Applying algebraic graph theory to some problems in incidence geometry

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(includes joint work with Valentina Pepe and Leo Storme)

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Outline

- Problems in incidence geometry and link with graph theory
- General (algebraic) graph theory
- Some problems in dual polar graphs attacked with algebraic combinatorics

- We study *incidence geometries* (finite or infinite).
- An *incidence geometry of rank n* is a pair $(\{P_1, \ldots, P_n\}, \mathrm{I})$ with n disjoint sets of objects $P_1, \ldots, P_n$,
 and a symmetric incidence relation I between objects of $\neq$ type.
- Objects are often referred to as *points*, *lines*, *planes*,....
- Plenty of examples: projective spaces, polar spaces, generalized polygons, unitals, inversive planes,....
- Many of these incidence structures are *spherical buildings* (Tits) (related to Coxeter groups)

The projective space $\mathrm{PG}(n, q)$

- We build a rank n incidence geometry from vector space $V(n+1, q)$.
- Objects in P_i are i-dimensional subspaces (or i-spaces), with $i \in \{1, \ldots, n\}$.
- Incidence is just symmetrized inclusion.

Example of problem in $\mathrm{PG}(n, q)$

A *partial spread of lines*: a subset of the 2-spaces in P_2,
all pairwise intersecting trivially (i.e. not incident with a common 1-space).
What is the maximum size for partial spreads of lines?

Constructing classical polar spaces

- Consider $V(n, q)$ and a non-singular quadratic, symmetric, alternating of hermitian form f.
- A subspace is *isotropic* if f vanishes on it.
- Let P_i be set of isotropic i-spaces.
- Incidence is just symmetrized inclusion I.
- This incidence structure is a *classical polar space*, with rank maximal dimension of isotropic subspaces.

Axiomatization

- Axioms for polar spaces exist.
- If rank is ≥ 3: finite polar space must be classical (Tits)

In a polar space of rank d:
one can study 1-spaces, 2-spaces,.... d-spaces (or *maximals*).

Example of problem for 1-spaces in polar spaces

- 2 distinct isotropic 1-spaces or *points* are either on unique isotropic 2-space or *line* (collinear) or on none.
- A *partial ovoid*: a set of points with no 2 collinear.
- What is the maximum size for a partial ovoid?

Graphs

A *graph* Γ consists of:

- a finite set Ω: the *vertices*
- a set E of pairs of vertices: the *edges*.

x and y are *adjacent* if $\{x, y\}$ is an edge of Γ.

Cliques and cocliques in graphs

- Clique: subset of vertices, all mutually adjacent.
- Coclique: subset of vertices, no two adjacent.

We can now translate some of our problems.

Projective spaces

Grassmann graph $J_q(n+1,2)$ in $V(n+1,q)$:

- 2-spaces as vertices
- x and y are adjacent if $dim(x \cap y) = 1$.

Finding partial spreads of lines in $\mathrm{PG}(n,q)$
$=$finding cocliques in $J_q(n+1,2)$.

Polar spaces

(Remember: polar space consists of isotropic subspaces in $V(n, q)$ with respect to form f.)

Polar graph:

- vertices: isotropic 1-spaces or points
- x and y are adjacent if x and y are collinear (i.e. span isotropic 2-space)

Finding partial ovoids in polar space = finding cocliques of polar graph.

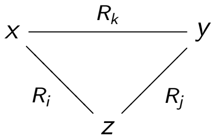
Geometries $\Longrightarrow$ Nice graphs for graph theory people!

Graph theory $\Longrightarrow$ Solutions for problems of geometry people!

Association schemes

$(\Omega, \{R_0, \dots, R_D\})$ is an association scheme if:

- $\{R_0, \dots, R_D\}$ partitions $\Omega \times \Omega$,
- R_0 is identity relation,
- $(\omega_1, \omega_2) \in R_i \iff (\omega_2, \omega_1) \in R_i$,
- there are *intersection numbers* p_{ij}^k:
  if $(x, y) \in R_k$, the number of elements z in Ω
  for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is p_{ij}^k.

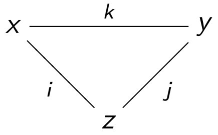


Consider graph $\Gamma = (\Omega, E)$.

- A *path of length k* is a sequence $(x_0, \ldots, x_k)$ with every 2 successive vertices adjacent.
- Distance between 2 vertices x, y: length of shortest path $(x, \ldots, y)$. (denoted by $d(x, y)$).
- Diameter: maximum distance in graph.
- We define R_i as $\{(x, y) \in \Omega \times \Omega | d(x, y) = i\}$.

Distance-regular graphs

A graph $\Gamma = (\Omega, E)$ is *distance-regular* if $(\Omega, \{R_0, \ldots, R_d\})$ is association scheme.
So number of z with $d(x, z) = i, d(y, z) = j$ only depends on i, j, k.



Regularity of graphs

- *Neighbours of vertex* x: all vertices adjacent to x
- A graph is *regular with valency* k if all vertices have k neighbours.
- In particular: distance-regular graphs are regular!

Consider distance-regular graph Γ with vertex set Ω and diameter d:

- Define $(|\Omega| \times |\Omega|)$-matrices $A_0, \ldots, A_d$:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases} .$$

- *Eigenvalues of* Γ: the $d+1$ eigenvalues of A_1.
- $\mathbb{R}^\Omega$ decomposes orthogonally into $d+1$ real eigenspaces of A_1:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d.$$

- These V_j are in fact eigenspaces for all A_i!
- Orthogonal projection $E_j : \mathbb{R}^\Omega \to V_j$ can be written as linear combination of $A_0, \ldots, A_d$. (these projections are *minimal idempotents*)

Characteristic vectors

For any subset of vertices S,
the *characteristic vector* $\chi_S \in \mathbb{R}^\Omega$ is $(0, 1, 0, 1, 1, \ldots)^T$
with $(\chi_S)_\omega = 1$ if $\omega \in S$ and $(\chi_S)_\omega = 0$ if $\omega \notin S$.

Linear programming bounds (Delsarte)

- Any minimal idempotent E_j is positive semidefinite: $v^T E_j v \geq 0, \forall v \in \mathbb{R}^\Omega$.
- Rewriting E_j as $\lambda_0^* A_0 + \ldots + \lambda_d^* A_d$ in $(\chi_S)^T E_j \chi_S$ can thus yield powerful information on subset S.

Metric and cometric schemes

- Association schemes $(\Omega, \{R_0, \dots, R_d\})$ with R_i i-distance relation of distance-regular graph are *metric* or *P-polynomial*.
- Dually, if the minimal idempotents $E_0, \dots, E_d$ can be ordered in a *nice way*, we call it *cometric* or *Q-polynomial*.

Remember: classical polar space of rank d
consists of isotropic subspaces in $V(n,q)$ w.r.t form f,
with d maximal dimension of isotropic subspaces.

Dual polar graph from classical polar space

- Vertices: isotropic d-spaces (or maximals)
- x and y are adjacent if $dim(x \cap y) = d - 1$

Distance-regularity of dual polar graph

- Geometric interpretation of distance between vertices:

$$d(x,y) = i \Longleftrightarrow dim(x,y) = d - i.$$

- Dual polar graph is distance-regular with diameter d!

Size: $(q^e+1)\ldots(q^{d-1}.q^e+1)$,
Valency: $q^e(q^d-1)/(q-1)$,
with q a prime power and e depending on type.

Types of dual polar graphs

			e
hyperbolic	$Q^+(2d-1,q)$	$D_d(q)$	0
unitary	$H(2d-1,q^2)$	${}^2A_{2d-1}(q)$	1/2
parabolic	$Q(2d,q)$	$B_d(q)$	1
symplectic	$W(2d-1,q)$	$C_d(q)$	1
unitary	$H(2d,q^2)$	${}^2A_{2d}(q)$	3/2
elliptic	$Q^-(2d+1,q)$	${}^2D_{d+1}(q)$	2

(Watch out for confusion between q and q^2 in unitary case...)

Remember: classical polar space of rank d consists of isotropic 1-spaces up to d-spaces.

Definition of partial spreads

- A partial spread is a set of isotropic d-spaces, all pairwise intersecting trivially.
- $(q^d - 1)/(q - 1)$ 1-spaces in each maximal, and $(q^{d-1}.q^e + 1)(q^d - 1)/(q - 1)$ in full polar space.
- So $q^{d-1}.q^e + 1$ is upper bound for partial spreads (in case of equality: *spread*)

Graph-theoretic approach

A partial spread is a set of vertices in the dual polar graph, all at maximal distance d from each other!

Applying linear programming bound

- If S is partial spread, then $\chi_S A_0 \chi_S = |S|, \chi_S A_d \chi_S = |S|(|S|-1)$ and $\chi_S A_i \chi_S = 0, \forall i \in \{1, \ldots, d-1\}$.
- Dual polar graph is *regular near* $2d$*-gon*, so

$$M = A_0 - \frac{A_1}{q^e} + \frac{A_2}{q^{2e}} - \ldots + \frac{(-1)^d A_d}{q^{de}}$$

  is minimal idempotent up to positive scalar and thus positive semidefinite.
- So $0 \leq (\chi_S)^T M \chi_S = |S| + \frac{(-1)^d |S|(|S|-1)}{q^{de}}$.
- If d is odd $\implies |S| \leq q^{de} + 1$.
- This beats spread bound $q^{d-1}.q^e + 1$ if $e \in \{0, \frac{1}{2}\}$, not if $e \in \{1, \frac{3}{2}, 2\}$.

Remember: partial spreads consist of vertices of dual polar graph, all at maximal distance d!

Partial spreads for $e = 1/2$ (the unitary dual polar graph ${}^2A_{2d-1}(q)$)

- Partial spreads in ${}^2A_{2d-1}(q)$ have size less than $(q^2)^{d-1}.q + 1$ (so no spreads) (Thas, 1990)
- Partial spreads of size $q^d + 1$ always exist in ${}^2A_{2d-1}(q)$ (Aguglia-Cossidente-Ebert, 2001)
- For $d = 3$: $q^d + 1$ is the maximum size (De Beule-Metsch, 2007)
- For all odd d: $q^d + 1$ is the maximum size (linear programming, 2009)

Partial spreads S in ${}^2A_{2d-1}(q)$ attaining bound q^d+1

- If $|S| = q^d + 1$, then every vertex adjacent to some element of S is at distance $d-1$ from q^{d-1} elements of S,
  and at distance d from $q^d - q^{d-1}$ elements of S.
  (we say S is a 1-regular code)
- This can be proved by elementary means using *variance trick*.
- Keeping $|S|$ variable in that proof actually yields $|S| \leq q^d + 1$ as well!

We now treat the converse problem.
(joint work with Valentina Pepe and Leo Storme)

The original Erdős-Ko-Rado-theorem (1961)

- An EKR-family is a family S of k-sets in a set of size n, all pairwise intersecting non-trivially.
- If $n \geq 2k$ then $|S| \leq \binom{n-1}{k-1}$.
- If $n > 2k$, equality $\iff$ S consists of all k-sets through fixed element.

Erdős-Ko-Rado-problem in dual polar graphs

- An EKR-family in a dual polar graph is a family of maximals (isotropic d-spaces) all pariwise intersecting non-trivially (so in at least 1-space).
- How big can they be?
- Is taking all maximals through fixed 1-space (unique?) optimal construction?

Remember about dual polar graph:

- vertices: isotropic d-spaces (or maximals) w.r.t form on vector space,
- x and y are at maximal distance d iff they intersect trivially.

So EKR-families in dual polar graphs:
families of vertices all at distance less than d!

Relevant works

- Brouwer-Godsil-Koolen-Martin: Width and dual width of subsets in polynomial association schemes (2004)
- Tanaka: Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs (2007)

So EKR-families in dual polar graphs= cocliques of R_d

Theorem (Hoffman)

In a regular graph $\Gamma = (\Omega, E)$ with valency k and minimal eigenvalue λ, a coclique S satisfies:

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda}.$$

If equality holds: $\chi_S - \frac{|S|}{|\Omega|}\chi_\Omega$ in eigenspace λ.

So we must study eigenvalues and eigenspaces of A_d!

Eigenspaces of dual polar graph Γ and A_d

- Define $V_0 = \langle \chi_\Omega \rangle$ and construct V_i $(1 \leq i \leq d)$ such that $V_0 \perp \ldots \perp V_i$ is image of incidence matrix from i-spaces to d-spaces.
- $$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d$$
  is orthogonal decomposition into eigenspaces of dual polar graph, with cometric ordering.
- These V_i are also eigenspaces for A_d..... or subspaces of eigenspaces!

What happens in most dual polar graphs for EKR-families S

- Coclique bound $\frac{|\Omega|}{1-k/\lambda}$ is number of maximals through fixed 1-space.
- Equality forces χ_S in $V_0 \perp V_1 \perp \cancel{V_2} \perp \ldots \perp \cancel{V_d}$.
- Tanaka's results (2007): equality $\iff$
  S is all maximals through fixed isotropic 1-space

We have to consider the different types of dual polar graphs separately!

			e
hyperbolic	$Q^+(2d-1,q)$	$D_d(q)$	0
unitary	$H(2d-1,q^2)$	$^2A_{2d-1}(q)$	1/2
parabolic	$Q(2d,q)$	$B_d(q)$	1
symplectic	$W(2d-1,q)$	$C_d(q)$	1
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Exception: bipartite $D_d(q)$ for odd d ($e = 0$)

- Coclique bound is $|\Omega|/2$
- Equality if and only if S is one of the bipartite halves.

Exception: bipartite $D_d(q)$ for even d ($e = 0$)

- Coclique bound on $|S|$ is number of maximals through fixed 1-space, equality: $\chi_S \in (V_0 \perp V_1 \perp \cancel{V_2} \ldots \perp \cancel{V_{d-2}} \perp V_{d-1} \perp \cancel{V_d})$.
- Solution: work in each bipartite half (half dual polar graph) and slightly adjust Tanaka's arguments to obtain classification.

Exception: $B_d(q)$ for odd d ($e = 1$)

- Coclique bound on $|S|$ number of maximals through fixed 1-space, equality: $\chi_S \in (V_0 \perp V_1 \perp \not{V_2} \perp \ldots \perp \not{V_{d-1}} \perp V_d)$.
- Solution : embed $B_d(q)$ in one half of $D_{d+1}(q)$ and use previous classification.
- Answer: equality iff S is all maximals through 1-space, half of an embedded $D_d(q)$, or extra construction for $d = 3$.

Exception: $C_d(q)$ for odd d and even q ($e = 1$)

$C_d(q)$ is isomorphic to $B_d(q)$!

Exception: $C_d(q)$ for odd d and odd q ($e = 1$)

- $C_d(q)$ is NOT isomorphic to $B_d(q)$..but has same parameters.
- Coclique bound on $|S|$ is number of maximals through fixed 1-space, but equality: $\chi_S \in (V_0 \perp V_1 \perp \not{V_2} \perp \ldots \perp \not{V_{d-1}} \perp V_d)$.
- Basic idea: if χ_S is orthogonal to more V_i, it is easier to control.
- V_d is kernel of incidence matrix between d-spaces (the vertices) and $(d-1)$-spaces
- Counting from point of view of $(d-1)$-space is easier as we can "filter" a bit (counting for d-spaces: *outer distribution* has rank 3...)
- Answer: EKR-families of maximum size are all vertices through 1-space, or one vertex and its neighbours for $d = 3$.

Exception: ${}^2A_{2d-1}(q)$ for odd d ($e = 1/2$)

- Coclique bound is $|S| \leq \frac{|\Omega|}{q^d+1}$, equality: $\chi_S \in (V_0 \perp \cancel{V_1} \ldots \perp \cancel{V_{d-1}} \perp V_d)$.
- Cannot be attained: S would be $(d-1)$-design: each $(d-1)$-space in exactly $(q+1)/(q^d+1)$ elements of S.
- Number of vertices through fixed 1-space: $\frac{|\Omega|}{q^{2d-1}+1}$.
- For $d = 3$, maximum size by taking one vertex and neighbours: $1 + q(q^4 + q^2 + 1)$:

$$\begin{array}{ccc} \text{all through 1-space} & < \text{maximum size} & < \text{eigenvalue bound} \\ (q+1)(q^3+1) & < 1 + q(q^4 + q^2 + 1) & < (q+1)(q^5+1). \end{array}$$

Some open problems

- Maximum size for partial spreads in ${}^2A_{2d-1}(q)$ for even d?
  (so all vertices at maximal distance d)
  (answer is at least $q^d + 1$)
- Maximum size for EKR-families in ${}^2A_{2d-1}(q)$ for odd $d \geq 5$?
  (so no 2 vertices at maximal distance d)

Thank you for your attention!

(Slides (and more) on `http://cage.ugent.be/~fvanhove`)