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Outline

- Introduction: distance-regular graphs, \(Q\)-polynomial, (regular) near \(2d\)-gons
- \(Q\)-polynomial near \(2d\)-gons: properties, classification
- Substructures in certain regular near \(2d\)-gons
Consider graph $\Gamma = (\Omega, E)$
($\Omega$: vertex set, $E$: set of edges (pairs of vertices))
($\Omega \neq \emptyset$, undirected, no loops or multiple edges)

- A walk of length $k$ is a sequence $(x_0, \ldots, x_k)$ with every 2 successive vertices adjacent.
- Distance between 2 vertices $x, y$ (denoted by $d(x, y)$): length of shortest walk $(x, \ldots, y)$.
- Diameter: maximum distance in graph.
- $\Gamma_i(x)$: set of vertices at distance $i$ from vertex $x$. 
**Definition**

A connected graph $\Gamma = (\Omega, E)$ is *distance-regular* if for $d(x, y) = k$ number of $z$ with $d(x, z) = i, d(y, z) = j$ is parameter $p_{ij}^k$ only depending on $i, j, k$.

![Diagram showing a triangle with vertices $x$, $y$, and $z$, and edges labeled $i$, $j$, and $k$.]
Equivalent definition

Connected graph $\Gamma$ is distance-regular if there are constants $a_i, b_i, c_i$, such that for $d(x, y) = i$

$$|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i, |\Gamma_i(x) \cap \Gamma_1(y)| = a_i, |\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i.$$ 

Some properties

- $c_1, \ldots, c_d$ and $b_0, \ldots, b_{d-1}$ already determine all parameters $p^k_{ij}$,
- number of vertices at distance $i$ from any vertex is constant $k_i$. 
Example: icosahedron

12 vertices, diameter $d = 3$, every vertex has 5 neighbours.

If $d(x, y) = 2$:

- $|\Gamma_1(x) \cap \Gamma_1(y)| = c_2 = 2$
- $|\Gamma_2(x) \cap \Gamma_1(y)| = a_2 = 2$
- $|\Gamma_3(x) \cap \Gamma_1(y)| = b_2 = 1$. 
**Definition**

Consider distance-regular graph $\Gamma$ with finite vertex set $\Omega$:

*Adjacency matrix* $A_i$: $(|\Omega| \times |\Omega|)$-matrix over $\mathbb{R}$:

$$(A_i)_{x,y} = \begin{cases} 
1 & \text{if } d(x, y) = i \\
0 & \text{if } d(x, y) \neq i 
\end{cases}.$$
Distance-regularity

**Definition**

Consider distance-regular graph \( \Gamma \) with finite vertex set \( \Omega \):

*Adjacency matrix* \( A_i \): \( (|\Omega| \times |\Omega|) \)-matrix over \( \mathbb{R} \):

\[
(A_i)_{x,y} = \begin{cases} 
1 & \text{if } d(x,y) = i \\
0 & \text{if } d(x,y) \neq i 
\end{cases}
\]

**Properties**

- \( A_i \) is symmetric,
- \( A_0 \) is identity matrix,
- \( A_0 + \cdots + A_d \) is all-one matrix,
- \( A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \).

**Definition**

Algebra generated by \( A_0, \ldots, A_d \) is *Bose-Mesner algebra*. 
Eigenvectors for distance-regular graph $\Gamma$

$\mathbb{R}^\Omega$ is real vector space with basis indexed by elements of finite vertex set $\Omega$.

$\mathbb{R}^\Omega$ uniquely decomposes as:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,$$

with every $v \neq 0 \in V_j$ an eigenvector of every relation $A_i$: $A_iv = \lambda_{ji}v$.

Idempotents

- Orthogonal projection $E_j$ onto $V_j$ is also in $\langle A_0, \ldots, A_d \rangle$.
- These minimal idempotents $E_j$ form second basis $\{E_0, \ldots, E_d\}$ for $\langle A_0, \ldots, A_d \rangle$. 
**Q-polynomial distance-regular graphs**

- Bose-Mesner algebra \( \langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle \) is also closed under entrywise multiplication “\( \circ \)”.  
- \( E_0, \ldots, E_d \) is a Q-polynomial or cometric ordering if every \( E_j \) is \( q_j(E_1) \) for some polynomial of degree \( j \) (entrywise multiplication!).  
- Q-polynomial ordering need not be unique, but is determined by \( E_1 \)!  
- Such ordering only depends on the \( p_{ij}^k \) (or: the \( b_i \) and \( c_i \)).  
- If \( \lambda \) is eigenvalue of \( \Gamma \) corresponding to \( E_1 \), we say \( \Gamma \) is Q-polynomial w.r.t to \( \lambda \).
**Q-polynomial distance-regular graphs**

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- Q-polynomial ordering need not be unique, but is determined by $E_1$!

- Such ordering only depends on the $p_{i,j}^k$ (or: the $b_i$ and $c_i$).

- If $\lambda$ is eigenvalue of $\Gamma$ corresponding to $E_1$, we say $\Gamma$ is Q-polynomial w.r.t to $\lambda$.

**Major problem**

Bannai-Ito (1984): classify all Q-polynomial distance-regular graphs (for large enough $d$)!
Definition

Suppose $S \subseteq \Omega$ is subset of vertices in $\Omega$. Its \textit{characteristic vector} $\chi_S \in \mathbb{R}^\Omega$ is the $(0, 1)$-vector with:

$$(\chi_S)_\omega = \begin{cases} 1 & \text{if } \omega \in S \\ 0 & \text{if } \omega \notin S \end{cases}.$$ 

Hence:

$$\chi_\emptyset = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \chi_\Omega = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$
Near 2\(d\)-gons: definition (Shult & Yanushka, 1980)

A graph \(\Gamma\) of diameter \(d \geq 2\) is a near 2\(d\)-gon when:

1. if \(x\) is adjacent to 2 vertices of clique of size 3 (not containing \(x\)):
   
   \[\implies x\ \text{is adjacent to all 3.}\]

2. \(\forall\) vertex \(x\) and \(\forall\) maximal clique \(\ell\), there is a unique vertex \(y \in \ell\) at minimal distance from \(x\) in the graph.
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   \[ \Rightarrow x \text{ is adjacent to all 3.} \]

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- Simple example: ordinary $2d$-gon!
- Near $2d$-gons with exactly 2 vertices in every maximal clique = bipartite graphs of diameter $d$
Near 2\(d\)-gons: definition (reminder)

A near 2\(d\)-gon is a graph of diameter \(d\) satisfying:

1. if \(x\) is adjacent to 2 vertices of clique of size 3 (not containing \(x\)):
   \[
   \Rightarrow \quad x \text{ is adjacent to all 3.}
   \]

2. \(\forall\) vertex \(x\) and \(\forall\) line \(\ell\), there is a unique vertex \(y \in \ell\) at minimal distance from \(x\) in the graph.

So 2 adjacent vertices are in unique maximal clique or line! \(\Rightarrow\) we draw cliques as **straight lines**.

Hence we can draw 2nd condition like this:
**Definition**

A near $2d$-gon is *regular* if it is distance-regular.

(i.e. if $d(x, y) = k$, then there are $p_{ij}^k$ vertices $z$ with $d(x, z) = i$, $d(z, y) = j$)

Frédéric Vanhove (UGent)
Definition

A near 2d-gon is regular if it is distance-regular.
(i.e. if \(d(x, y) = k\), then there are \(p_{ij}^k\) vertices \(z\) with \(d(x, z) = i, d(z, y) = j\))

- Then it has an order \((s, t)\), \(s, t \geq 1\):
  - \(s + 1\) vertices on each line, \(t + 1\) lines through each vertex.
- If \(d(x, y) = i\) then through \(y\):
  - \(t_i + 1\) lines with 1 point at distance \(i - 1\) from \(x\),
    and its \(s\) other points at distance \(i\) from \(x\),
  - \(t - t_i\) lines at distance \(i\) from \(x\),
  with \(c_i = t_i + 1, a_i = (s - 1)(t_i + 1), b_i = s(t - t_i)\).

\((s, t_2, \ldots, t_d = t)\) determines \(a_i, b_i, c_i\), all \(p_{ij}^k\) and Q-pol. property!
Example: generalized $2d$-gons of order $(s, t)$ (Tits, 1959)

A regular near $2d$-gon of order $(s, t)$ is a generalized $2d$-gon if

$$t_1 = \ldots = t_{d-1} = 0.$$ 

Properties:

- **Duality**: graph on lines is generalized $2d$-gon of order $(t, s)$.
- $d = 2$: regular near $4$-gons of order $(s, t)$
  - = generalized quadrangles of order $(s, t)$.
- Feit-Higman (1964): if $s, t \geq 2$ then $d \in \{2, 3, 4\}$. 
Example: Hamming graphs

Consider alphabet $X$ of size $q \geq 2$

- vertices: codewords in $X^d$
- adjacency: differing in exactly one position.

Regular near $2d$-gon with $s = q - 1$ and $c_i = t_i + 1 = i$. 
Example: Hamming graphs

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- vertices: codewords in $X^d$
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Regular near $2d$-gon with $s = q - 1$ and $c_i = t_i + 1 = i$.

Example: dual polar graphs

Consider vector space $V(n, q)$ with non-degenerate quadratic/alternating/Hermitian form of Witt index $d$.

- vertices: totally isotropic $d$-spaces
- adjacency: intersecting in $(d - 1)$-space

Regular near $2d$-gon with $s = q^e$ and $c_i = t_i + 1 = \frac{q^i - 1}{q - 1}$.
**Theorem**

If $\Gamma$ is a regular near $2d$-gon of order $(s, t)$, satisfying:

$$d \geq 4, s \geq 2, c_2 = t_2 + 1 \geq 3,$$

then $\Gamma$ is a Hamming graph or dual polar graph.

**Proof (history):**

Suppose $\Gamma$ is regular near $2d$-gon of order $(s,t)$.

**Eigenvalues of $\Gamma$**

- Largest eigenvalue: valency $s(t + 1)$.
  Eigenspace: spanned by all-one vector $\chi_{\Omega}$
- Smallest eigenvalue: $-(t + 1)$.
  Eigenspace: kernel incidence matrix between vertices and lines

Corresponding idempotent is (up to positive scalar):

$$M = \frac{A_0}{1} + \frac{A_1}{-s} + \ldots + \frac{A_d}{(-s)^d}.$$
Our goal is to classify $Q$-polynomial regular near $2d$-gons!
The case $d = 2$: here all regular near 4-gons are $Q$-polynomial!

**Theorem (Brouwer-Cohen-Neumaier, 1989)**

If $\Gamma$ is a regular near $2d$-gon of order $(s, t)$ with $d \geq 3$, $s \geq 2$, and is $Q$-polynomial with respect to $\theta$:

1. $\Gamma$ is a Hamming graph or dual polar graph, or
2. $\theta = -(t + 1)$ and $t_i + 1 = -\frac{(-s)^i - 1}{s + 1} \left(1 + \frac{s + t_2}{s - 1} \frac{(-s)^i - 1}{s + 1}\right)$.

(Both can in fact hold for the same graph!)
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**Theorem (Weng, 1997)**

If $\Gamma$ is a $Q$-polynomial regular near $2d$-gon of order $(s, t)$ with $d \geq 4, s \geq 2$ and $c_2 = t_2 + 1 \geq 2$, then $\Gamma$ is a Hamming graph or dual polar graph.

So if $s \geq 2$, we have to consider $d = 3$ or $d \geq 4, c_2 = 1$!
The case $d = 3$

**Theorem (Terwilliger, 1995)**

If $\Gamma$ is a regular near 6-gon of order $(s, t)$ with $s \geq 2$, then:

$$c_3 = t + 1 \leq (s^2 - s + 1)(t_2 + s + 1),$$

and equality holds iff $\Gamma$ is $Q$-polynomial w.r.t. $-(t+1)$, and then $\forall x, y$ with $d(x, y) = 3$:

$$M \left( s(t_2 + s + 1) \left( \chi_{\{x\}} - \chi_{\{y\}} \right) + (\chi_X - \chi_Y) \right) = 0,$$

with $X = \Gamma_1(x) \cap \Gamma_2(y)$ and $Y = \Gamma_1(y) \cap \Gamma_2(x)$. 
The case \( d = 3 \)

In a regular near hexagon with parameters \((s, t_2, t)\) with \(s \geq 2\),
\((t + 1) = (s^2 - s + 1)(t_2 + s + 1)\),
if \(d(x, z) = 2\) and \(d(x, y) = d(y, z) = 3\), then:

\[
|\Gamma_2(x) \cap \Gamma_2(z) \cap \Gamma_1(y)| = s^2 + s \left( t_2 + 1 - |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| \right) + t_2 + 1.
\]

This follows by using \((\chi_{\{z\}})^T A_i \chi_S = |\Gamma_i(z) \cap S|\) for any subset \(S\), expressing \(M\) in terms of \(A_i\), and working out

\[
(\chi_{\{z\}})^T M \left( s(t_2 + s + 1) \left( \chi_{\{x\}} - \chi_{\{y\}} \right) + (\chi_X - \chi_Y) \right) = 0.
\]
Introduction

Q-polynomial near 2d-gons

Substructures

The case \( d = 3 \)

Definition

In a regular near 2d-gon, a \textit{quad} is a subset of vertices \( Q \):

- Every line intersecting \( Q \) in two points is in \( Q \) (subspace)
- if \( d(x, y) + d(y, z) = d(x, z) \) with \( x, z \in Q \) \( \implies \) \( y \in Q \) (convexity),
- induced subgraph has diameter two,
- no element of \( Q \) is adjacent to all others.

Theorem (Shult & Yanushka, 1980)

In regular near 2d-gon of order \((s, t)\) with \( s \geq 2, c_2 = t_2 + 1 \geq 2 \), every \( x, y \) with \( d(x, y) = 2 \) are in a unique quad, which must induce a generalized quadrangle of order \((s, t_2)\).
Example: quads in Hamming graphs

- $(0,0,0,\ldots,0)$ and $(1,1,0,\ldots,0)$ are at distance 2.
- Quad containing them consists of words $(x_1, x_2, 0, \ldots, 0)$
- Induced subgraph is isomorphic to Hamming graph of diameter 2.
Example: quads in Hamming graphs

- $(0,0,0,\ldots,0)$ and $(1,1,0,\ldots,0)$ are at distance 2.
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Example: quads in dual polar graphs

- Two isotropic $d$-dim. spaces are at distance 2 when intersecting in $(d-2)$-space.
- Quad containing them consists of vertices through this $(d-2)$-space.
- Induced subgraph is isomorphic to dual polar graph of diameter 2.
Suppose $Q$ is a quad in a regular near hexagon with parameters $(s, t_2, t), s \geq 2$.

If $x \notin Q$, then one can prove that either:

- $x$ is classical w.r.t. $Q$: $x \in \Gamma_1(Q)$, and
  \[ \exists! \pi_Q(x) \in \Gamma_1(x) \cap Q \text{ with } \]
  \[ d(x, y) = d(x, \pi_Q(x)) + d(\pi_Q(x), y), \forall y \in Q, \]

- or $x$ is ovoidal w.r.t. $Q$: $x \in \Gamma_2(Q)$, and $\Gamma_2(x) \cap Q$ is a (subtended) ovoid in $Q$ (i.e. every line in $Q$ contains one of its vertices).
Lines with respect to quads (if $s \geq 2$)

Suppose $\ell$ is a line, and $Q$ a quad with $\ell \subseteq \Gamma_2(Q)$. One can prove: $\ell \subseteq \Gamma_2(Q)$ and the subtended ovoids partition $Q$. 

\[
\begin{array}{c}
\text{Lines with respect to quads (if } s \geq 2) \\
\text{Suppose } \ell \text{ is a line, and } Q \text{ a quad with } \ell \subseteq \Gamma_2(Q). \text{ One can prove: } \ell \subseteq \Gamma_2(Q) \text{ and the subtended ovoids partition } Q.
\end{array}
\]
Suppose \( \Gamma \) is regular near hexagon with parameters \((s, t_2, t)\), 
\( s \geq 2 \) and 
\( t + 1 = (s^2 - s + 1)(t_2 + s + 1) \).

\( \implies \) Higman’s bound for generalized quadrangles (1971):
\( 0 \leq t_2 \leq s^2 \) (in fact: \( t_2 = s^2 \) or \( 0 \leq t_2 \leq s^2 - s \)).
Suppose $\Gamma$ is regular near hexagon with parameters $(s, t_2, t)$, $s \geq 2$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$. 

$\implies$ Higman’s bound for generalized quadrangles (1971): 
$0 \leq t_2 \leq s^2$ (in fact: $t_2 = s^2$ or $0 \leq t_2 \leq s^2 - s$).

**Extremal case: $t_2 = 0$**

- If $t_2 = 0$, then $\Gamma$ is generalized hexagon of order $(s, s^3)$.
- Tits (1959): construction for prime powers $s$.

**Extremal case: $t_2 = s^2$**

If $t_2 = s^2$, then $s$ is a prime power and $\Gamma \cong^2 A_5(s)$
(unitary dual polar graph, which has two $Q$-polynomial orderings!)
Lemma

Suppose $\Gamma$ is a regular near hexagon with parameters $(s, t_2, t)$, $s \geq 2$, $t_2 \geq 1$, $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$. If $u, v, w \in Q$ are mutually at distance 2 in quad $Q$, then the number of vertices $p \in \Gamma_2(Q)$ subtending an ovoid of $Q$ containing $u, v, w$ is:

$$\frac{s(t - t_2)(s + t_2)(s + 1 - \beta)}{t_2 + 1},$$

where $\beta = |\Gamma_1(u) \cap \Gamma_1(v) \cap \Gamma_1(w)|$, the number of “centers”.

![Diagram of a near hexagon with labeled vertices and an ovoid](image-url)
Suppose $\Gamma$ is a regular near hexagon with parameters $(s, t_2, t), s \geq 2, t_2 \geq 1, t + 1 = (s^2 - s + 1)(t_2 + s + 1)$.

Consider $x, y$ in a quad $Q$ with $d(x, y) = 2$.

For every $\beta \in \{0, \ldots, t_2 + 1\}$, let $T_\beta$ be number of $z \in Q$, not adjacent to $x$ or $y$, such that $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| = \beta$.

\[
\sum_{\beta} T_\beta = s^2 t_2 - st_2 - s + t_2
\]

\[
\sum_{\beta} T_\beta \beta = s(t_2 + 1)(t_2 - 1)
\]

\[
\sum_{\beta} T_\beta \beta(\beta - 1) = t_2(t_2 + 1)(t_2 - 1)
\]

Note that $T_\beta = 0$ if $\frac{s(t-t_2)(s+t_2)(s+1-\beta)}{t_2+1}$ is not a non-negative integer!
Theorem (De Bruyn & V., 201?)

There are no (necessarily $Q$-polynomial) regular near hexagons with parameters $(s, t, t) \in \{(8, 4, 740), (92, 64, 1314560), (95, 19, 1027064), (105, 147, 2763012)\}.

Proof.

In each case, we have $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$. Consider $x, y$ in a quad $Q$, with $d(x, y) = 2$. The previous system of equations in the $T_\beta$ has no solutions.
Problem: $t_2 = 1$

Which regular near hexagons have parameters $(s, t_2, t)$, $s \geq 2$, $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$ (and are thus $Q$-pol. w.r.t $-(t + 1)$) if $t_2 = 1$?
The case $d = 3$

**Problem:** $t_2 = 1$

Which regular near hexagons have parameters $(s, t_2, t)$, $s \geq 2$, $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$ (and are thus $Q$-pol. w.r.t $-(t + 1)$) if $t_2 = 1$?

- Here $s \in \{2, 3, 4, 5, 8, 11, 18, 23, 32, 53, 158\}$.

- Quads have order $(s, 1)$, so grids:

- $(s, t_2, t) = (2, 1, 11)$: unique such regular near hexagon (Brouwer, 1982).

- What about $(s, t_2, t) = (3, 1, 34)$?
The case $d = 3$

Suppose $\Gamma$ is regular near hexagon with $(s, t_2, t) = (3, 1, 34)$. Suppose $Q$ is quad, and $x, y$ are on line $\ell$ in $\Gamma_2(Q)$. Either:

- "double transposition case":

- "cycle case":
Suppose \( \Gamma \) is regular near hexagon with \((s, t_2, t) = (3, 1, 34)\).
Suppose \( Q \) is quad, and suppose \( x \) is on line \( \ell \) in \( \Gamma_2(Q) \). Either:

1. \( \times \) double transposition
   - (orange),
2. \( \times \) cycle (green and blue):
   - \( \times \) double transposition,
   - no cycles:
The case $d = 3$

**Theorem (De Bruyn & V., 201?)**

*There are no (necessarily $Q$-polynomial) regular near hexagons with $(s, t_2, t) = (3, 1, 34)$.*

**Proof.**

Suppose $Q$ is a quad, and $x \in \Gamma_2(Q)$.
Consider vertices on lines through $x$ in $\Gamma_2(Q)$.
Counting w.r.t. triples: $21 \times$ double transposition, $60 \times$ cycle.
But only 27 lines through $x$ in $\Gamma_2(Q)$, all containing at most 2 vertices in cycle case!
What about $Q$-polynomial regular near $2d$-gons with $s \geq 2$, $d \geq 4$ and $c_2 = t_2 + 1 = 1$?

**Theorem (Hiraki, 1999)**

If $\Gamma$ is a regular near $2d$-gon with parameters $(s, t_2, \ldots, t_d)$ with $s \geq 2$ and $0 = t_1 = \ldots = t_r < t_{r+1}$, where $d \geq 2r + 1$, then there is a regular near $2q$-gon with parameters $(s, t_2, \ldots, t_q)$ as convex subgraph if $r + 1 \leq q \leq d - r$. 
The case \( d \geq 4 \) and \( c_2 = 1 \)

**Theorem (De Bruyn & V,201?)**

There is no (necessarily \( Q \)-polynomial) regular near \( 2d \)-gon of order \( (s,t) \), \( s \geq 2 \), with \( t_i + 1 = -\frac{(-s)^i - 1}{s+1} \left( 1 + \frac{s}{s-1} \frac{(-s)^i - 1}{s+1} \right) \) if \( d \geq 4 \).

**Proof**

- If \( d = 4 \), then only \( s = 2 \) is feasible but \( (s,t_2,t_3,t) = (2,0,8,24) \) was ruled out by De Bruyn (2010).
- If \( d = 5 \), then only \( s = 2 \) is feasible, and Hiraki implies existence sub-near 6-gon with \( (s,t_2,t_3) = (2,0,8) \). Such subset \( S \) doesn’t survive LP-bound \( \forall \) idempotent \( E_j \):

\[
(\chi_S)^T E_j \chi_S \geq 0 \iff (\chi_S)^T (Q_0 j A_0 + \ldots Q_5 j A_5) \chi_S \geq 0
\]

\[
\Rightarrow Q_0 j k_0 + Q_1 j k_1 + Q_2 j k_2 + Q_3 j k_3 \geq 0.
\]

- If \( d \geq 6 \), then Hiraki implies existence sub-near 8-gon with impossible parameters (due to above). □
Remember:

**Definition of near $2d$-gons**

A near $2d$-gon is a graph of diameter $d$ satisfying:

1. if $x$ is adjacent to 2 vertices of clique of size 3 (not containing $x$): $x$ is adjacent to all 3.

2. $\forall$ vertex $x$ and $\forall$ maximal clique $\ell$, there is a unique vertex $y \in \ell$ at minimal distance from $x$ in the graph.

**Definition of regular near $2d$-gons**

A near $2d$-gon is *regular* if it is distance-regular.

**Definition of the $Q$-polynomial property**

A distance-regular graph is *$Q$-polynomial* if for some ordering $E_0, E_1, \ldots, E_d$, every idempotent $E_j$ is $q_j(E_1)$ using entrywise multiplication.
Introduction

The case $d \geq 4$ and $c_2 = 1$

Theorem (overview)

If $\Gamma$ is a $Q$-polynomial regular near 2$d$-gon $S$ of order $(s, t)$ with $d \geq 3$, $s \geq 2$, then precisely one of the following cases occurs:

1. $\Gamma$ is a Hamming graph.
2. $\Gamma$ is a dual polar graph.
3. $\Gamma$ is a generalized hexagon of order $(s, s^3)$, $s \geq 2$.
4. $\Gamma$ is near hexagon from the extended ternary Golay code.
5. $\Gamma$ is near hexagon from Steiner system $S(5, 8, 24)$.
6. $\Gamma$ is a regular near hexagon with parameters $(s, t_2, t)$, with $s \geq 4$, $1 \leq t_2 \leq s^2 - s$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$.

In the last case, $(s, t_2) \in \{(4, 1), (5, 1), (5, 5), (8, 1), (10, 5), (11, 1), (18, 1), (23, 1), (32, 1), (33, 11), (53, 1), (129, 27), (158, 1), (221, 169), (285, 75), (558, 216), (2093, 91)\}$ or $s > 10^7$. 
Definition

In regular near $2d$-gon of order $(s, t)$, an *ovoid* is a set of vertices $S$ such that each line contains one element, i.e. $S$ is a coclique of size $|\Omega|/(s+1)$.

So distance 1 is impossible within $S$, while distances 2, 3 are possible in regular near hexagons!
Definition

In regular near $2d$-gon of order $(s, t)$, an ovoid is set of vertices $S$ such that each line contains one element, i.e. $S$ is a coclique of size $|\Omega|/(s+1)$.

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Theorem (De Bruyn & V., 2013)

Suppose $S$ is finite regular near hexagon of order $(s, t), s \geq 2$, with $c_3 = t + 1$ attaining upper bound $(s^2 - s + 1)(t_2 + s + 1)$. If $S$ is ovoid, then $S$ induces in distance-2-graph an srg($v, k, \lambda, \mu$):

\[
\begin{align*}
    v &= (s^2 - s + 1) \left(1 + s + t_2 s + s^2\right) \left(-s^2 + 1 + t_2 + s^4 + t_2 s^3 - t_2 s^2\right)/(t_2 + 1) \\
    k &= (s^2 - s + 1) \left(s + t_2 + 1\right) \left(s^3 + t_2 s^2 - t_2 s + t_2\right)s/(t_2 + 1) \\
    \lambda &= (s^2 - s + 1) \left(s^4 + 2 t_2 s^3 + 2 s^3 + t_2^2 s^2 + 2 t_2 s^2 - s - t_2 s - 1 - t_2\right)/(t_2 + 1) \\
    \mu &= (s^2 - s + 1) \left(s^3 + t_2 s^2 - s - t_2 s + t_2 + 1\right) \left(s + t_2 + 1\right)/(t_2 + 1)
\end{align*}
\]
Theorem (reminder)

Suppose $S$ is finite regular near hexagon of order $(s, t), s \geq 2,$ with $c_3 = t + 1$ attaining upper bound $(s^2 - s + 1)(t^2 + s + 1)$. If $S$ is ovoid, then $S$ induces in distance-2-graph an srg$(v, k, \lambda, \mu)$.

Proof again relies on Terwilliger’s identity and description eigenspace for smallest eigenvalue.
Theorem (reminder)

Suppose $S$ is finite regular near hexagon of order $(s, t), s \geq 2,$
with $c_3 = t + 1$ attaining upper bound $(s^2 - s + 1)(t_2 + s + 1)$.
If $S$ is ovoid, then $S$ induces in distance-2-graph an srg($v, k, \lambda, \mu$).

- Proof again relies on Terwilliger’s identity
  and description eigenspace for smallest eigenvalue.
- srg also follows from Brouwer-Godsil-Koolen-Martin (2003),
  since here ovoids have “degree” $s = 2$ and “dual width” $w^* = 1$. 
Consider ovoid in finite regular near hexagon $\Gamma$ of order $(s, t), s \geq 2,$ with $c_3 = t + 1 = (s^2 - s + 1)(s + t_2 + 1)$. Only 3 known feasible parameter sets:

- $s = 2, t_2 = 1, t = 11$:  
  $\Gamma$ is near hexagon from extended ternary Golay code with 729 vertices,  
  $\text{srg}(243, 132, 81, 60)$ is $Delsarte graph$ (1972).

- $s = 2, t_2 = 2, t = 14$:  
  $\Gamma$ is near hexagon from $5 - (24, 8, 1)$ design with 759 vertices,  
  $\text{srg}(253, 140, 87, 65)$ is graph on 253 blocks of $4 - (23, 7, 1)$ design.

- $s = 4, t_2 = 1, t = 77$:  
  $\Gamma$ would have 235625 vertices,  
  ovoid would yield $\text{srg}(47125, 12012, 3575, 2886)$.  

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But on the other hand:

**Corollary (De Bruyn & V., 2013)**

*No generalized hexagon of order \((s, s^3), s \geq 2,\) can have an ovoid.*
Selected references


Thank you for your attention!

Slides (and more) on http://cage.ugent.be/~fvanhove