_L)$ and $p \mathrm{~I~} M_i$. This yields

$$(s+1)^2(t-1) = \left(\frac{t(st+\alpha)}{\alpha} - (s+1)\right)\alpha.$$

This equation simplifies to $(s-t+1)(s(t-1)+(\alpha-1)) = 0$. Hence $t = s+1$. ∎

Definition

Assume that Φ is a pg-spread of a pg$(s, s+1, \alpha)$ such that every line is regular with respect to Φ. Then $\mathcal{L} \setminus \Phi$ is partitioned in $\frac{s(s+1)}{\alpha} + 1$ sets $\mathcal{L}_i$ $(i = 1, \ldots, \frac{s(s+1)}{\alpha} + 1)$ each containing $s+1$ mutually skew lines. The spread Φ will be called a *replaceable* pg-*spread* for reasons that will become clear very soon. This definition generalizes the definition given in [85]; they restrict their definition to spreads of pg$(2\alpha - 1, 2\alpha, \alpha)$ and they call them *regular spreads*. We prefer to use another terminology as *regular spreads* were earlier introduced in another context.

Remark

If Φ is a replaceable pg-spread of a pg$(s, s+1, \alpha)$, then the incidence structure $\mathcal{D}(\Phi)$ with as points the elements of Φ and as blocks the sets Φ_L, incidence being the natural incidence, is a symmetric $2-(\frac{s(s+1)}{\alpha}+1, s+1, \alpha)$ design. This yield extra conditions on the parameters s and α. We will prove in the sequel that $\mathrm{PQ}^+(4n-1,q)$ ($q=2$ or 3) has replaceable spreads, which yield $2-(2^{2n}-1, 2^{2n-1}, 2^{2n-2})$ designs in the case $q=2$ and $2-(\frac{3^{2n}-1}{2}, 3^{2n-1}, 2\cdot 3^{2n-2})$ designs in the case $q=3$. These designs have the parameters of the complement of the designs of points and hyperplanes of a $\mathrm{PG}(2n-1,q)$, ($q=2$ or 3).

The construction

Let $\mathcal{S}$ be a $\mathrm{pg}(s, s{+}1, \alpha)$ with a replaceable pg-spread Φ. Define the following incidence structure $\mathcal{S}_\Phi = (\mathcal{P}_\Phi, \mathcal{L}_\Phi, \mathrm{I}_\Phi)$. The elements of $\mathcal{P}_\Phi$ are on the one hand the points of $\mathcal{S}$ and on the other hand the sets $\mathcal{L}_i$, $i=1,\ldots,s(s+1)/\alpha+1$; the set of lines $\mathcal{L}_\Phi = \mathcal{L}\setminus\Phi$. Finally $p\ \mathrm{I}_\Phi\ L$ is defined by $p\ \mathrm{I}\ L$ if $p\in\mathcal{P}$ and by $L\in p$ if $p\in\{\mathcal{L}_i \| i=1,\ldots,s(s+1)/\alpha+1\}$.

Theorem 3.4.1
$\mathcal{S}_\Phi$ is a $\mathrm{pg}(s+1, s, \alpha)$

Proof. It is clear from the construction that $\mathcal{S}_\Phi$ is a partial linear space of order $(s+1, s)$. We only prove here that for each anti-flag (p, L) the incidence number equals α. Let p be a point of $\mathcal{S}$ and let L_p be the line of the pg-spread Φ through p. If L_p is not intersecting L in the partial geometry $\mathcal{S}$, then the α lines of $\mathcal{S}$ through p and intersecting L are all elements of $\mathcal{S}_\Phi$ while the point of type $\mathcal{L}_i$ defined by L is not collinear with p. However if L_p is intersecting L in the partial geometry $\mathcal{S}$, then there are $\alpha-1$ lines of $\mathcal{S}$ (being also lines of $\mathcal{S}_\Phi$) through p and intersecting L. Let $\mathcal{L}_i$ be the unique set defined by L and let $L_i(p)$ be the line of $\mathcal{S}$ through p and contained in $\mathcal{L}_i$, then $p\ \mathrm{I}_\Phi\ L_i(p)\ \mathrm{I}_\Phi\ \mathcal{L}_i\ \mathrm{I}_\Phi\ L$. Hence also in the case the incidence number $\alpha(p, L)$ equals α. Assume $p\in\{\mathcal{L}_i\| i=1,\ldots,s(s+1)/\alpha+1\}$ then as each line L of $\mathcal{S}_\Phi$ not contained in p intersects the point set of p in α points, it follows that the incidence number is again α. ∎

Remark

It has been checked by computer by Mathon and Street [85] (and also by V. Tonchev, private communication) that $\mathrm{PQ}^+(7,2)$ has, up to isomorphism, exactly 3 replaceable spreads, yielding (after dualizing) 3 non-isomorphic partial geometries $\mathrm{pg}(7,8,4)$. One of these $\mathrm{pg}(7,8,4)$ contains replaceable spreads too which yield again partial geometries $\mathrm{pg}(7,8,4)$, non-isomorphic to the former ones. In total Mathon and Street have found by this technique (they call this construction *switching*) 7 partial geometries $\mathrm{pg}(7,8,4)$ that are not isomorphic to $\mathrm{PQ}^+(7,2)$. We will give in the next section a geometric construction of the 3 replaceable spreads of $\mathrm{PQ}^+(7,2)$, a construction which is even valid in the general class $\mathrm{PQ}^+(4n-1,q)$, $q=2$ or 3.

Replaceable spreads of $\mathrm{PQ}^+(4n-1,q)$ **(**$q=2$ **or** $q=3$**)**

Assume that $\mathcal{S}$ is a partial geometry of type $\mathrm{PQ}^+(4n-1,q)$ ($q=2$ or 3). It is easy to check (see [40]) that two lines L and M are intersecting lines of $\mathcal{S}$ iff on the quadric $\mathrm{Q}^+(4n-1,q)$, $L\cap M^* = \emptyset$ iff $L^*\cap M=\emptyset$. Hence any subset of elements $\mathrm{PG}(2n-1,q)$ of Ω contained in an element ω_i of the orthogonal spread Σ yields a set of mutually disjoint lines of $\mathcal{S}$. If H is a subspace of ω_i, and if a pg-spread of $\mathcal{S}$ contains 2 elements through H then all elements through H are elements of the pg-spread. In [40], the following theorem has been proved for $q=2$, but the proof can easily be modified for $q=3$.

Theorem 3.4.2
Suppose that z is a point on $\mathrm{Q}^+(4n-1,q)$ ($q=2$ or 3) . The set of lines V of $\mathrm{PQ}^+(4n-1,q)$ intersecting as $(2n-1)$-dimensional subspaces of $\mathrm{Q}^+(4n-1,q)$ in a point z of the quadric is contained in exactly 2 pg-spreads.

As before, assume that Ω is the set of all hyperplanes of the elements ω_i, $i=0,\ldots,q^{2n-1}$ of the orthogonal spread Σ. Let $z\in\omega_0$. One of the pg-spreads, which we will denote by Φ_1, consists of all hyperplanes of ω_0. The other pg-spread, which we will denote by Φ_2, equals $V\cup W$, with $W=\{M_i = z^*\cap\omega_i \| i=1,\ldots,q^{2n-1}\}$.
We construct a third type of pg-spread of $\mathrm{PQ}^+(7,q)$ ($q=2$ or 3). Let ω' be an element of $\mathcal{D}_1\setminus\Sigma$. Then $\omega'\cap\omega_i$, $i=0,\ldots,q^3$ is either empty or is a line. Without loss of generality we may assume that $\omega'\cap\omega_i$, $i=0,\ldots,q^2$, is a line l_i. The set $\Phi_3=\{\pi_{i,j}\|i=0,\ldots,q^2; j=0,\ldots q\}$, with $\pi_{i,j}$, $j=0,\ldots,q$ a plane of ω_i through l_i is a pg-spread of $\mathrm{PQ}^+(7,q)$ ($q=2$ or 3) which is clearly not isomorphic to the other two.

Theorem 3.4.3 ([37])
The partial geometry $\mathrm{PQ}^+(7,q)$ ($q=2$ or 3) has up to isomorphism exactly 3 pg-spreads, namely Φ_1, Φ_2 and Φ_3 each of them being replaceable.

Remarks

1. There are 9 pg-spreads of type Φ_1, 135 of type Φ_2 and 126 of type Φ_3 in $\mathrm{PQ}^+(7,2)$) .

2. It is known that $\mathrm{Aut}(\mathrm{PQ}^+(7,2))=\mathrm{Alt}(9)$. This can easily be proved by regarding $\mathrm{Aut}(\mathrm{PQ}^+(7,2))$ acting on the 9 elements of the orthogonal spread Σ. The group is transitive on the points as well as on the lines of $\mathrm{PQ}^+(7,2)$.

3. The spread $\Phi\cong\Phi_1$ is a replaceable pg-spread of $\mathrm{PQ}^+(4n-1,q); \forall n\geq 2$. For every line $L\in\mathcal{L}\setminus\Phi$, the $s+1$ elements of Φ_L are the hyperplanes of ω_0 not containing $z=L^*\cap\omega_0$. Hence there is a canonical bijection

from the sets $\mathcal{L}_i$, $i = 1, \ldots, \frac{q^2 n-1}{q-1}$ to the points of ω_0. From this it follows that the symmetric design $\mathcal{D}(\Phi_1)$ is indeed isomorphic to the complement of the design of points and hyperplanes of $\mathrm{PG}(2n-1, q)$. The partial geometry $\mathcal{S}_\Phi$ is the geometry with as point set the set of points of $\mathcal{S}$ union the set of points of a fixed element ω_0 of the orthogonal spread Σ. The line set is the set of hyperplanes contained in the other elements ω_i $(i = 1, \ldots q^{2n-1})$ of the orthogonal spread Σ. A point p is incident with a line L, if and only if the line L (as subspace on the quadric Q^+) is contained in the polar hyperplane p^* of p with respect to the quadric. Note that the point graph as well as the block graph of this geometry were known before, see [13].

3.5 The sporadic partial geometry of Van Lint and Schrijver [128]

They constructed the following sporadic proper partial geometry. We will sketch two constructions of this geometry. Let β be a primitive element of GF(81). Then $\gamma = \beta^{16}$ is a primitive 5-th root of the unity. Let $\mathcal{P} = \mathrm{GF}(81)$, let $\mathcal{L}$ be the set $\{(b, 1+b, \gamma+b, \gamma^2+b, \gamma^3+b, \gamma^4+b) \| b \in \mathrm{GF}(81)\}$. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \in)$ is a pg$(5, 5, 2)$. The point graph of this geometry is a srg$(81, 30, 9, 12)$, and is a graph which was not known before.
Another construction of this geometry is given by Cameron and van Lint [28]. Let C be the ternary repetition code of length 6, i.e.

$$C = \{(0,0,0,0,0,0), (1,1,1,1,1,1), (2,2,2,2,2,2)\}.$$

Any coset of C in $\mathrm{GF}(3)^6$ has a well-defined type i in GF(3), i.e. the sum i of the coordinates of any vector in the coset. Let $\mathcal{A}_i$ be the set of cosets of type i. Define a tripartite graph Γ by joining the coset $C + v$ to the coset $C + v + w$ for each vector w of weight 1. Any element in $\mathcal{A}_i$ has 6 neighbours in $\mathcal{A}_{i+1}$ and 6 in $\mathcal{A}_{i+2}$ (indices taken mod 3).
Consider the incidence structure with point set $\mathcal{A}_i$, and line set $\mathcal{A}_{i+1}$, in which incidence is defined by adjacency in Γ. Then this incidence structure is the partial geometry of van Lint and Schrijver.
For example suppose $i = 0$, and let p be the point with coset representative $(0,0,0,0,0,0)$. The six lines incident with p have representatives of the form $(1,0,0,0,0,0)$, hence the 30 points $(\neq p)$ collinear with p have representatives of the form $(1,2,0,0,0,0)$. It is immediately clear that two points are incident with at most one line. A line not incident with p has a representative of the form $(2,2,0,0,0,0)$ or $(2,1,1,0,0,0)$; in both cases, it is incident with two points collinear with p.

Remark

Assume $\mathcal{S}$ is a proper symmetric partial geometry with $\alpha = 2$; then the numerical conditions of Theorem 3.1.1 yield $s = 5$. It is not known whether this geometry is unique. There is a great doubt that this geometry is a member of an infinite family. However the point graph is a member of an infinite family of so-called cyclotomic type (see [22]).

3.6 The sporadic partial geometry of Haemers [60]

Haemers [60] constructed a proper partial geometry pg$(4, 17, 2)$ starting from the following graph Γ (see for instance [73]). The vertices of Γ are the 175 edges of the Hoffman–Singleton graph $HoS(50)$. Two vertices of Γ are adjacent whenever the corresponding edges of $HoS(50)$ have distance two (i.e. the two edges are disjoint and there exists an edge connecting both). One can prove that this graph is a srg(175,72,20,36); moreover Γ is a pseudo–geometric $(4, 17, 2)$–graph. Haemers proved that Γ is indeed geometric. First of all we remark that a line of the partial geometry will be a set of 5 disjoint edges pairwise at distance two in the Hoffman–Singleton graph $HoS(50)$. It is easy to see that in a Petersen graph there are 6 such sets. If one can find 105 Petersen graphs in the Hoffman–Singleton graph, then one has the right number of lines. However there are more than 105 Petersen graphs in $HoS(50)$. W. Haemers was able to find a good subset of 105 special Petersen graphs in the Hoffman–Singleton graph, such that every pentagon of $HoS(50)$ is contained in exactly one such special Petersen graph. Note that any two edges at distance two in $HoS(50)$ are in a unique pentagon, so in a unique special Petersen graph, hence they define a unique set of 5 disjoint edges pairwise at distance two. In other words, the incidence structure of the 175 vertices of Γ and the 630 so-called 1-factors of the special Petersen graphs of $HoS(50)$ has the property that any two adjacent vertices define a unique line. This is enough to conclude that the pseudo–geometric graph Γ is indeed geometric.

Remarks

1. The point graph of the dual of this geometry has parameters $v = 630, k = 85, \lambda = 20, \mu = 10$ and was not known before.

2. Another construction of this geometry is related to the Steiner system S(5,8,24) and was given by Calderbank and Wales [23].

3. There are reasons enough to conjecture that this partial geometry is not a member of an infinite family.

3.6.1 The partial geometries of Mathon

This infinite family of geometries was constructed in [84].
The *Hermitian graph* $\mathcal{H}(q)$, also called Taylor graph is defined for odd prime powers q, in the affine plane $\mathrm{AG}(2, q^2)$. The vertices of $\mathcal{H}(q)$ are pairs (x, y) satisfying $x^{q+1} + y + y^q = 0$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if $x_1 x_2^q + y_1 + y_2^q$ is a square or a non-square in $\mathrm{GF}(q^2)$ according as $q \equiv 3$ or $1 \pmod 4$.
One can show that $\mathcal{H}(q)$ is a

$$\mathrm{srg}\left(q^3, \frac{(q-1)(q^2+1)}{2}, \frac{(q-1)^3}{4} - 1, \frac{(q-1)(q^2+1)}{4}\right)$$

and this is a pseudo- geometric $(q-1, \frac{q^2-1}{2}, \frac{q-1}{2})$-graph. The automorphism group $\mathcal{G}(q)$ of $\mathcal{H}(q)$ is a subgroup or the classical group $\mathrm{P\Gamma U}(3, q^2)$ and acts 2-transitively on the points (x, y) of $\mathcal{H}(q)$. By theorem 3.2.2, for $\mathcal{H}(q)$ to be geometric, we must be able to cover its edges by mutually edge-disjoint maximal q-cliques which then form the lines of a geometry. Also this set of lines must be fixed under the automorphism group $\overline{\mathcal{G}}(q)$ of the corresponding partial geometry. Unfortunately, $\mathcal{H}(q)$ contains a large number of q-cliques belonging to a number of orbits under its automorphism group $\mathcal{G}(q)$ (which obviously is a subset of $\overline{\mathcal{G}}(q)$).
If $q = 3^{2n}$, $n \geq 1$, there is a special orbit of q-cliques which can be used to construct a partial geometry.

Theorem 3.6.1 ([84])
The Hermitian graph $\mathcal{H}(3^{2n})$, $n \geq 1$ is the point graph of a partial geometry $\mathcal{M}_3(n)$, a

$$\mathrm{pg}\left(3^{2n} - 1, \frac{3^{4n} - 1}{2}, \frac{3^{2n} - 1}{2}\right).$$

Remark

The graph $\mathcal{H}(3)$ is the Schläfli graph and is uniquely defined by its parameters, it is the point graph of the generalized quadrangle of order $(2, 4)$. It has been proved by Spence [106] (computer search) that $\mathcal{H}(5)$ and $\mathcal{H}(7)$ are not geometric. Nothing is known for the other values.

3.7 Some characterization theorems for partial geometries

3.7.1 The graphs with $\alpha = s - 1$

If $\mathcal{S}$ is a partial geometry with $\alpha = s - 1$, then the complement of its point graph has a negative eigenvalue -2. All the strongly regular graphs with smallest eigenvalue -2 however are classified by Seidel [101]. Using this theorem a classification of the pseudo–geometric $(s, t, s-1)$–graphs can be given. Moreover a lot of information on geometric $(s, t, s-1)$–graphs is known.

Theorem 3.7.1 ([36])
If Γ is a pseudo–geometric $(s, t, s-1)$–graph, then one of the following cases occurs:

1. *$s = 2$ and $t = 1$, 2 or 4. Then Γ is geometric and the corresponding generalized quadrangles are unique* (see Section 2.5)*.*
2. *$s = 3$ and $t = 1$, 2 or 4.*
   (a) *If $t = 1$, then Γ is geometric and the corresponding partial geometry is the unique dual 2–design with these parameters.*
   (b) *If $t = 2$, then Γ is geometric and there are exactly two possibilities, i.e. the corresponding nets are the two nets of order 4 and degree 3.*
   (c) *If $t = 4$, then Γ never is geometric.*
3. *$s > 3$, $t = s-1$ and Γ is geometric if and only if there exists an affine plane of order $s+1$. The corresponding net is obtained by deleting two parallel classes of lines from the affine plane.*
4. *$s > 3$, $t = 2(s-1)$. If there exists a hyperoval $\mathcal{O}$ in some projective plane of order $2s$ then Γ is geometric, the geometry being the dual of $\mathcal{S}(\mathcal{O})$.*

3.7.2 Partial geometries embedded in projective and affine spaces

We first introduce the *Axiom of Veblen* for a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$.

> **Axiom of Veblen.** *If $L_1 \mathrm{~I~} x \mathrm{~I~} L_2, L_1 \neq L_2, M_1 \not\mathrm{I}\, x \not\mathrm{I}\, M_2$, and if L_i is concurrent with M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2.*

Sometimes this axiom is also called the axiom of Pasch. The dual of this axiom is sometimes called *The diagonal axiom* (see chapter 4.2). Note that for a generalized quadrangle Veblen and its dual are trivially satisfied. The incidence structure of points and lines of a projective space satisfies of course the axiom of Veblen.
The only known partial geometry with $\alpha \notin \{1, t+1\}$ and satisfying the axiom of Veblen is the dual net (i.e. the incidence number $\alpha(x, L) = s$) $H_q^n, n > 2$, which is constructed as follows: the points of H_q^n are the points

of $\mathrm{PG}(n,q)$ not in a given subspace $\mathrm{PG}(n-2,q) \subset \mathrm{PG}(n,q)$, the lines of H_q^n are the lines of $\mathrm{PG}(n,q)$ which have no point in common with $\mathrm{PG}(n-2,q)$, the incidence in H_q^n is the natural one. By the following theorem these dual nets H_q^n are characterized by the Axiom of Veblen.

Theorem 3.7.2 ([120])
Let $\mathcal{N}^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1 > s$. If $\mathcal{N}^*$ satisfies the Axiom of Veblen, then $\mathcal{N}^* \cong H_q^n$ with $n > 2$ (hence $s = q$ and $t+1 = q^{n-1}$).

Of course, partial geometries embedded in a projective space will satisfy the axiom of Veblen. There exists however a complete classification of partial geometries embedded in a projective space.

Theorem 3.7.3 ([41])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a projective partial geometry $\mathrm{pg}(s,t,\alpha)$ with ambient space $\mathrm{PG}(n,s)$, then only the following cases may occur:

1. *$\alpha = s+1$, and $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n,s)$;*
2. *$\alpha = 1$, and $\mathcal{S}$ is a classical generalized quadrangle* (see [20])*;*
3. *$\alpha = t+1, n = 2$ and $\mathcal{S}$ is a dual design in $\mathrm{PG}(2,s)$;*
4. *$\alpha = s$ and $\mathcal{S} = H_s^n (n \geq 3)$.*

There also exists a complete classification of partial geometries embedded in an affine space by Thas [112]. For the case of a generalized quadrangle some sporadic embeddings can occur. We will however restrict ourselves here to the case of proper partial geometries.

Theorem 3.7.4 ([112])
If $\mathcal{S}$ is an affine proper partial geometry with ambient space $\mathrm{AG}(n,s+1)$, then $n = 3$ and $\mathcal{S} = T_2^*(\mathcal{K})$ with $\mathcal{K}$ a maximal arc in the plane at infinity.

Corollary

If $T_n^*(\mathcal{K})$ $(n > 1)$ is a linear representation of a partial geometry of order (s,t), then either $\mathcal{K}$ is the complement of a hyperplane (hence $\alpha = s$), or $n = 2$.
For more characterization theorems, especially regarding geometries of type $\mathcal{S}(\mathcal{K})$ and of type $T_2^*(\mathcal{K})$, we refer to [38], [56] and [40].

Chapter 4

Semipartial geometries

4.1 Definitions

A *semipartial geometry* $\mathrm{spg}(s,t,\alpha,\mu)$ with parameters s,t,α,μ is a partial linear space $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ of order (s,t), such that for each anti-flag (x,L), the incidence number $\alpha(x,L)$ equals 0 or a constant $\alpha(>0)$ and such that for any two points which are not collinear, there are μ $(\mu>0)$ points collinear with both (μ-condition). Semipartial geometries were introduced by Debroey and Thas [46].

Remarks

1. A semipartial geometry $\mathrm{spg}(s,t,\alpha,\mu)$ is a $(0,\alpha)$-geometry $\mathcal{S}$ such that, because of the μ-condition, the point graph $\Gamma(\mathcal{S})$ is a
$$\mathrm{srg}(1+\frac{(t+1)s(\mu+t(s-\alpha+1))}{\mu},(t+1)s,s-1+t(\alpha-1),\mu).$$

2. Semipartial geometries generalize at the same time the partial quadrangles of Cameron [24] and the partial geometries. It is immediately clear that a semipartial geometry is a partial geometry if and only if $\mu=(t+1)\alpha$. If we want to exclude the partial geometries we will speak about *proper* semipartial geometries. In any case, for the rest of this section we will suppose, unless the contrary is stated, that $\mathcal{S}$ is not a 2-design, hence that $\alpha\leq\min(t+1,s)$.

3. The dual of a semipartial geometry again is a semipartial geometry if and only if either $s=t$ or $\mathcal{S}$ is a partial geometry (see [46]).

Using the fact that the point graph is strongly regular, and using other counting arguments, one can deduce a lot of conditions between the parameters of a semipartial geometry. We will give a summary of the most important ones in the next theorem.

Theorem 4.1.1 ([46])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a proper semipartial geometry with parameters s, t, α, μ, then

1. *$t \geq s$, hence $|B| = b = \frac{v(t+1)}{s+1} \geq v$;*
2. *$D = (t(\alpha - 1) + s - 1 - \mu)^2 + 4((t+1)s - \mu)$ is either a square or equals 5 (then $\mathcal{S}$ is isomorphic to the pentagon) and*
$$\frac{2(t+1)s + (v-1)(t(\alpha-1) + s - 1 - \mu + \sqrt{D})}{2\sqrt{D}}$$
is an integer;
3. *$\alpha^2 \leq \mu \leq (t+1)\alpha$ and $\alpha|\mu$;*
4. *$\mu|(t+1)st(s+1-\alpha)$;*
5. *$\alpha|ts(t+1)$ and $\alpha|ts(s+1)$;*
6. *$\alpha^2|\mu st$;*
7. *$\alpha^2|t((t+1)\alpha - \mu)$;*
8. *$2|v(t+1)s$;*
9. *$3|v(t+1)s(s-1)$ and $3|v(t+1)st(\alpha-1)$;*
10. *$8|v(t+1)s(s-1)(s-2)$;*
11. *$8|v(t+1)s(t(\alpha-1)((t-1)(\alpha-1)-(\alpha-2))+t(s+1-\alpha)(\mu-2\alpha+1))$;*

Remark

The Krein inequalities for strongly regular graphs also yield some extra conditions, but these are rather complicated formulae.

4.2 A first list of examples of proper semipartial geometries

4.2.1 The thin partial quadrangles

Let Γ be a strongly regular graph with $\lambda = 0$. Then this graph is a partial quadrangle with $s = 1$ and $t = k - 1$, hence a thin geometry. Up to now the only known examples of such graphs are the pentagon $Pn(5)$, the Petersen graph $Pe(10)$, the Clebsch graph $Cl(16)$, the Hoffman–Singleton graph $HoS(50)$, and the graphs from the Higman–Sims family (i.e. $Gew(56)$, $HS(77)$ and $HS(100)$). The parameter sets (v, k, μ) for these graphs are resp. equal to (5,2,1), (10,3,1), (16,5,2), (50,7,1), (56,10,2), (77,16,4), (100,22,6). All these graphs are uniquely defined by their parameters.

4.2.2 The semipartial geometries $\overline{M(k)}$, $k \in \{2,3,7,57\}$

The three thin partial quadrangles with $\mu = 1$ are better known as *Moore graphs*. These graphs are the graphs with valency $k > 1$, girth 5 (i.e. they have no 3-cycles nor 4-cycles but they do have 5-cycles) and with the minimum number of vertices, which is $k^2 + 1$. It is known that necessarily $k \in \{2,3,7,57\}$. However a Moore graph with $k = 57$ is not known to exist. With each Moore graph Γ there is associated another semipartial geometry, which we will denote by $\overline{M(k)}$. The point set $\mathcal{P}$ is the set of vertices Γ, the line set $\mathcal{L}$ is the set $\{\Gamma(x) \| x \in \mathcal{P}\}$, I is the natural incidence relation. Then $\overline{M(k)} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry $\mathrm{spg}(k-1, k-1, k-1, (k-1)^2)$ ([46]).

4.2.3 The semipartial geometries $U_{2,3}(n)$

Let U be a set of cardinality n, $n > 4$. Let $\mathcal{P}$ be the set of pairs of U, let $\mathcal{L}$ be the set of unordered triples of U, and let I be the inclusion relation. Then $U_{2,3}(n) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry $\mathrm{spg}(2, n-3, 2, 4)$ ([46]). The point graph of this geometry is the triangular graph $T(n)$.

4.2.4 The semipartial geometries $LP(n, q)$

Define $\mathcal{P}$ as the set of lines of $\mathrm{PG}(n, q)$ $(n \geq 4)$, $\mathcal{L}$ as the set of planes of $\mathrm{PG}(n, q)$, and I as the inclusion relation. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = q(q+1)$, $t = \frac{q^{n-1}-1}{q-1} - 1$, $\alpha = q+1$, $\mu = (q+1)^2$ ([46]). Remark that for $n = 3$ this construction yields the dual design of lines and planes of $\mathrm{PG}(3, q)$.

4.2.5 The semipartial geometries $\overline{W(2n+1, q)}$

Let σ be a symplectic polarity of $\mathrm{PG}(2n+1, q)$, $n \geq 1$. Let $\mathcal{P}$ be the point set of $\mathrm{PG}(2n+1, q)$, $\mathcal{L}$ the set of lines which are not totally isotropic (i.e. are hyperbolic) with respect to σ, and I the incidence relation of $\mathrm{PG}(2n+1, q)$. Then $\overline{W(2n+1, q)} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a $\mathrm{spg}(q, q^{2n} - 1, q, q^{2n}(q-1))$ ([46]).

4.2.6 The semipartial geometries $\mathrm{NQ}^{\pm}(2n-1, 2)$

Let Q be a (non-singular) hyperquadric in $\mathrm{PG}(2n-1, 2)$, ($n \geq 2$ if Q is elliptic, and $n \geq 3$ if Q is hyperbolic). Let $\mathcal{P}$ be the set of points off the quadric, let $\mathcal{L}$ be the set of non-intersecting lines of Q, and let I be the incidence of $\mathrm{PG}(2n-1, 2)$. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = 2^{2n-3} - \varepsilon 2^{n-2} - 1$, $\mu = 2^{2n-3} - \varepsilon 2^{n-1}$, where $\varepsilon = +1$ for the hyperbolic quadric (we will denote this geometry by $\mathrm{NQ}^{+}(2n-1, 2)$) and where $\varepsilon = -1$ for the elliptic quadric (we will denote this geometry by $\mathrm{NQ}^-(2n-1,2)$). This was first remarked by H. Wilbrink (private communication).

4.2.7 The semipartial geometries H_q^{n*}

This semipartial geometry is defined by taking as point set $\mathcal{P}$ the set of lines of a projective space $\Sigma \cong \mathrm{PG}(n,q)$, $n \geq 4$, skew to a fixed projective space $H \cong \mathrm{PG}(n-2,q)$ and as line set $\mathcal{L}$ the set of the planes of Σ which intersect H in exactly one point. This semipartial geometry has parameters $s = q^2 - 1$, $t = \frac{q^{n-1}-1}{q-1} - 1$, $\alpha = q$, $\mu = q(q+1)$.

Remark

On page 42 the axiom of Veblen has been introduced. The dual of this axiom is better known as the *Diagonal Axiom*:

> **The Diagonal Axiom** $(\mathcal{D})$ *If* $x_1 \mathrel{\mathrm{I}} L \mathrel{\mathrm{I}} x_2, x_1 \neq x_2, y_1 \mathrel{\not\mathrm{I}} L \mathrel{\not\mathrm{I}} y_2$, *and if* x_i *is collinear to* y_j *for all* $i, j \in \{1,2\}$, *then* y_1 *is collinear to* y_2.

The semipartial geometry H_q^{n*} satifies the diagonal axiom and is clearly related to the dual net H_q^n defined on page 42.
Indeed a $\mathrm{spg}(s,t,\alpha,\mu)$ $\mathcal{S}$ satisfying the diagonal axiom, gives rise to a semipartial geometry $\overline{\mathcal{S}}$ satisfying the diagonal axiom (see [41] and [45]) by the following construction. Let x and y be two collinear points. We denote by $D^1_{x,y}$ the set of points collinear with x and y but not on the line $L_{x,y}$ joining x and y and by $D^2_{x,y}$ the set of points of $L_{x,y}$ which are collinear with a point (hence with all points) of $D^1_{x,y}$. Then $D_{x,y} = D^1_{x,y} \cup D^2_{x,y}$ is a maximal set of pairwise collinear points, and any such set $D_{x,y}$ is called a *diagonal clique*. The incidence structure $\overline{\mathcal{S}}$ with the same point set as $\mathcal{S}$ and with line set the set of diagonal cliques of $\mathcal{S}$, is a semipartial geometry with parameters $\overline{s} = (t+1)(\alpha-1)$, $\overline{t} = s/(\alpha-1)-1$, $\overline{\alpha} = \alpha$, $\overline{\mu} = \mu$ which satisfies the diagonal axiom. Note that $\overline{\mathcal{S}}$ has the same point graph as $\mathcal{S}$ and that $\overline{\overline{\mathcal{S}}} \cong \mathcal{S}$. This describes the relation between the dual of H_q^n and the semipartial geometry H_q^{n*}.

4.3 The linear representations of semipartial geometries

If $T_n^*(\mathcal{K})$ is a linear representation of a semipartial geometry, then $\mathcal{K}$ has to be a set of points in $\mathrm{PG}(n,q)$ such that each line of the projective space is either an exterior line, a tangent or an $(\alpha+1)$-secant and such that, because of the μ-condition, each point of $\mathrm{PG}(n,q) \setminus \mathcal{K}$ is on $\frac{\mu}{\alpha(\alpha+1)}$ $(\alpha+1)$-secants. The following examples are known.

4.3.1 Linear representations of proper partial quadrangles

In this case $\mathcal{K}$ is a $(t+1)$-cap with the property that each point not in $\mathcal{K}$ is on $t+1-\mu$ tangents. Calderbank [21] has given an almost complete classification of partial quadrangles with a linear representation. His proof is a number theoretic proof. He lists the possible parameter values of the associated strongly regular graph.
Only the following cases may occur.

1. $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid of the projective space PG$(3,q)$. It is a partial quadrangle with parameters $s=q-1,\ t=q^2,\ \mu=q(q-1)$ and was first constructed by Cameron [24].

2. Suppose $q=3$ and assume that $\mathcal{K}$ is not an ovoid. Then $\mathcal{K}$ is either an 11-cap in PG(4,3), see for instance [34] and [98] for a description, and the partial quadrangle $T_4^*(\mathcal{K})$ has parameters $s=2, t=10, \mu=2$, or $\mathcal{K}$ is the unique 56-cap in PG(5,3) in which case the partial quadrangle has parameters $s=2, t=55, \mu=20$. This 56-cap was first constructed by Segre [99] but was also studied by several other authors, see for instance [6], [17], [66], [86] and [114].

3. Suppose $q=4$. Then either $\mathcal{K}$ is an ovoid in PG$(3,4)$, or it is a 78-cap in PG$(5,4)$ such that each external point is on 7 secants, or a 430-cap in PG$(6,4)$ such that each external point is on 55 secants. If $\mathcal{K}$ is a 78-cap, the partial quadrangle $T_5^*(\mathcal{K})$ has parameters $s=3, t=77, \mu=14$. At least one example exists and was discovered by Hill [67]. If $\mathcal{K}$ is a 430-cap then the partial quadrangle has parameters $s=3, t=429, \mu=110$. Up to now however, the existence of such a cap is not known.

4. Suppose $q \geq 5$. Then it was proved in [126] that the partial quadrangle has to be $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid.

Remarks

1. If $T_n^*(\mathcal{K})$ is a linear representation of a partial quadrangle with $q=2$, then the partial quadrangle coincides with its point graph; there is only one solution, the strongly regular graph srg$(16,5,0,2)$ which is the Clebsch graph $Cl(16)$, and the geometry is of type $T_3^*(\mathcal{O})$ with $\mathcal{O}$ the elliptic quadric in PG$(3,2)$.

2. The existence of a cap $\mathcal{K}$ in PG(n,q), such that every exterior point is on a constant number of tangents, implies the existence of a uniformly packed $[|\mathcal{K}|, |\mathcal{K}|-n-1, 4]$ code C, and so the dual code $C^\perp$ is an $[|\mathcal{K}|, n+1]$ code over GF(q) with exactly 2 weights (see [21] and [22]).

4.3.2 Linear representations of proper semipartial geometries with $\alpha > 1$

In this case the following models are known.

1. The set $\mathcal{K}$ is a unital $\mathcal{U}$ in the projective plane Π_∞=PG(2,q^2) at infinity, and $T_2^*(\mathcal{U})$ has parameters $s = q^2-1, t = q^3, \alpha = q, \mu = q^2(q^2-1)$.
2. If $\mathcal{K}$ is a Baer subspace $\mathcal{B}$ of the projective space $\Pi_\infty = \mathrm{PG}(n, q^2)$, $n \geq 2$, at infinity, then $T_n^*(\mathcal{B})$ has parameters $s = q^2-1$, $t = \dfrac{q^{n+1}-1}{q-1} - 1$, $\alpha = q$, $\mu = q(q+1)$.

4.4 Semipartial geometries and generalized quadrangles

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized quadrangle of order (s,t), $s \geq 2$, then one can construct in the following way a $(0,1)$-geometry. Let p be any point of $\mathcal{S}$, let $p^\perp$ be the set of all points of $\mathcal{S}$ collinear with p (the trace of p) and let $\mathcal{L}(p)$ be the set of lines of $\mathcal{S}$ through p, then the incidence structure $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ with $\mathcal{P}_p = \mathcal{P} \setminus p^\perp$, $\mathcal{L}_p = \mathcal{L} \setminus \mathcal{L}(p)$, and with I_p=I $\cap((\mathcal{P}_p \times \mathcal{L}_p) \cup (\mathcal{L}_p \times \mathcal{P}_p))$ is clearly a $(0,1)$-geometry of order $(s-1,t)$. Moreover $\mathcal{S}_p$ satisfies the following property.

(*) If L and M are two disjoint lines of $\mathcal{S}_p$ then there are either 0, $s-1$, or s lines of $\mathcal{S}_p$ concurrent to both L and M.

Note that this property of course is trivial in the case $s = 2$. The point graph $\Gamma(\mathcal{S}_p)$ of $\mathcal{S}_p$ will be a strongly regular graph with parameter μ if and only if for any 2 non-collinear points x and y in $\mathcal{P}_p$, the set $\{p,x,y\}^\perp$ of points in $\mathcal{S}$ collinear with p, x and y has cardinality $t+1-\mu$. However in a generalized quadrangle $\mathcal{S}$, $|\{x,y,z\}^\perp|$ is a constant for any triad $\{x,y,z\}$ of non-collinear points, if and only if $\mathcal{S}$ has order (s,s^2); moreover in this case $|\{x,y,z\}^\perp| = s+1$ (see Theorem 2.2.3). Hence the only partial quadrangles of type $\mathcal{S}_p$ are the ones of order $(s-1,s^2)$ with $\mu = s(s-1)$.
In Chapter 2 we have encountered a lot of generalized quadrangles of order (s,s^2). For all of them s is a prime power q and we will therefore in the sequel use q instead of s.
First of all there is the non-classical example $T_3(\mathcal{O})$, constructed by Tits, see Section 2.4.2. If p is the special point (∞) in $T_3(\mathcal{O})$ then the resulting partial quadrangle has a linear representation in AG$(4,q)$; it is the partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the hyperplane Π_∞. If p is any other point of $T_3(\mathcal{O})$ then the resulting partial quadrangle might be non-isomorphic to $T_3^*(\mathcal{O})$. On the other hand any flock of a cone in PG$(3,q)$ implies the existence of a generalized quadrangle of order (q,q^2) (see Theorem 2.4.3), and

these generalized quadrangles give rise to a lot of non-isomorphic partial quadrangles with parameters $(s,t,\mu) = (q-1, q^2, q(q-1))$.
Ivanov and Shpectorov [74] prove that every partial quadrangle with parameters $(s,t,\mu) = (q-1, q^2, q(q-1))$ is of type $\mathcal{S}_p$ and is uniquely extendible to a generalized quadrangle $\mathcal{S}$. For this they prove that such a partial quadrangle always satisfies property (*). In fact they even prove a more general result: every strongly regular graph $\mathrm{srg}(q^4, (q^2+1)(q-1), q-2, q(q-1))$, such that every 2 adjacent vertices are contained in a clique of order q, is the point graph of a partial quadrangle of type $\mathcal{S}_p$, and this partial quadrangle is uniquely extendible to a generalized quadrangle of order (q, q^2). Remark that this implies that the srg(81,20,1,6) is unique; see also [11] where another proof of that result is given.
Anyhow, it follows from the theorem of Calderbank that the linear representation of a partial quadrangle with parameters $(s,t,\mu) = (q-1, q^2, q(q-1))$ should be in $\mathrm{AG}(4,q)$, hence it should be $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in Π_∞.
In [44] the following theorem is proved.

Theorem 4.4.1
Let $T_n^*(\mathcal{K})$ $(n \geq 3)$ be a linear representation of a $(0,1)$-geometry of order $(q-1,t)$, $q > 2$, that satisfies (*). If $\mathcal{K}$ spans the hyperplane Π_∞, then $T_n^*(\mathcal{K})$ is the partial quadrangle $T_3^*(\mathcal{O})$.

Remark

It is clear that the $(0,1)$-geometries of order $(q-1,t)$ of type $T_1^*(\mathcal{K})$ should have $t=1$, hence are the grids of order $q-1$. If $\mathcal{S}$ is a $(0,1)$-geometry of type $T_2^*(\mathcal{K})$, then $\mathcal{K}$ is a set of points in the plane Π_∞ such that every line intersects $\mathcal{K}$ in 0,1 or 2 points, i.e. $\mathcal{K}$ is an arc in Π_∞. $T_2^*(\mathcal{K})$ satisfies (*) if and only if $|\mathcal{K}|$ is $q+1$ or $q+2$.

There is another way to construct semipartial geometries from generalized quadrangles. Suppose that $\overline{\mathcal{S}}$ is a generalized quadrangle embedded in a projective space $\mathrm{PG}(n,q)$, hence $\overline{\mathcal{S}}$ is classical. Suppose that p is a point of $\mathrm{PG}(n,q)$ and that Π is a hyperplane of $\mathrm{PG}(n,q)$ not containing p. Let $\overline{\mathcal{P}}_1$ be the projection of the point set of $\overline{\mathcal{S}}$ from p on Π and let $\overline{\mathcal{P}}_2$ be the set of points of Π on a tangent of $\overline{\mathcal{S}}$ through p. Let $\mathcal{S}$ be the geometry with point set $\mathcal{P} = \overline{\mathcal{P}}_1 \setminus \overline{\mathcal{P}}_2$, whereas the line set $\mathcal{L}$ is the set of all projections on Π of the lines of $\overline{\mathcal{S}}$, excluding the projections completely contained in $\overline{\mathcal{P}}_2$. The incidence is the one of the projective space. If $\overline{\mathcal{S}}$ is either $Q^-(5,q)$ or $H(4,q^2)$ then $\mathcal{S}$ is a semipartial geometry. Of course the parameters depend on whether p is a point on $\overline{\mathcal{S}}$ or not.
If $\overline{\mathcal{S}} = Q^-(5,q)$ and p is a point on the quadric, the semipartial geometry $\mathcal{S}$ is $T_3^*(\mathcal{O})$, with $\mathcal{O}$ the elliptic quadric in $\mathrm{PG}(3,q)$. However if p is not on the quadric, it yields a semipartial geometry with parameters $s = q-1$, $t = q^2$, $\alpha = 2$, $\mu = 2q(q-1)$. This construction is due to

Hirschfeld and Thas [70]. If $\overline{\mathcal{S}} = H(4, q^2)$ and p is a point on the Hermitian variety, the semipartial geometry $\mathcal{S}$ is $T_2^*(\mathcal{U})$, with $\mathcal{U}$ the Hermitian unital in $\mathrm{PG}(2, q^2)$. However if p is not on $H(4, q^2)$, it yields a semipartial geometry with parameters $s = q^2 - 1$, $t = q^3$, $\alpha = q + 1$, $\mu = q(q + 1)(q^2 - 1)$. This example is due to J. A. Thas (private communication).

Remark

Another construction of the $\mathrm{spg}(q - 1, q^2, 2, 2q(q - 1))$ of [70] was given by R. Metz (private communication). Let $Q(4, q)$ be a non-singular quadric of the projective space $\mathrm{PG}(4, q)$. If we define $\mathcal{P}$ as the set of the elliptic quadrics $Q^-(3, q)$ on $Q(4, q)$, $\mathcal{L}$ as the set of all pencils of such elliptic quadrics which are pairwise tangent in a common point, and I as the natural incidence relation, then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is isomorphic to the semipartial geometry constructed by Hirschfeld and Thas [70]. It has been observed by Brown [15] that one does not need necessarily the GQ $Q^-(5, q)$ for this construction. Indeed if a GQ $\mathcal{S}$ of order (s, s^2) contains a subquadrangle $\mathcal{S}'$ of order s, then every point x of $\mathcal{S} \setminus \mathcal{S}'$ is collinear with the $s^2 + 1$ points of an ovoid, denoted by $\mathcal{O}_x$, of $\mathcal{S}'$. The ovoid $\mathcal{O}_x$ is said to be *subtended by* x. If it happens to be that every such subtended ovoid $\mathcal{O}_x$ is also a subtended ovoid $\mathcal{O}_y$ for another point $y \in \mathcal{S} \setminus \mathcal{S}'$, then the ovoid is called *doubly subtended* and is denoted by $\mathcal{O}_{x,y}$. If every subtended ovoid of $\mathcal{S}'$ is doubly subtended, then the subGQ $\mathcal{S}'$ is called *doubly subtended* in the GQ $\mathcal{S}$.

Theorem 4.4.2 ([15])
Assume $\mathcal{S}$ is a GQ *of order (s, s^2) containing a sub*GQ *$\mathcal{S}'$ that is doubly subtended in $\mathcal{S}$; then the incidence structure with points the subtended ovoids of $\mathcal{S}'$, lines the rosettes of subtended ovoids (a rosette is a set of s subtended ovoids containing a common point x and having two by two just x in common), incidence the natural incidence, is a* $\mathrm{spg}(s - 1, s^2, 2, 2s(s - 1))$.

The generalized quadrangle $Q(4, q)$ is indeed doubly subtended in $Q^-(5, q)$ and this yields the construction of R. Metz. However Brown [15] remarks that $Q(4, q)$ is also doubly subtended in the GQ of order (q, q^2) (q odd) related to the flock $K1$ of Kantor, and hence yields a semipartial geometry.

4.5 Semipartial geometries and spg reguli

In [116] a new construction method for semipartial geometries is introduced.
An spg *regulus* is a set R of m-dimensional subspaces $\mathrm{PG}^{(1)}(m, q), \ldots, \mathrm{PG}^{(r)}(m, q)$, $r > 1$ of $\mathrm{PG}(n, q)$, satisfying:

1. $\mathrm{PG}^{(i)}(m, q) \cap \mathrm{PG}^{(j)}(m, q) = \emptyset$, for all $i \neq j$.

2. If $\mathrm{PG}(m+1,q)$ contains $\mathrm{PG}^{(i)}(m,q)$, then it has a point in common with either 0 or α $(\alpha > 0)$ spaces in $R \setminus \{\mathrm{PG}^{(i)}(m,q)\}$. If this $\mathrm{PG}(m+1,q)$ has no point in common with $\mathrm{PG}^{(j)}(m,q)$ for all $j \neq i$, then it is called a *tangent* $(m+1)$*-space of* R *at* $\mathrm{PG}^{(i)}(m,q)$.

3. If the point x of $\mathrm{PG}(n,q)$ is not contained in an element of R, then it is contained in a constant number θ $(\theta \geq 0)$ of tangent $(m+1)$-spaces of R.

By considering all the $(m+1)$-dimensional spaces through $\mathrm{PG}^{(i)}(m,q)$ one finds that $\alpha(q-1)$ has to divide $(r-1)(q^{m+1}-1)$, and that the number of tangent $(m+1)$-spaces of R at $\mathrm{PG}^{(i)}(m,q)$ equals

$$\frac{q^{n-m}-1}{q-1} - \frac{r-1}{\alpha} \cdot \frac{q^{m+1}-1}{q-1}.$$

By counting the number of ordered pairs (M,x), with M a tangent $(m+1)$-space $\mathrm{PG}(m+1,q)$ of R, and x a point of M which is not in an element of R, one obtains:

$$\theta = \frac{\big(\alpha(q^{n-m}-1) - (r-1)(q^{m+1}-1)\big) r q^{m+1}}{\alpha\big((q^{n+1}-1) - r(q^{m+1}-1)\big)}.$$

Note that by $r > 1$ and by the first condition in the definition of R one has $n \geq 2m+1$. If $n = 2m+1$, then there are no tangent $(m+1)$-spaces, and $\alpha = r-1$. If $n = 2m+2$, then any two tangent $(m+1)$-spaces at distinct elements of R intersect.
Given an spg regulus R, one can construct a semipartial geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows. Embed $\mathrm{PG}(n,q)$ as a hyperplane in $\mathrm{PG}(n+1,q)$. The points of $\mathcal{S}$ are the points in $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$. The lines of $\mathcal{S}$ are the $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ which contain an element of R but are not contained in $\mathrm{PG}(n,q)$. Incidence is that of $\mathrm{PG}(n+1,q)$. Then $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(q^{m+1}-1, r-1, \alpha, (r-\theta)\alpha)$, see [116].

Remarks

1. This geometry is a partial geometry if and only if $\theta = 0$; hence if $\theta \neq 0$, which implies that $r \geq q^{m+1}$, then $\mathcal{S}$ is a proper semipartial geometry.

2. If $n = 2m+1$, then $\mathcal{S}$ is a net of order $s+1 = q^{m+1}$ and degree $t+1 = r$.

spg reguli and polar spaces

A spread R of the non-singular elliptic quadric $\mathrm{Q}^-(2m+3,q)$ $(m \geq 0)$ contains $q^{m+2}+1$ elements (of dimension m) and is always an spg regulus. It yields a $\mathrm{spg}(q^{m+1}-1, q^{m+2}, q^m, q^{m+1}(q^{m+1}-1))$. For $m = 0$, this is the

partial quadrangle $T_3^*(\mathcal{O})$. For $m = 1$, it yields a spg$(q^2 - 1, q^3, q, q^2(q^2 - 1))$ with the same parameters as the semipartial geometry $T_2^*(\mathcal{U})$. Indeed $T_2^*(\mathcal{U})$ is isomorphic to the semipartial geometry arising from a regular spread R (see for instance [119]) of $Q^-(5, q)$. However if the spread is non-regular, then the associated semipartial geometry is not isomorphic to $T_2^*(\mathcal{U})$. If $m > 1$, and q is even, then the quadric $Q^-(2m + 3, q)$ has spreads, hence this yields new semipartial geometries. If q is odd, no spread of the quadric $Q^-(2m + 3, q)$ $(m > 1)$ is known.
If the non-singular quadric $Q(2m + 2, q)$ (of $PG(2m + 2, q)$), $m \geq 0$, has a spread R, then it is not an spg regulus.
If R is a spread of the quadric $Q^+(2m + 1, q)$, $m \geq 1$, then necessarily m is odd, and moreover this spread is an spg regulus, but the associated semipartial geometry is a net.
Let $H(n, q^2)$ be a non-singular Hermitian variety of $PG(n, q^2)$, $n \geq 2$. If n is odd, the Hermitian variety has no spread (see [18] for the case $n = 3$ and [118] for $n \geq 5$). Assume that n is even. Then R is always an spg regulus with $m = \frac{1}{2}n - 1$ and $|R| = q^{n+1} + 1$. Hence there corresponds a semipartial geometry spg$(q^n - 1, q^{n+1}, q^{n-1}, q^n(q^n - 1))$. However if $n = 2$ then this semipartial geometry is $T_2^*(\mathcal{U})$. Unfortunately for $n > 2$ no spread of $H(n, q^2)$, n even, is known. Brouwer (private communication, [1981]) proved that $H(4, 4)$ has no spread. For more details on spreads of polar spaces we refer to [119].

4.6 Some characterization theorems for semipartial geometries

Theorem 4.6.1 ([46])
If $\mathcal{S}$ is a proper semipartial geometry with $\alpha = t$, then $\mathcal{S} \cong \overline{M(t+1)}$, hence $s = t$, $\mu = \alpha^2$ and $t \in \{1, 2, 6, 56\}$.

Theorem 4.6.2 ([45], [132])
Let $\mathcal{S}$ be a proper semipartial geometry with $\mu = \alpha^2$.

1. *If $\alpha = 2$, then $\mathcal{S} \cong U_{2,3}(n)$;*
2. *if $2 < \alpha = s$ then $\alpha = t \in \{1, 2, 6, 56\}$ and $\mathcal{S} \cong \overline{M(t+1)}$;*
3. *if $2 < \alpha < s$ then $\mathcal{S} \cong LP(n, q)$.*

Theorem 4.6.3 ([45])
Let $\mathcal{S}$ be a proper semipartial geometry with parameters s, t, $\alpha(> 1)$ and $\mu = \alpha(\alpha + 1)$. If $\mathcal{S}$ satisfies the diagonal axiom $(\mathcal{D})$, then $\mathcal{S}$ is isomorphic to a semipartial geometry of type H_q^{n*}.

Remark

In [132] it is proved that all proper semipartial geometries with $\mu = \alpha^2$ and $2 \leq \alpha < s$ satisfy the diagonal axiom. Moreover, they proved that up to possibly a finite number of exceptions, all proper semipartial geometries with $\mu = \alpha(\alpha+1)$ satisfy the diagonal axiom. Cuypers [35] observed that by adding some extra combinatorial conditions, their proofs can even be generalized to $(0,\alpha)$-geometries.

4.7 Semipartial geometries embedded in projective and affine spaces

On the embedding of semipartial geometries in projective and affine spaces the following results are known.

Theorem 4.7.1
If $\mathcal{S}$ is a proper projective semipartial geometry $\mathrm{spg}(s,t,\alpha,\mu)$*, $s > 2$, $\alpha > 1$, with ambient space* $\mathrm{PG}(n,s)$*, $n \geq 3$, then n is odd and $\mathcal{S}$ is the semipartial geometry $\overline{W(n,s)}$.*

Remark

This theorem was proved by Debroey and Thas [48] for the case $n = 3$ and by De Clerck, Debroey and Thas [121] for $n > 3$. If $\mathcal{S}$ is any semipartial geometry with $\alpha = s = 2$, then $\mathcal{S}$ is a cotriangle space and those are classified (see theorem 4.8.1). A complete classification of the embedded cotriangle spaces exists for $n = 3$ ([48]) and for $n = 4$ ([121]). In [82] an embedding of $U_{2,3}(n+2)$ in $\mathrm{PG}(n,2)$ is given. The lines of this geometry are hyperbolic lines of some symplectic polarity. Also an embedding of $U_{2,3}(n+3)$ in $\mathrm{PG}(n,2)$ is described. The lines of this geometry are hyperbolic for some symplectic polarity if and only if n is odd. The problem of determining all embeddings of $U_{2,3}(n)$ in $\mathrm{PG}(d,2)$ is equivalent to determining (up to equivalence) all binary codes of length n with all weights even and minimum weight greater than 4, see [63].

Theorem 4.7.2 ([42])
Assume that $\mathcal{S}^D$ is the dual of a semipartial geometry $\mathcal{S}$ with $\alpha > 1$. If $\mathcal{S}^D$ is projective with ambient space $\mathrm{PG}(n,s)$*, $n \geq 3$, then $n = 3$ and $\mathcal{S}^D$ is the design of points and lines in* $\mathrm{PG}(3,q)$*, or $\mathcal{S}^D = H_s^3$ or $\mathcal{S}^D = NQ^-(3,2)$.*

Open question

Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(3,q^2)$. The incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, defined by taking as point set $\mathcal{P}$ the point set of $\mathcal{H}$ and as line set $\mathcal{L}$ the set of lines of $\mathcal{H}$ minus all the lines concurrent with a given line

L, is a dual partial quadrangle embedded in $\mathrm{PG}(3, q^2)$. One can prove that the dual of this geometry is isomorphic to $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an elliptic quadric. It is not known whether this is the only proper dual partial quadrangle embedded in a projective space.

A complete classification of all proper semipartial geometries embedded in affine spaces is still open. However the problem is solved by De Clerck and Thas [47] for dimensions 2 and 3.

Theorem 4.7.3
If $\mathcal{S}$ is a proper affine $\mathrm{spg}(s, t, \alpha, \mu)$ with ambient space $\mathrm{AG}(n, s+1)$, then $n \geq 3$. If $n = 3$ then $\mathcal{S}$ is either the pentagon embedded in $\mathrm{AG}(3, 2)$ (trivial case) or $\mathcal{S}$ is a linear representation and $\mathcal{S} = T_2^*(\mathcal{U})$ or $\mathcal{S} = T_2^*(\mathcal{B})$ (hence $s+1$ is a square).

4.8 Copolar Spaces

A *copolar space* (see [62]) is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ such that for each anti-flag (x, L), the incidence number $\alpha(x, L)$ equals 0 or $|L| - 1$.
It is easily seen that the copolar spaces of order (1,t) are precisely those graphs which contain no triangles. A copolar space of order (2,t) is better known as a *cotriangle space*.
The copolar space $\mathcal{S}$ is called *indecomposable* if and only if $\mathcal{S}$ is not the union of two or more copolar spaces on disjoint point sets. A *reduced* copolar space is an indecomposable copolar space such that for all vertices x and y in the point graph $\Gamma(\mathcal{S})$, $\Gamma(x) = \Gamma(y)$ implies $x = y$.
Remark that a semipartial geometry with parameters s, t, $\alpha = s$ is indeed a copolar space of order (s, t). Of course the dual of a net is a copolar space, and since there is no hope to classify them, we assume from now on that there exists at least one anti-flag (x, L) such that $\alpha(x, L) = 0$.
In [62] the finite reduced copolar spaces of order (s, t), $s \geq 2$, are classified up to isomorphism. It turns out that a reduced copolar space of order (s, t), $s \geq 2$, is a (proper) semipartial geometry.
We summarize the results in the next theorem.

Theorem 4.8.1 ([62])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, is a finite reduced copolar space of order (s, t), $s \geq 2$, then $\mathcal{S}$ is isomorphic to one of the following semipartial geometries:

1. $\overline{M(k)}$, $k \in \{2, 3, 7, 57\}$,
2. $U_{2,3}(n)$, $n > 4$,
3. $\overline{W(2n+1, q)}$,
4. $NQ^{\pm}(2n-1, 2)$.

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